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STRONGLY CLEAN ELEMENTS THAT ARE ONE-SIDED INVERSES

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ABSTRACT. A longstanding open question is whether every strongly clean ring is Dedekind-finite (definitions recalled below). We give an example of a ring with two strongly clean elements that are one-sided, but not two-sided inverses of one another, suggesting that the answer to that question may be negative.

We discuss possible ways of strengthening this result to give a full negative answer. We end with some brief observations on related topics, in particular, uniquely strongly clean rings.

1. The example

Here is some standard usage that we will follow.

Definition 1.1. *In this note, all rings are understood to be associative and unital.*

An element x of a ring R is called clean *if it can be written as the sum $e+u$ of an idempotent e and a unit u of R, and* strongly clean *if it can be so written with e and u commuting with each other, equivalently, commuting with x. A ring R is called clean, respectively strongly clean, if all of its elements have the property named* ([5], [7]).

A ring R is called Dedekind-finite *if every one-sided-invertible element of R is invertible.*

The question of whether every strongly clean ring is Dedekind-finite was posed in 1999 by W. K. Nicholson [5, Question 2, p. 3590]. For k a field, we construct below a k-algebra R having a pair of elements that are strongly clean, and are one-sided but not two-sided inverses of one another. Thus, if this ring can somehow be embedded in a ring where *all* elements are strongly clean, the resulting example would answer Nicholson's question in the negative.

Here is some notation that will be used in describing our example.

Definition 1.2. *Throughout this section k will be a field, and V will denote the underlying k-vector-space of the subalgebra of the rational function field $k(t)$ comprising elements having denominator not divisible by t; i.e., of the localization $k[t]_{(t)}$ of the polynomial ring $k[t]$ at the prime ideal (t). We denote by R the k-algebra $\mathrm{End}_k(V)$ of all k-linear endomaps of the vector space V, written on the left of their arguments and composed accordingly.*

(We could equally well take for R the subalgebra of $\mathrm{End}_k(V)$ generated by the five maps that will be denoted y, x, $(y-1)^{-1}$, e, and $(x-e)^{-1}$ below. But it will be convenient to have a ring that we can refer to when defining these elements.)

Let y be the endomorphism of V given by multiplication by t in $k[t]_{(t)}$:

(1.1) $\quad y(f(t)) \;=\; t\,f(t) \quad \text{for } f(t) \in V,$

and x the endomorphism of V given by multiplying by t^{-1} and, in power series notation, dropping the t^{-1} term if any. In rational function notation, this is

(1.2) $\quad x(f(t)) \;=\; t^{-1}(f(t)-f(0)) \quad \text{for } f(t) \in V.$

It is clear that

(1.3) $\quad x\,y \;=\; 1.$

On the other hand, $y\,x$ has the effect of dropping the t^0 term from the power series representing an element of V, so

(1.4) $\quad y\,x \;\neq\; 1.$

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We will now prove x and y strongly clean in R.

Note that $y-1$ is invertible in R: it takes $f(t)$ to $(t-1)f(t)$, and $t-1$ is invertible in $k[t]_{(t)}$. Thus y is the sum of the invertible element $y-1$ and the idempotent element 1, which commute, showing that

(1.5) y is strongly clean in R.

In proving x strongly clean, we can't take for e one of the idempotents 1 or 0: $x-1$ is not invertible since x fixes $(1-t)^{-1}$ (easily seen by looking at power series expansions); and $x-0$ is not invertible since x annihilates 1. So to get a decomposition $x=e+u$, we will have to use a nontrivial idempotent e.

To construct e, note that

(1.6) $V \;=\; V_0 \oplus V_1$,

where

(1.7) V_0 is the subspace $k[t] \subseteq V$,

and

(1.8) $V_1 \subseteq V$ is the space of rational functions whose numerators have lower degrees than their denominators.

Indeed, given an element $p(t)/q(t) \in V$, where $p(t),\, q(t) \in k[t]$, its decomposition as in (1.6) arises from the decomposition of $p(t)$ as a multiple of $q(t)$ in $k[t]$, plus a remainder of degree less than that of $q(t)$. (Intuitively, V_1 is the set of $f(t) \in V$ such that $f(\infty)=0$.) We now

(1.9) Let $e \in R$ be the projection of V onto V_0 under the decomposition (1.6).

Clearly, x, defined in (1.2), carries V_0 into itself. A little thought shows that it also carries V_1 into itself: Although when (1.2) is applied to an element $f \in V_1$, the expression $f(t)-f(0)$ has in general lost the property that f had, of having value 0 at ∞, it still has a finite value there, namely $-f(0)$; so multiplying by t^{-1} again brings the value at ∞ to 0. (The reader can translate all this into reasoning about degrees and values at 0 of numerators and denominators.)

Hence,

(1.10) x commutes with e.

Now since V_1 contains no nonzero constants, we see from (1.2) that x has trivial kernel on V_1. It is also surjective on V_1: Given $f(t) \in V_1$, if we let $c \in k$ be the value of $tf(t)$ at ∞, we find that $tf(t)-c \in V_1$ and x carries that element to $f(t)$. (This is like the construction of the preceding paragraph, with the roles of 0 and ∞ interchanged.) Thus x acts invertibly on V_1. Since e annihilates V_1, this tells us that

(1.11) $x-e$ acts invertibly on V_1.

How does $x-e$ behave on V_0? For every $n \geq 0$, we see from (1.2) that the action of x on polynomials in t of degree $\leq n$ is nilpotent, while by (1.9), e acts as the identity endomorphism on V_0; so

(1.12) $x-e$ acts invertibly on V_0.

Together, (1.11) and (1.12) say that $x-e$ acts invertibly on V; so x is the sum of the idempotent e and the invertible element $x-e$, which commute by (1.10). So

(1.13) x is strongly clean.

We have thus proved

Proposition 1.3. *In the k-algebra R of all k-vector-space endomorphisms of the vector space V defined in Definition 1.2, the elements x and y described by* (1.1) *and* (1.2) *are strongly clean, and satisfy* $x\,y=1$ *but* $y\,x \neq 1$. □

2. Thoughts on possible stronger results

I wonder whether one can push the approach of the above construction further, and get an example actually answering Nicholson's question:

Question 2.1. *As in Definition 1.2, let k be a field and V the underlying k-vector-space of $k[t]_{(t)}$; and as in* (1.6)-(1.8)*, let $V=V_0 \oplus V_1$, where $V_0=k[t]$ and V_1 is the space of elements of $k[t]_{(t)}$ whose numerators have lower degrees than their denominators. Further,*

(2.1) *Let R be the algebra of k-linear endomorphisms x of V such that there exist subspaces $V_0' \subseteq V_0$ and $V_1' \subseteq V_1$, each of finite k-codimension in the indicated space, and elements $r_0,\, r_1 \in k(t)$, such that x carries V_0' into V_0 by multiplication by r_0, and carries V_1' into V_1 by multiplication by r_1.*

Is R strongly clean?

Note that if an element x satisfies the condition of (2.1), the elements r_0 and r_1 referred to will be uniquely determined by x, but the subspaces V_0' and V_1' will not, though there will, of course, be maximal subspaces on which x acts as multiplication by the indicated elements. Given V_0' and V_1', the element x is uniquely determined by r_0, r_1, and the behavior of x on vector-space complements of V_0' in V_0 and of V_1' in V_1. On those complements, x is allowed to act as an arbitrary k-linear map into V; it is not required to carry elements of V_0 or V_1 into that same subspace, nor to respect multiplication by members of $k[t]_{(t)}$. (And indeed, the y of (1.1) carries $(1-t)^{-1} \in V_1$ to $t\,(1-t)^{-1}$, whose numerator is not of lower degree than its denominator, as would be required for it to belong to V_1. Its $V_0 \oplus V_1$ decomposition is $-1+(1-t)^{-1}$.)

It is easy to see that the set of maps x satisfying the condition of (2.1) is closed under addition, and not hard to check that it is closed under composition.

In studying Question 2.1, it might help to look at a more general construction of which (2.1) is a particular case. If V is an infinite-dimensional vector space over a field k, suppose we define a "partial endomorphism" of V to mean a k-linear map from a subspace of finite codimension in V into V, and a "quasi-endomorphism" of V to mean an equivalence class of partial endomorphisms under the equivalence relation of agreeing on a subspace of finite codimension. (*Have these concepts been studied? If so, what are they called?*) The quasi-endomorphisms of V form a k-algebra in a natural way. Now suppose V_0 and V_1 are infinite-dimensional vector spaces over k, and D_0, D_1 are division subalgebras of the algebras of quasi-endomorphisms of V_0 and V_1 respectively. Let R be the ring of all endomorphisms of $V_0 \times V_1$ which, when restricted to each V_i, carry a subspace of finite codimension back into V_i, and such that the induced quasi-endomorphism is a member of D_i. (Thus, R has a natural homomorphism to $D_0 \times D_1$, whose kernel is the ideal of endomorphisms of V of finite rank.)

Sadly, my best guess is that the answers to Question 2.1 and its generalization to algebras of the sort described in the preceding paragraph are, at least as presently formulated, negative. This is based on the following observation. We have used two spaces V_0 and V_1 in Question 2.1 only to set up a framework in which to construct pairs of elements which are one-sided but not two-sided inverses to one another; but the strong cleanness conclusion, if it holds, should apply to the corresponding construction on a single vector space. Now let V be the underlying k-vector space of the polynomial ring $k[t]$, which has, as above, a natural algebra of quasi-endomorphisms isomorphic to $k(t)$, and let $x \in \mathrm{End}(V)$ be the operation of multiplication by t. Then given any $y \in \mathrm{End}(V)$ centralizing x, we can regard $y(1)$ as a polynomial $p(t)$, and it is easy to verify that y must act on all of V as multiplication by $p(t)$; so the centralizer of x is isomorphic as a k-algebra to $k[t]$, and x is not clean in that k-algebra. (Restricting our search for endomorphisms centralizings x to the subalgebra of elements of $\mathrm{End}(V)$ which induce quasi-endomorphisms belonging to $k(t)$ obviously does not improve things.)

On the other hand, if we let V be the underlying vector space of $k[t]_{(t)}$ or of $k[[t]]$, then the endomorphism corresponding to multiplication by t does become strongly clean in that over-ring; indeed, again calling that operation x, we see that $x-1$ is invertible in those two algebras. So perhaps some variant of the idea of Question 2.1 will give a strongly clean ring containing elements x and y as in Proposition 1.3.

Perhaps a much stronger sort of result is true than what is asked for above:

Question 2.2. *For every* (*associative unital*) *algebra R over a field k, and every $x \in R$, does there exist a k-algebra $R' \supseteq R$ in which x is strongly clean?*

(*If so, then by a transfinite induction one can embed every k-algebra R in a strongly clean k-algebra.*)

To prove such a result, one might start with an arbitrary faithful action of R on a vector space V, and try to obtain induced faithful actions on new vector spaces that would make the action of x strongly clean in the endomorphism rings of these spaces. I have looked at the case where $W = V^*$, the vector-space dual of V (with the induced action of R on W written on the right if we have been writing the action on V on the left, to handle the contravariance), but not gotten anywhere. Variants which I haven't looked at would be to let W be an infinite direct sum or direct product of copies of V, or an ultrapower of V.

A different approach to Question 2.2, which I have also attempted without success, but which others might try is, assuming a k-algebra R and an element $x \in R$ given, to see what happens when one adjoins to R two elements u and u^{-1}, universal for satisfying the five relations saying that they are inverses to each other, that they commute with x, and that $x-u$ is idempotent. A slight variant would be to first adjoin a universal idempotent e commuting with x, study what one can say about the structure of the resulting ring, *then* adjoin a universal inverse to $x-e$, and see whether one can prove that R embeds in the resulting ring. If we had a nice normal form for elements of R, then the Diamond Lemma [1] might be used to get a normal form for the extended ring, and see whether the natural map of R into it is indeed an embedding.

The Fitting Decomposition Theorem [4, p. 299 (19.16)] implies that the endomorphism ring of any module of *finite length* over any ring is strongly clean. Such endomorphism rings are Dedekind-finite, but conceivably, that theorem might be useful in some multi-step construction of a strongly clean non-Dedekind-finite ring.

For some results relating strong cleanness to other ring-theoretic conditions, including Dedekind finiteness, see [2].

3. Notes on related topics

An element x of a ring R is called *uniquely clean* if there is a *unique* decomposition $x = e + u$ with e idempotent and u invertible, and *uniquely strongly clean* if there is a unique such decomposition in which e and u commute with each other. Rings in which all elements have these properties are called *uniquely clean rings,* respectively, *uniquely strongly clean rings.* There has been considerable research on these classes of rings, e.g., [3], [6], [7].

An open question, [3, Question 19], [7, Question 5.1], is whether every homomorphic image of a uniquely strongly clean ring is again uniquely strongly clean. This is known to be true for uniquely strongly clean rings in which all idempotents are central (which are precisely the uniquely clean rings. For that equivalence, see [6, Lemma 4], and for the result on homomorphic images, [6, Theorem 22]). As a possible way to look for a counterexample if idempotents are not required to be central, one might first note that though the matrix ring $M_2(\mathbb{Z}/2\mathbb{Z})$ is strongly clean, it is not uniquely strongly clean, since $x = e_{12} + e_{21} + e_{22}$ has the two strongly clean decompositions $0 + x$ and $1 + (x - 1)$, and then try to find a uniquely strongly clean ring R, and a surjective homomorphism $R \to M_2(\mathbb{Z}/2\mathbb{Z})$ under which distinct elements p and q both map to x, with the (unique) strongly clean decompositions of p and q mapping to the distinct decompositions of that common image.

Another topic: Observe that a nonzero uniquely strongly clean ring R can never be an algebra over a field k with more than two elements, since if u is an element of k other than 0 and 1, it has the two strongly clean decompositions $0 + u$ and $1 + (u - 1)$. Even if a uniquely strongly clean ring R is not assumed an algebra over a field, image rings of characteristic 2 occur throughout the study of such rings. For example, the quotient of R by its Jacobson radical is a Boolean ring [6, Theorem 20, (1) $\Longrightarrow$ (3)].

One might get interesting results not restricted to rings with important homomorphic images of characteristic 2 if, in an algebra R over a field k, one defined a "metaidempotent" (*is there an existing term?*) to mean a k-linear combination r of a family of mutually commuting idempotents; equivalently, an element r that satisfies a polynomial relation of the form $(r-a_1)\dots(r-a_n) = 0$, where the a_i are *distinct* members of k; and one might study k-algebras R which satisfy the generalization of unique strong cleanness saying that every element has a unique decomposition as the sum of a metaidempotent and a member of the Jacobson radical $J(R)$. There should, again, be versions of this condition for rings which are not algebras over fields.

We end by noting a result of a similar flavor to the existence result we tried unsuccessfully to prove in section 2, but which is, in contrast, trivial to prove.

Lemma 3.1. *Every ring R can be embedded in a ring R' such that every element of R' is a sum of two commuting units. Namely, the formal Laurent series ring $R' = R((t)) = R[[t]][t^{-1}]$ has this property.*

Proof. Given $x \in R'$, choose an integer N such that $x \in t^{N+1}R[[t]]$. Let $u = t^N + x$ and $u' = -t^N$. Since u and u' both have invertible leading terms, they are both units. They commute, since t is central; and clearly $u + u' = x$. □

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