A FREE EPIC SUBALGEBRA OF A FREE ASSOCIATIVE ALGEBRA *

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We shall show here that any free associative algebra $R = k < X_1, X_2, \ldots >$ in more than one indeterminate over a commutative ring k contains a proper subalgebra S such that the inclusion $S \subseteq R$ is an epimorphism of the category of associative rings. In fact, S can also be taken to be free. We shall do this by first constructing such an example in the 3-indeterminate case, $R = k < X, Y, Z >$ and then observing how it can be adapted to the cases of two, or more than three indeterminates.

The idea of the example will be as follows. We take a certain invertible matrix A over R, and let S_0 denote the subalgebra generated by the entries of A, and S_1 the subalgebra generated by the entries of A and A^{-1}. Then the inclusion of S_0 in S_1 is certainly an epimorphism. We shall then find an element which when adjoined to S_1 generates all of R, but when adjoined to S_0 gives a proper subalgebra S.

To choose the form of A, we recall that if k is a field, then R has weak algorithm, and it is known that any invertible 2x2 matrix over a ring with weak algorithm is a product of matrices of the form $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ with α and β invertible (in which we are not interested) and matrices $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$. Now an easy computation shows that the subalgebra generated by the entries of $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}$ is the whole subalgebra generated by x and y, and hence also contains the entries of the inverse of that matrix. So we need a product of at least three factors. Let us take for R the free associative algebra in three indeterminates X, Y, Z, and let

$$A = A(X,Y,Z) = \begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ Y & 1\end{pmatrix}\begin{pmatrix}1 & Z\\ 0 & 1\end{pmatrix} = \begin{pmatrix}XY+1 & XYZ+X+Z\\ Y & YZ+1\end{pmatrix}.$$

*I have no present plans of publishing these observations. Anyone writing a relevant paper is welcome to include them, with appropriate attribution. Just let me know, so that I can keep track of whether they're in print anywhere.

Clearly we have

$$A^{-1} = A(-Z, -Y, -X) = \begin{pmatrix} ZY+1 & -ZYX-Y-Z \\ -Y & YX+1 \end{pmatrix}.$$

Let us regard $k < X, Y, Z >$ as the semigroup algebra on the free semigroup $< X, Y, Z >$. Let us also make the convention that when we say that a family of elements of a semigroup or algebra generates a free subalgebra, this will be short for saying that they constitute a free generating set for the algebra they generate.

Given a set of elements of $k < X, Y, Z >$, if we can find a semigroup ordering on $< X, Y, Z >$ such that the leading terms of the given elements with respect to that ordering all occur with coefficient 1, and generate a free subsemigroup of $< X, Y, Z >$, then the given elements will generate a free subalgebra of $k < X, Y, Z >$; and unless these leading terms are precisely X, Y, and Z, this subalgebra will be proper. Now one can verify that the terms of highest degree in the four entries of A (which will be their leading terms under any ordering which respects the length function) do indeed generate a free semigroup. Our first problem is to find a fifth element which, with them, still generates a free subsemigroup. After trying a few elements which do not, such as YX and ZYX, we come to $YXXY$ and find:

Lemma. The elements $Y, XY, YZ, XYZ, YXXY \in < X, Y, Z >$ are free generators of a free subsemigroup.

Proof. If a word in X, Y, Z is a product of these five elements, its factorization into these terms can be reconstructed uniquely using the following observations: Any occurrence of Z must come from the same factor as the letter to its left. Any X must similarly come from the same factor as the letter to its right. If the sequence XX occurs, the letter to its left also comes from the same factor.

If we "join" letters which by the above observations must come from the same factors, we see that the "connected components" into which the given word falls are precisely the original factors. $\|$

Now let S denote the subalgebra of R generated by

(1) $Y,\ XY,\ YZ,\ XYZ+X+Z,\ YXXY+X.$

From the above Lemma and our preceding observations, S is free on these generators.

We see that S contains the entries of A, hence S is epimorphically included in the subalgebra of R generated by S and the entries of A^{-1}. Note that this contains $YX+1$, hence YX, hence $(YXXY+X) - (YX)(XY) = X$. Hence it also contains $(XYZ+X+Z) - X(YZ+1) = Z$. Since it also contains Y, it is precisely R. Thus we have:

<u>Proposition.</u> Let k be a commutative ring, $R = k < X, Y, Z >$, and let S be the subring of R generated by (1). Then S is free on this generating set, and the inclusion $S \subseteq R$ is an epimorphism of associative algebras. ||

If we start with a free algebra on more than three indeterminates, we can simply perform the above construction with three of the indeterminates and throw all the others into S as they are, getting a proper free epic subalgebra of this R.

If we have only two indeterminates we can make use of the fact that the free semigroup on three indeterminates, $< X, Y, Z >$ is embeddable in the free semigroup on two indeterminates, $< U, V >$. We use the embedding given by

$$\overline{X} = UV, \quad \overline{Y} = U, \quad \overline{Z} = UVV.$$

Now let $S \subseteq k < U, V >$ be the subalgebra generated by the entries of $A(\overline{X}, \overline{Y}, \overline{Z})$ and the element $\overline{YXXY} + V$. Then we see as above that S is free on these five elements, while if we adjoin the entries of $A(\overline{X}, \overline{Y}, \overline{Z})^{-1}$, the resulting ring contains $(\overline{YXXY} + V) - (\overline{YX})(\overline{XY}) = V$. As it also contains U, it is all of $k < U, V >$.

Let us note that the inclusion of a proper subsemigroup S in a free semigroup F can never be epic. For if we let T be the semigroup with three elements, 1 (the identity), 0 (a zero), and an element x satisfying $x^2 = 0$, then we can map F into T by the homomorphism f_1 taking all generators (and hence all elements but 1) to 0, and also by the homomorphism f_2 taking those generators not in S to x, and all other nonidentity elements to 0. These homomorphisms are distinct, but agree on S, so $S \subseteq F$ is not epic.

I do not know whether an epic subring of a commutative polynomial ring can be proper.