THE CLASS OF FREE SUBALGEBRAS OF A FREE ASSOCIATIVE ALGEBRA IS NOT CLOSED UNDER TAKING UNIONS OF CHAINS, OR PAIRWISE INTERSECTIONS

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The results stated in the title, which will be shown by two examples below, answer in the negative two questions asked by A. T. Kolotov ([1], end of §1). We motivate both examples. (We shall also show that the the analogs of these two examples work for commutative polynomial algebras; though the union-of-a chain example requires polynomials in infinitely many indeterminates, and neither example is as naturally motivated. But I doubt that the existence of such examples will be news to commutative ring theorists.)

We fix a field k. All rings and k-algebras will be associative, with 1. We begin with the simpler case:

1. Pairwise intersections. Recall that if R is a ring, a relation

$$(1) \qquad a_1 b_1 + \dots + a_n b_n = 0, \text{ equivalently}$$

$$a \cdot b = 0, \text{ where } a = (a_1, \dots, a_n),\ b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

among elements $a_i, b_i \in R$ is said to be trivial if for every i, either $a_i = 0$ or $b_i = 0$. The ring R is an n-fir if for every relation (1) of $m \leq n$ terms, there exists an $m \times m$ matrix U over R which trivializes (1), i.e. such that the equivalent relation $aU^{-1} \cdot Ub = 0$ is trivial. A free associative algebra is an n-fir for all n ([2], Cor. to Prop.2.4.2, p.80.)

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Given a finite family $a_1, \ldots, a_n$ of elements of a ring R, let $\mathrm{rt.rk.}_R(a)$ denote the least number of generators of the right ideal $a_1R + \ldots + a_nR$, and similarly for $b_1, \ldots, b_n \in R$ let $\mathrm{lt.rk.}_R(b)$ denote the minimum number of generators of the left ideal $Rb_1 + \ldots + Rb_n$. If (1) is a trivial relation, we clearly have $\mathrm{rt.rk.}_R(a) + \mathrm{lt.rk.}_R(b) \leq n$. It is also easy to see that if U is an invertible $n \times n$ matrix, the terms of aU^{-1} generate the same right ideal as those of a, and the terms of Ub the same left ideal as those of b. Hence in the preceding statement, "trivial" can be weakened to "trivializable". We deduce

Lemma 1. If R is an n-fir, then for any n-term relation (1) holding over R one has $\mathrm{rt.rk.}_R(a) + \mathrm{lt.rk.}_R(b) \leq n.$ ||

If S is a subring of R, and $a_1, \ldots, a_n \in S$, it is easy to see that $\mathrm{rt.rk.}_R(a) \leq \mathrm{rt.rk.}_S(a)$, and similarly for left ranks.

We shall get our example by starting with a 3-term relation

$$(2) \qquad a_1b_1 + a_2b_2 + a_3b_3 = 0$$

in a free algebra F, which can be brought by an invertible matrix U to the "very trivial" form

$$(3) \qquad (*, 0, 0)\begin{pmatrix}0\\0\\*\end{pmatrix} = 0,$$

so that $\mathrm{rt.rk.}_R(a) = \mathrm{lt.rk.}_R(b) = 1$. We then find a certain free subalgebra $G_1 \subseteq F$ which contains the entries of a and b, but not all of those of U. Then (2) will still be trivializable, using a matrix U_1 over G_1, but we shall find that such a matrix can only bring it to the form

$$(*, 0, 0)\begin{pmatrix}0\\ *\\ *\end{pmatrix} = 0,$$

because $\mathrm{lt.rk.}_R(b) = 2$. We will similarly find a free subalgebra G_2 containing the entries of a and b such that rt.rk.$_{G_2}(a) = 2$. Hence all entries of a and b will lie in $G_1 \cap G_2$, but rt.rk.$_{G_1 \cap G_2}(a)$ + lt.rk.$_{G_1 \cap G_2}(b) \geq 2 + 2 > 3$, so $G_1 \cap G_2$ is not a 3-fir, and in particular, not a free algebra.

The details now. To get a relation in a free algebra which is not trivial, but can be brought to the form (3), we simply take a relation of the form (3) and "de-trivialize" it by applying an invertible matrix U. The group of invertible matrices over a free algebra is generated by the invertible upper triangular and lower triangular matrices (|2|, Theorem 2.2.4). So let us try the simplest case, where U is itself triangular.

Let F be the free associative algebra on 5 generators, $k\langle x, y_1, y_2, y_3, z\rangle$, let us start with the relation

$$(x, 0, 0)\begin{pmatrix}0\\0\\z\end{pmatrix} = 0,$$

and let us apply the invertible matrix $U = \begin{pmatrix}1 & y_1 & y_3\\0 & 1 & y_2\\0 & 0 & 1\end{pmatrix}$ to the first factor, and $U^{-1} = \begin{pmatrix}1 & -y_1 & y_1y_2-y_3\\0 & 1 & -y_2\\0 & 0 & 1\end{pmatrix}$ to the second. We get the relation

$$(4) \qquad a\cdot b = (x, xy_1, xy_3)\begin{pmatrix}(y_1y_2-y_3)z\\-y_2z\\z\end{pmatrix} = 0.$$

We want a subalgebra $G_1 \subseteq F$ over which a can still be reduced by an invertible matrix to the form $(*, 0, 0)$, i.e. necessarily to $(x,0,0)$. So G_1 should contain , y_1 and y_3. We see that once we have these elements, (4) can be trivialized by the matrix $U_1 = \begin{pmatrix}1 & y_1 & y_3\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$, to

$$(5) \qquad (x, 0, 0)\begin{pmatrix}0\\-y_2z\\z\end{pmatrix} = 0.$$

So let G_1 denote the subalgebra of F generated by the entries of U_1 and those of (4), equivalently, of (5), i.e. by the elements

(6) $$x,\ y_1,\ y_3,\ y_2z,\ z.$$

It is clear that these elements are independent (i.e. are free generators of a free subalgebra), and if we write $p = y_2z$, we see that the left ideal of G_1 generated by the entries of b, equivalently by the entries of the column vector in (5), is $G_1 p + G_1 z$, so $\text{lt.rk.}_{G_1}(b) = 2$.

We now go through the right-left dual procedure, and find that the subalgebra G_2 generated by the elements

(7) $$x,\ xy_1,\ y_1y_2-y_3,\ y_2,\ z,$$

is free, contains the entries of (5), and satisfies $\text{rt.rk.}_{G_2}(a) = 2$. By the arguments given above we now have

Proposition 2. Let k be any field, F the free associative k-algebra $k\langle x, y_1, y_2, y_3, z\rangle$, and $G_1, G_2 \subseteq F$ the subalgebras generated by the families of elements (6) and (7) respectively. Then G_1 and G_2 are free on these generating sets, but $G = G_1 \cap G_2$ is not a free algebra, in fact not a 3-fir. ||

2. Remarks on the above construction. One can show that the intersection ring $G_1 \cap G_2$ is precisely the subalgebra of F generated by the six entries of (4). I will just sketch the idea. Let us grade F by the free semigroup S on 4 generators x, y_1, y_2, z, giving y_3 degree y_1y_2, (and all other generators the obvious degrees). Then G_1 and G_2 are clearly homogeneous subalgebras of F. Note also that given $s \in S$, a necessary condition for there to exist nonzero elements of G_1 homogeneous of degree s is that every occurrence of y_2 in s either be followed by z, or preceded by y_1 (y_3 gives the latter case). Similarly, for G_2 to have

such an element, every occurrence of y_1 must either be preceded by x or followed by y_2. If we take an s satisfying both these conditions, write it out, and put a "dividing mark" between any two letters: $\ldots u|v\ldots$ unless uv is the degree of a generator of G_1 or G_2, then s falls into segments, such that the analysis of the forms of elements of $G_1 \cap G_2$ can essentially be performed segment by segment. The only segments that can occur are $x\, y_1\, y_2\, z$ and subsegments thereof, and the verification that the only elements occurring are expressions in the terms of (4) becomes fairly easy.

It is easy to show that the algebra generated by these six elements is presented by the single relation (4).

In contrast with the preceding example, one can prove

Lemma 3. If F is a free associative k-algebra, then the intersection of any family of free k-subalgebras of F is a 2-fir. (More generally, if F is any ring with 2-term weak algorithm |2|, then the intersection of any family of subrings which contain all the units of F, and are themselves strong GE_2-rings, is again a strong GE_2-ring.)

Sketch of proof. In a ring with 2-term weak algorithm, any relation (1) with $n = 2$ can be trivialized by multiplying by a sequence of matrices $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ which is essentially unique subject to some nondegeneracy conditions. ([2] Prop. 2.7.1 p. 94.) Hence if the terms of such a relation lie in a strong GE_2-subring (a subring over which one can trivialize any 2-term relation (1) by a product of upper and lower triangular matrices,) all the "u's" and "v's" involved in the [illegible] trivialization of (1) also lie in this subring, and will thus lie in the intersection of any family of such subrings. ||

Cf. |3| Prop. 2.1, where the same method gives an analogous result for fixed rings of endomorphisms. It might now be worth trying to [illegible]

[illegible]

see whether the ideas of the preceding example can be used to answer the long-open question of whether the fixed ring of an automorphism (or a group of automorphisms) of a free algebra must again be one.

Note that the subalgebras G_1, G_2 of the preceding example are not rationally closed in F, e.g. y_2z, $z \in G_1$, but $y_2 \notin G_1$. This might just be an artifact of our using the simplest possible invertible matrix U; if instead we had taken a product of an upper and a lower triangular one, we might well have gotten an example with rationally closed G_1 and G_2. On the other hand, if for some reason any such counterexample necessarily involves non-rationally closed subalgebras, then the fixed-ring problem may have a positive answer, since fixed rings of automorphism groups of rings without zero-divisors are necessarily rationally closed.

3 The analogous commutative ring. Let F now denote the commutative polynomial algebra $k[x, y_1, y_2, y_3, z]$, and G_1, G_2 the subalgebras of F generated by the families (6) and (7) respectively. We see easily that these two sets are each algebraically independent over k, so G_1 and G_2 are each also polynomial rings in 5 indeterminates. (They also clearly have the same fields of fractions as F). We can show that $G = G_1 \cap G_2$ is not a polynomial ring, but by very different methods from those of §1.* Let $\alpha: F \to k$ be the k-algebra homomorphism taking all the indeterminates to 0. We shall show that there are 6 linearly independent α-derivations $G \to k$ — geometrically, 6 linearly independent tangent vectors to Spec G at the point corresponding to α. But if G were a polynomial ring, it would have to be on ≤ 5 indeterminates, since $\text{tr.deg.}_k G \leq \text{tr.deg.}_k F = 5$, and could not have > 5 linearly independent α-derivations. (Geometrically, what this means in that the point α of Spec G is singular; a polynomial ring has smooth spectrum.)

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*As noted at the end of this subsection (bottom of p.7) there are simpler examples in the commutative case, so the reader may wish to skip the rest of this section.

We can show our six derivations are linearly independent by evaluating them on the 6-tuple of elements comprising the entries of (4),

(8) $x,\ xy_1,\ xy_3,\ (y_1y_2-y_3)z,\ -y_2z,\ z.$

In fact, because our situation is "symmetric", via an automorphism of F interchanging x and z, y_1 and $-y_2$, and y_3 and $y_1y_2-y_3$, it will suffice to display three α-derivations $G \to k$ which are linearly independent on x, xy_1, xy_2, and zero on the last three terms of (8).

Two of these are easily described: take $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial q}$: $G_2 \to G_2$ (where $q = xy_1$, one of the free generators of G_2), compose with α: $G_2 \to k$, and restrict to $G = G_1 \cap G_2$. These take on the values 1, 0, 0, 0, 0, 0 and 0, 1, 0, 0, 0, 0 on (8), respectively. To get the third, let us define β: $G_2 \to k$ to take $y_1y_2 - y_3$ to 1, and all other elements of (7) to 0. Now $\beta | G_1 \cap G_2 = \alpha | G_1 \cap G_2$. (This may be seen by grading F by the free abelian semigroup on x, y_1, y_2, z, of §2, and noting that if an element of $G_1 \cap G_2$, expressed in terms of (6) had a y_3^n term, then taking its $y_1^ny_2^n$-homogeneous component and comparing representations in terms of (6) and (7), we would get $y_3^n = (y_3 - y_1y_2)^n$, a contradiction.) Hence if we take the derivation $\frac{\partial}{\partial x}$: $G_2 \to G_2$, compose with β, and restrict to $G_1 \cap G_2$, we get an α-derivation, and this can be seen to take (8) to 1, 0, -1, 0, 0, 0, completing our argument.

(A simpler example of polynomial subrings of a polynomial ring, with non-polynomial intersection, is $k[x, xy] \cap k[x^2, y]$ in $k[x,y]$. So the above example is not of independent interest, unless the property that the subrings have the same field of fractions as the original ring is hard to obtain.)

4. Ascending chains. Let us first note that it is easy to find a k-algebra which is the union of an ascending chain of free subalgebras, but not itself free. An example is the commutative k-algebra $k[x, x^{1/2}, \ldots, x^{1/2^n}, \ldots]$. But Kolotov asked, in effect, whether such a subalgebra could be found within a free algebra.

The ideas that led to the example I shall give were the following. In trying to prove that the union of any such chain was free, it would be convenient to know that if $G \subseteq F$ are free algebras, and x is a member of a free generating set for F, and lies in G, then x is also a member of some free generating set for G. Since I was not able to prove this, I began looking for counterexamples, and the following approach suggested itself.

One knows how to find all solutions in a free algebra to the equation

$$ab - cd = 1. \tag{9}$$

They form a chain of parametric solutions of increasing degrees of complexity. (|2| p.90, formula (26). The approach is similar to that alluded to in §1, with (9) in place of (1).) Now if s is one of our indeterminates, we can multiply a solution to (9) by s to get

$$(sa)b - (sc)d = s.$$

We may hope that if we choose our solution to (9) sufficiently nondegenerate, the elements

$$p = sa, \quad b, \quad q = sc, \quad d \tag{10}$$

will be independent in F, i.e. free generators of a free subalgebra G, but that $s = pb - qd$ will not belong to a set of free generators of G.

Note that, assuming that a family of elements (10) can be proved to be independent, the problem of showing that s is not a member of a generating set reduces to the abstract question: in a free associative algebra on four generators p, b, q, d, will pb-qd be a member of a free generating set? The answer is no, and this can be proved in several different ways. One of these requires a preparatory result of interest for its own sake.

Lemma 4. Let R be a ring having weak algorithm with respect to a filtration-function v. For any 2-sided ideal $I \subseteq R$, let $\text{min-}v(I) = \inf_{a \in I-\{0\}} v(a)$. Then for two ideals $I, J \subseteq R$, $\text{min-}v(IJ) = \text{min-}v(I) + \text{min-}v(J)$.

Proof. Let X be a weak v-basis for I as a right ideal (|2| p.72 bottom.) Then $I = \oplus_X xR$, and elements coming from different summands are v-independent. Hence $IJ = \oplus_X xJ$, and the result follows immediately. ||

Corollary 5. An element of a free associative algebra which lies in the product of two proper 2-sided ideals is not a member of any free generating set for the algebra.

Proof. An element belonging to a free generating set has degree 1 with respect to the associated filtration. But if $x \in IJ$, then $v(x) \geq \text{min-}v(I) + \text{min-}v(J) \geq 2$. ||

We can now prove:

Lemma 6. Let F be the free associative algebra on 4 indeterminates, $k\langle s,x,y,z \rangle$. Then the 4 elements

(11) $$p = s(xy + 1), \qquad b = zy + 1$$
$$q = s(xyz + x + z), \qquad d = y$$

are independent, and the subalgebra G which they generate contains $s = pb - qd$, but s is not a member of any free generating set for G.

Proof. To show G is free on p, q, b, d, let us map F into the group algebra on the free group on four generators s, u, y, v by sending

$$s \mapsto s, \quad x \mapsto (s^{-1}u - 1)y^{-1}, \quad y \mapsto y, \quad z \mapsto (v-1)y^{-1}.$$

Then we find

$$p \mapsto u \qquad b \mapsto v$$

$$q \mapsto (uv-s)y^{-1} \qquad d \mapsto y.$$

The subalgebra $k\langle u, v, s, y, y^{-1}\rangle$ of that group algebra, in which the image of G now lies, can in turn be mapped into itself by taking s to $uv-sy$, and all the other generators to themselves. The images of the indicated generators of G are now carried to u, s, v, y, proving their independence.

One proof that s does not belong to any free generating set for G is given by applying Corollary 6, with $I = J =$ ideal generated by the indeterminates. Another comes by noting that if s did belong to such a set, $G/(pb-qd)$ would be a free k-algebra. In particular, the relation $\bar{p}\bar{b} - \bar{q}\bar{d} = 0$ would be trivializable in this algebra. But since G is free on p, q, b, d, the algebra $G/(pb-qd)$ is universal for having elements $\bar{p}\bar{b}-\bar{q}\bar{d}$ satisfying such a relation. Hence if that relation were trivializable, any such relation in any ring would be so, which is not so. ||

We shall now obtain our counterexample to the original question by iterating this construction: At each stage, the "p's, q's, b's and d's" of the preceding stage will be treated like "s", and made not to lie in any free generating set of the next larger subalgebra. (This does not itself assure that the union of the resulting chain of subalgebras is not free, but we shall in fact find this easy to prove from the details of the construction.)

Note that the process we are suggesting is a repeatedly "branching" one, since <u>each</u> p, q, b or d will require its own p, q, b and d at the next stage. This "branching" can be conveniently expressed by using as an index-set for our indeterminates the free semigroup with 1 on four symbols p, q, b, d, which will be denoted $<p, q, b, d>$. (In fact, the symbols p, q, b, d will never appear except as indices!) The length of a word $W \in <p, q, b, d>$ will be written $|W|$. (In particular, $|1| = 0$.) We can now state and prove:

<u>Theorem 7.</u> Let $F = k < s;\ x_W, y_W, z_W \mid W \in <p, q, b, d>>$, a free associative algebra on a set of indeterminates indexed as shown. Let us inductively define elements s_W ($W \in <p, q, b, d>$) as follows:

(12) $\qquad s_1 = s,$

(13)
$$s_{Wp} = s_W(x_W y_W + 1) \qquad s_{Wb} = z_W y_W + 1$$
$$s_{Wq} = s_W(x_W y_W z_W + x_W + z_W), \quad s_{Wd} = y_W \qquad (W \in <p,q,b,d>).$$

For each $n \geq 0$, let $S_n = \{s_W \mid |W| = n\}$ (a set of 4^n elements), and let $G_n \subseteq F$ be the subalgebra generated by S_n. Then,

(14) for each n, S_n is an independent set, i.e. G_n is free on S_n,

(15) $G_0 \subseteq G_1 \subseteq \ldots$, but

(16) $G = \bigcup G_n$ is not a free k-algebra, in fact, not even an $\aleph_0$-fir.

<u>Proof.</u> We shall prove (14) by induction. The case n=0 is clear. Assuming S_n is independent, we shall show S_{n+1} independent. Note that the elements of S_n may all be expressed in terms of s and indeterminates x_U, y_U, z_U with $|U| < n$. It follows that $\{s_W, x_W, y_W, z_W \mid |W| = n\}$ is independent. Let us call the free subalgebra of F generated by this set H_n. We now forget about F, and regard the free algebra H_n as the coproduct of 4^n copies of the free k-algebra $k < s, x, y, z >$ of Lemma 6, indexed by $\{W \mid |W| = n\}$.

Then applying the statement of that Lemma to each copy, we easily see that S_{n+1} is independent, establishing (14).

(15), of course, is seen by noting

(17) $$s_W = s_{Wp}\, s_{Wb} + s_{Wq}\, s_{Wd}.$$

Now for each $n \geq 0$, let $I_n \subseteq G_n$ denote the ideal generated by all the generators s_W ($|W| = n$). By (17) we have

(18) $$I_n \subseteq I^2_{n+1}.$$

In particular, $I_1 \subseteq I_2 \subseteq \ldots$, so $I = \bigcup I_n$ will be an ideal of $G = \bigcup G_n$. But applying (18) to I we get

(19) $$I = I^2.$$

We can now finish off in a number of ways. The one requiring no technical knowledge of free algebras, firs, etc. is to note that for each n, $G_n/I_n \cong k$, hence $G/I \cong k$. Now if G were <u>free</u> on a generating set X, then for each $x \in X$, I would have to contain $x-c_x$ for some $c_x \in k$. Hence with respect to the new free generating set $\{x-c_x \mid x \in X\}$, I would be the ideal generated by the indeterminates. But that ideal certainly does not satisfy (19). Alternatively, we can see from Lemma 6 that no right ideal in a ring with weak algorithm satisfies (19). In fact, by the results of |4| no ideal I in an $\aleph_0$-fir satisfies (19). ||

5. <u>Remarks.</u> The analogs of Lemma 6 and Theorem 7 are also true for commutative polynomial rings. In Lemma 6 one proves that s is not a member of any generating set for G by verifying that $G/(s)$ is not a polynomial ring — its spectrum has the singular point $p=q=a=b=0$. In Theorem 10 the first proof that G is not free that we gave goes over word-for-word to the commutative case. (We don't know any analog to "not an $\aleph_0$-fir.")

It is well-known that the free associative algebra on two indeterminates contains subalgebras free on countably many indeterminates; hence the example of Theorem 7 can be embedded in the free algebra on two indeterminates. This does not apply to the commutative analog, of course.

It would be interesting to know whether one can get an example like that of Theorem 7, but where the number of free generators of G_n is bounded.

Lemma 4 and the application we have made of it suggest

<u>Definition 8.</u> If a is an element of a ring R, then $h(a)$ will denote the supremum of numbers n such that x lies in a product $I_1 \ldots I_n$ of n proper 2-sided ideals in R.

By Lemma 4, $h(a)$ is a lower bound on the values $v(a)$ for filtration functions v with respect to which R has weak algorithm. In particular, if R is free, the condition $h(a) = 1$ is necessary for a to lie in a free generating set. It is not sufficient: for instance, from the fact that $k\langle x,y\rangle/(xy-yx-1)$ is simple, we see that $xy-yx-1$ is contained in only one proper 2-sided ideal, so $h(xy-yx-1) = 1$; but since this element goes to 1 modulo the commutator ideal, it can't belong to a free generating set. The condition $\forall \alpha \in k$, $h(a-\alpha) = 1$ has more chance of being sufficient for a to belong to a free generating set, but I don't have high hopes. Nevertheless, as a function on R invariant under automorphisms yet related to the values of filtration-functions, $h(a)$ might be of use in the study of the automorphism groups of free algebras.

Note that for $a \in R \subseteq S$, there is no necessary relation between $h_R(a)$ and $h_S(a)$. In particular, in the situation of Lemma 6, the proper ideals of G that gave the value $h_G(pb-qd) = 2$ generate improper ideals in F.

One might ask whether an ascending chain of free subalgebras G_n of a free algebra F, will have free union G if for every $a \in G$, the values of $h_{G_n}(a)$ are bounded.

Afterthought: In Lemma 4 we really only need to have I a right ideal and J a left ideal. IJ will then just be an additive subgroup, but the the conclusion still holds. So likewise, we might define h'(a) by removing from Definition 8 the condition that I_1 be closed under left multiplication by elements of R, and I_n under right multiplication. This h' might not behave as elegantly as h (e.g., $h(ab) \geq h(a)+h(b)$, but not so for h'), but nontheless, the condition $h'(a) = 1$ (i.e. a does not lie in the product of a proper right ideal and a proper left ideal of R) is a much stronger one than $h(a) = 1$, and so might be more useful in studying generators of free algebras. (We should set $h'(u) = 0$ for u a unit, just to put that trivial case aside.) Cf. [5]. However, one still finds that $h'(xy-yx-1) = 1$. For if an element of degree 2 lies in a product IJ, one can see from the proof of Lemma 4 that it can be expressed in terms of elements of I and J of degree 1. But elements of degree 1 generating a proper 1-sided ideal also generate a proper 2-sided ideal.

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