Von Neumann regular rings with tailor-made ideal lattices

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An algebraic distributive lattice L in which the greatest element is compact can be represented as the ideal lattice of a Neumann regular algebra R over an arbitrary field k if either (a) every element of L is a (possibly infinite) join of join-irreducible compact elements, i.e. if L is the lattice $D(P)$ of lower subsets of a partially ordered set P, or (*known*) if (b) every compact element of L is complemented, i.e. if L is the lattice of ideals of a Boolean ring, or if (c) L has only countably many compact elements.

In each case, the ring R is a direct limit of finite products of full matrix algebras over k.

[2013 Addendum: I do not plan to publish this note; the main results are covered in Goodearl and Wehrung, [**2**].]

1. Introduction. Recall that an element a of a complete lattice L is said to be *compact* if whenever a is majorized by the join of a family of elements of L, it is majorized by the join of some finite subfamily thereof. The compact elements of the subalgebra lattice of an algebra (in the sense of universal algebra, with finitary operations) are the finitely generated subalgebras; thus in such a subalgebra lattice, every element is a (generally infinite) join of compact elements. Any complete lattice with the latter property is called an *algebraic* lattice.

The lattice of 2-sided ideals of a ring R is the subalgebra lattice of R as an (R,R)-bimodule, so it is an algebraic lattice, and it clearly has the further property that the greatest element is compact. If R is von Neumann regular, this ideal lattice is known to be distributive.

We shall here obtain partial converses to the above observation, showing that under any of three hypotheses, an algebraic distributive lattice L whose greatest element 1 is compact can be represented as the ideal lattice of a von Neumann regular ring. Two of the hypotheses (see abstract above) are genuine restrictions on the structure of L, the third is a cardinality condition.

2. Lower subsets of partially ordered sets. An important subclass of the algebraic distributive lattices L are those in which every element is a join of *join-irreducible* compact elements. Such lattices can be characterized up to isomorphism as the lattices $D(P)$ of *lower subsets* of partially ordered sets P; i.e. subsets A such that $p<q\in A \Rightarrow p\in A$. Indeed, given a partially ordered set P it is immediate that $D(P)$ is a complete distributive lattice, that the principal lower subsets $\{q\in P \mid q\leq p\}$ $(p\in P)$ are the join-irreducible compact elements, and that every element is a join of a family of these. Conversely, given a lattice L in which every element is a join of join-irreducible compact elements, if we let P denote the set of these elements, we obtain a natural isomorphism $L\cong D(P)$.

We see that the greatest element of $D(P)$ will be compact if and only if P has only finitely many maximal elements, and every element of P is majorized by one of these.

Given a partially ordered set P with this property, we shall construct in the next two sections a von Neumann regular ring R whose ideal lattice is isomorphic to $D(P)$. This was inspired by hearing of an unpublished result of Handelman, who constructs such a ring whenever P has cardinality $\leq\aleph_1$, generalizing in turn a result of Kim and Roush [**2**] who do this when P is finite.

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3. Constructing R from P. Let P be (as above) a partially ordered set whose set $P_{\max}$ of maximal elements is finite, and such that every element of P is majorized by a member of $P_{\max}$. Let k be any field.

We shall obtain from P a set X, and construct R as a certain k-algebra of endomorphisms of the vector-space V with basis X.

Let X consist of all strings $[p_0, n_1, p_1, \ldots, n_i, p_i]$ where $i \geq 0$, $p_0 > p_1 > \ldots > p_i \in P$, $p_0 \in P_{\max}$, and $n_1, \ldots, n_i$ are arbitrary nonnegative integers. (The latter could equally well be elements of any infinite set.) For each $p \in P$, X_p will denote the set of elements of X with last component p.

Given $p > q$ in P, let us write $U_{p,q}$ for the set of strings $[m_1, q_1 \ldots, m_j, q_j]$ with $j \geq 1$ and $p > q_1 > \ldots > q_j = q$. These strings are not members of X (since they don't begin with a member of P), but rather, the things that one can ''tack on at the end'' of a member of X_p to get a member of X_q. We shall write this tacking-on operation as juxtaposition, letting xu denote the member of X_q obtained by tacking onto $x \in X_p$ the terms of $u \in U_{p,q}$.

Let V be the k-vector-space with basis X. For each $p \in P$, V_p will denote the subspace with basis X_p. The operators on V which will comprise our algebra R will in fact carry each of the spaces V_p into itself; thus this algebra R could equally be described as a subalgebra of the direct product over P of the endomorphism algebras of the spaces V_p; and this is the way I think of it, but it is less cumbersome to speak of an algebra of maps than an algebra of P-tuples of maps.

To construct R, we define for every $p \in P$ and $x, y \in X_p$ the operator $f_{x,y}: V \to V$ by

$$\begin{aligned} &f_{x,y}(y) = x, \\ &f_{x,y}(yu) = xu \text{ for every } u \in U_{p,q} \ (p > q \in P), \\ &f_{x,y}(z) = 0 \text{ for } z \in X \text{ not of the form } y \text{ or } yu. \end{aligned} \tag{1}$$

The set of operators $f_{x,y}$ with $x, y \in X_p$ will be denoted F_p; the full family of operators $\cup_P F_p$ will be written F.

Lemma 1. *The set of linear operators F is linearly independent over k.*

Proof. Given a nontrivial linear combination $r = \Sigma\, c_{x,y} f_{x,y}$ $(c_{x,y} \in k)$, choose $f_{x,y}$ with $c_{x,y} \neq 0$ so as to maximize the element p such that $f_{x,y} \in F_p$. Then in $r(y) \in V$, the coefficient of x is $c_{x,y}$, so $r \neq 0$. □

Lemma 2. *The span of F is a unital subalgebra R of the algebra of endomorphisms of V.*

Namely, the identity endomorphism of V is given by

$$1 = \Sigma_{p \in P_{\max}} f_{[p],[p]}, \tag{2}$$

and composition of operators is described by

$$\begin{aligned} &f_{w,x} f_{x,y} = f_{w,y} \\ &f_{w,x} f_{xu,yu} = f_{wu,yu} \\ &f_{wu,xu} f_{x,y} = f_{wu,yu} \\ &f_{w,x} f_{y,z} = 0 \ \textit{in all other cases.} \end{aligned} \tag{3}$$

Proof. By calculation from (1). □

4. Properties of R. The next Lemma describes R locally.

Lemma 3. *Let Q be any finite subset of P containing $P_{\max}$, and let N be any positive integer. Let $X_{Q,N}$ denote the finite subset of X consisting of those strings whose components from P all lie in Q, and whose integer components are all $\leq N$.*

Then the subalgebra $R_{Q,N}$ of R spanned by the elements $f_{x,y}$ with $x, y \in X_{Q,N}$ is isomorphic to a finite direct product *of full matrix algebras over* k*.*

Proof. For $p \in Q$ let us abbreviate $X_{Q,N} \cap X_p$ to $X_{Q,N,p}$. Given $x, y \in X_{Q,N,p}$, define $e_{x,y} = f_{x,y} - \Sigma f_{xu,yu}$, where the sum is over all u of the form $[i, q]$ with $i \leq N$ and $p > q \in Q$. Writing $E_{Q,N,p} = \{e_{x,y} \mid x, y \in X_{Q,N,p}\}$, it is easy to verify from (3) that the elements of this set multiply like matrix units (in the algebra of square matrices of size $\mathrm{card}(X_{Q,N,p})$), that they have zero product with elements of $E_{Q,N,q}$ for $q \in Q$, $q \neq p$, and that $\cup_{p \in Q} E_{Q,N,p}$ spans $R_{Q,N}$.

Thus $R_{Q,N}$ is the direct product over $p \in Q$ of a family of full matrix algebras. □

Since R is the direct limit of these subalgebras as we enlarge Q and N, we have

Corollary 4. R *is von Neumann regular.* □

To determine the ideal lattice of R, we begin with

Lemma 5. *The ideal* RrR *generated by an element* $r \in R$ *contains all elements of the basis* F *occurring with nonzero coefficient in* r*.*

Proof. Let $f_{x,y} \in F_p$ be an element occurring with nonzero coefficient in r, chosen among such elements so as to minimize $p \in P$. We shall show that $f_{x,y}$ belongs to RrR; the same conclusion for arbitrary elements occurring with nonzero coefficient in r follows easily by induction on the number of summands in r.

Consider the element $f_{x,x}\, r\, f_{y,y} \in RrR$. One sees from (3) (and the minimality of p) that this is a scalar multiple of $f_{x,y}$, the scalar factor being the sum of the coefficients in r of the basis elements

$f_{x,y}$ itself

and

(4) those $f_{x',y'}$ $(x', y' \in X_q,\ q > p)$ for which there exists $u \in U_{q,p}$ such that $x = x'u,\ y = y'u$.

If the sum of all these coefficients is nonzero, then RrR contains a nonzero scalar multiple of $f_{x,y}$ and we are done. In the contrary case, the family of elements (4) must be nonempty; choose such an element $f_{x',y'} \in F_q$ so as to maximize q. Now let $v = [n, p] \in U_{q,p}$, where n is chosen so that $f_{x'v,y'v}$ does *not* appear with nonzero coefficient in r. (This is where we use the fact that there are infinitely many possibilities for n.) Then the element $f_{x,x'v}\, r\, f_{y'v,y}$ will also be a scalar multiple of $f_{x,y}$, the scalar coefficient being the coefficient of $f_{x',y'}$ in r (no other summand in r but $f_{x',y'}$ makes a nonzero contribution), and hence nonzero. Thus the ideal of R generated by r contains $f_{x,y}$ in either case. □

It follows that the ideal lattice of R is generated under (generally infinite) joins by the ideals $R f_{x,y} R$. The information about these ideals needed to determine this lattice is collected in

Lemma 6. (i) *For* $p \in P$*, all elements* $f_{x,y} \in F_p$ *generate the same ideal, which we shall denote* I_p*.*

(ii) *If* $p < q$ *in* P*, then* $I_p \subseteq I_q$*.*

(iii) *For any* $p \in P$*,* $I_p \not\subseteq \Sigma_{p \not\leq q} I_q$*.*

Proof. (i) It suffices to show that given $f_{w,x}$ and $f_{y,z}$ in F_p, the ideal generated by $f_{w,x}$ contains $f_{y,z}$. And indeed, $f_{y,z} = f_{y,w}\, f_{w,x}\, f_{x,z}$.

(ii) Taking $f_{w,x} \in F_q$, $f_{y,z} \in F_p$ and $u \in U_{q,p}$, we have $f_{y,z} = f_{y,wu}\, f_{w,x}\, f_{xu,z}$.

(iii) Consider the action of R on the invariant subspace $V_p \subseteq V$. We find from (1) that all elements of F_p have nontrivial action, but that elements of F_q with $q \not\geq p$ act trivially. □

We easily deduce

Proposition 7. *The lattice of ideals of* R *is isomorphic to* $D(P)$, *the lattice of lower subsets of* P. □

5. Ideal lattices of commutative regular rings. Let us look briefly at *commutative* von Neumann regular rings. The results we shall describe are doubtless known, but we include them for perspective on our other results.

If a is a compact element of the ideal lattice of a commutative von Neumann regular ring R, it is generated by an idempotent e, and one sees that $1-e$ will generate an ideal c such that

$$a \vee c = 1, \ a \wedge c = 0.$$

Given any element a in a bounded lattice (a lattice with a least element, 0, and a greatest element, 1), such a c is called a *complement* of a, and an element a having a complement is said to be *complemented*.

In any bounded distributive lattice L, an element a with a complement c induces a direct product decomposition

(5) $$L \cong [0, a] \times [0, c].$$

It is easy to deduce that in such a lattice complements are unique when they exist, and that the complemented elements form a sublattice which is in fact a Boolean ring. If furthermore L is complete and 1 is compact, we can deduce from (5) that every complemented element a is also compact.

Returning to the lattice L of ideals of a commutative von Neumann regular ring, in which we have seen that every compact element is complemented, we conclude that the compact elements are precisely the complemented elements, and that these form a Boolean ring B. Every element $a \in L$, being a join of compact elements, will be determined by the *ideal* of the Boolean ring B comprising those compact elements which it majorizes; moreover, it is not hard to verify that this is the only ideal of B whose join is $a \in L$, hence L can be identified with the lattice of all ideals of B. One can also describe this as the lattice of open subsets of the Stone space $\mathrm{Spec}(B)$.

A Boolean ring B is itself a von Neumann regular ring, so conversely the lattice of ideals of any Boolean ring is the ideal lattice of a commutative von Neumann regular ring. But we can say more: for any field k the k-algebra R of locally constant k-valued functions on $\mathrm{Spec}(B)$ has this same lattice of ideals. (B itself is given by the case $k = Z_2$ of this construction.) This k-algebra B is generated by the characteristic functions of the open-closed subsets of $\mathrm{Spec}(B)$, and the subalgebra generated by any finite family of these is a finite product of copies of k, so R is a direct limit of such product algebras. Hence as in the preceding section, though degenerately, the algebra we have found is a direct limit of finite products of full matrix algebras over k.

In summary,

Lemma 8. (reference?) *Let* k *be a field. Then the following conditions on a lattice* L *are equivalent:*

(i) L *is isomorphic to the lattice of ideals of a commutative von Neumann regular k-algebra* R.

(ii) L *is an algebraic distributive lattice in which all compact elements are complemented.*

(iii) L *is isomorphic to the lattice of ideals of a Boolean ring.*

(iv) L *is isomorphic to the lattice of open subsets of a Stone space* (*i.e., a totally disconnected compact Hausdorff space*).

Moreover, in (i) *the k-algebra* R *can always be taken to be a direct limit of a system finite product algebras* $k \times \ldots \times k$. □

The intersection of the class of lattices characterized here with the class of lattices of the form $D(P)$ with P as in the preceding section is quite small; for in a lattice $D(P)$, a necessary condition for the lower subset generated by an element $p \in P$ to be complemented is that p be maximal. Hence if all compact elements are complemented, we have $P = P_{\max}$, a finite set, and the lattice will have the form 2^n for some integer n.

6. Ideals of semilattices. In any complete lattice L, the join of two compact elements is compact, hence the compact elements form an upper semilattice S (not generally complete). If L is algebraic, an arbitrary element of L will be a possibly infinite join of compact elements, hence the join of an upward directed system of compact elements. It is easy to deduce that L is isomorphic to the lattice $I(S)$ of *ideals* of the semilattice S containing the least element, 0. An ideal in a semilattice means a lower subset closed under finite joins; the finitely generated ideals are all principal (generated by the join of any finite generating set), and correspond bijectively to the elements of S, while the general ideal is a directed union of principal ideals.

Any semilattice S is, of course, the direct limit of its finite subsemilattices S', and in a finite semilattice, every element is a join of join-irreducible elements. The idea of the next ring-theoretic construction is, very roughly, to imitate our first construction, using instead of the single partially ordered set P the system of partially ordered sets of nonzero join-irreducible elements of these finite subsemilattices S', going to the limit over larger and larger $S' \subseteq S$. We shall be able to bypass some of the complications of that construction, and in particular, drop the integers n_i that we used there, because the nature of our limit process itself will provide the needed ‘‘multiple copies’’ of our vector space basis elements. Of course, the new construction will create some complications of its own. I have not been able to overcome the technical difficulties that would be involved in setting up the direct limit over a general directed system; in particular, of making the required diagrams of algebra maps commute. Hence we shall require S to be countable, so that we can take our direct limit over a system indexed by the natural numbers.

7. The construction. Let L be an algebraic distributive lattice whose greatest element 1 is compact, and such that the semilattice S of compact elements of L is countable. Let us write S as the union of a chain of finite subsemilattices,

(6) $$\{0, 1\} = S_0 \subseteq S_1 \subseteq \ldots$$

For $n = 0, 1, \ldots,$ let

$$P_n = \{\text{join-irreducible elements of } S_n\}.$$

Here we do not count 0 as join-irreducible (it is the join of the empty family), so it does not belong to any of the P_n. Observe that every element of S_n will be the join of a subset of P_n.

For $n \geq 0$ and $p \in P_n$, we define the finite set

$$X_{n,p} = \{\, [p_0, p_1, \ldots, p_n] \mid p_0 \geq \ldots \geq p_n = p,\ p_i \in P_i\ (0 \leq i \leq n)\,\}.$$

We shall again use juxtaposition to denote extension of such strings. In this case this will mean that for $x = [p_0, \ldots, p_{n-1}] \in X_{n-1,p}$ and $q \in P_n$ with $p \geq q$, we will write

$$xq = [p_0, \ldots, p_{n-1}, q] \in X_{n,q}.$$

We now form the vector space $V_{n,p}$ on each set $X_{n,p}$, and for each n we write $X_n = \bigcup_{p \in P_n} X_{n,p}$ and $V_n = \bigoplus_{p \in P_n} V_{n,p}$. We define R_n to be the k-algebra of those endomorphisms of V_n which take each $V_{n,p}$ into itself, i.e. the direct product over $p \in P_n$ of the full matrix algebras on $\mathrm{card}(X_{n,p})$ generators. A basis for R_n consists of the elements

$$e_{x,y} \quad (x, y \in X_{n,p},\ p \in P_n),$$

defined to carry x to y, and all other elements of X_n to zero.

Finally, we define the linear maps

$$f_n : R_n \to R_{n+1}$$

by

(7) $$f_n(e_{x,y}) = \Sigma_{q \in P_{n+1},\, p \geq q}\, e_{xq,yq} \in R_{n+1}.$$

It is straightforward to verify that these maps are homomorphisms. They are also one-to-one; this follows from the observation that every element $p \in P_n$ can be written as a join of elements of P_{n+1}, so in particular, p majorizes some element q in that set.

For every $n \geq 0$ and $p \in P_n$, let $J_{n,p}$ denote the ideal of R_n generated by the elements $e_{x,y}$ with $x, y \in X_{n,p}$. Thus, every ideal of R_n can be written uniquely as the sum of some subset of this family of ideals. From the definition of f_n it is easy to see

Lemma 9. *For $p \in P_n$, $q \in P_{n+1}$, the ideal of R_{n+1} generated by $f_n(J_{n,p})$ contains $J_{n+1,q}$ if and only if $p \geq q$.* □

We shall take for our ring R the direct limit of the chain of embeddings f_n. But in order to make this ring have the properties we want, we must (sorry!) go back to (6), i.e. to our choice of the chain of semilattices $S_n \subseteq L$, and impose an additional condition. To state this condition, let us, for each natural number n and each positive integer $i \leq \mathrm{card}(S_n)$, define $a_{n,i}$ to be the number of elements of P_{n+1} which are majorized by exactly i elements of S_n. Then by enlarging our semilattices if necessary, we can clearly achieve a situation where, for each n,

(8) We cannot, by keeping S_n fixed and enlarging S_{n+1}, *decrease* the sequence $(a_{n,1}, a_{n,2}, \ldots, a_{n,\mathrm{card}(S_n)})$ with respect to lexicographic ordering.

We shall henceforth assume (8). The purpose of this peculiar requirement is to give us

Lemma 10. *Let $p_1, p_2 \in S_n$ and $q \in P_{n+1}$, and suppose that $p_1 \vee p_2 \geq q$. Then either $p_1 \geq q$ or $p_2 \geq q$.*

Proof. From the distributivity of L we see that in L, $q = (p_1 \wedge q) \vee (p_2 \wedge q)$. We want to show that one of these meets equals q. Assume the contrary. Now the intersections $p_1 \wedge q$ and $p_2 \wedge q$ may not be compact in L, but by the assumption of algebraicity they will be joins of families of compact elements. Using the compactness of q, we deduce that there will exist elements $r_1 \leq p_1 \wedge q$ and $r_2 \leq p_2 \wedge q$ which still satisfy $r_1 \vee r_2 = q$, given by joins of *finite* subfamilies of those families of compact elements, and hence themselves compact. If we extend S_{n+1} by adjoining r_1 and r_2, then the resulting semilattice has no *new join-irreducible* elements except perhaps these elements themselves; moreover, q has ceased to be join-irreducible. Now r_1 and r_2 are each majorized by *more* elements of S_n than q was (because one of them is majorized by p_1 and the other by p_2), hence we have reduced the number of join-irreducible elements majorized by the number of members of S_n that majorized q, while possibly increasing only the numbers majorized by larger families. This contradicts (8), proving the Lemma. □

This allows us to generalize Lemma 9 to

Lemma 11. *For $m < n$, $p \in P_m$, $q \in P_n$, the ideal of R_n generated by $f_{n-1} \ldots f_m(J_{m,p})$ contains $J_{n,q}$ if and only if $p \geq q$.*

Proof. ‘‘Only if’’ is straightforward, because the maps f_i take elements $e_{x,y}$ such that x and y have last component $p \in L$ only to sums of elements for which the corresponding element of L is $\leq p$ (7).

To get the converse, suppose $p \geq q$ and assume inductively that the ideal of R_{n-1} generated by the image of $J_{m,p}$ contains every $J_{n-1,r}$ $(r \in P_{n-1})$ such that $p \geq r$. Now in the semilattice S_{n-1}, the element p is a join of join-irreducible elements $r \in P_{n-1}$, hence this join majorizes q in S_n, hence by the preceding Lemma, one such $r \in P_{n-1}$ majorizes q. By our inductive assumption the ideal generated by the image of $J_{m,p}$ contains $J_{n-1,r}$, and by Lemma 9 the ideal generated by the image of $J_{n-1,r}$ contains $J_{n,q}$, giving the desired conclusion. □

We now let R denote the direct limit of the chain of embeddings f_n.

Lemma 12. *Let* m, n *be natural numbers,* $p_1, \ldots, p_g \in P_m$, *and* $q_1, \ldots, q_h \in P_n$. *Then the ideal of* R *generated by the image of* $J_{m,p_1} + \ldots + J_{m,p_g}$ *contains the ideal generated by the image of* $J_{n,q_1} + \ldots + J_{n,q_h}$ *if and only if* $\vee p_i \geq \vee q_j$ *in* L.

Proof. Suppose the ideal of R generated by the image of $J_{m,p_1} + \ldots + J_{m,p_g}$ contains the ideal generated by the image of $J_{n,q_1} + \ldots + J_{n,q_h}$. Since these ideals are finitely generated, the same inclusion must hold between their images in R_N for some integer N, which we can take greater than both m and n. We can deduce from the preceding Lemma (and the "Boolean" nature of addition of ideals in a product of matrix rings over a field) that every $r \in P_N$ which is majorized by some q_j is also majorized by some p_i. But each q_j is a join of join-irreducible elements $r \in P_N$, and by the above observation each of these elements r is majorized by some p_i, so each q_j is majorized by $\vee p_i$; hence so is $\vee q_j$, as required.

Conversely, suppose $\vee p_i \geq \vee q_j$, and let us take any $N > m, n$. Then $\{p_i\} \subseteq S_{N-1}$, hence by Lemma 10, the set of elements of P_N majorized by $\vee p_i$ is the union of the sets majorized by the separate p_i's. Combining this with the analogous characterization of the union of the sets of elements of P_N majorized by the q_j's, we conclude that the former union contains the latter. Hence from the characterization in the preceding Lemma of the ideals of R_N generated by the images of the J_{m,p_i} and J_{n,q_j}, we get the desired inclusion of sums of ideals. □

It follows that the semilattice of finitely generated ideals of R is isomorphic to the semilattice S of all compact elements of L. This in turn implies

Proposition 13. *The ideal lattice of* R *is isomorphic to* L. □

8. Remarks The existence of such a von Neumann regular ring R was proved by Kim and Roush [**2**] under the additional hypothesis that L was the ideal lattice of a countable distributive lattice. This is equivalent to requiring that the upper semilattice S of compact elements be closed under finite meets; if we had assumed that, we could have taken our S_n to be distributive lattices themselves. It was to get around the fact that they were not that we imposed condition (8) on these semilattices, leading to Lemma 10, which establishes a relation between successive S_n that a finite distributive lattice has with respect to itself.

Like Kim and Roush, I had to put a countability hypothesis on S. If we drop this hypothesis, there is a reasonable candidate for a directed partially ordered set over which to perform our limit construction, namely (assuming for simplicity that L is infinite) the set of all finite subsemilattices of S containing 0 and 1, ordered by writing $S' < S''$ if S' is a proper subsemilattice of S'', and they satisfy the analog of condition (8). The problem, however, is what to use for the sets $X_{S'}$, and for analogs of the "extensions" $xu \in X_{n+1}$ of elements $x \in X_n$, which we used in defining our ring homomorphisms f_n.

9. Remarks on infinite distributivity. It is interesting to consider the three classes of algebraic distributive lattices we have looked it in terms of the infinite meet-distributivity condition

(9) $$\wedge_I (a \vee b_i) = a \vee (\wedge_I b_i).$$

This condition is satisfied for *all* a and b_i in a lattice of the form $D(P)$, since in such a lattice arbitrary joins and meets are given by unions and intersections of subsets of P. In the lattice of open subsets of a Stone space X, condition (9) does not hold in general: If x is a non-isolated point, (b_i) the set of all open neighborhoods of x, and a the open set $X - \{x\}$, then all of the joins on the left-hand side of (9) equal X, hence so does their meet, while the meet on the right is the interior of $\{x\}$, hence empty, so the right-hand join equals a. However, in any complete distributive lattice, (9) holds whenever a is *complemented*. (Use the decomposition (5).) Hence in the lattice of open subsets of a Stone space, we can say that it holds whenever a is *compact*.

Even this result for compact elements does not hold in lattices of the sort considered in the preceding section. Consider, for instance, the lattice L of ideals of the ring of integers $\mathbf{Z}$. This is certainly an algebraic lattice, and easily seen to be distributive. Because $\mathbf{Z}$ has ascending chain condition, every element of L is compact, and L is countable. However, letting a be the ideal $2\mathbf{Z}$ and b_i the ideal $3^i\mathbf{Z}$, we see that on the left-hand side of (9) each join is the unit ideal, so the left hand side is the unit ideal, while on the right the meet is zero, so we get the ideal a.

Since it is difficult to picture the algebra associated to this lattice by our construction, let me also give an explicit von Neumann regular ring for which (9) fails for compact a. Let k be any field, and R the k-algebra of all sequences of 2×2 matrices over k which are eventually constant, and whose eventual value is diagonal (with not necessarily equal diagonal entries scalar). Let a denote the ideal of those sequences of matrices whose eventual scalar value has 0 as its $(2,2)$ entry, and b_i the ideal of sequences whose eventual scalar values have 0 as their $(1,1)$ entry, and which have zero for *all* entries in all terms up to the ith. The reader can easily evaluate the two sides of (9) and see that again the left hand side is the unit ideal while the right hand side is a.

On the other hand, the infinite join-distributivity condition

$$\vee_I \, (a \wedge b_i) = a \wedge (\vee_I \, b_i), \qquad (10)$$

holds in any algebraic distributive lattice. (Idea of proof: If not, take a compact element majorized by the right hand side but not by the left.)

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