The question considered here isn’t terribly important, but you may find the counterexample interesting. No present plans of publishing this. Comments welcomed as always.

When $C^{pt}$ has products that are not products in $C$

George M. Bergman

If $C$ is a category having a final object $F$, then $C^{pt}$, the category of “pointed” objects of $C$, has for objects all pairs $(\varepsilon, X)$ where $X$ is an object of $C$ and $\varepsilon$ a morphism $F \to X$, and has for morphisms all morphisms of second components which make commuting triangles with $F$. It is easy to verify that if $(\varepsilon_i, X_i)$ $(i \in I)$ are objects of $C^{pt}$, and the objects $X_i$ have a product $\prod X_i$ in $C$, then, writing $\varepsilon$ for the morphism $F \to \prod X_i$ induced by the $\varepsilon_i$, the pair $(\varepsilon, \prod X_i)$ will be a product of the objects $(\varepsilon_i, X_i)$ in $C^{pt}$.

Suppose, on the other hand, that we know that the objects $(\varepsilon_i, X_i)$ have a product $(\varepsilon, X)$ in $C^{pt}$. Must $X$ be a product of the $X_i$ in $C$? From the preceding observation this will be so if and only if the $X_i$ have a product in $C$, but a little reflection produces no reason to expect that such a product must exist. However, a counterexample was unexpectedly difficult to find. I shall develop below a rather curious counterexample, then show how, with hindsight, one can get a less exotic one.

We begin with the following general construction. Suppose $A$ is an abelian monoid. We define a category $\text{Set}_A$ whose objects are sets $S$ given with maps $p: S \to A$ (which for brevity will not be shown in writing these objects), and in which a morphism $f: S \to T$ in $\text{Set}_A$ means a finitely-many-to-one set map $f: S \to T$ such that

$$\text{for all } t \in T, \sum_{f(s)=t} p(s) = p(t).$$

It is immediate that set-theoretic composition of maps makes this a category.

We shall be interested in objects of $\text{Set}_A$ with finite underlying sets. The full subcategory of $\text{Set}_A$ consisting of these objects is not in general connected; its connected components are the full subcategories $\text{Set}_{A,a}$ $(a \in A)$ composed of all objects $S$ satisfying $\sum_S p(s) = a$. Each such component has a final object, $F_a$, a one-element set such that $p$ takes the one element to $a$.

Now take for $A$ the group $\mathbb{Z}_2$. If $s$ is an element of an object $S$ of $\text{Set}_{\mathbb{Z}_2}$, we shall call $p(s)$ the parity of $s$, and call $s$ even or odd according as $p(s)$ is 0 or 1. Let $C$ be the full subcategory of $\text{Set}_{\mathbb{Z}_2,0}$ consisting of those objects all of whose members have the same parity. In other words, an object of $C$ consists either of an arbitrary finite number of even elements, or of an even number of odd elements.

We see that $C$ has a final object $F_0$, consisting of a single even element. This admits morphisms only to objects of $C$ which have even elements; by choice of $C$ such objects consist entirely of even elements. It is easy to deduce that $C^{pt}$ is isomorphic to the category of pointed finite sets, which of course has finite products. We claim, however, that these do not constitute products of these same objects.

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in $\mathbf{C}$. For example, let $X$ be the object of $\mathbf{C}^{\text{pt}}$ consisting of two even elements $x$ and $y$, with $x$ the basepoint (the image of the unique element of $F_0$ under the map $\varepsilon$). The product in $\mathbf{C}^{\text{pt}}$ of two copies of $X$ is their set-theoretic product $W$, with all four elements even, and $(x, x)$ as basepoint. Now let $Y$ be an object of $\mathbf{C}$ consisting of four odd points, $s, t, u, v$, and define morphisms $f, g: Y \to X$, letting the first take $s$ and $t$ to $x$, and take $u$ and $v$ to $y$, while the second takes $s$ and $u$ to $x$, and takes $t$ and $v$ to $y$. If $W$ with its two projections to $X$ were the product of two copies of $X$ in $\mathbf{C}$, we would have a morphism $Y \to W$ carrying each of $s, t, u, v$ to a distinct element of $W$; but this map would not satisfy (1). So we have obtained the desired counterexample.

(Does anyone know whether the construction $\mathbf{Set}_{Z_2}$ has been used anywhere else?)

The key properties that made this work are that the full subcategory of $\mathbf{C}$ consisting of objects admitting pointed structures has finite products, but that outside of this category, there is an object $Y$ such that the set-valued functor $\mathbf{C}(Y, -)$ does not respect those products. With this observation in mind one can find a simpler example. Let $\mathbf{C}$ be the category obtained from $\mathbf{Set}$ by adjoining one additional object $Y$, setting $\mathbf{C}(Y, Y) = \{\text{Id}_Y\}$ and $\mathbf{C}(X, Y) = \emptyset$, for $X \in \text{Ob}(\mathbf{Set})$, but defining $\mathbf{C}(Y, X)$ to be the set of all nonempty subsets of $X$ (or if we prefer, the set of all one- and two-element subsets of $X$), and letting composition with any $f \in \text{Set}(X, X')$ carry $X_0 \in \mathbf{C}(Y, X)$ to the image-set $f(X_0) \in \mathbf{C}(Y, X')$. It is straightforward to verify that this makes $\mathbf{C}$ a category, which has for final object the final object of $\mathbf{Set}$. The full subcategory of $\mathbf{C}$ consisting of elements admitting pointed structures, i.e., the nonempty objects of $\mathbf{Set} \subseteq \mathbf{C}$, has small products; but $\mathbf{C}(Y, -)$ does not respect these products.

Incidentally, returning for a moment to the definition of the categories $\mathbf{Set}_A$, let us note (though this is irrelevant to our examples) that one can, if one likes, weaken the condition that all morphisms be finitely-many-to-one on underlying sets. On first sight, it might appear that we could replace it by the condition that for each $t \in T$, we have $p(s) = 0$ for all but finitely many elements $s \in f^{-1}(t)$, since this allows us to make sense of (1). However, this condition is not respected by composition of set-maps. A condition that does respect composition, and generalizes the conjunction of “finitely-many-to-one” and (1), is

(2) For all $t \in T$, there exists a way of partitioning $f^{-1}(t)$ into finite subsets, such that for all but one of these subsets, the sum of $p(s)$ over the elements of the subset is 0, while the sum over the one remaining subset equals $p(t)$.

I have no idea what if anything this construction may be good for, however.

Department of Mathematics
University of California
Berkeley CA 94720

Electronic mail address: gbergman@math.berkeley.edu

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