

CRITERIA FOR EXISTENCE OF SEMIGROUP HOMOMORPHISMS AND PROJECTIVE RANK FUNCTIONS

GEORGE M. BERGMAN

ABSTRACT. Let P , S , and T be semigroups, $f : P \rightarrow S$ and $g : P \rightarrow T$ semigroup homomorphisms, and X a generating set for S (possibly infinite). Clearly, a *necessary* condition for there to exist a homomorphism $S \rightarrow T$ making a commuting triangle with f and g is that for every relation $f(p) = w(x_1, \dots, x_n)$ holding in S , with $p \in P$, w a semigroup word, and $x_1, \dots, x_n \in X$, there exist $t_1, \dots, t_n \in T$ satisfying $g(p) = w(t_1, \dots, t_n)$.

Under what assumptions will that also be sufficient? We show that one such family of assumptions is that (i) every element of S is a divisor some element of $f(P)$, (ii) T is right and left cancellative, (iii) T is power-cancellative, i.e. $x^d = y^d \implies x = y$ for $d > 0$, and (iv) a certain technical condition which, in particular, holds if T admits a semigroup ordering with the order-type of the natural numbers.

As an application, we obtain an elementary criterion for the existence of an integer-valued rank function on finitely generated projective modules over a ring.

1. MAIN RESULTS

Here is a bit of notation and terminology that we will use:

Definition 1.1. *If S is a semigroup, S^1 will denote the monoid obtained by adjoining an identity element to S .*

An element s of a semigroup S will be called a divisor of an element $t \in S$ if $t = asb$ for some $a, b \in S^1$.

An element s of a semigroup S will be called a weak divisor of a subset $A \subseteq S$ if there exists a positive integer d such that s^d is a divisor of some element of A^d (i.e., of some product $a_1 \dots a_d$ with $a_i \in A$).

My only excuse for the peculiar concept of “weak divisor”, and for condition (1.4) of the next theorem, which uses it, is that these are what were needed to abstract an argument I discovered in considering rank functions on projective modules over a ring. That application will be made in §2. In §3 we will, inter alia, look at some simpler conditions that imply (1.4). In the meantime, I will mention that (1.4) holds frequently, e.g., whenever T is a free semigroup or a free abelian semigroup. Experience may eventually show that one of the stronger conditions mentioned in §3 covers all cases of interest.

Theorem 1.2. *Let P , S and T be semigroups, $f : P \rightarrow S$ and $g : P \rightarrow T$ semigroup homomorphisms, and X a generating set for S . Suppose that*

(1.1) *every element of S is a divisor of some element of $f(P)$,*

(1.2) *T is right and left cancellative (i.e., for $x, y \in T$ and $a, b \in T^1$, $axb = ayb \implies x = y$),*

(1.3) *T is power cancellative (i.e., $x^d = y^d \implies x = y$ for $x, y \in T$, $d > 0$),*

and

(1.4) *every finite subset of $g(P) \subseteq T$ has only finitely many weak divisors in T (Definition 1.1 above).*

Then the following conditions are equivalent:

(1.5) *For every relation $f(p) = w(x_1, \dots, x_n)$ holding in S , with $p \in P$, w a semigroup word, and $x_1, \dots, x_n \in X$, there exist $t_1, \dots, t_n \in T$ satisfying $g(p) = w(t_1, \dots, t_n)$.*

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(1.6) *There exists a homomorphism $h : S \rightarrow T$ such that $g = hf$.*

Proof. Clearly, (1.6) implies (1.5) (with $t_i = h(x_i)$); so what we must prove is that, under the stated hypotheses, the converse holds.

Observe that a homomorphism $h : S \rightarrow T$ is determined by its restriction to X ; and clearly, given any set map

$$(1.7) \quad \eta : X \rightarrow T,$$

the necessary and sufficient condition for η to extend to a homomorphism $S \rightarrow T$ is that

$$(1.8) \quad \text{for every relation } w_1(x_1, \dots, x_m) = w_2(x_1, \dots, x_m) \text{ holding in } S, \text{ where } w_1 \text{ and } w_2 \text{ are semigroup words and } x_1, \dots, x_m \in X, \text{ the corresponding relation } w_1(\eta(x_1), \dots, \eta(x_m)) = w_2(\eta(x_1), \dots, \eta(x_m)) \text{ holds in } T.$$

Given homomorphisms $f : P \rightarrow S$ and $g : P \rightarrow T$ as in the statement of the theorem, and a map (1.7), when will that map determine a homomorphism h such that $g = hf$? I claim that if

$$(1.9) \quad \text{for every relation } f(p) = w(x_1, \dots, x_n) \text{ holding in } S, \text{ with } p \in P \text{ and } x_1, \dots, x_n \in X, \text{ one has } g(p) = w(\eta(x_1), \dots, \eta(x_n)) \text{ in } T,$$

then assuming (1.1)-(1.4), this will imply that (1.8) holds, yielding a homomorphism $h : S \rightarrow T$; and clearly, (1.9) then shows that h will satisfy the desired relation $g = hf$.

To prove (1.8) assuming (1.9), consider any relation as in the hypothesis of (1.8). Let us write s for the common value in S of the two sides of this relation. Then by (1.1), s is a divisor of $f(p)$ for some $p \in P$. Expressing the right and left factors that carry s to $f(p)$ in terms of the generating set X (using a list $x_1, \dots, x_n \in X$ extending x_1, \dots, x_m , and allowing empty words if either or both factors are 1), we get a relation

$$(1.10) \quad u(x_1, \dots, x_n) w_1(x_1, \dots, x_m) v(x_1, \dots, x_n) = f(p) = u(x_1, \dots, x_n) w_2(x_1, \dots, x_m) v(x_1, \dots, x_n)$$

in S . Regarding this as two equations of the sort appearing in the hypothesis of (1.9), applying (1.9) to each, and, since the resulting left-hand sides are both $g(p)$, equating the right-hand sides, we get an equation in T which, by the cancellativity condition (1.2), yields the conclusion of (1.8), as desired.

So,

$$(1.11) \quad \text{To prove the Theorem, it will suffice to show that assuming (1.5), there will exist a map (1.7) for which (1.9) holds.}$$

In constructing such an η , the following notation will be useful:

$$(1.12) \quad \text{We shall denote by } \Sigma \text{ the family of equations } g(p) = w(\eta(x_1), \dots, \eta(x_n)) \text{ arising (as in the conclusion of (1.9)) from all relations } f(p) = w(x_1, \dots, x_n) \text{ that hold in } S, \text{ but with the symbols } \eta(x_1), \dots, \eta(x_n) \text{ now regarded as an } n\text{-tuple of } T\text{-valued unknowns (though the symbols } g(p) \text{ will, as in (1.9), still denote the indicated constants in } T).$$

Below, we shall first deduce from (1.5) that every *finite* subset of Σ has a solution in T^X , then show from this that the whole family Σ has such a solution.

To get the result on finite subsets of Σ , let us start by showing that given any two equations in Σ ,

$$(1.13) \quad g(p) = u \quad \text{and} \quad g(p') = u',$$

there exists a nonnegative integer d such that the equation

$$(1.14) \quad g(p^d p') = u^d u',$$

which also belongs to Σ , has the same solution-set in T^X as the pair of equations (1.13).

For this purpose, let $\eta(x_1), \dots, \eta(x_n)$ be the finitely many variables which appear in one or both of the words u and u' , and consider assignments of values to $\eta(x_1), \dots, \eta(x_n)$ such that (1.14) holds for at least one value of d . Every value of $u(\eta(x_1), \dots, \eta(x_n))$ that such an assignment leads to will, in view of (1.14), be a weak divisor (Definition 1.1) of the 2-element set $\{g(p), g(p)g(p')\}$, so by (1.4),

$$(1.15) \quad \text{there are only finitely many values in } T \text{ that } u(\eta(x_1), \dots, \eta(x_n)) \text{ can have in solutions to (1.14), as } d \text{ ranges over all nonnegative integers.}$$

Moreover, I claim that for any such value of $u(\eta(x_1), \dots, \eta(x_n))$ that is *not* equal to $g(p)$, the equality (1.14) can only hold for one value of d . Indeed, assuming that for some d and some choice of values for $\eta(x_1), \dots, \eta(x_n)$ the equation (1.14) holds, with $g(p) \neq u$, we see by power-cancellativity of T (1.3) that that choice will make $g(p)^c \neq u^c$ for every $c > 0$, so if we multiply this inequality on the right by (1.14), right cancellativity of T (condition (1.2)) shows that $g(p)^{c+d}g(p') \neq u^{c+d}u'$. So taking for d the least value for which the given choice of values for $\eta(x_1), \dots, \eta(x_n)$ make (1.14) hold, we see that it is the only such

value. Hence, as $u(\eta(x_1), \dots, \eta(x_n))$ ranges over the finitely many values referred to in (1.15), only finitely many values of d allow solutions of (1.14) with $g(p) \neq u(\eta(x_1), \dots, \eta(x_n))$.

So let us choose a d that is *not* one of those finitely many values; so that all solutions of (1.14) for that choice of d satisfy $g(p) = u$. Then by left cancellativity (hypothesis (1.2)), those solutions also satisfy $g(p') = u'$. Thus, for that choice of d , the values of $\eta(x_1), \dots, \eta(x_n)$ that make (1.14) hold are, as claimed, precisely those that make both equalities of (1.13) hold.

Repeatedly applying this result, we see that the solution-set in T^X of any *finite* subset of Σ is equal to the solution-set of a single element of Σ . By (1.5), every member of Σ has a nonempty solution-set; hence the above argument shows that every finite subset of Σ does.

Finally, let us show from this that the whole set Σ has a nonempty solution-set.

We begin by noting that by (1.1), each $x \in X$ is a divisor of some element $f(p_x)$ ($p_x \in P$), hence

$$(1.16) \quad \text{for each } x \in X \text{ there is some member of } \Sigma, \text{ say } g(p_x) = u_x, \text{ such that the word } u_x \text{ involves the variable } \eta(x).$$

For each $x \in X$, let us fix such an equation $g(p_x) = u_x$ in Σ , and denote by T_x the set of divisors in T of the element $g(p_x)$. By (1.4) each T_x is finite (since every divisor of an element $g(p_x)$ is, in particular, a weak divisor of the singleton set $\{g(p_x)\}$). Thus, in any solution in the semigroup T of any subsystem of Σ which includes the equation $g(p_x) = u_x$, the value given to $\eta(x)$ must be a member of the finite set T_x .

Now let Σ_0 be any finite subset of Σ , let $\eta(x_1), \dots, \eta(x_n)$ be the variables occurring in the right-hand sides of the equations in Σ_0 , and let Σ_1 be the finite set obtained by adjoining to Σ_0 the additional equations $g(p_{x_i}) = u_{x_i}$ for $i = 1, \dots, n$. Since Σ_1 is a finite subset of Σ , it will, as we have shown, have solutions; and since it contains the equations $g(p_{x_i}) = u_{x_i}$ for $i = 1, \dots, n$, those solutions will have x_i -coordinate in T_{x_i} for $i = 1, \dots, n$. But solutions to Σ_1 are also solutions to its subfamily Σ_0 , so there are solutions to the latter family with x_i -coordinate in T_{x_i} for $i = 1, \dots, n$. And since the equations in Σ_0 involve no variables but $\eta(x_1), \dots, \eta(x_n)$, we can, in such solutions, modify the values assigned to all other coordinates $\eta(x)$ ($x \in X$) in any way; in particular, replace each such coordinate by a member of the corresponding set T_x . This shows that

$$(1.17) \quad \text{the solution-set in } T_x \text{ of every finite subset } \Sigma_0 \subseteq \Sigma \text{ has nonempty intersection with } \prod_{x \in X} T_x.$$

Let us now regard $\prod_X T_x$ as a compact topological space, under the product of the discrete topologies on the finite factors T_x , and consider each semigroup word in the variables $\{\eta(x) \mid x \in X\}$ as defining a map $\prod_X T_x \rightarrow T$. Regarding T as a discrete space, each such map is continuous, since it depends on only finitely many coordinates of $\prod_X T_x$, so the set of solutions in $\prod_X T_x$ to each equation in Σ is closed.

The class of finite subsets of Σ is closed under finite unions, hence the class of their solution-sets in $\prod_X T_x$ is closed under finite intersections. By the preceding results, all such finite intersections are nonempty closed sets, hence the compactness of $\prod_X T_x$ shows that the solution-set of the full family Σ is nonempty. A member of this intersection will be a map (1.7) satisfying (1.9), which by (1.11) completes the proof of the Theorem. \square

2. APPLICATION TO RANK FUNCTIONS ON PROJECTIVE MODULES

The motivation for the above theorem came from the study of integer-valued projective rank functions on rings R , i.e., functions taking isomorphism classes of finitely generated projective left R -modules to natural numbers, which carry direct sums of modules to the sums of the corresponding natural numbers, and carry the module R to 1; cf. [2]. Let us apply our theorem to that case.

Theorem 2.1. *Let R be an associative ring, and X a set of finitely generated nonzero projective left R -modules, such that every finitely generated projective left R -module is isomorphic to a direct sum of members of X .*

Then there exists a nonnegative-integer-valued projective rank function for R if and only if whenever one has a module isomorphism

$$(2.1) \quad R^c \cong (P_1)^{c_1} \oplus \dots \oplus (P_n)^{c_n} \quad (P_1, \dots, P_n \in X, c, c_1, \dots, c_n \geq 0),$$

the integer c is a linear combination of c_1, \dots, c_n , with nonnegative integer coefficients.

This rank function can be taken to be nondegenerate (to carry nonzero projective modules to positive integers) if and only if in all such cases, the integer c can in fact be written as a linear combination of c_1, \dots, c_n with positive integer coefficients.

Proof. To get the first assertion, apply the preceding theorem, taking for both P and T the additive semigroup \mathbb{N} of nonnegative integers, for S the semigroup of isomorphism classes of nonzero finitely generated

projective R -modules (mumbling the words needed to replace these proper classes by genuine sets), for f the homomorphism sending 1 to the isomorphism class of the free R -module of rank 1, and for g the identity map. Condition (1.1) of the theorem holds (with “divisor” understood to mean “direct summand”, in view of the choice of semigroup S) because every finitely generated projective module is a direct summand in a free module of finite rank. Conditions (1.2) and (1.3) (again, translated to additive language) clearly hold in the semigroup $T = \mathbb{N}$, and we see (1.4) by noting that an element of that additive semigroup is a “divisor” of another if and only if it is majorized by that element under the natural ordering of the integers, and is a “weak divisor” of a family if and only if it is majorized by some member of that family; and that every nonnegative integer majorizes only finitely many others. The criterion of Theorem 1.2 now assumes the desired form (the “integer coefficients” in the statement of the present theorem corresponding to the t_i of that theorem).

The final assertion is obtained by the same argument, using the semigroup $\mathbb{N} \setminus \{0\}$ rather than \mathbb{N} for P and T . \square

In [3], generalized projective rank functions, with values in semigroups $(1/n)\mathbb{N}$, are used to study homomorphisms of R into $n \times n$ matrix rings over division rings. The same method as above shows that R admits a $(1/n)\mathbb{N}$ -valued projective rank function if and only if for every isomorphism (2.1), the integer nc can be written as a linear combination of the c_i with nonnegative (or, if we require the rank function to be nondegenerate, positive) integer coefficients.

3. VARIANT CONDITIONS AND EXAMPLES

In the proof of Theorem 1.2, the full strength of the condition (1.4) on weak divisors was only used in showing that every pair of members of Σ has the same solution-set as a single member of Σ ; and that argument was also the only place where we used the power-cancellativity condition (1.3). The one later use of the weak divisor hypothesis, in showing that if every finite subset of Σ could be realized, then so could the whole family Σ , it was merely used to show that the set T_x of divisors of an element $g(p_x)$ was finite. Hence dropping the conditions not used there, we get.

Corollary 3.1 (to proof of Theorem 1.2). *Assume the hypothesis of Theorem 1.2, but without condition (1.3) (power cancellativity), and with (1.4) weakened to merely say that every element of $g(P)$ has only finitely many divisors in T . Then a necessary and sufficient condition for there to exist a homomorphism $S \rightarrow T$ making a commuting triangle with the given maps from P is that for every finite set of relations*

$$(3.1) \quad f(p_i) = w_i(x_1, \dots, x_n) \quad (i = 1, \dots, k)$$

satisfied in S by elements $x_1, \dots, x_n \in X$, there exist $t_1, \dots, t_n \in T$ satisfying the corresponding relations $g(p_i) = w_i(t_1, \dots, t_n) \quad (i = 1, \dots, k)$. \square

In the opposite direction, there are conditions stronger than (1.4) but less complicated; so the statement of Theorem 1.2 with (1.4) replaced by such a condition remains true, though weaker. One such condition is gotten by assuming the conclusion of (1.4) for *all* finite sets of elements of T , not just those contained in $g(P)$:

(3.2) Every finite subset of T has only finitely many weak divisors.

A much stronger condition, which does not require the concept of weak divisor, and implies all of (1.2), (1.3) and (3.2) (hence (1.4)) is

$$(3.3) \quad T \text{ admits a total ordering having the order-type of the natural numbers, and satisfying} \\ (\forall a, b, c \in T) \quad a < b \implies ca < cb \quad \text{and} \quad ac < bc.$$

That a semigroup T satisfying (3.3) satisfies (1.2), (1.3) and (3.2) will be easy to see once we note a few elementary properties that (3.3) implies. First, any idempotent $e \in T$ must be an identity element. Indeed, if for any a the product ea were $< a$ or $> a$, then from (3.3) we would get the same strict inequality between e^2a and ea , contradicting the idempotence of e ; so e is a left identity element, and by the symmetric argument it is a right identity element. Second, for all a, b with a not an identity element, we have $ab > b$. For if we had $b > ab$ or $b = ab$, we would get $ab > a^2b$, respectively $ab = a^2b$. In the former case, we could go on to get an infinite descending chain $b > ab > a^2b > a^3b \dots$, contradicting our order-type hypothesis, while in the latter we would get $a = a^2$, so by our previous observation, a would be an identity element. Again, we similarly have $ba > b$. It follows that if a is a divisor of b , then $a \leq b$, and more generally, that a weak divisor of any set is one of the finitely many elements of T less than or equal to the largest element of that set. Given these properties, (1.2), (1.3) and (3.2) are easily deduced.

Easy examples of such ordered semigroups are given by the subsemigroups of the additive group of real numbers generated by unbounded increasing sequences of positive real numbers, such as $2\frac{1}{2}, 3\frac{1}{3}, \dots, n + n^{-1}, \dots$. If we take the additive semigroup generated by such a sequence whose members are also linearly independent, such as the powers of π , that semigroup will be free abelian; thus, free abelian semigroups on countably many generators satisfy (3.3). The same is true of free (nonabelian) semigroups on countably many generators; the interested reader can easily find an appropriate ordering. But instead of establishing (1.2)-(3.2) for that class of semigroups in this way, let us, after noting that (1.2) and (1.3) clearly hold for free semigroups, give another pair of tools for establishing (1.4) and (3.2):

(3.4) If, for P, S, T, f, g as in the first sentence of Theorem 1.2, T admits a finitely-many-to-one homomorphism j to a semigroup which satisfies (1.4) with respect to its subsemigroup $fg(P)$, then T satisfies (1.4) with respect to its subsemigroup $g(P)$.

(3.5) If a semigroup T admits a finitely-many-to-one homomorphism j to a semigroup which satisfies the condition of (3.2), then T also satisfies (3.2).

These statements are trivial to verify. Now the free semigroup T on countably many generators x_1, x_2, \dots can be mapped to the free semigroup on one generator x (which we have seen satisfies (3.2)) by sending x_i to x^i , and this map is finitely-many-to-one; hence by (3.5), T satisfies (3.2). The same argument applies to free semigroups in any semigroup variety V such that

(3.6) V does not satisfy any identity $x^m = x^n$ with $m \neq n$,

(equivalently, varieties in which the free object on one generator is isomorphic to the semigroup of positive integers. Some varieties satisfying (3.6) have, and some do not have the property that their free semigroups also satisfy (1.2) and (1.3). An example which satisfies neither is the variety defined by the identity saying that all products of two elements are central: $xyz = zxy$.)

Of course, in any nontrivial variety V of semigroups, free objects on *uncountably* many generators do not admit finitely-many-to-one homomorphisms to the free object on one generator. However, it is not hard to see that a semigroup satisfies (3.2) if and only if all of its countable subsemigroups do. Hence, *all* free semigroups in varieties satisfying (3.6) satisfy (3.2).

For an example showing that condition (3.3) on an ordered semigroup, of having the order-type of the natural numbers, is stronger than that of being countable and well-ordered, and also stronger than being generated by a subset having the order-type of the natural numbers, let T be the additive subsemigroup of \mathbb{R} generated by $1/2, 2/3, 3/4, \dots, n/(n+1), \dots$. Since T is generated by a well-ordered set of positive real numbers, it is well-ordered [1, Theorem III.2.9, p. 123]. But for any $p \leq q$ in this semigroup, it is not hard to show that p is a weak divisor of $\{q\}$. (For the d in the definition of weak divisor, take a common denominator of p and q , and remember that since T contains $1/2$, it contains all positive integers, so every integer is a “divisor” in T of every larger integer.) Hence, the singleton set $\{1\}$ has infinitely many weak divisors in this semigroup. I do not know whether there is a counterexample to the conclusion of Theorem 1.2 with this T in place of a T satisfying (1.4).

We remark that in a semigroup T satisfying (1.2)-(1.4), the operation “the set of weak divisors of” on subsets of T need not be idempotent. For example, within the free semigroup on two generators x and y , let T be the subsemigroup generated by x^2, x^3 and y . Thus, T consists of those words in x and y in which every occurrence of x is adjacent to another occurrence of x . Consider the weak divisors in T of the singleton set $\{x^3y\}$. We find that x^2 is not a weak divisor of this set: no power of x^3y can be factored in such a way that one of the factors is x^2 and the others belong to T . However, x^3 , being a divisor of x^3y in T , is a weak divisor of our set $\{x^3y\}$; and x^2 is a weak divisor of any subset of T containing x^3 , since $(x^2)^2$ divides $(x^3)^2$ in T . Hence x^2 does lie in the set of weak divisors of the set of weak divisors of $\{x^3y\}$.

On the other hand, it is not hard to show that on subsets of a *commutative* semigroup, the “set of weak divisors” operation *is* idempotent; and that in a free semigroup in the variety of all semigroups, the weak divisors of a set are simply the divisors of its members, so that there, too, the “set of weak divisors” operation is idempotent.

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