

WHICH GROUP ALGEBRAS CANNOT BE MADE ZERO BY IMPOSING A SINGLE NON-MONOMIAL RELATION?

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ABSTRACT. For which groups G is it true that for all fields k , every non-monomial element of the group algebra kG generates a *proper* 2-sided ideal? The only groups for which we know this to be true are the torsion-free abelian groups. We would, in particular, like to know whether it is true for all *free* groups.

We show that the above property fails for wide classes of groups: for every group G that contains an element $g \neq 1$ whose image in $G/[g, G]$ has finite order (in particular, every group containing a $g \neq 1$ that has finite order, or that satisfies $g \in [g, G]$); and for every group containing an element g which commutes with a distinct conjugate $h^{-1}gh \neq g$ (in particular, for every nonabelian solvable group).

Closure properties of the class of groups satisfying the desired condition are noted. Further questions are raised; in particular, a plausible Freiheitssatz for group algebras of free groups is noted, which would imply the hoped-for result for such group algebras.

1. STARTING POINT: THE QUESTION FOR FREE GROUPS

Let us give a name to the property we will be considering.

Definition 1. A group G will be called *resistant* if for every field k and every element $r = \sum_{g \in G} c_g g$ of the group algebra kG whose support, $\text{supp}(r) = \{g \mid c_g \neq 0\}$, has cardinality > 1 , the 2-sided ideal of kG generated by r is proper. (Henceforth, “2-sided ideal” will be shortened to “ideal”.)

The term ‘resistant’ is sometimes used in the theory of p -groups with a different meaning [18].

Clearly, every torsion-free abelian group is resistant; these are the only cases that I know. The present note arises out of

Question 2. [19] [1, Q 3(i)] *Are all free groups resistant?*

This suggests the corresponding question for groups free in various varieties, such as free solvable groups of a given derived length, and free nilpotent groups of given nilpotency class. If either of those questions had a positive answer (for all derived lengths, respectively all nilpotency classes), one could deduce a positive answer to Question 2; but it turns out that both have negative answers, and the easy arguments proving this show very large classes of groups to be non-resistant. We develop these in §§2-4 below. We note in §5 some closure properties of the class of resistant groups; these imply inter alia that if the free group on two generators is resistant, then so is the free product of any family of torsion-free abelian groups. In §6 we suggest a plausible Freiheitssatz for group algebras of free groups, which, if proved, would imply a positive answer to Question 2.

2. WHICH BINOMIAL ELEMENTS GENERATE THE IMPROPER IDEAL?

For G a group and k a field, elements of kG with support of cardinality 1 (monomials) are invertible, hence are excluded in the statement of Definition 1. Given an element with support of cardinality 2 (a binomial) $c_1g_1 + c_2g_2$ ($c_1, c_2 \in k - \{0\}$, $g_1 \neq g_2 \in G$) we can divide by the unit c_1g_2 , and write the resulting element as $g - c$ ($g \in G - \{1\}$, $c \in k - \{0\}$).

Note that imposing the relation $g - c = 0$ on kG makes the image of g central, so all its conjugates $h^{-1}gh$ fall together, and any relation already satisfied in G by these conjugates becomes equivalent to a relation $g^n = 1$. But since g becomes identified with c , if g^n falls together with 1 but $c^n \neq 1$ in k , the resulting factor-algebra collapses.

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Theorem 3 below characterizes those $g \in G$ such that this does *not* happen for any $c \neq 0$ in any field k (condition (iv) of that theorem). Theorem 4 gives the details of what happens for other g .

Writing

$$(1) \quad [g, h] = g^{-1}h^{-1}gh,$$

we shall denote by $[g, G]$ the subgroup of G generated by the elements $[g, h]$ ($h \in G$). This is a normal subgroup of G , since $h'^{-1}[g, h]h' = [g, h'^{-1}[g, hh']$.

Theorem 3. *For an element $g \neq 1$ of a group G , the following conditions are equivalent.*

- (i) *The image \bar{g} of g in $G/[g, G]$ has infinite order.*
- (ii) *For every equation $\prod_{i=1}^N (h_i^{-1}gh_i)^{\varepsilon_i} = 1$ holding in G , where $h_i \in G$ and $\varepsilon_i = \pm 1$ ($i = 1, \dots, N$), one has $\sum_{i=1}^N \varepsilon_i = 0$.*
- (iii) *For every equation $\prod_{i=1}^M h_i^{-1}gh_i = \prod_{j=1}^{M'} h_j'^{-1}gh_j'$ holding in G with $h_i, h_j' \in G$, one has $M = M'$.*
- (iv) *For every field k and element $c \in k - \{0\}$, the element $g - c$ generates a proper ideal of kG .*
- (v) *Either for some element c of infinite multiplicative order in some field k , or for a family of elements c of infinitely many distinct finite multiplicative orders in (possibly various) fields k , the elements $g - c$ generate proper ideals in the algebras kG .*

Proof. We shall show (i) \iff (ii) \iff (iii), and then (i) \implies (iv) \implies (v) \implies (i).

(i) \implies (ii) is immediate, since a relation $\prod_{i=1}^N (h_i^{-1}gh_i)^{\varepsilon_i} = 1$ as in (ii) gives $\bar{g}^{\sum_{i=1}^N \varepsilon_i} = 1$ in $G/[g, G]$. To see the converse, note that a relation $\bar{g}^n = 1$ in $G/[g, G]$ corresponds to a relation $g^n = \prod_{i=1}^N [g, h_i]^{\pm 1}$ in G , and since each $[g, h_i]$ has the form $g^{-1}(h_i^{-1}gh_i)^{\pm 1}$, this yields a relation as in (ii) with the sum of the exponents equal to n , forcing $n = 0$ if (ii) holds, and so giving (ii) \implies (i).

(ii) \implies (iii) is clear, since from any relation as in (iii), we get, on right-dividing by the right-hand side, a relation as in (ii) with $\sum \varepsilon_i = M - M'$. To get the converse, note that in any relation as in (ii), if we have successive terms $(h_i^{-1}gh_i)^{-1}(h_{i+1}^{-1}gh_{i+1})^{+1}$, we can rearrange these as $(h_{i+1}^{-1}gh_{i+1})^{+1}(h_i^{-1}gh_i)^{-1}$, where $h'_i = h_i(h_{i+1}^{-1}gh_{i+1})$. In this way, we can recursively turn any product as in (ii) into a product of the same form, and with the same $\sum \varepsilon_i$, in which all the terms with exponent -1 occur to the right of all terms with exponent $+1$. That relation is clearly equivalent to a relation as in (iii) with $M - M' = \sum \varepsilon_i$.

To get (i) \implies (iv), note that assuming (i), $k(G/[g, G])$ will be free as a module over its central Laurent polynomial subalgebra $k\langle \bar{g} \rangle$; hence tensoring over $k\langle \bar{g} \rangle$ with $k\langle \bar{g} \rangle / (\bar{g} - c) \cong k$, we get a nonzero homomorphic image of kG in which $g - c$ goes to zero, establishing (iv). Clearly, (iv) \implies (v). The proof of (v) \implies (i) uses the idea sketched in the second paragraph of this section: if we had $\bar{g}^n = 1$ for some $n > 1$, then applying a case of (v) in which $c^n \neq 1$, we would get a contradiction. \square

(A special case of the argument for (i) \implies (iv), where G is a free nilpotent group, so that (i) automatically holds, was pointed out to me a few years ago by Dave Witte Morris in a discussion of Question 2.)

Turning to elements g not satisfying the above conditions, we have

Theorem 4. *Let $g \neq 1$ be an element of a group G which does not satisfy the equivalent conditions of Theorem 3. Then there exists an integer $n \geq 1$ characterized by the following equivalent conditions:*

- (i) *The image \bar{g} of g in $G/[g, G]$ has order exactly n .*
- (ii) *On the set of all relations of the form $\prod_{i=1}^N (h_i^{-1}gh_i)^{\varepsilon_i} = 1$ holding in G , the values assumed by $\sum_{i=1}^N \varepsilon_i$ are precisely the multiples of n .*
- (iii) *On the set of all relations of the form $\prod_{i=1}^M h_i^{-1}gh_i = \prod_{j=1}^{M'} h_j'^{-1}gh_j'$ holding in G , the values assumed by $M - M'$ are precisely the multiples of n .*
- (iv) *For k a field and c an element of $k - \{0\}$, the element $g - c$ of kG generates the improper ideal if and only if $c^n \neq 1$.*

Sketch of proof. Take n as in (i). It is not hard to show that the sets of integers described in (ii) and in (iii) are additive subgroups of \mathbb{Z} , since relations can be ‘‘multiplied’’ and ‘‘inverted’’; call their positive generators n' and n'' respectively. Arguments essentially analogous to those in the proof of Theorem 3 show that $n = n' = n''$. The proof that the n of (i) satisfies (iv) is likewise analogous to the proof of (i) \iff (iv) in Theorem 3, and (iv) uniquely determines n , because for all positive integers m there exist fields with elements of multiplicative order m . \square

The next corollary gives some applications of this result. Case (b) below, for $\alpha > 0$, demolished my hope that group algebras of 2-sided orderable groups might all be resistant, since the group of order-preserving affine endomaps of the real line is so orderable, by the ordering that makes $g_1 \geq g_2$ whenever $g_1(t) \geq g_2(t)$ in a neighborhood of $+\infty$. Cases (c) and (d) both show that a free product of free groups with amalgamation of a common subgroup need not be resistant: the former implies this for the groups $\langle h, h' \mid h^n = h'^n \rangle$ ($n > 1$), the latter, proved with the help of the former, gives the more general case $\langle h, h' \mid h^n = h'^{n'} \rangle$ ($n, n' > 1$).

Note that where in Theorem 4, n denoted the least positive integer with various properties, in this corollary it denotes any such positive integer, i.e., any positive multiple of the n of that theorem.

Corollary 5. *For each of the following classes of groups G , choices of $g \in G - \{1\}$, and positive integers n , every element $c \neq 0$ of a field k satisfying $c^n \neq 1$ has the property that $g - c \in kG$ generates the improper ideal.*

- (a) G any group having a nonidentity element g of finite order, and any $n > 1$ such that $g^n = 1$.
- (b) G the group of affine maps of the real line generated by the elements g and h , where $g(t) = t + 1$ and $h(t) = \alpha t$ ($t \in \mathbb{R}$) for α any rational number $m/m' \neq 1, 0$, with $n = |m - m'|$.
- (c) G any group containing elements $h \neq h'$ such that $h^n = h'^n$ for some $n > 1$, with $g = h^{-1}h'$.
- (d) G any group containing elements h_1, h_2 which do not commute, but such that for some $n > 1$, h_1 commutes with h_2^n , with $g = [h_2, h_1]$.

Hence no group with any of the indicated properties is resistant.

Proof. We shall denote by “(i)-(iv)” the conditions so named in Theorem 4. In each case of this corollary, we shall see that one of (i)-(iii) is satisfied, thus establishing (iv), which gives the desired conclusion.

In case (a) it is clear that (i) is satisfied, with the n of (i) a divisor of the given n .

In case (b), note that $h^{-1}gh$ carries t to $t + (m'/m)$, hence $(h^{-1}gh)^m = g^{m'}$, so the n of (iii) will be a divisor of $|m' - m|$.

In case (c), for g as defined there, note that in $G/[g, G]$, since \bar{h} must commute with the central element $\bar{g} = \bar{h}^{-1}\bar{h}'$, it commutes with \bar{h}' . Hence $\bar{g}^n = (\bar{h}^{-1}\bar{h}')^n = \bar{h}^{-n}\bar{h}'^n = 1$, so we can apply (i).

In case (d), we can apply (c) above with $h = h_2$, $h' = h_1^{-1}h_2h_1$, noting that indeed $h \neq h'$, but that $h^n = h_2^n = h_1^{-1}h_2^n h_1$ (by hypothesis), which equals h'^n . \square

Remarks: The final inequality of Theorem 4(iv) can never hold if $c = 1$; hence that case of the result says that $g - 1$ can never generate the improper ideal – which is certainly true, since $g - 1$ lies in the augmentation ideal of kG .

The case $g = 1$ is excluded in the statements of the above results for a different reason: in that case, $g - c$, if not zero, is a monomial element of kG , which we are not interested in, though the conclusion that $g - c$ generates the improper ideal unless $c = 1$ is true.

Variants of Corollary 5(b) give further interesting examples. For instance, though replacing the α of that example with a transcendental real number will not lead to a group with the indicated properties, if we take *any* real number α , and let G be generated by the g of that example together with both $h(t) = \alpha t$ and $h'(t) = (\alpha + 1)t$, we find that $g(hgh^{-1}) = h'gh'^{-1}$, so case (iii) of Theorem 4 is again applicable. If we take α to be the positive root of the equation $\alpha^2 = 1 + \alpha$ (“the golden section”), then with G again generated just by g and h , we get $h^2gh^{-2} = g(hgh^{-1})$, with the same consequence.

3. SOME TRINOMIALS

We do not have a general result on when a trinomial element of a group algebra generates the improper ideal; but the particular result given by the next theorem, though technical in its statement, is easy to prove, and quickly allows us to show that no nonabelian solvable group is resistant. The theorem does not involve a choice of $c \in k$, so our group algebras could be taken over any commutative ring, but to keep our context consistent we will assume them taken over a specified field k .

Theorem 6. *Suppose a group G has elements g, h such that the normal subgroup $N \subseteq G$ generated by the commutator $[g, h]$ contains an element $f \neq 1$ which commutes with g in G . Then in the group algebra kG , the trinomial $1 + h - f$ generates the improper ideal.*

Proof. Let us check first that $1+h-f$ is indeed a trinomial; i.e., that $1, h, f$ are distinct. By hypothesis, $f \neq 1$. If h were equal to either 1 or f , it would commute with g , so the normal subgroup N generated by $[g, h]$ would be trivial, so since $f \in N$ we would have $f = 1$, again contradicting our hypothesis.

The proof that the ideal generated by $1+h-f$ is improper uses the same trick: Modulo that ideal, h falls together with $f-1$, hence commutes with g , hence $[g, h]$ becomes 1 , hence all elements of N fall together with 1 . Hence $f-1$ becomes zero, so h becomes zero. Since h is invertible, our algebra collapses. \square

In particular,

Corollary 7. *If in a group G an element g commutes with a conjugate $h^{-1}gh \neq g$, then in the group algebra kG , the trinomial $1+h-[g, h]$ generates the improper ideal.*

Proof. If g commutes with $h^{-1}gh \neq g$, then it also commutes with $g^{-1}(h^{-1}gh) \neq 1$, i.e., with $[g, h]$, and we can apply the preceding theorem. \square

Corollary 8. *Any nonabelian solvable group G has an element g which commutes with a conjugate $h^{-1}gh \neq g$; hence its group algebra kG contains a trinomial element which generates the improper ideal.*

Proof. Let h_1 and h_2 be non-commuting elements of the next-to-last nontrivial term of the derived series of the solvable group G .

If h_1 commutes with $h_2^{-1}h_1h_2$, then we have the asserted relation, with $g = h_1, h = h_2$.

On the other hand, if h_1 does not commute with $h_2^{-1}h_1h_2$, then it does not commute with $h_1^{-1}(h_2^{-1}h_1h_2) = [h_1, h_2]$. Letting $g = [h_1, h_2]$ and $h = h_1^{-1}$, we see that the distinct elements g and $h^{-1}gh$ both lie in the last nontrivial term of the derived series, so they commute with each other, giving the desired relation.

Corollary 7 gives the final conclusion. \square

Though Corollary 7 was aimed at getting this result, not every group to which that corollary applies need have a nonabelian solvable subgroup. Indeed, though the commutativity of g with $h^{-1}gh$ implies that in the sequence of elements $h^{-i}gh^i$ ($i \in \mathbb{Z}$), adjacent elements commute, this does not force terms which are not adjacent to commute, as would be needed to get solvability in an obvious way. That the universal example of Corollary 7, $\langle g, h \mid [h, h^{-1}gh] = 1 \rangle$, contains no nonabelian solvable subgroup would be lengthy to prove, but I will sketch an easier example.

Let F_0 and F_1 be free groups on countably infinite families of generators, written $(g_{2i})_{i \in \mathbb{Z}}$ and $(g_{2i+1})_{i \in \mathbb{Z}}$ respectively, let H be an infinite cyclic group $\langle h \rangle$, and let G be the semidirect product $H \ltimes (F_0 \times F_1)$ with H acting on $F_0 \times F_1$ by $h^{-1}g_jh = g_{j+1}$. It is not hard to verify that the subgroup of G generated by any two noncommuting elements contains two noncommuting elements g, g' of $F_0 \times F_1$. Such elements must have noncommuting projections either in F_0 or in F_1 ; but two noncommuting elements of a free group are free generators of a free subgroup. Hence any noncommutative subgroup of G contains a free subgroup on two generators, and so is not solvable.

Curiously, though Corollary 8 dashed the hope that one might be able to prove a positive answer to Question 2 by showing that groups free in appropriate varieties of solvable or nilpotent groups are resistant, one can nonetheless use the theory of free nilpotent groups (the F/F_{i+1} in the proof below) to get a positive result:

Proposition 9. *No free group F contains elements f, g, h as in Theorem 6.*

Proof. Suppose a free group F has such elements f, g, h . Clearly, they will have the same property in some finitely generated subgroup of F , so since a subgroup of a free group is again free, we may assume F finitely generated.

Now the subgroup of F generated by f and g , being commutative but free, has to be free on a single generator g_0 . Let g_0 lie in the i -th term of the lower central series of F [10, §10.2], which we shall write F_i , but not in F_{i+1} . By [10, Theorem 11.2.4, p.175], F_i/F_{i+1} is free abelian, hence in particular, torsion-free, so f and g , being nonzero powers of g_0 , have nonidentity images in that group; i.e., neither lies in F_{i+1} .

But since f lies in the normal subgroup of F generated by $[g, h]$, a commutator involving g , it *must* lie in F_{q+1} . This contradiction completes the proof. \square

Let us show using a similar argument that no *binomial* element of kF can generate the improper ideal. By the observations and results of §2, it suffices to prove that if $g \in F - \{1\}$, then its image $\bar{g} \in F/[g, F]$ has infinite order. To this end, let F_i be the last term of the lower central series of F which contains g , and note that the

image of g in F_i/F_{i+1} has infinite order, since that group is free abelian. Moreover, $[g, F] \subseteq [F_i, F] = F_{i+1}$, so the infinite order condition goes over to $\bar{g} \in F/[g, F]$, as required. Hence

Proposition 10. *In a free group, every nonidentity element satisfies the equivalent conditions of Theorem 3. Hence in the group algebra of a free group over a field, no binomial element generates the improper ideal.* \square

(Strictly speaking, to call on [10, Theorem 11.2.4] we should, as in the proof of Proposition 9, have reduced to the case of F free of finite rank. That reduction is, again, straightforward. Alternatively, the theorem cited is easily seen to imply the corresponding statement for free groups of infinite ranks.)

4. DIGRESSION: IF G HAS ELEMENTS OF FINITE ORDER, kG ALMOST ALWAYS HAS INVERTIBLE NON-MONOMIALS

(The reader might choose to skip or skim this section. In the remainder of the note, there is only one passing reference to a result from it, namely to Theorem 14 in the paragraph following Question 23.)

In §2 we found that a binomial element $g - c$ of a group algebra generates the improper ideal if and only if the image of g in $G/[g, G]$ has finite order n , and c does not have order dividing n . Now, when an element g itself has finite order, group algebras $k\langle g \rangle$ tend to have many non-monomial invertible elements. We shall show in this section that with (up to isomorphism) precisely three exceptions, if k is a field and G a group containing an element g of finite order, then the group algebra kG has a non-monomial invertible element, even if not one of the form $g - c$. (The proofs will be independent of the results of §2. At a couple of points we could have called on results from that section, but the versions of those steps given below are easy enough.)

We begin with the three exceptions. We shall write $\mathbb{Z}/p\mathbb{Z}$ for the field of p elements, Z_p for the cyclic group of order p .

Lemma 11. *In the group algebras $(\mathbb{Z}/2\mathbb{Z})Z_2$, $(\mathbb{Z}/2\mathbb{Z})Z_3$, and $(\mathbb{Z}/3\mathbb{Z})Z_2$, no non-monomial element is invertible, hence (since these algebras are commutative), no non-monomial element generates the improper ideal.*

Proof. We first consider the two cases where the base field is $\mathbb{Z}/2\mathbb{Z}$. If an element r of one of these group algebras generates the improper ideal, then r cannot lie in the augmentation ideal; hence since the elements of the support of r all have coefficient 1, that support must have odd cardinality; moreover, if r is to be non-monomial, that cardinality must be > 1 . This immediately excludes the case $(\mathbb{Z}/2\mathbb{Z})Z_2$, and leaves one possibility in $(\mathbb{Z}/2\mathbb{Z})Z_3$, the sum of the three group elements. But this sum is an idempotent $\neq 1$, hence non-invertible.

In the remaining case $(\mathbb{Z}/3\mathbb{Z})Z_2$, let us write Z_2 as $\{1, g\}$, and note that every non-monomial in the group algebra has the form $c_1 + c_2g$ ($c_1, c_2 \in (\mathbb{Z}/3\mathbb{Z}) - \{0\}$). Hence, up to a scalar factor, every non-monomial can be written $g + c$, where $c \in \{\pm 1\}$. But $(g + 1)(g - 1) = g^2 - 1 = 1 - 1 = 0$, so both these elements are zero-divisors, hence non-invertible; hence so is every non-monomial. \square

In showing that in all other cases we have the opposite conclusion, we begin with the cases where for some $g \in G$ of prime order, we can in fact find an invertible element within the subalgebra $k\langle g \rangle \subseteq kG$.

Lemma 12. *Suppose a group G has an element g of prime order p , and k is a field such that the pair (k, Z_p) is not (up to isomorphism) one of the three cases covered by Lemma 11. Then $k\langle g \rangle$ contains a non-monomial invertible element.*

Proof. We first consider the case where the order p of G is also the characteristic of the field k . Then the element $s = 1 + g + \cdots + g^{p-1}$ has square zero, so every element of the form $1 - cs$ ($c \in k$) is invertible. Since for $p \neq 2$, the assumption $\text{char}(k) = p$ implies that k contains $\geq p > 2$ elements, while for $p = 2$, the hypothesis of the lemma excludes the one field $k \cong \mathbb{Z}/2\mathbb{Z}$ that has only two elements, we can always choose $c \in k - \{0, 1\}$, and then $1 - cs$ will have for support all of $\langle g \rangle$, hence be non-monomial, as desired.

If $\text{char}(k) \neq p$, then $(1 + g + \cdots + g^{p-1})/p$ is an idempotent $e \neq 0, 1$, and for any $c \in k - \{0\}$, the element $e + c(1 - e)$ will have inverse $e + c^{-1}(1 - e)$. If $c = 1$, this element will be 1, a monomial; so the case $k \cong \mathbb{Z}/2\mathbb{Z}$, where that is the only choice, has to be handled by a different construction; we shall do this in the next paragraph. So let $c \neq 0, 1$. Then the coefficients of g, \dots, g^{p-1} in $e + c(1 - e)$ will all be equal and nonzero; so if $p > 2$, this gives us our desired invertible non-monomial element. If $p = 2$, then to get a non-monomial we also have to avoid making the coefficient of $1 = g^0$ zero. We see that there is a unique value c_0 which makes it zero (a quick calculation shows $c_0 = -1$, but we don't need to know this). Now since the hypothesis of the lemma in this

case excludes $k \cong \mathbb{Z}/3\mathbb{Z}$, we can choose $c \neq 0, 1, c_0$, and get $e + c(1 - e)$ with support all of $\langle g \rangle$, hence again non-monomial.

Finally, if $k \cong \mathbb{Z}/2\mathbb{Z}$, we can by hypothesis assume $p > 3$. For such p , here is a construction that in fact works for any field k of characteristic not dividing $p - 2$. Let $r = 1 + g + \cdots + g^{p-3}$, a non-monomial. Note that $g^2(1 - g)r = g^2(1 - g^{p-2}) = g^2 - 1$, so in $k\langle g \rangle/(r)$, we have $\bar{g}^2 = 1$. Since g has order p , we also have $\bar{g}^p = 1$, and together these imply $\bar{g} = 1$. Hence the relation $1 + \bar{g} + \cdots + \bar{g}^{p-3} = 0$ becomes $p - 2 = 0$; but by assumption, $p - 2$ is invertible in k , so $1 = 0$ in $k\langle g \rangle/(r)$. Thus, 1 belongs to the ideal (r) of $k\langle g \rangle$, so r is invertible in that commutative ring, hence also in the overring kG . \square

There remains the case where G has a *proper* subgroup of the sort that Lemma 11 shows us does *not* by itself yield a non-monomial invertible element. In handling this case, we shall use the following more general result (more general because it does not assume the existence of an element of finite order in G).

Lemma 13. *Let k be a field, and G a group having a proper subgroup G_0 such that kG_0 contains either a nonzero element with square zero, or an idempotent other than 0 and 1. Then kG contains a non-monomial invertible element.*

Proof. Let h be any element of $G - G_0$.

Part of our proof can be done with the nilpotent-or-idempotent assumption weakened on the one hand to say that kG_0 contains nonzero elements a, b with $ba = 0$, but strengthened, on the other, to say that $ahb \neq 0$. Then we see that ahb is a nonzero square-zero element of kG whose support does not contain 1; hence $1 + ahb$ is a non-monomial invertible element of kG .

On the other hand, assuming that for our chosen h no such a and b exist, then restoring our nilpotent-or-idempotent hypothesis, kG_0 must either contain a square-zero element a , which (by that assumption) satisfies $aha = 0$, or an idempotent $e \neq 0, 1$ which similarly satisfies $eh(1 - e) = 0 = (1 - e)he$.

In the former case, we see that ah is a nonzero square-zero element whose support does not contain 1, so $1 + ah$ gives the desired invertible element.

In the latter case, note that the given equations are equivalent to $eh = ehe = he$, so e commutes with h , hence it also commutes with h^{-1} . We then find that the element $(1 - e) + he$, which is non-monomial (since its support contains at least one element of G_0 and at least one in the coset hG_0), has 2-sided inverse $(1 - e) + h^{-1}e$. \square

The first two paragraphs of the proof of Lemma 12 show (even in the cases that had to be excluded from the statement of that lemma) that every group algebra $k\langle g \rangle$ with g an element of prime order has either a nonzero element with square zero or an idempotent other than 0 or 1, so Lemma 13 is applicable to groups having such $\langle g \rangle$ as proper subgroups. Putting together the above results, we get

Theorem 14. *If k is a field, and G a group having an element of finite order, then kG contains a non-monomial invertible element unless $k \cong \mathbb{Z}/2\mathbb{Z}$ and $G \cong Z_2$ or Z_3 , or $k \cong \mathbb{Z}/3\mathbb{Z}$ and $G \cong Z_2$. In those three cases there is no such element.* \square

Whether nonmonomial invertible elements in group algebras kG occur *only* when G has elements of finite order is a longstanding open question.¹ Of course, Theorem 4 shows that the weaker property that a nonmonomial element generates the improper 2-sided ideal occurs in a much wider class of cases. For that condition, Theorem 14 and Theorem 4 between them leave open

Question 15. *Suppose G is a group containing an element g which has infinite order, but whose image in $G/[g, G]$ has finite order n , and that k is a finite field such that $\text{card}(k - \{0\})$ divides n . Must kG contain a non-monomial element which generates the improper ideal?*

¹The statement that nonmonomial units occur only in that case is often called ‘‘Kaplansky’s unit conjecture’’. However, Kaplansky [13, p.448, next to last parag.] only notes in passing, in connection with Problem 6, his zero-divisor question, that the analogous question for units was asked as Problem V.2 in a list [8] put out in connection with a symposium at Kishinev (to which I had apparently directed his attention). The only version of that list which I am presently able to access, [15], uses a different numbering system; the relevant question there is 1.135, posed by D. M. Smirnov and A. A. Bovdi, which asks whether for G a group without elements of finite order, every unit of $\mathbb{Z}G$ has the form $\pm g$ ($g \in G$).

5. CLOSURE PROPERTIES OF THE CLASS OF RESISTANT GROUPS

To examine the class of resistant groups and some related classes more closely, we make

Definition 16 (cf. Definition 1). *If G is a group and k a field, G will be called k -resistant if for every $r \in kG$ with support of cardinality > 1 , the ideal of kG generated by r is proper.*

If \mathcal{K} is a class of fields, a group G will be called \mathcal{K} -resistant if it is k -resistant for all $k \in \mathcal{K}$.

A key fact will be

Proposition 17. *There exists a set of universal sentences of the form*

$$(2) \quad (\forall g_1, \dots, g_n \in G, \forall c_1, \dots, c_{n'} \in k) P(g_1, \dots, g_n, c_1, \dots, c_{n'}),$$

where in each such sentence, P is a Boolean expression in equations in g_1, \dots, g_n under group operations and equations in $c_1, \dots, c_{n'}$ under field operations (with P , n and n' depending on the sentence (2) in question), such that for a group G and a field k , G is k -resistant if and only if G and k satisfy all the conditions (2) in this set.

Idea of proof. We want to examine all possible ways that the ideal of a group algebra kG generated by a non-monomial element

$$(3) \quad r = \sum_{i=1}^m c_i g_i$$

($c_i \in k$, $g_i \in G$, $m \geq 2$) might contain $1 \in kG$, and construct a set of sentences which together say that none of these possibilities occurs. The general element of the ideal generated by r has the form

$$(4) \quad \sum_{j=1}^{m'} c_{m+j} g_{m+j} r g_{m+m'+j}$$

(where m is as in (3), $m' \geq 0$, $c_i \in k$ for $m < i \leq m + m'$, and $g_i \in G$ for $m < i \leq m + 2m'$). Clearly, this will equal $1 \in kG$ for given choices of m' , $c_{m+1}, \dots, c_{m+m'}$, and $g_{m+1}, \dots, g_{m+2m'}$ if and only if, when we expand (4) using (3), and note which of the products

$$(5) \quad g_{m+j} g_i g_{m+m'+j} \quad (1 \leq i \leq m, 1 \leq j \leq m')$$

in the expansion are equal to which others – an equivalence relation on product-expressions of this form – it turns out that the common value of (5) for one of these equivalence classes is $1 \in G$, and the sum of the coefficients $c_{m+j} c_i$ for that equivalence class is $1 \in k$, while for each of the other equivalence classes, the corresponding sum is 0.

So let us construct the sentences (2) as follows. Each will be determined by a choice of $m, m' \geq 2$ (since only in such cases can (3) and (4) describe a non-monomial element r yielding 1 in the ideal it generates). Given these, we let $n = m + 2m'$, $n' = m + m'$, and let these index symbols $g_1, \dots, g_n, c_1, \dots, c_{n'}$. We then list all equivalence relations on the set of product-expressions (5), and for each such equivalence relation, all choices of one distinguished equivalence class. For each such equivalence relation and distinguished class, we write down the conjunction of the list of group-theoretic and field-theoretic equalities and negations-of-equalities saying that (i) g_1, \dots, g_m are distinct, (ii) c_1, \dots, c_m are nonzero, (iii) the products $g_{m+j} g_i g_{m+m'+j}$ in each equivalence class are equal, and are unequal to those in other equivalence classes, (iv) for the distinguished equivalence class, the common value of these products is the group element 1, (v) the sum over the distinguished equivalence class of the field elements $c_{m+j} c_i$ is 1, and (vi) the corresponding sum over each of the other equivalence classes is 0.

For each choice of m and m' , we form the conjunction P , over all choices of equivalence relations and distinguished equivalence classes as above, of the *negations* of the resulting conjunctions of formulas. The resulting sentence (2), applied to any group G and field k , says that no m -term element $r \in kG$ and m' -term expression (4) constitute a counterexample to the k -resistance of G . Doing this for each choice of $m, m' \geq 2$, we get the desired family of sentences (2).

(Incidentally, not all of the equivalence relations referred to above will necessarily be consistent with a group structure, nor need all the systems of equations and negations of equations in the c 's be consistent with the properties of fields. This does not matter: if a conjunction so used in P cannot actually occur, its presence (in negated form, as above) will have no effect on the set of (G, k) satisfying (2).) \square

We now modify slightly the form of our conditions (2). (And though I have spoken above of “negations of equalities” to emphasize that we are working with Boolean expressions in equalities, I will call them inequalities from here on.)

Proposition 18. *Any sentence (2) as described in the statement of Proposition 18 is equivalent to a finite conjunction of sentences of the form*

$$(6) \quad ((\forall g_1, \dots, g_n \in G) P'(g_1, \dots, g_n)) \vee ((\forall c_1, \dots, c_{n'} \in k) P''(c_1, \dots, c_{n'})),$$

where each P' is a disjunction of equations and inequalities of group-theoretic expressions in g_1, \dots, g_n , and each P'' a disjunction of equations and inequalities of field-theoretic expressions in $c_1, \dots, c_{n'}$.

Proof. Any Boolean expression in a set of relations is equivalent to a conjunction of disjunctions of those relations and their negations. Applying this to the expression $P(g_1, \dots, g_n, c_1, \dots, c_{n'})$ in (2), and noting that universal quantification respects conjunctions, we see that each instance of (2) is equivalent to a finite family of formulas $(\forall g_1, \dots, g_n \in G, \forall c_1, \dots, c_{n'} \in k) P^*(g_1, \dots, g_n, c_1, \dots, c_{n'})$, in each of which $P^*(g_1, \dots, g_n, c_1, \dots, c_{n'})$ is a disjunction of equations and inequalities. Each such equation or inequality involves only the group elements or only the field elements, so sorting them accordingly, we can rewrite each $P^*(g_1, \dots, g_n, c_1, \dots, c_{n'})$ as $P'(g_1, \dots, g_n) \vee P''(c_1, \dots, c_{n'})$; so that (2) is equivalent to a family of sentences

$$(7) \quad (\forall g_1, \dots, g_n \in G, \forall c_1, \dots, c_{n'} \in k) P'(g_1, \dots, g_n) \vee P''(c_1, \dots, c_{n'}).$$

Universal quantification does not in general respect disjunction. (E.g., the statement about integers, that for all n , either n is even or n is odd, does not entail that either all integers are even or all are odd.) But the fact that $P'(g_1, \dots, g_n)$ and $P''(c_1, \dots, c_{n'})$ involve disjoint sets of variables implies that here one *can* pass the disjunction through the quantification, and rewrite (7) as (6). \square

We can now move the conditions on field elements entirely out of our formulas.

Theorem 19. *For every class \mathcal{K} of fields, there is a set of sentences of the form*

$$(8) \quad (\forall g_1, \dots, g_n \in G) P(g_1, \dots, g_n),$$

where each P is a disjunction of equations and inequalities of group-theoretic expressions in g_1, \dots, g_n , under symbolic group operations, such that the \mathcal{K} -resistant groups are precisely the groups that satisfy all these sentences (8).

In particular, taking for \mathcal{K} the class of all fields, one gets a set of sentences (8) characterizing the resistant groups.

Proof. To obtain the desired system of sentences (8), one goes through the sentences (6) of Proposition 18 and asks, for each, whether the second part, $(\forall c_1, \dots, c_{n'} \in k) P''(c_1, \dots, c_{n'})$, holds for all $k \in \mathcal{K}$. If it does, we ignore that instance of (6), while if it does not, we include the other part, $(\forall g_1, \dots, g_n \in G) P'(g_1, \dots, g_n)$, in our set of sentences (8). That the resulting set of sentences characterizes the \mathcal{K} -resistant groups then follows from the fact that the sentences (6) characterize pairs (G, k) such that G is k -resistant. \square

Calling on some standard facts about classes of objects characterized by universally quantified sentences, this gives us

Corollary 20. *For every class \mathcal{K} of fields (including the class of all fields), the class of \mathcal{K} -resistant groups*

- (a) *is closed under passing to subgroups,*
- (b) *is closed under taking direct limits,*
- (c) *is closed under taking inverse limits, and*
- (d) *is closed under taking ultraproducts.*

Proof. (a)-(c) are instances of general results on classes of structures defined by *universal sentences*: that such classes satisfy (a) (which is fairly obvious) is the group-theoretic case of [12, Corollary 2.4.2, p. 49]; (b) appears as [3, Exercise 5.2.25, p. 243] (it is only stated there for direct limits over the natural numbers, but that restriction is not needed); (c) appears as [12, Theorem 2.4.6] and [3, Exercise 5.2.24, p. 243]. Property (d) holds for structures defined by arbitrary first-order sentences (which can involve arbitrarily long strings of alternating universal and existential quantifiers); this is Łoś's Theorem [2, Theorem 2.9, and Theorem 2.16 (a) \implies (b)]. \square

Remarks: Conditions (b) and (c) can be deduced from (a) and (d): Given a directed or inversely directed partially ordered set (I, \leq) and an I -indexed commutative diagram of groups $(G_i)_{i \in I}$ and group homomorphisms $f_{ij} : G_i \rightarrow G_j$ ($i \leq j$), we consider the filter on I generated by the principal up-sets, respectively the principal down-sets. Letting \mathcal{U} be an ultrafilter on I refining this filter, we find that the ultraproduct $(\prod_I G_i)/\mathcal{U}$ contains

a copy of $\varinjlim_I G_i$, respectively $\varprojlim_I G_i$. Hence if each G_i is \mathcal{K} -resistant, that ultraproduct will be so by (d), and thus the direct or inverse limit will be so by (a).

Here is a nice consequence.

Proposition 21. *If, for some class \mathcal{K} of fields, the free group on two generators is \mathcal{K} -resistant, then so is the free product of any family of torsion-free abelian groups.*

Sketch of proof. We shall see that \mathcal{K} -resistance of each of the groups or sorts of groups to be listed as (i)-(vii) below implies that of the next. When we refer in our arguments to “(a)”, “(b)”, etc., these are the four cases of Corollary 20 above.

Here are the seven (sorts of) groups. In (ii)-(vi), n denotes a fixed positive integer, assumed the same in successive cases; but \mathcal{K} -resistance for case (i) will imply the same for case (ii) for all n , and \mathcal{K} -resistance of (vi) for all n will be used in deducing \mathcal{K} -resistance of (vii).

- (i) The free group on two generators,
- (ii) the free group on n generators,
- (iii) a nonprincipal ultrapower over a countable index set of the free group on n generators,
- (iv) the free product of n copies of a nonprincipal ultrapower of the infinite cyclic group over a countable index set,
- (v) the free product of every family of n free abelian groups of finite ranks,
- (vi) the free product of every family of n torsion-free abelian groups,
- (vii) the free product of every family (finite or infinite) of torsion-free abelian groups.

Under our assumption that (i) is \mathcal{K} -resistant, the fact that the free group on two generators contains subgroups free on any finite number n of generators implies the \mathcal{K} -resistance of (ii), by (a). By (d), (ii) gives us (iii). Using the fact that the free group on n generators is the free product of n copies of the infinite cyclic group, and using the normal form for free products of groups, it is not hard to show that an ultrapower of the free group on n generators contains the free product of n copies of the corresponding ultrapower of the infinite cyclic group, giving (iv). Now a nonprincipal countable ultrapower of the infinite cyclic group Z is uncountable, but is, like Z , torsion-free abelian, hence it must contain infinitely many linearly independent elements, hence in particular, it must contain subgroups that are free abelian of all finite ranks; so the free product of n copies of that group will contain subgroups as in (v). Note next that every finitely generated subgroup of a torsion-free abelian group is free abelian of finite rank, hence every torsion-free abelian group is a direct limit of free abelian groups of finite rank; hence the corresponding statement holds for free products of n such groups, so by (b), (v) gives (vi). Similarly, the free product of a general family of torsion-free abelian groups is a direct limit of free products of finite families of such groups, so another application of (b) gives (vii). \square

For the group theorist who finds the use of ultrapowers a bit esoteric, let me indicate, very roughly, in a different way, what is behind Proposition 21. Suppose we know the free group $F = \langle x, y \rangle$ is k -resistant, and want to show the same for, say, $G = \langle x, y, y' \mid [y, y'] = 1 \rangle$, i.e., the free product of the free abelian group on one generator x , and the free abelian group on two generators y, y' . So let $r(x, y, y')$ be a non-monomial element of kG , and consider the group homomorphism $G \rightarrow F$ carrying x to x , y to y , and y' to y^N for some large integer N . The key fact (which I leave it to the reader to convince himself or herself of) is that for N large enough, the image $r(x, y, y^N)$ will also be non-monomial – in fact, that the distinct terms in the support of $r(x, y, y')$ will have distinct images in F . Hence by our assumption on F , $r(x, y, y^N)$ generates a proper ideal of kF ; hence $r(x, y, y')$, which maps to it, must generate a proper ideal of kG . The ultrapower argument gets the same conclusion by showing that the above G is a subgroup of an ultrapower of F .

Some further results on the class of groups that can be obtained as direct or inverse limits of free products of finite families of free abelian groups of finite ranks are noted in an appendix, §7.

Let us ask about a possible result stronger than that asked for in Question 2.

Question 22. *Is the class of resistant groups closed under taking free products of groups?*

A question which looks more elementary, but which I also don't know how to approach, is

Question 23. *Is the class of resistant groups closed under taking direct products of groups?*

If we replace “resistant” with “ \mathcal{K} -resistant” for an arbitrary class \mathcal{K} of fields, both questions have negative answers, as shown by Theorem 14, which tells us that the groups Z_2 and Z_3 are \mathcal{K} -resistant for $\mathcal{K} = \{\mathbb{Z}/2\mathbb{Z}\}$, but

that this is not true of any groups properly containing them, though such groups can be formed by taking direct products or free products of these groups with themselves, with each other, or with other \mathcal{K} -resistant groups, such as Z . I know of no counterexamples other than these (up to isomorphism of groups and fields); but these cases suggest that positive answers to Questions 22 and 23 are not likely to be true for trivial reasons.

But here is a special case of Question 23 which has an easy positive answer.

Proposition 24. *If G is a resistant group and A a torsion-free abelian group, then $G \times A$ is again resistant.*

More generally, this is true with “resistant” replaced by “ \mathcal{K} -resistant” for any class \mathcal{K} of fields closed under passing to extension fields.

Proof. We will prove the general statement. Our proof will use the observation that $k(G \times A) \cong kG \otimes_k kA$.

Let r be a non-monomial element of $k(G \times A)$, where G and A are as in the hypothesis, and $k \in \mathcal{K}$.

Suppose first that not all elements of $\text{supp}(r) \subseteq G \times A$ have the same G -component. Then letting k' be the field of fractions of the commutative integral domain kA , we see that the natural embedding

$$(9) \quad k(G \times A) \cong kG \otimes_k kA \rightarrow kG \otimes_k k' \cong k'G$$

carries r to a non-monomial element of this k' -algebra. Since $k' \in \mathcal{K}$ and G is \mathcal{K} -resistant, that element generates a proper ideal, whose inverse image in $kG \otimes_k kA$ will be a proper ideal containing r .

On the other hand, if all elements of $\text{supp}(r)$ have the same G -component $g \in G$, we can write $r = ag$ where a is a non-monomial element of kA . Thus, a is non-invertible in the commutative ring kA , and so is contained in a proper maximal ideal M . Letting k'' denote the field $(kA)/M$, we see that the natural map

$$(10) \quad k(G \times A) \cong kG \otimes_k kA \rightarrow kG \otimes_k k'' \cong k''G$$

carries r to zero, so the kernel of (10) is a proper ideal containing r . □

Returning to Proposition 21, the method used to prove that result can be used to show that still more groups are \mathcal{K} -resistant under the same assumption. For instance, let w be any nonidentity element of the free group on two generators, $\langle x, y \rangle$. Suppose we take a nonprincipal ultrapower of $\langle x, y \rangle$, and within it, let G be the subgroup generated by the canonical image of $\langle x, y \rangle$, together with an element z of the ultrapower of the free abelian subgroup $\langle w \rangle$, such that, in that ultrapower, z and w are linearly independent. The centralizer of z in $\langle x, y \rangle$ will be the same as the centralizer of w therein; and if w is not a proper power in $\langle x, y \rangle$, that centralizer will be precisely $\langle w \rangle$. The resulting group will then have the structure $\langle x, y, z \mid [w, z] = 1 \rangle$. If w happens to belong to some free generating set $\{v, w\}$ of the free group $\langle x, y \rangle$, then G is just the free product of the free abelian groups on $\{v\}$ and on $\{w, z\}$, and so will have the form described in Proposition 21; but if w , in addition to not being a proper power, does not belong to such a free generating set (e.g., if $w = x^m y^n$ with $m, n > 1$, or $w = [x, y]$), then G will lie outside the class of groups named in that proposition, though it will be resistant by the reasoning used there. Another group which one could show resistant in the same way is $\langle x, y, z_1, z_2, z_3 \mid 1 = [x, z_1] = [y, z_2] = [xy, z_3] \rangle$.

If we knew that Questions 22 and 23 had positive answers, then starting with the infinite cyclic group and iterating the operations of product and free product, we could conclude that a large class of groups presented by sets of generators together with commutativity relations among *some* pairs of these generators were resistant. But this would not cover all groups having presentations of that description, so let us ask

Question 25. *Is every right-angled Artin group – that is, every group having, for some set I and some set J of 2-element subsets of I , the presentation*

$$(11) \quad G_{I,J} = \langle x_i \ (i \in I) \mid x_i x_j = x_j x_i \ \text{for all } \{i, j\} \in J \rangle$$

resistant?

Examples of groups (11) that cannot be gotten by iterating the constructions of Questions 22 and 23 are those with $I = \{1, \dots, n\}$ for any $n \geq 4$ and $J = \{\{i, i+1\} \mid 1 \leq i < n\}$, and the cyclically indexed analog of these, with $I = Z_n$ for any $n \geq 5$, and $J = \{\{i, i+1\} \mid i \in I\}$. (See [4] for a general survey of the subject of right-angled Artin groups.)

A positive answer to Question 25 would lead, via the application of ultraproducts and subgroups, to further classes of resistant groups.

We remark that right-angled Artin groups are a special case of *Artin-Tits groups* [9], these being groups with presentations in which one again starts with a generating set $\{x_i \mid i \in I\}$ and a set J of 2-element subsets of I , but

then imposes for each $\{i, j\} \in J$ a relation of one of the forms $(x_i x_j)^{d_{ij}} = (x_j x_i)^{d_{ij}}$ or $(x_i x_j)^{d_{ij}} x_i = (x_j x_i)^{d_{ij}} x_j$, with $d_{ij} \geq 1$ in each case.

However, among such groups G , *only* the right-angled Artin groups, i.e., those whose relations are all of the first form and have $d_{ij} = 1$, can be resistant. For if an Artin-Tits group has among its defining relations a relation $(x_i x_j)^{d_{ij}} = (x_j x_i)^{d_{ij}}$ with $d_{ij} > 1$, then Corollary 5(c) applies with $h = x_i x_j$, $h' = x_j x_i$, while if it has a defining relation $(x_i x_j)^{d_{ij}} x_i = (x_j x_i)^{d_{ij}} x_j$, then taking $g = x_i x_j^{-1}$, we see that in $G/[g, G]$, \bar{x}_j commutes with $\bar{g} = \bar{x}_i \bar{x}_j^{-1}$, hence with \bar{x}_i , hence the given relation reduces to $\bar{x}_i = \bar{x}_j$, which says $\bar{g} = 1$; hence Theorem 3(i) fails, hence so does Theorem 3(iv). (The above arguments used implicitly the facts that in the former situation, the relations defining G do not imply $x_i x_j = x_j x_i$, and that in the latter, they do not imply $x_i = x_j$. These can be deduced from the result of [17] that given a family of relations as in the definition of an Artin-Tits group, the monoid that those relations define embeds in the group that they define, together with the observation that since each such defining relation involves the same set of generators on both sides, any monoid relation $a = b$ that they imply will also involve the same set of generators on both sides, and be a consequence of the subset of our defining relations that involve no generators not occurring in a and b .)

6. A POSSIBLE FREIHEITSSATZ FOR GROUP ALGEBRAS

A standard group-theoretic result, the Freiheitssatz [14], says roughly (we will be more precise soon) that if one divides the free group F on generators $(x_i)_{i \in I}$ by the normal subgroup N generated by a single relator w , and if x_{i_0} is one of the generators involved in w , then the subgroup of F/N generated by the images of all the other generators x_i ($i \in I - \{i_0\}$) is free on those generators. For instance, in the group $\langle x, y, z \mid x^{-2} y^{-3} x^2 y^3 = 1 \rangle$, since the relation involves x , the subgroup generated by y and z is free on those generators, and similarly, since the relation involves y , the elements x and z generate a free subgroup. On the other hand, since z is not involved in our relator $x^{-2} y^{-3} x^2 y^3$, we cannot say the same about the subgroup generated by x and y ; clearly it is not free on x and y .

However, one has to be careful about what sort of relator one uses. The group described above could also be written $\langle x, y, z \mid z(x^{-2} y^{-3} x^2 y^3) z^{-1} = 1 \rangle$, where the relator now formally involves z , but the subgroup generated by x and y is, of course, still not free. The statement of the Freiheitssatz excludes such cases by requiring ‘‘cyclically reduced’’ relators: words w in the generators such that not only do symbols x_i and x_i^{-1} never appear adjacent within w , but such that they also do not occur one as the first, and the other as the last term of w . Clearly, any relation is equivalent to one given by a cyclically reduced relator.

Might some sort of Freiheitssatz hold regarding one-relator *factor algebras* of group algebras of free groups?

Suppose F is the free group on generators $(x_i)_{i \in I}$, and $r \in kF$ is a relator we wish to divide out by. What are the obstacles to hoping that for any x_{i_0} occurring in r , the subalgebra of our factor algebra generated by the other x_i is isomorphic to the group algebra on the free group on those generators? In this case, there are several. Note that in the free group on generators x, y, z , the ideal generated by $x^2 - y^3$ is also generated by $zx^2 - zy^3$ and likewise by $x^2 z - y^3 z$; so we should forbid any generator-symbol (or inverse of a generator-symbol) from appearing simultaneously as the first letter of all members of the support of our relator, or as the last letter of all these elements. This condition automatically excludes the result of conjugating a relator r by a generator x_i which r does not involve – unless the support of r contains the element 1. Bringing in that further case, let us make

Definition 26. For F the free group on generators $(x_i)_{i \in I}$, and k a field, we shall call an element $r \in kF$ strongly reduced if, when the elements of $\text{supp}(r)$ are written in normal form,

- (a) there is no symbol $x_i^{\pm 1}$ with which all elements of $\text{supp}(r)$ begin,
- (b) there is no symbol $x_i^{\pm 1}$ with which all elements of $\text{supp}(r)$ end, and
- (c) if $1 \in \text{supp}(r)$ (so that (a) and (b) hold trivially), there is no symbol $x_i^{\pm 1}$ such that all elements of $\text{supp}(r)$ other than 1 both begin with $x_i^{\pm 1}$ and end with the inverse symbol, $x_i^{\mp 1}$.

For any $r \in kF$, let us say that a generator x_{i_0} is ‘‘involved in’’ r if x_{i_0} or $x_{i_0}^{-1}$ occurs anywhere in the normal form of any of the elements of $\text{supp}(r)$. We can now pose

Question 27. Let F be the free group on a set $(x_i)_{i \in I}$, k a field, r a strongly reduced element of kF , and $(r) \subseteq kF$ the ideal it generates. Then must the following equivalent conditions hold?

- (i) For every x_{i_0} involved in r , the subalgebra of $kF/(r)$ generated by the images of the $x_i^{\pm 1}$ ($i \in I - \{i_0\}$) is (up to natural isomorphism) the group algebra over k of the free group on $\{x_i \mid i \in I - \{i_0\}\}$.

- (ii) For every x_{i_0} involved in r , the ideal (r) has zero intersection with the subalgebra of kF generated by $\{x_i^{\pm 1} \mid i \in I - \{i_0\}\}$.
- (iii) Every nonzero element $s \in (r)$ involves (at least) all generators x_i which r involves.

We see from formulation (ii) above that a positive answer to this question would imply a positive answer to Question 2. Formulation (iii) seems to give the best insight as to what would be needed to come up with a proof or a counterexample.

Some further observations: If we could prove the suggested Freiheitssatz for elements whose supports contain 1 as in condition (c), then the whole statement would follow. Indeed, given an r' which, rather, satisfies (a) and (b), and whose support does not contain 1, let u be an element of minimal length in that support, and let v and w be (not necessarily distinct) elements of that support such that the first and last terms in their normal form expressions differ respectively from the corresponding terms in the expression for u . The element $r = u^{-1}r'$ certainly generates the same ideal as does r' and has 1 in its support. The element $u^{-1}v$ of its support has normal form beginning with the normal form of u^{-1} , hence involves all the generators that u involves, from which it is not hard to deduce that r involves exactly the same set of generators as r' . Finally, the initial factor in the normal form of $u^{-1}v$ is the inverse of the final factor in that of u , while the final factor of $u^{-1}w$ is, by assumption, *not* the final factor of u , from which it can be seen that r' satisfies (c); so if the Freiheitssatz holds for elements as in (c), the result for elements as in (a) and (b) whose support does not contain 1 also holds.

(I claimed in an earlier preprint of this note that, likewise, if one knew the result for all elements r satisfying (a) and (b) and not having 1 in their supports, one could prove it for elements r as in (c); but looking more closely, I do not see how to prove this. A situation where there is in general no hope of doing this by applying (a) and (b) to a 2-sided associate of r is when r involves only one of our generators; but in that case, it is not hard to show directly that r satisfies (i) above. For two examples where one can indeed get from an r as in (c) with 1 in its support an element $r' = urv$ ($u, v \in F$) as in (a) and (b) not having 1 in its support – but not in any evident systematic way – consider on the one hand $r = 1 + x + y + xy$, and on the other the very similar expression, $r = 1 + x + y + xy + yx$. Then $r' = y^{-1}rx^{-1}$ works for the former but not the latter; while $x^{-1}rx^{-1}$ works for the latter but not the former. (Does any such expression work for both? Yes, it happens that $y^{-1}x^{-1}rx^{-1}$ does.) But I don't see a way to find such u and v for general r .)

Note that a nonzero member of (r) can have far fewer elements in its support than r itself does. For a familiar case, take any $g \in F - \{1\}$ let $r = 1 + g + \cdots + g^{n-1}$, for some large n , and note that $(1 - g)r = 1 - g^n$, with only two terms. For another sort of example, let x and y be two members of the given free generating set of F and let r have the form $p(x) + y$, where p is any polynomial (or more generally, any Laurent polynomial) in one variable. Then $xr - rx = xy - yx$, again with only two terms. Nevertheless, in each of these cases the set of free *generators* involved in the indicated element of (r) still includes all those involved in r ; so they do not give counterexamples to property (iii) above.

Analogues of Questions 27 and 2 for *free associative algebras* (monoid algebras of free monoids) were posed by P. M. Cohn in [5, Conjectures 1 and 3, pp.121-122]. Since the free monoid has no invertible elements other than 1, the desired Freiheitssatz did not involve any conditions analogous to the “cyclically reduced” assumption of the group-theoretic Freiheitssatz, or the “strongly reduced” assumption used above. L. G. Makar-Limanov [16, §5] subsequently proved such a Freiheitssatz for k of characteristic 0, and thus a positive answer to the analog to Question 2 for free associative algebras over such k .

7. APPENDIX: DIRECT AND INVERSE LIMITS OF FREE PRODUCTS OF FREE ABELIAN GROUPS OF FINITE RANK

We saw in Proposition 21 that if the free group on two generators is resistant, then so is the free product of any family of torsion-free abelian groups. The resistance of free products of finite families of free abelian groups of finite ranks was a key step in this deduction, and a key tool was Corollary 20, saying that the class of resistant groups is closed under subgroups, direct and inverse limits, and ultraproducts. We shall show below that for

$$(12) \quad \mathcal{G} = \text{the class of all free products of finite families of free abelian groups of finite ranks,}$$

what we can get from \mathcal{G} by direct limits alone are precisely the groups all of whose finitely generated subgroups lie in \mathcal{G} , while what we can get by inverse limits alone form (curiously) a proper subclass of those direct limits.

We begin with a general observation, which we will subsequently apply to the \mathcal{G} of (12).

Proposition 28. *Let \mathcal{G} be any class of finitely generated groups which is closed under isomorphisms, and under passing to finitely generated subgroups (not necessarily the class described in (12)).*

(a) *Suppose further that every chain of surjective non-one-to-one homomorphisms $G_1 \rightarrow G_2 \rightarrow \dots$ among members of \mathcal{G} is finite. Then the direct limits of directed systems of groups in \mathcal{G} are precisely the groups whose finitely generated subgroups all belong to \mathcal{G} .*

(b) *Suppose, rather, that every “reverse chain” of surjective non-one-to-one homomorphisms, $\dots \rightarrow G_{-2} \rightarrow G_{-1}$, among members of \mathcal{G} that admits a common finite bound on the number of generators required by the G_i is finite. Then every inverse limit of groups in \mathcal{G} has the property that all its finitely generated subgroups belong to \mathcal{G} . (Hence, such inverse limits are also direct limits of groups in \mathcal{G} .)*

Proof. We shall show, under the respective hypotheses of (a) and (b), that if G is a direct limit, respectively an inverse limit, of groups in \mathcal{G} , then every finitely generated subgroup of G belongs to \mathcal{G} . The fact that every group is the directed union of its finitely generated subgroups yields the converse to this result in the situation of (a), and hence the parenthetical observation in (b).

Suppose first that G is the direct limit of a system of groups $G_i \in \mathcal{G}$ ($i \in I$) and maps $f_{ij} : G_i \rightarrow G_j$ for $i \leq j$, under a directed partial ordering \leq on I , and let H be a subgroup of G generated by finitely many elements g_1, \dots, g_n . By the directedness of I , there will be some $i_0 \in I$ such that G_{i_0} contains elements $g_{i_0,1}, \dots, g_{i_0,n}$ mapping to $g_1, \dots, g_n \in G$. Now consider for each $i \geq i_0$ the subgroup $H_i \subseteq G_i$ generated by the images of $g_{i_0,1}, \dots, g_{i_0,n}$. These subgroups will form a directed system of members of \mathcal{G} (in view of our assumption that \mathcal{G} is closed under passing to finitely generated subgroups); so assuming the hypothesis of (a), after some point i_1 in that directed system, the induced maps among the H_i will all be one-to-one, hence be isomorphisms. Hence $H_{i_1} \subseteq G_{i_1}$ is isomorphic to $H \subseteq G$; moreover, being a finitely generated subgroup of $G_{i_1} \in \mathcal{G}$, the group H_{i_1} lies in \mathcal{G} , so $H \in \mathcal{G}$.

On the other hand, suppose G is the inverse limit of a system of groups $G_i \in \mathcal{G}$ ($i \in I$) and maps $G_i \rightarrow G_j$ ($i \leq j$) under an inversely directed partial ordering on I , and again let H be a finitely generated subgroup of G . This time, we look at the images $H_i \subseteq G_i$ ($i \in I$) of H , and the induced homomorphisms among these. If H is generated by n elements, the same will be true of all the H_i , so by (b), we can find an i_1 before which all these homomorphisms are isomorphisms, and conclude that $H \cong H_{i_1}$; hence again, $H \in \mathcal{G}$. \square

We now turn to the particular \mathcal{G} of (12).

That this \mathcal{G} is closed under passing to finitely generated subgroups follows from the Kurosh Subgroup Theorem [7, Theorem 7.8]. (That theorem says that any subgroup H of a free product G of groups is a free product of copies of subgroups of some of those groups, together with a free group. In our situation, the latter free group, a free product of infinite cyclic groups, can be regarded simply as bringing additional free abelian groups of rank 1 into the description of H as a free product; so H is, as required, a member of \mathcal{G} .)

Note that if $G \in \mathcal{G}$ is the free product of free abelian groups A_1, \dots, A_N of ranks $d_1, \dots, d_N \geq 1$, then it cannot be generated by fewer than $d_1 + \dots + d_N$ elements, since its abelianization is the free abelian group of that rank. Moreover, d_1, \dots, d_N determine the structure of G , so there exist, up to isomorphism, only finitely many groups in \mathcal{G} generated by a given finite number of elements.

Now in the chains of surjective homomorphisms $G_1 \rightarrow G_2 \rightarrow \dots$ and $\dots \rightarrow G_{-2} \rightarrow G_{-1}$ considered in Proposition 28, the numbers of generators of the groups in a given chain is always bounded: in the chains of the former sort, by the number of generators of G_1 ; in those of the latter sort, by the hypothesis of statement (b). Hence if a chain of either sort in our present \mathcal{G} were infinite, it would have to contain two isomorphic groups, which would lead to a non-one-to-one surjective endomorphism of some member of \mathcal{G} . But the groups in \mathcal{G} are known to be *Hopfian*, i.e., not isomorphic to proper homomorphic images of themselves [6]. (That paper shows that a free product of finitely many finitely generated Hopfian groups is Hopfian.) Hence the existence of such infinite chains would lead to a contradiction; so \mathcal{G} satisfies the hypotheses of both (a) and (b).

For this \mathcal{G} , the inclusion of the class of inverse limits in the class of direct limits (noted parenthetically at the end of (b)) is proper: The additive group of rational numbers, though a direct limit of free abelian groups, has no nontrivial homomorphisms into groups in \mathcal{G} , hence it is not an inverse limit of such groups.

In summary,

Theorem 29. *For \mathcal{G} as in (12), the direct limits of groups in \mathcal{G} are those groups all of whose finitely generated subgroups belong to \mathcal{G} , while the inverse limits of groups in \mathcal{G} form a proper subclass thereof. \square*

We have noted that the class of direct limit groups characterized in the first part of the above result includes all free products of families of torsion-free abelian groups. A member of that class which is not itself such a free product is the direct limit of the chain of inclusions of free groups on two generators

$$(13) \quad \langle x_1, x_2 \rangle \subseteq \dots \subseteq \langle x_{n-1}, x_n \rangle \subseteq \langle x_n, x_{n+1} \rangle \subseteq \dots, \text{ where } x_{n-1} = [x_n, x_{n+1}].$$

This group is nontrivial but has trivial abelianization, which is clearly not the case for any free product of abelian groups. An example of an *inverse* limit of groups in \mathcal{G} which is not a free product of abelian groups (and so gives another example of a direct limit with this property) is the inverse limit G of the chain of surjections of free groups

$$(14) \quad \dots \rightarrow \langle x_1, \dots, x_n \rangle \rightarrow \dots \rightarrow \langle x_1, x_2 \rangle \rightarrow \langle x_1 \rangle,$$

where each map $\langle x_1, \dots, x_n \rangle \rightarrow \langle x_1, \dots, x_{n-1} \rangle$ takes x_n to 1, and fixes the other free generators. Indeed, one can show that every abelian subgroup of G is cyclic, so if G were a free product of abelian groups, the free factors would be infinite cyclic, i.e., G would be free; but by [11, Corollary to Theorem 1] it is non-free.

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