The class of free subalgebras of a free associative algebra is not closed under taking unions of chains, or pairwise intersections

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The results stated in the title, which will be shown by two examples below, answer in the negative two questions asked by A. T. Kolofov ([1], end of §1). We motivate both examples. (We shall also show that the analogs of these two examples work for commutative polynomial algebras; though the union-of-a chain example requires polynomials in infinitely many indeterminates, and neither example is as naturally motivated. But I doubt that the existence of such examples will be news to commutative ring theorists.)

We fix a field k. All rings and k-algebras will be associative, with 1.

We begin with the simpler case:

1. **Pairwise intersections.** Recall that if R is a ring, a relation

\[(1) \quad a_1 b_1 + \ldots + a_n b_n = 0, \quad \text{equivalently} \]

\[a \cdot b = 0, \quad \text{where} \quad a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n)\]

Among elements a_i, b_i \in R is said to be trivial if for every i, either a_i = 0 or b_i = 0. The ring R is an n-fir if for every relation (1) of min terms, there exists an \(m \times m\) matrix U over R which trivializes (1), i.e. such that the equivalent relation \(aU^{-1}Ub = 0\) is trivial. A free associative algebra is an n-fir for all n ([2], Cor. to Prop. 2.4.2, p. 80.)

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Given a finite family $a_1, \ldots, a_n$ of elements of a ring $R$, let
\[ \text{rt.rk.}_R(a) \] denote the least number of generators of the right ideal $a_1R + \ldots + a_nR$, and similarly for $b_1, \ldots, b_n \in R$ let \[ \text{lt.rk.}_R(b) \] denote the minimum number of generators of the left ideal $Rb_1 + \ldots + Rb_n$. If (1) is a trivial relation, we clearly have \[ \text{rt.rk.}_R(a) + \text{lt.rk.}_R(b) \leq n. \] It is also easy to see that if $U$ is an invertible $n \times n$ matrix, the terms of $aU^{-1}$ generate the same right ideal as those of $a$, and the terms of $Ub$ the same left ideal as those of $b$. Hence in the preceding statement, "trivial" can be weakened to "trivializable". We deduce

**Lemma 1.** If $R$ is an $n$-fir, then for any $n$-term relation (1) holding over $R$

one has \[ \text{rt.rk.}_R(a) + \text{lt.rk.}_R(b) \leq n. \]

If $S$ is a subring of $R$, and $a_1, \ldots, a_n \in S$, it is easy to see that \[ \text{rt.rk.}_R(a) \leq \text{rt.rk.}_S(a), \] and similarly for left ranks.

We shall get our example by starting with a 3-term relation.

\[ a_1b_1 + a_2b_2 + a_3b_3 = 0 \]  

in a free algebra $F$, which can be brought by an invertible matrix $U$ to the "very trivial" form

\[ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} = 0, \]

so that \[ \text{rt.rk.}_R(a) = \text{lt.rk.}_R(b) = 1. \] We then find a free subalgebra $G_1 \subseteq F$ which contains the entries of $a$ and $b$, but not all of those of $U$. Then (2) will still be trivializable, using a matrix $U_1$ over $G_1$, but we shall find that such a matrix can only bring it to the form

\[ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \end{pmatrix} = 0, \]

because \[ \text{lt.rk.}_R(b) = 2. \] We will similarly find a free subalgebra $G_2$ containing
the entries of $a$ and $b$ such that $rt(rk_a G_2)(a) = 2$. Hence all entries of $a$ and $b$ will lie in $G_1 \cap G_2$, but $rt(rk_a G_1 \cap G_2) + lt(rk_a G_1 \cap G_2) \geq 2 + 2 > 3$, so $G_1 \cap G_2$ is not a 3-fir, and in particular, not a free algebra.

The details now. To get a relation in a free algebra which is not trivial, but can be brought to the form (3), we simply take a relation of the form (3) and "de-trivialize" it by applying an invertible matrix $U$. The group of invertible matrices over a free algebra is generated by the invertible upper $(\mid 2 \mid$, Theorem 2.2.4) triangular and lower triangular matrices. So let us try the simplest case, where $U$ is itself triangular.

Let $F$ be the free associative algebra on 5 generators, $k< x, y_1, y_2, y_3, z>$, let us start with the relation

$$(x, 0, 0) \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = 0,$$

and let us apply the invertible matrix $U = \begin{pmatrix} 1 & y_1 & y_3 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$ to the first factor, and $U^{-1} = \begin{pmatrix} 1 & -y_1 & y_2 \\ 0 & 1 & -y_3 \\ 0 & 0 & 1 \end{pmatrix}$ to the second. We get the relation

$$(4) \quad a \cdot b = (x, x y_1, x y_3) \begin{pmatrix} y_1 y_2 & y_3 \\ -y_2 & z \end{pmatrix} = 0.$$

We want a subalgebra $G_1 \subseteq F$ over which $a$ can still be reduced by an invertible matrix to the form $(*, 0, 0)$, i.e. necessarily to $(x, 0, 0)$. So $G_1$ should contain $x, y_1$ and $y_3$. We see that once we have these elements, (4) can be trivialized by the matrix $U_1 = \begin{pmatrix} 1 & y_1 & y_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, to denote

$$(5) \quad (x, 0, 0) \begin{pmatrix} 0 \\ -y_2 z \\ z \end{pmatrix} = 0.$$

So let $G_1$, the subalgebra of $F$ generated by the entries of $U_1$ and those of (4), equivalently, of (5), i.e. by the elements...
It is clear that these elements are independent (i.e. are free generators of a free subalgebra), and if we write \( p = y_2 z \), we see that the left ideal of \( G_1 \) generated by the entries of \( b \), equivalently by the entries of the column vector in (5), is \( G_1 p + G_1 z \), so \( \text{lt.rk.} G_1(b) = 2 \).

We now go through the right-left dual procedure, and find that the subalgebra \( G_2 \) generated by the elements

\[
(7) \quad x, xy_1, y_1 y_2, y_3, y_2, z.
\]

is free, contains the entries of (5), and satisfies \( \text{rt.rk.} G_2(a) = 2 \). By the arguments given above we now have

**Proposition 2.** Let \( k \) be any field, \( F \) the free associative \( k \)-algebra \( k < x, y_1, y_2, y_3, z > \), and \( G_1, G_2 \subseteq F \) the subalgebras generated by the families of elements (6) and (7) respectively. Then \( G_1 \) and \( G_2 \) are free on these generating sets, but \( G = G_1 \cap G_2 \) is not a free algebra, in fact not a 3-fir. ||

2 \ Remarks on the above construction. One can show that the intersection ring \( G_1 \cap G_2 \) is precisely the subalgebra of \( F \) generated by the six entries of (4). I will just sketch the idea. Let us grade \( F \) by the free semigroup \( S \) on 4 generators \( x, y_1, y_2, z \), giving \( y_3 \) degree \( y_1 y_2 \) (and all other generators the obvious degrees). Then \( G_1 \) and \( G_2 \) are clearly homogeneous subalgebras of \( F \). Note also that given \( s \in S \), a necessary condition for there to exist nonzero elements of \( G_1 \) homogeneous of degree \( s \) is that every occurrence of \( y_2 \) in \( s \) either be followed by \( z \), or preceded by \( y_1 \) (\( y_3 \) gives the latter case). Similarly, for \( G_2 \) to have
such an element, every occurrence of $y_1$ must either be preceded by $x$ or both followed by $y_2$. If we take an $s$ satisfying these conditions, write it out, and put a "dividing mark" between any two letters: $u\bar{v}...$, unless $uv$ is the degree of a generator of $G_1$ or $G_2$, then $s$ falls into segments, such that the analysis of the forms of elements of $G_1 \cap G_2$ can essentially be performed segment by segment. The only segments that can occur are $x y_1 y_2 z$ and subsegments thereof, and the verification that the only elements occurring are expressions in the terms of (4) becomes fairly easy.

It is easy to show that the algebra generated by these six elements is presented by the single relation (4).

In contrast with the preceding example, one can prove

**Lemma 3.** If $F$ is a free associative $k$-algebra, then the intersection of any family of free $k$-subalgebras of $F$ is a 2-fir. (More generally, if $F$ is a ring with 2-term weak algorithm [2], then the intersection of any family of subrings which contain all the units of $F$, and are themselves strong, $G_E^2$-rings, is again a strong $G_E^2$-ring.)

**Sketch of proof.** In a ring with 2-term weak algorithm, any relation (1) with $n = 2$ can be trivialized by multiplying by a sequence of matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ which is essentially unique subject to some nondegeneracy conditions. ([2, Prop. 2.7.1 p. 94.]) Hence if the terms of such a relation lie in a strong $G_E^2$-subring (a subring over which one can trivialize any 2-term relation (1) by a product of upper and lower triangular matrices) all the "$u$'s" and "$v$'s" involved in the trivialization of (1) also lie in this subring, and will thus lie in the intersection of any family of such subrings.

Cf. [3] Prop. 2.1, where the same method gives an analogous result for fixed rings of endomorphisms. It might now be worth trying to apply the lemma to give free $G_E^2$-algebras with an endomorphism plane...
see whether the ideas of the preceding example can be used to answer the long-
open question of whether the fixed ring of an automorphism (or a group of
automorphisms) of a free algebra must again be one.

Note that the subalgebras $G_1, G_2$ of the preceding example are not
rationally closed in $F$, e.g. $y_2 z, z \in G_1$, but $y_2 \notin G_1$. This might just
be an artifact of our using the simplest possible invertible matrix $U$; if
instead we had taken a product of an upper and a lower triangular one, we might
well have gotten an example with rationally closed $G_1$ and $G_2$. On the
other hand, if for some reason any such counterexample necessarily involves
non-rationally closed subalgebras, then the fixed-ring problem may have a
positive answer, since fixed rings of automorphisms groups of rings without
zero-divisors are necessarily rationally closed.

3 The analogous commutative ring. Let $F$ now denote the commutative polynomial
algebra $k[x, y_1, y_2, y_3, z]$, and $G_1, G_2$ the subalgebras of $F$ generated by
the families (6) and (7) respectively. We see easily that these two sets are
each algebraically independent over $k$, so $G_1$ and $G_2$ are each also polynomial
rings in 5 indeterminates. (They also clearly have the same fields of fractions
as $F$). We can show that $G = G_1 \cap G_2$ is not a polynomial ring, but by very
different methods from those of §1. Let $\alpha : F \to k$ be the $k$-algebra homomorphism
taking all the indeterminates to 0. We shall show that there are 6 linearly
independent $\alpha$-derivations $G \to k$ — geometrically, 6 linearly independent
tangent vectors to $\text{Spec } G$ at the point corresponding to $\alpha$. But if $G$
were a polynomial ring, it would have to be on $\leq 5$ indeterminates, since
$\text{tr.deg.}_k G \leq \text{tr.deg.}_k F = 5$, and could not have $> 5$ linearly independent
$\alpha$-derivations. (Geometrically, what this means is that the point $\alpha$ of
$\text{Spec } G$ is singular; a polynomial ring has smooth spectrum.)

*As noted at the end of this subsection (bottom of p. 7) there are simpler
examples in the commutative case, so the reader may wish to skip the rest
of this section.*
We can show our six derivations are linearly independent by evaluating them on the 6-tuple of elements comprising the entries of (4),

\[(8) \quad x, xy_1, xy_2, (y_1y_2-y_3), y_2, z.\]

In fact, because our situation is "symmetric," via an automorphism of \(F\) interchanging \(x\) and \(z\), \(y_1\) and \(-y_2\), and \(y_3\) and \(y_1y_2-y_3\), it will suffice to display three \(\alpha\)-derivations \(G \to k\) which are linearly independent on \(x, xy_1, xy_2\), and zero on the last three terms of (8).

Two of these are easily described: take \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial y_1} : G_2 \to G_2\) (where \(q = xy_1\), one of the free generators of \(G_2\)), compose with \(\alpha : G_2 \to k\), and restrict to \(G = G_1 \cap G_2\). These take on the values 1, 0, 0, 0, 0 and 0, 1, 0, 0, 0 on (8), respectively. To get the third, let us define \(\beta : G_2 \to k\) to take \(y_1y_2-y_3\) to 1, and all other elements of (7) to 0. Now \(\frac{\partial}{\partial x} : G_1 \cap G_2 = \alpha \frac{\partial}{\partial y_1} G_1 \cap G_2\). (This may be seen by grading \(F\) by the free abelian semigroup on \(x, y_1, y_2, z\), and noting that if an element of \(G_1 \cap G_2\) expressed in terms of (6) had a \(y^n_3\) term, then taking its \(y_1y_2^n\)-homogeneous component and comparing representations in terms of (6) and (7), we would get \(y^n_3 = (y_3-y_1y_2)^n\), a contradiction.) Hence if we take the derivation \(\frac{\partial}{\partial x} : G_2 \to G_2\), compose with \(\beta\), and restrict to \(G_1 \cap G_2\), we get an \(\alpha\)-derivation, and this can be seen to take (8) to 1, 0, -1, 0, 0, 0, completing our argument.

(A simpler example of polynomial subrings of a polynomial ring, with non-polynomial intersection, is \(k[x, xy] \cap k[x^2, y] \subset k[x,y]\). So the above example is not of independent interest, unless the property that the subrings have the same field of fractions as the original ring is hard to obtain.)
4. Ascending chains. Let us first note that it is easy to find a k-algebra which is the union of an ascending chain of free subalgebras, but not itself free. An example is the commutative k-algebra $k[x, x^{1/2}, \ldots, x^{1/2^n}, \ldots]$. But Kolotov asked, in effect, whether such a subalgebra could be found within a free algebra.

The ideas that led to the example I shall give were the following. In trying to prove that the union of any such chain was free, it would be convenient to know that if $G \subseteq F$ are free algebras, and $x$ is a member of a free generating set for $F$, and lies in $G$, then $x$ is also a member of some free generating set for $G$. Since I was not able to prove this, I began looking for counterexamples, and the following approach suggested itself.

One knows how to find all solutions in a free algebra to the equation

$$ab - cd = 1.$$  

They form a chain of parametric solutions of increasing degrees of complexity (|2| p.90, formula (26). The approach is similar to that alluded to in §1, with (9) in place of (1).) Now if $s$ is one of our indeterminates, we can multiply a solution to (9) by $s$ to get

$$(sa)b - (sc)d = s.$$  

We may hope that if we choose our solution to (9) sufficiently nondegenerate, the elements

$$p = sa, \quad b, \quad q = sc, \quad d$$

will be independent in $F$, i.e. free generators of a free subalgebra $G$, but that $s = pb - qd$ will not belong to a set of free generators of $G$.

Note that, assuming that a family of elements (10) can be proved to be independent, the problem of showing that $s$ is not a member of a generating
set reduces to the abstract question: in a free associative algebra on four generators \( p, b, q, d \), will \( pb - qd \) be a member of a free generating set? The answer is no, and this can be proved in several different ways. One of these requires a preparatory result of interest for its own sake.

**Lemma 4.** Let \( R \) be a ring having weak algorithm with respect to a filtration-function \( v \). For any \( 2 \)-sided ideal \( I \subseteq R \), let \( \min_v(I) = \inf_{a \in I \setminus \{0\}} v(a) \). Then for two \( 2 \)-sided ideals \( I, J \subseteq R \), \( \min_v(IJ) = \min_v(I) + \min_v(J) \).

**Proof.** Let \( X \) be a weak \( v \)-basis for \( I \) as a right ideal (\cite{2}, p.72 bottom.) Then \( I = \oplus_X xR \), and elements coming from different summands are \( v \)-independent. Hence \( IJ = \oplus_X xJ \), and the result follows immediately. \( \| \)

**Corollary 5.** An element of a free associative algebra which lies in the product of two proper \( 2 \)-sided ideals is not a member of any free generating set for the algebra.

**Proof.** An element belonging to a free generating set has degree 1 with respect to the associated filtration. But if \( x \in IJ \), then \( v(x) \geq \min_v(I) + \min_v(J) \geq 2 \). \( \| \)

We can now prove:

**Lemma 6.** Let \( F \) be the free associative algebra on 4 indeterminates, \( k \langle s, x, y, z \rangle \).

Then the 4 elements

\[
(11) \quad p = s(xy + 1), \quad b = xy + 1 \\
q = s(xyz + x + z), \quad d = y
\]

are independent, and the subalgebra \( G \) which they generate contains \( s = pb - qd \), but \( s \) is not a member of any free generating set for \( G \).
Proof. To show \( G \) is free on \( p, q, b, d \), let us map \( F \) into the group algebra on the free group on four generators \( s, u, y, v \) by sending

\[
s \mapsto s, \quad x \mapsto (s^{-1}u - 1)y^{-1}, \quad y \mapsto y, \quad z \mapsto (v-1)y^{-1}.
\]

Then we find

\[
p \mapsto u \quad b \mapsto v
\]

\[
q \mapsto (uv-s)y^{-1} \quad d \mapsto y.
\]

The subalgebra \( k\langle u, v, s, y, y^{-1} \rangle \) of that group algebra, in which the image of \( G \) now lies, can in turn be mapped into itself by taking \( s \) to \( uv-sy \), and all the other generators to themselves. The images of the indicated generators of \( G \) are now carried to \( u, s, v, y \), proving their independence.

One proof that \( s \) does not belong to any free generating set for \( G \) is given by applying Corollary 6, with \( I = J \) ideal generated by the indeterminates. Another comes by noting that if \( s \) did belong to such a set, \( G/(pb-qd) \) would be a free \( k \)-algebra. In particular, the relation \( \bar{pb} - \bar{qd} = 0 \) would be trivializable in this algebra. But since \( G \) is free on \( p, q, b, d \), the algebra \( G/(pb-qd) \) is universal for having elements \( \bar{pb}, \bar{qd} \) satisfying such a relation. Hence if that relation were trivializable, any such relation in any ring would be so, which is not so. 

We shall now obtain our counterexample to the original question by iterating this construction: At each stage, the "\( p \)'s, \( q \)'s, \( b \)'s and \( d \)'s" of the preceding stage will be treated like "\( s \)", and made not to lie in any free generating set of the next larger subalgebra. (This does not itself assure that the union of the resulting chain of subalgebras is not free, but we shall in fact find this easy to prove from the details of the construction.)

Note that the process we are suggesting is a repeatedly "branching" one, since each \( p, q, b \) or \( d \) will require its own \( p, q, b \) and \( d \) at the next
stage. This "branching" can be conveniently expressed by using as an index-set for our indeterminates the free semigroup with 1 on four symbols p, q, b, d, which will be denoted \(<p, q, b, d>\). (In fact, the symbols p, q, b, d will never appear except as indices!) The length of a word \(W \in <p, q, b, d>\) will be written \(|W|\). (In particular, \(|1| = 0\).) We can now state and prove:

**Theorem 7.** Let \(F = k < s; x_W, y_W, z_W | W \in <p, q, b, d>\) a free associative algebra on a set of indeterminates indexed as shown. Let us inductively define elements \(a_W (W \in <p, q, b, d>)\) as follows:

\[
\begin{align*}
(12) \quad & s_1 = s, \\
(13) \quad & a_W = s_{W} (x_W y_W + 1) \\
& a_{Wb} = x_W y_W + 1 \\
& a_{Wq} = s_W (x_W y_W z_W + x_W + z_W), \\
& a_{Wd} = y_W z_W \\
(W \in <p, q, b, d>).
\end{align*}
\]

For each \(n \geq 0\), let \(S_n = \{a_W | |W| = n\}\) (a set of \(4^n\) elements), and let \(G_n \subseteq F\) be the subalgebra generated by \(S_n\). Then,

\[
\begin{align*}
(14) \quad & \text{for each } n, \ S_n \text{ is an independent set, i.e. } G_n \text{ is free on } S_n, \\
(15) \quad & G_0 \subseteq G_1 \subseteq \ldots, \text{ but} \\
(16) \quad & G = \bigcup G_n \text{ is not a free } k\text{-algebra, in fact, not even an } S_0\text{-fir.}
\end{align*}
\]

**Proof.** We shall prove (14) by induction. The case \(n=0\) is clear. Assuming \(S_n\) is independent, we shall show \(S_{n+1}\) independent. Note that the elements of \(S_n\) may all be expressed in terms of \(s\) and indeterminates \(x_U, y_U, z_U\) with \(|U| < n\). It follows that \(\{a_W, x_W, y_W, z_W | |W| = n\}\) is independent.

Let us call the free subalgebra of \(F\) generated by this set \(H_n\). We now forget about \(F\), and regard the free algebra \(H_n\) as the coproduct of \(4^n\) copies of the free \(k\)-algebra \(k < s, x, y, z >\) of Lemma 6, indexed by \(\{W | |W| = n\}\).
Then applying the statement of that Lemma to each copy, we easily see that $S_{n+1}$ is independent, establishing (14).

(15), of course, is seen by noting

$$s_W = s_{Wp} s_{Wb} s_{Wq} s_{Wd}. \tag{17}$$

Now for each $n \geq 0$, let $I_n \subseteq G_n$ denote the ideal generated by all the generators $s_W (\mid W \mid = n)$. By (17) we have $I_n \subseteq I_{n+1}$.

$$I_n \subseteq I_{n+1}. \tag{18}$$

In particular, $I_1 \subset I_2 \subset \ldots$, so $I = \bigcup I_n$ will be an ideal of $G = \bigcup G_n$.

But applying (18) to $I$ we get

$$I = I^2. \tag{19}$$

We can now finish off in a number of ways. The one requiring no technical knowledge of free algebras, firs, etc. is to note that for each $n$, $G_n/I_n \cong k$, hence $G/I \cong k$. Now if $G$ were free on a generating set $X$, then for each $x \in X$, I would have to contain $x-o_x$ for some $o_x \in k$. Hence with respect to the new free generating set $\{x-o_x \mid x \in X\}$, I would be the ideal generated by the indeterminates. But that ideal certainly does not satisfy (19).

Alternatively, we can see from Lemma 6 that no right ideal in a ring with weak algorithm satisfies (19). In fact, by the results of $|4|$ no ideal $I$ in an $S_0$-fir satisfies (19).

5. Remarks. The analogs of Lemma 6 and Theorem 7 are also true for commutative polynomial rings. In Lemma 6 one proves that $k$ is not a member of any generating set for $G$ by verifying that $G/(s)$ is not a polynomial ring — its spectrum has the singular point $p=q=a=b=0$. In Theorem 10 the first proof that $G$ is not free that we gave goes over word-for-word to the commutative case. (We don't know any analog to "not an $S_0$-fir.")
It is well-known that the free associative algebra on two indeterminates contains subalgebras free on countably many indeterminates; hence the example of Theorem 7 can be embedded in the free algebra on two indeterminates. This does not apply to the commutative analog, of course.

It would be interesting to know whether one can get an example like that of Theorem 7, but where the number of free generators of $G_n$ is bounded.

Lemma 4 and the application we have made of it suggest

**Definition 5.** If $a$ is an element of a ring $R$, then $h(a)$ will denote the supremum of numbers $n$ such that $x$ lies in a product $I_1 ... I_n$ of $n$ proper 2-sided ideals in $R$.

By Lemma 4, $h(a)$ is a lower bound on the values $w(a)$ for filtration functions $w$ with respect to which $R$ has weak algorithm. In particular, if $R$ is free, the condition $h(a) = 1$ is necessary for $a$ to lie in a free generating set. It is not sufficient; for instance, from the fact that $k < x, y > (xy - yx - 1)$ is simple, we see that $xy - yx - 1$ is contained in only one proper 2-sided ideal; so $h(xy - yx - 1) = 1$; but since this element goes to 1 modulo the commutator ideal, it can't belong to a free generating set. The condition $w(a) = k$, $h(a - e) = 1$ has more chance of being sufficient for $a$ to belong to a free generating set, but I don't have high hopes. Nevertheless, as a function on $R$ invariant under automorphisms yet related to the values of filtration functions, $h(a)$ might be of use in the study of the automorphism groups of free algebras.

Note that for $a \in R \subseteq S$, there is no necessary relation between $h_R(a)$ and $h_S(a)$. In particular, in the situation of Lemma 6, the proper ideals of $G$ that give the value $h_G(pb - qd) = 2$ generate improper ideals in $F$.

One might ask whether an ascending chain of free subalgebras $G_n$ of a free algebra $F$, will have free union $G$ if for every $a \in G$, the values of $h_G(a)$ are bounded.
Afterthought: In Lemma 4 we really only need to have I a right ideal and J a left ideal. IJ will then just be an additive subgroup, but the conclusion still holds. So likewise, we might define $h'(a)$ by removing from Definition 8 the condition that $I_1$ be closed under left multiplication by elements of $R$, and $I_n$ under right multiplication. This $h'$ might not behave as elegantly as $h$ (e.g., $h(ab) > h(a) + h(b)$, but not so for $h'$), but the condition nonetheless, $h'(a) = 1$ (i.e., $a$ does not lie in the product of a proper right ideal and a proper left ideal of $R$) is a much stronger one than $h(a) = 1$, and so might be more useful in studying generators of free algebras. (We should set $h'(u) = 0$ for $u$ a unit, just to put that trivial case aside.) Cf. [5].

However, one still finds that $h'(xy-yx-1) = 1$. For if an element of degree 2 lies in a product $IJ$, one can see from the proof of Lemma 4 that it can be expressed in terms of elements of $I$ and $J$ of degree 1. But elements of degree 1 generating a proper 1-sided ideal also generate a proper 2-sided ideal.

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