A NOTE ON ENGEL'S THEOREM

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Jacobson [1, Ch.II §§1-3] gives a development of Engel's Theorem in which the assumption that the class T of linear transformations considered be a Lie algebra is weakened: The conditions that T be closed under addition and scalar multiplication are dropped, and the condition of closure under Lie bracket is weakened to an assumption that for every $s, t \in T$, T also contains an element, denoted $s \times t$, of the form $st + \alpha ts$, where α is a scalar depending on s and t. Thus that development simultaneously includes the corresponding results for Jordan algebras, associative algebras, and semigroups of linear transformations, as well as Lie algebras.

In the following development we allow the still more general expression

$$(1) \qquad s \times t = st + as$$

where a is an arbitrary member of the subalgebra of linear transformations (with 1) generated by s and t. Our proof is similar to but shorter than that of [1].

We shall use the following key observation. If $s \times t$ is as in (1), and u is some vector annihilated by the linear transformation s, then $(s \times t)u = stu$, since the last term of (1) kills u. Hence more generally

$$(2) \qquad \text{If } s_1, \ldots, s_r \in \operatorname{Ann}(u), \text{ then } (s_1 \times (\ldots \times (s_r \times t) \ldots))u = s_1 \ldots s_r tu.$$

In what follows, capital letters will denote <u>sets</u> of linear transformations, of vectors, etc.. When we write an expression like AB, this will stand for $\{ab \mid a \in A, b \in B\}$; similarly $A \times B$ will mean $\{a \times b \mid a \in A, b \in B\}$; A^2 will mean AA.

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__Theorem.__ Let T be a set of endomorphisms of a finite-dimensional vector space V over a field k. Suppose that T is closed under an operation $\times$ as in (1), and that all elements of T are nilpotent. Then $T^n = 0$ for some n.

__Proof.__ The result is trivial for $V = 0$, so assume inductively that $V \neq 0$, and that the Theorem is true for all vector spaces of smaller dimension than V.

Let $U \subseteq V$ be a __nonzero__ subspace of V, which is the intersection of the annihilators in V of some family of elements of T, and which is __minimal__ for this property. Let S denote the set of elements of T which annihilate U. Thus, $U = \mathrm{Ann}\, S$. Clearly, S is closed under $\times$. Also, S will act on V/U, and by applying our inductive hypothesis to this action, we see that for some m, $(S_{V/U})^m = 0$.* Hence taking $n = m+1$ we see that $S^n = 0$.

Now if $TU = 0$, then $T = S$ and we are done. We shall assume $TU \neq 0$ and obtain a contradiction. Note that by (2), $(S\times(\ldots\times(S\times T)\ldots))U$, with r S's, equals $S^r T\, U$. Hence for $r = n$, this is zero, while we have assumed that for $r = 0$ it is nonzero; so we can choose a largest $r \geq 0$ such that this set is nonzero. Take q in this set $(S\times(\ldots\times(S\times T)\ldots))$ such that $qU \neq 0$. Then by (2) again, $S\, q\, U = (S\times q)\, U$, which is zero by choice of r. This means $q\, U \subseteq \mathrm{Ann}(S) = U$. Thus $q|U$ is a __nonzero__ endomorphism of U. Also, $q \in T$, so q is nilpotent, so $q|U$ has a nonzero kernel U'. $U' = \mathrm{Ann}(S \cup \{q\})$ is smaller than U, contradicting the choice of U. This completes the proof of the Theorem.

Remark: I at first hoped that the opposite generalization of Engel's Theorem might be true: that a __vector-space__ of nilpotent linear transformations on a finite-dimensional vector-space might be nilpotent. But this is false. The vector-space of 3×3 matrices spanned by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is easily shown to consist of nilpotent matrices, but the product of those two matrices

*To verify that $S_{V/U}$ satisfies the same hypothesis as S, we must know that the action of $s \times s'$ on V/U ($s, s' \in S$) is still of the form $\bar{s}\,\bar{s}' + \bar{a}\,\bar{s}$; in particular, that the a in the expression of $s \times s'$ induces an endomorphism of V/U. It was to get this that we assumed $a \in$ the algebra generated by s, t in (1).

is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which is not nilpotent.

In considering the above question I was led to another. Suppose A and B are finite-dimensional vector spaces, and we choose $A_0 \subseteq A$, $B_0 \subseteq B$ such that $\dim B_0 < \dim A_0$. Clearly, the set of all linear maps from A to B that carry A_0 into B_0 forms a <u>vector-subspace</u> of $\mathrm{Hom}(A,B)$, whose members are all <u>singular</u>. Question: is the converse true? I.e., if T is a vector subspace of $\mathrm{Hom}(A,B)$ whose members are all singular, must T take some subspace $A_0 \subseteq A$ into a subspace $B_0 \subseteq B$ of smaller dimension? I can prove this if T is 2-dimensional. (Might exterior algebra be somehow applicable?)

REFERENCE

[1] Nathan Jacobson, Lie Algebras, Interscience tracts in pure and applied mathematics, Number 10, John Wiley and Sons, 1962.

Addendum: the answer to the question in the last paragraph is no. For instance, let L be any rank 1 simple Lie algebra, and consider the space of maps of the form $\mathrm{ad}(x): L \to L$ $(x \in L)$. Each is singular. If L had a subspace M that was carried by all these into some N of smaller dimension than L, then M could not equal L, because $[L,L] = L \not\subseteq N$. But if $M \neq L$, take $x \in L - M$; then $\mathrm{ad}(x)$ is nonsingular on L (because L has rank 1) hence cannot take L into any N of smaller dimension. The easiest version of this example, as Chernoff points out, is the cross-product on $\mathbb{R}^3$.