Comments, corrections, and related references welcomed, as always!

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ON SEMIGROUPS THAT ARE PRIME IN THE SENSE OF TARSKI, AND GROUPS PRIME IN THE SENSES OF TARSKI AND OF RHODES

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ABSTRACT. If $\mathcal{C}$ is a category of algebras closed under finite direct products, and $M_{\mathcal{C}}$ the commutative monoid of isomorphism classes of members of $\mathcal{C}$, with operation induced by direct product, A. Tarski defines a nonidentity element p of $M_{\mathcal{C}}$ to be *prime* if, whenever it divides a product of two elements in that monoid, it divides one of them, and calls an object of $\mathcal{C}$ prime if its isomorphism class has this property.

McKenzie, McNulty and Taylor [6, p. 263] ask whether the category of nonempty semigroups has any prime objects. We show in §2 that it does not. However, for the category of monoids, and some other subcategories of semigroups, we obtain examples of prime objects in §§3-4. In §5 two related questions from [6], open so far as I know, are recalled.

In §6, which can be read independently of the rest of this note, we recall two related conditions that are called primeness by semigroup theorists, and obtain results and examples on the relationships among those two conditions and Tarski's, in categories of groups. §7 notes an interesting characterization of one of those conditions on finite algebras in an arbitrary variety.

Several questions are raised.

1. Note on terminology.

As noted in the abstract above, Alfred Tarski's definition of "prime" is different from a pair of senses currently common in semigroup theory. In §6 below the latter two conditions are recalled, and their relationship with each other and with Tarski's on groups is examined. But until that section, *prime algebra* will be understood in Tarski's sense, as indicated in Definition 2.1(ii) below.

2. The category of nonempty semigroups has no prime objects.

Here are two general usages that we will follow through §5:

Definition 2.1. (i) *As in* [6], *"semigroup" will be understood to mean "nonempty semigroup". The category of all semigroups will be denoted* **Semigp**.

(ii) *Also as in* [6] (*p. 263, top paragraph*), *if* $\mathcal{C}$ *is a category admitting finite direct products* (*e.g.,* **Semigp**), *an object* X *of* $\mathcal{C}$ *other than the final object* (*the product of the empty set, corresponding to the identity element of the monoid of isomorphism classes*) *will be called* prime *if, whenever an object* Y *of* $\mathcal{C}$ *admits* X *as a direct factor, and* $Y = Y_0 \times Y_1$ *is another direct product decomposition of* Y, *then one of* Y_0, Y_1 *admits* X *as a direct factor.*

Now for some bits of language and notation that will be useful in the arguments of this section.

Definition 2.2. (i) *By a* null semigroup *we shall mean a semigroup in which all pairs of elements have the same product. For any cardinal* κ, *the null semigroup of cardinality* $\kappa+1$ (*i.e., having exactly* κ *elements which are not products, in addition to the one which is*) *will be denoted* $\mathrm{Null}(\kappa)$.

(ii) *Two elements* x *and* x' *of a semigroup* S *will be called* action equivalent *if for all* $y \in S$, $x\,y = x'\,y$ *and* $y\,x = y\,x'$. (*Clearly, this is an equivalence relation on* S.)

(iii) *An element* x *of a semigroup* S *will be called a* product element *if it can be written* $x = y\,z$ *for* $y, z \in S$, *and a* non-product element *otherwise.*

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(iv) *Given a semigroup* S, *a* skeleton S' *of* S *will mean any subsemigroup of* S *which consists of all the product elements, and one and only one representative of each action equivalence class that contains no product elements.*

The definition of "skeleton" may look strange; a more natural characterization is that it is a minimal subsemigroup that contains at least one element from every action equivalence class. This guarantees that it contains every product element $x\,y$, since given x and y, it will contain some x' product-equivalent to x and some y' product-equivalent to y, and hence will contain their product, $x'y' = xy' = xy$. But since descending chains of nonempty sets can have empty intersection, the existence of such minimal examples is not self-evident, while the existence of S' as in (iv) above is.

Clearly,

(2.1) For S a semigroup, all skeleta of S are isomorphic to one another.

We also observe that

(2.2) A semigroup S is null if and only if it has one and only one action equivalence class.

Indeed, $\Rightarrow$ is clear. Conversely, if all elements are action equivalent, then for any $x,\, x', y,\, y' \in S$ we have $x'y' = x\,y' = x\,y$, showing that all products are equal.

It is also easy to see that if S and T are semigroups, then

(2.3) Elements (s,t), $(s',t') \in S\times T$ are action equivalent if and only if $s, s' \in S$ are action equivalent and $t, t' \in T$ are action equivalent.

(2.4) An element $(s,t) \in S \times T$ is a product element if and only if $s \in S$ is a product element and $t \in T$ is a product element.

The next lemma gives lots of isomorphisms among products of semigroups, which will be a tool in proving our no-prime-objects result.

Lemma 2.3. *If* S *and* T *are semigroups having isomorphic skeleta, and* κ *is an infinite cardinal greater than or equal to the cardinality of every action equivalence class in* S *and of every action equivalence class in* T, *then* $S \times \mathrm{Null}(\kappa) \cong T \times \mathrm{Null}(\kappa)$.

Moreover, every action equivalence class in that product semigroup has cardinality κ, *and contains* κ *non-product elements.*

Proof. We can assume without loss of generality that S and T have a common skeleton; hence, in particular, that their sets of product elements are the same. So, in view of (2.4), $S \times \mathrm{Null}(\kappa)$ and $T \times \mathrm{Null}(\kappa)$ have the same product elements.

Since the common skeleton of S and T contains at least one representative of each action equivalence class, and since the behavior of the semigroup operation on an element is determined by which action equivalence class it belongs to, we can get an isomorphism between $S\times\mathrm{Null}(\kappa)$ and $T\times\mathrm{Null}(\kappa)$ if, for each action equivalence class of the common skeleton of S and T, we can define a bijection between the sets of non-product elements of the corresponding action equivalence classes of $S\times\mathrm{Null}(\kappa)$ and $T\times\mathrm{Null}(\kappa)$.

But each of those non-product sets has cardinality κ. E.g., in $S \times \mathrm{Null}(\kappa)$, such a set has cardinality *at most* κ because it is contained in the direct product of a $\leq \kappa$-element action equivalence class of S and the $\kappa{+}1$-element semigroup $\mathrm{Null}(\kappa)$, and since κ is infinite, $\kappa(\kappa{+}1) = \kappa$; while it has *at least* κ elements occurring as pairs (s,α), where s is any member of the given equivalence class of S, and α ranges over the κ non-product elements of $\mathrm{Null}(\kappa)$. Hence the desired bijections can be chosen. $\square$

We can now prove

Lemma 2.4. *No non-null semigroup is prime in* **Semigp**.

Proof. Let S be a non-null semigroup. Take an infinite cardinal κ larger than the cardinality of any action equivalence class of S, and form a semigroup $S^+ \supset S$ by attaching, to one arbitrarily chosen action equivalence class, $A \subset S$, κ additional elements. (I.e., add κ additional elements to the underlying set of S, and define multiplication on the resulting set so that all products involving those elements are the same as the products one gets by putting members of A in their place. In particular, the product of two of the added elements will be the same as the product of any two members of A.)

By Lemma 2.3, we have

(2.5) $\quad S\times \mathrm{Null}(\kappa)\ \cong\ S^+\times \mathrm{Null}(\kappa).$

If S were prime, it would be isomorphic to a direct factor of one of the factor semigroups on the right-hand side of (2.5). That factor cannot be $\mathrm{Null}(\kappa)$, because S, being non-null, has more than one action equivalence class. On the other hand, if it were S^+, so that we could write

(2.6) $\quad S^+\ \cong\ S\times T,$

then since S has no action equivalence classes of cardinality κ, but S^+ does, the factor T must have one; but the products of that class with two different action equivalence classes in S would give two action equivalence classes in S^+ of cardinality κ, contradicting our construction of S^+. So S is not prime. □

To show that semigroups $\mathrm{Null}(\kappa)$ are also non-prime, we need a criterion for when a semigroup can be written as a direct product with $\mathrm{Null}(\kappa)$ as one of the factors.

Lemma 2.5. *Let κ be a cardinal* (*finite or infinite*), *and S a semigroup. Then the following two conditions are equivalent:*

(i) *There exists a semigroup S_0 such that $S\ \cong\ S_0\times \mathrm{Null}(\kappa)$.*

(ii) *Each action equivalence class A of S can be written as a disjoint union, $A=A_0\cup A_1$, such that* $\mathrm{card}(A_1)=\kappa\ \mathrm{card}(A_0)$, *and all* product *elements of S lying in A belong to A_0.*

Moreover, if S and κ are finite, (ii) *can be rephrased:*

(ii$'$) *Each action equivalence class A of S has cardinality $\kappa{+}1$ times an integer greater than or equal to the number of product elements in A.*

Proof. If (i) holds, let us assume without loss of generality that $S=S_0\times\mathrm{Null}(\kappa)$. Note that for each $s\in S_0$, the $\kappa{+}1$ elements (s,x) $(x\in\mathrm{Null}(\kappa))$ are action equivalent, and at most one (the one with x the product element of $\mathrm{Null}(\kappa)$) can be a product element, so taking the union of these families over the elements x of an action equivalence class of S_0, and writing A_0 for the set of (s,x) in this union with x the product element of $\mathrm{Null}(\kappa)$, and A_1 for the set of (s,x) with non-product x, we get the desired decomposition of the induced action equivalence class of S.

Conversely, if we have a decomposition of each action equivalence class of S as in (ii), let S_0 be the union over these classes A of their chosen subsets A_0. This is easily seen to be a subsemigroup of S such that $S\cong S_0\times\mathrm{Null}(\kappa)$.

When S and κ are finite, the equivalence of (ii) and (ii$'$) is straightforward. □

As promised, we can now prove

Lemma 2.6. *No semigroup* $\mathrm{Null}(\kappa)$ *is prime.*

Proof. For each κ, we shall construct a semigroup S which does *not* satisfy Lemma 2.5(ii), and a positive integer n such that $\mathrm{Null}(n)$ also does not satisfy that condition, but such that $S\times\mathrm{Null}(n)$ does; so in view of Lemma 2.5 (i) $\Longrightarrow$ (ii), this product is a counterexample to primeness of $\mathrm{Null}(\kappa)$.

If κ is infinite, let S consist of elements x_{ij} $(i\in\kappa,\ j\in\kappa{+}1)$, y_i $(i\in\kappa)$ and z, with multiplication given by

(2.7) $\quad x_{ij}\,x_{ij'}\ =\ y_i$ for $i\in\kappa,\ j,j'\in\kappa{+}1,$ while all products not of this form have value z.

We see that S is associative, since all 3-fold products equal z. Note that the action equivalence classes of S are, on the one hand, the sets

(2.8) $\quad \{x_{ij}\,|\,j\in\kappa{+}1\}$, one for each $i\in\kappa$,

and one more equivalence class, comprising all the remaining elements:

(2.9) $\quad \{y_i\,|\,i\in\kappa\}\,\cup\,\{z\}.$

Of these, the classes (2.8) clearly satisfy the condition of Lemma 2.5(ii), with A_0 the singleton $\{x_{i0}\}$, and A_1 consisting of the remaining κ elements, but the class (2.9) does not, since all of its elements are products.

Now take any positive integer n (e.g., $n=1$). Since κ is infinite, the finite semigroup $\mathrm{Null}(n)$ cannot satisfy condition (ii) of Lemma 2.5. But in the product semigroup $S\times\mathrm{Null}(n)$, we have effectively adjoined $n\,\kappa=\kappa=\kappa^2$ non-product elements (s,i) $(i\in n)$ to each of the κ-element action equivalence classes (2.8) and (2.9) of S; so the product semigroup does satisfy Lemma 2.5(ii), giving the desired example.

Finally, suppose κ is finite.

If $\kappa = 0$, then $\mathrm{Null}(\kappa)$ is the 1-element semigroup, corresponding to the identity element of the monoid of isomorphism classes, which by definition is not prime.

If $\kappa > 0$, let S be the semigroup of $2(\kappa{+}1)$ elements,

(2.10) $\quad x_i \ \ (i \in \kappa{+}1), \ \ y_i \ \ (i \in \kappa), \ \ \text{and} \ \ z,$

with multiplication

(2.11) $\quad x_i\, x_{i'} \ = \ y_0, \ \ \text{and all other products equal to} \ \ z.$

Here there are two action equivalence classes, each of cardinality $\kappa{+}1$, namely $\{x_i\}$ and $\{y_i\} \cup \{z\}$. Analogously to the infinite case, we see that the first satisfies Lemma 2.5(ii′), but the second does not, in this case because it has two product elements, y_0 and z, so that any decomposition as in Lemma 2.5(ii′) would require it to have at least $2(\kappa{+}1)$ elements, while it has only $\kappa{+}1$.

However, if we take the direct product of S with any semigroup $\mathrm{Null}(n)$ $(n > 0)$, the total number of elements of that equivalence class will be at least doubled, while the number of product elements will not be changed, and we see that condition (ii′) then holds; so $S \times \mathrm{Null}(n)$ has $\mathrm{Null}(\kappa)$ as a direct factor. Moreover, if we choose that n so that $n{+}1$ is not a multiple of $\kappa{+}1$, then $\mathrm{Null}(n)$ cannot itself have $\mathrm{Null}(\kappa)$ as a direct factor, completing the proof of non-primeness. □

Combining Lemmas 2.4 and 2.6, we get

Theorem 2.7. *The category* **Semigp** *of* (*nonempty*) *semigroups has no prime objects.* □

Note that in proving that arbitrary semigroups are not prime, the auxiliary semigroups that we used – mainly of the form $\mathrm{Null}(\kappa)$, but also a few of other sorts, e.g., the S in the proof of Lemma 2.6 – were all commutative, and had a zero element (an element z satisfying $zx = xz = z$ for all x). Hence we can similarly deduce

Corollary 2.8 (of proof)**.** *None of the following full subcategories of* **Semigp** *has any prime objects:*
(i) *The category of commutative semigroups.*
(ii) *The category of semigroups with zero element.*
(iii) *The category of commutative semigroups with zero element.*

On the other hand, I do not know the answer to

Question 2.9. *Does the category of* finite *semigroups have any prime objects?*

In our proof of Lemma 2.6, the argument showing that the finite semigroups $\mathrm{Null}(n)$ are not prime used only finite semigroups, so that conclusion is still true in the category of finite semigroups; but our proof of Lemma 2.4 used semigroups $\mathrm{Null}(\kappa)$ in a way that requires κ to be infinite even when S is finite, so if the same result holds in the category of finite semigroups, a different proof is needed.

3. Preparation for some positive results.

I obtained the results of the preceding section after a long grueling attempt to prove that the additive semigroup of positive integers was prime in **Semigp**. Since the argument I eventually found showing that this was not true makes strong use of action-equivalent elements, it seemed plausible that such results might still hold in the category of semigroups with cancellation.

And, indeed, the arguments I was trying to use do work for such semigroups, and can be adapted to several weaker and variant hypotheses.

Some of the preliminary results we will develop to help us prove these conclusions are true in more general contexts than those in which we will use them. In particular, though we will eventually use the next result with G the additive group of integers, the proof does not require commutativity, so I will use multiplicative notation till we specialize further.

Lemma 3.1. *Let S be a semigroup, G a group, and*

(3.1) $\quad \pi : S \to G$

a semigroup homomorphism. Then if S' is an oversemigroup of S such that left and right multiplication by members of S' carry S into itself, then π extends in a unique manner to a homomorphism $\hat{\pi} : S' \to G$. Indeed, for any $a \in S'$ and $s \in S$, $\hat{\pi}(a)$ can be described both as $\pi(as)\,\pi(s)^{-1}$ and as $\pi(s)^{-1}\pi(sa)$.

Proof. Given $a \in S'$, choose any $s_0 \in S$, and define $\hat{\pi}(a) = \pi(a\, s_0)\, \pi(s_0)^{-1}$, so that

(3.2) $\quad \pi(a\, s_0) = \hat{\pi}(a)\, \pi(s_0).$

Now for any $s \in S$, if we left-multiply (3.2) by $\pi(s)$, and recall that the argument of π on the left-hand side of (3.2) is a member of S, we get

(3.3) $\quad \pi(s\, a\, s_0) = \pi(s)\, \hat{\pi}(a)\, \pi(s_0).$

But now regarding the argument of π on the left-hand side of (3.3) as the product of $s\, a$ and s_0 in S, we can rewrite that expression as $\pi(sa)\, \pi(s_0)$, and cancel the $\pi(s_0)$'s, getting

(3.4) $\quad \pi(s\, a) = \pi(s)\, \hat{\pi}(a)$ for all $s \in S$.

Since we assumed nothing about the element $a \in S'$, we can obtain such an element $\hat{\pi}(a)$ for all such a, and so add "and all $a \in S'$" to (3.4).

Now consider any $a,\, a' \in S'$, and right-multiply (3.4) by $\hat{\pi}(a')$. Then applying to the left-hand side of the equation the case of (3.4) with $s\, a$ in the role of s, and a' in the role of a, we get

(3.5) $\quad \pi(s\, a\, a') = \pi(s)\, \hat{\pi}(a)\, \hat{\pi}(a'),$

whence, expanding the left-hand side as in (3.4), and cancelling the elements $\pi(s)$, we get

(3.6) $\quad \hat{\pi}(a\, a') = \hat{\pi}(a)\, \hat{\pi}(a').$

So $\hat{\pi}$ is indeed a homomorphism $S' \to G$, and from the case of (3.2) where $a \in S$, it is easily seen to extend π.

The final sentence of the Lemma clearly holds for any homomorphism $\hat{\pi}$ extending π. □

(The above result fails if the group G is replaced by a semigroup, or even a monoid. For instance, for any $n > 1$, let S be the additive semigroup of integers $\geq n$, S' the additive semigroup of all positive integers, G the additive monoid $S \cup \{0\}$, and $\pi : S \to S'$ the inclusion map.)

Corollary 3.2. *Let $(S_i)_{i \in I}$ be a family of semigroups.*

For each $i \in I$, let $S'_i = S_i \cup \{e\}$, the monoid obtained by adjoining an identity element to S. Then for any group G, every semigroup homomorphism $\pi : \prod_I S_i \to G$ has a unique extension to a monoid homomorphism $\hat{\pi} : \prod_I S'_i \to G$.

Proof. Let $S = \prod_I S_i$, and $S' = \prod_I S'_i$, regarded as an oversemigroup of S, and apply Lemma 3.1. □

Corollary 3.3. *If S and T are semigroups, G a group, and $\pi : S \times T \to G$ a semigroup homomorphism, then there exist unique semigroup homomorphisms $\pi_S : S \to G$ and $\pi_T : T \to G$ such that our original homomorphism π is given by*

(3.7) $\quad \pi(s, t) = \pi_S(s)\, \pi_T(t) \quad (s \in S,\ t \in T).$

Moreover, $\pi_S(S)$ and $\pi_T(T)$ centralize one another in G.

Proof. Apply Corollary 3.2 with $I = \{0, 1\}$, $S_0 = S$, $S_1 = T$, and for $s \in S$, $t \in T$, define $\pi_S(s) = \hat{\pi}(s, e)$ and $\pi_T(t) = \hat{\pi}(e, t)$. Then (3.7) and the mutual centralization assertion both follow from the fact that $\hat{\pi}$ respects multiplication. □

(Remark: The corresponding statement with "*monoids*" in place of "*semigroups*" is trivial – just define $\pi_S(s) = \pi(s, e)$ and $\pi_T(t) = \pi(e, t)$; and this even works with G also assumed a monoid, not necessarily a group. But for S and T semigroups not assumed to have neutral elements e, we need the above more roundabout development, including the assumption that G is a group.)

We now bring in another assumption. (In applying Lemma 3.1 with G an additive subgroup of the real numbers, we switch to additive notation for formulas involving π and $\hat{\pi}$.)

Lemma 3.4. *If, in the context of Lemma 3.1, G is an additive subgroup of the real numbers, and $\pi : S \to G$ assumes only nonnegative values, then $\hat{\pi} : S' \to G$ also assumes only nonnegative values.*

Hence, likewise, in the contexts of Corollaries 3.2 and 3.3, if G is an additive subgroup of the reals and π assumes only nonnegative values, then $\hat{\pi}$, respectively π_S and π_T, also assume only nonnegative values.

Proof. Suppose, in the context of Lemma 3.1 with G a subgroup of the real numbers, that some $a \in S'$ had $\hat{\pi}(a) < 0$. Then taking any $s \in \prod S$, there would be a positive integer n such that $\pi(a^n s) = n\,\hat{\pi}(a) + \pi(s) < 0$, contradicting our hypothesis on π.

Applying this in the contexts of Corollaries 3.2 and 3.3, we get the final sentence. □

Let us now hone in on the class of cases we are interested in.

(3.8) For the remainder of this section, k will be a fixed nonnegative integer and $\mathbb{N}$ the additive semigroup of all positive integers, so that $\mathbb{N}{+}k$ will denote the additive semigroup of natural numbers $\geq k$.

We can now give necessary and sufficient conditions for a semigroup to have $\mathbb{N}{+}k$ as a direct factor.

Lemma 3.5. *Let S be a semigroup given with a surjective semigroup homomorphism $\pi : S \to \mathbb{N}{+}k$, and a set-map $\nu : S \to S$ satisfying the following three conditions:*

(3.9) *For all $s \in S$, $\pi(\nu(s)) \;=\; \pi(s) + 1$.*

(3.10) *ν gives a bijection $S \to \{s \in S \mid \pi(s) \;\geq\; k{+}1\}$.*

(3.11) *For all $s,\, s' \in S$, $\nu(s)\, s' \;=\; \nu(s\, s') \;=\; s\, \nu(s')$.*

Then there exists an isomorphism of S with the direct product $(\mathbb{N}{+}k) \times U$ of the additive semigroup $\mathbb{N}{+}k$ and a semigroup U, such that π corresponds to the projection to $\mathbb{N}{+}k$, and ν to the map $(i, s) \mapsto (i{+}1, s)$.

Hence, a semigroup S has a direct product decomposition with $\mathbb{N}{+}k$ as a factor if and only if it admits maps π and ν as above.

Proof. Given S, π, and ν as above, note that (3.9) and (3.10) together imply that for every $i \geq 0$, we have the same statements with ν replaced by ν^i and "$+1$" replaced by "$+i$"; and likewise for (3.11).

Now define

(3.12) $U \;=\; \{s \in S \mid \pi(s) = k\}$.

Since π is a semigroup homomorphism, its value on a product of two elements of U will be $2k$, hence by the preceding observation with $i = k$, we can use the "k-version" of (3.10) to define a binary operation on U:

(3.13) $u \cdot v \;=\; \nu^{-k}(u\, v)$.

From (3.11) and the associativity of S, it is easy to check that for all $u, v, w \in U$, $(u{\cdot}v){\cdot}w = \nu^{-2k}(u\, v\, w) = u \cdot (v \cdot w)$, so (3.13) defines a semigroup structure; and to verify that as a semigroup, $S \cong (\mathbb{N}{+}k) \times U$ via the map

(3.14) $s \;\mapsto\; (\pi(s),\, \nu^{k-\pi(s)}(s))$.

This proves the main assertion of the lemma, which is the "if" direction of the final sentence. The forward implication of that final sentence is straightforward. □

Below, by a *direct factor* of a semigroup S we understand a semigroup P such that S has a direct product decomposition $S = \; P \times Q$. In particular, Lemma 3.5 above characterizes the semigroups having $\mathbb{N}{+}k$ as a direct factor.

Convention 3.6. *For the remainder of this section we shall assume that S and T are semigroups such that $S \times T$ has $\mathbb{N}{+}k$ as a direct factor. We will not write down that factorization, but simply assume we are given maps $\pi : S \times T \to \mathbb{N}{+}k$ and $\nu : S \times T \to S \times T$ satisfying the conditions of Lemma 3.5.*

In view of Corollary 3.3 and Lemma 3.4, the map π has the form

(3.15) $\pi(s,t) = \pi_S(s) + \pi_T(t)$, where $\pi_S : S \to \mathbb{N}$ and $\pi_T : T \to \mathbb{N}$ are semigroup homomorphisms.

We now turn to the properties of the ν of Convention 3.6. For the remainder of this section

(3.16) for $(s,t) \in S \times T$, we shall write $\nu(s,t) = (\nu_t(s),\, \nu_s(t)) \in S \times T$, and

(3.17) S' (resp. T') will denote the subsemigroup of elements of S (resp. T) that can be written as products of two elements of that semigroup.

(No connection with the earlier use of S' in this section to denote an over-semigroup of S.)

Lemma 3.7. *Assuming the above Convention and the notations* (3.15)-(3.16)*,*

(3.18) *for all* $s \in S',\ t_0, t_1 \in T',$ *one has* $\nu_{t_0}(s) = \nu_{t_1}(s),$ *and their common value again lies in* S'*, and likewise*
for all $s_0, s_1 \in S',\ t \in T',$ *one has* $\nu_{s_0}(t) = \nu_{s_1}(t),$ *and their common value again lies in* T'*.*

Hence we will make the simplification of notation

(3.19) *for* $s \in S'$*, we shall denote by* $\nu(s)$ *the common value of* $\nu_t(s) \in S'$ *for all* $t \in T'$*, and*
for $t \in T'$*, we shall denote by* $\nu(t)$ *the common value of* $\nu_s(t) \in T'$ *for all* $s \in S'$*,*

so that

(3.20) *for all* $s \in S',\ t \in T',\ \nu(s,t) = (\nu(s),\, \nu(t)).$

Moreover,

(3.21) *for all* $s, s' \in S$ *and* $t \in T,$ $\nu_t(s)\, s' \ = \ \nu(s\, s') \ = \ s\, \nu_t(s'),$ *and*
for all $s \in S$ *and* $t, t' \in T,$ $\nu_s(t)\, t' \ = \ \nu(t\, t') \ = \ t\, \nu_s(t').$

Proof. For any $s, s' \in S$ and $t, t' \in T,$ we have, by Convention 3.6 and (3.11), $\nu(ss', tt') = \nu(s,t)\,(s',t').$ Applying (3.16) to the first factor on the right-hand side, this gives $\nu(ss', tt') = (\nu_t(s)\, s',\, \nu_s(t')\, t').$ Taking first components, we have

(3.22) $\nu_{tt'}(s\, s') \ = \ \nu_t(s)\, s'.$

Note that the right-hand side does not depend on $t',$ hence neither does the left-hand side; and by the right-left dual argument, it likewise does not depend on $t,$ so we get the same value on replacing the subscript $t\,t'$ by any element of T'. Moreover, the argument of $\nu,$ a product of two arbitrary element of $S,$ can be any element of S'. This proves the first line of (3.18), allowing us to adopt the notation in the first line of (3.19). The first equality of the first line of (3.21) follows; and by interchanging right and left multiplication, and the roles of S and T as appropriate, we get (3.20) and the remaining equalities of (3.18) and (3.21). □

At this point, we cannot assert (3.18) without the restriction of the arguments to S' and T'. But we can deduce a strong result on the behavior of π_S and π_T on arbitrary elements:

Lemma 3.8. *For* $S,\ T,\ \pi$ *and* ν *as in Convention 3.6, we have* either

(3.23) *for all* $s \in S,\ t \in T,\ \pi_S(\nu_t(s)) \ = \ \pi_S(s) + 1$ *and* $\pi_T(\nu_s(t)) \ = \ \pi_T(t),$ or
for all $s \in S,\ t \in T,\ \pi_S(\nu_t(s)) \ = \ \pi_S(s)$ *and* $\pi_T(\nu_s(t)) \ = \ \pi_T(t) + 1.$

Proof. Let us fix arbitrary elements

(3.24) $s_0 \in S',\ t_0 \in T',$

and let m and n be the integers such that

(3.25) $\pi_S(\nu(s_0)) \ = \ \pi_S(s_0) + m$ and $\pi_T(\nu(t_0)) \ = \ \pi_T(t_0) + n.$

Applying (3.9) (with $S \times T$ in the role of S) and (3.15) to $(s_0,\, t_0)$ we see that

(3.26) $m + n \ = \ 1.$

Now note that for any $s \in S,\ t \in T,$ we have the following equalities (using, at the first and third steps, the fact that π_S is a homomorphism, at the second, (3.21), and at the last, (3.25)):

(3.27) $\pi_S(\nu_t(s)) + \pi_S(s_0) = \pi_S(\nu_t(s)\, s_0) = \pi_S(s\, \nu(s_0)) = \pi_S(s) + \pi_S(\nu(s_0)) = \pi_S(s) + \pi_S(s_0) + m.$

Cancelling $\pi_S(s_0)$ at the beginning and end, we get

(3.28) $\pi_S(\nu_t(s)) \ = \ \pi_S(s) + m$ for all $s \in S,\ t \in T.$

Reversing the roles of S and $T,$ we likewise have

(3.29) $\pi_T(\nu_s(t)) \ = \ \pi_T(t) + n$ for all $s \in S,\ t \in T.$

Now by (3.15), π_S assumes only nonnegative values, so we can choose an $s \in S$ minimizing $\pi_S(s),$ and applying (3.28) to this $s,$ we see that m must be nonnegative; and the analogous argument shows the same for n. From (3.26), it follows that one of m and n is 0 and the other 1. Substituting into (3.28) and (3.29) we get (3.23). ☐

Corollary 3.9. *If the first alternative of* (3.23) *holds, then the function* π_T *is identically* 0 *and* $\pi_S(S) = \mathbb{N}{+}k$, *while if the second holds,* π_S *is identically* 0 *and* $\pi_T(T) = \mathbb{N}{+}k$.

Proof. Assume without loss of generality that the first alternative of (3.23) holds, and suppose π_T were not identically zero. Then it would assume some positive value, and being a semigroup homomorphism, would assume arbitrarily large values.

Let $s \in S$ be an element at which π_S takes on its least value. By the above observation, we can choose $t \in T$ such that the value of $\pi(s,\,t) = \pi_S(s) + \pi_T(t)$ is $\geq k{+}1$.

Then by (3.10) applied to $S \times T$, this implies that for some $(s', t') \in S \times T$, $(s,t) = \nu(s',t') = (\nu_{t'}(s'), \nu_{s'}(t'))$ By the case of (3.23) we are assuming, this gives $\pi_S(s) = \pi_S(\nu_{t'}(s')) = \pi_S(s') + 1$, so $\pi_S(s') < \pi_S(s)$, contradicting our choice of s.

So π_T is indeed identically zero. Substituting this into (3.15) and observing that $\pi(S \times T) = \mathbb{N}{+}k$ by Convention 3.6 and the conditions of Lemma 3.5 it refers to, we conclude that $\pi_S(S) = \mathbb{N}{+}k$, as claimed. ☐

4. Primeness results, at last.

We are at last ready to note some subcategories of **Semigp** in which we can use the above results to prove some or all of the semigroups $\mathbb{N}{+}k$ prime. These categories cannot, of course, contain the semigroups $\mathrm{Null}(\kappa)$ that were used to prove *non*-primeness in §2.

Here is one family of conditions on semigroups that will suffice.

Definition 4.1. *We shall call a semigroup* S

(i) cancellative *if it satisfies*

(4.1) *for all* $s \neq s' \in S$ *and* $s'' \in S$, *one has* $s\,s'' \neq s'\,s''$, *and*

(4.2) *for all* $s \neq s' \in S$ *and* $s'' \in S$, *one has* $s''\,s \neq s''\,s'$,

(ii) right *cancellative if it merely satisfies* (4.1), *respectively,* left *cancellative if it satisfies* (4.2),

and,

(iii) weakly *cancellative if it satisfies*

(4.3) *for all* $s \neq s' \in S$ *there* exists *an* $s'' \in S$ *such that one of the inequalities* $s\,s'' \neq s'\,s''$ *or* $s''\,s \neq s''\,s'$ *holds*

(*i.e. if, in the language of Definition 2.2*(ii), *no two distinct elements of* S *are action equivalent*).

Theorem 4.2. *For all nonnegative integers* k, *the semigroup* $\mathbb{N}{+}k$ *is prime in the category of cancellative semigroups, and, more generally, in the categories of right cancellative and left cancellative semigroups, and still more generally, in the category of weakly cancellative semigroups.*

Proof. It will suffice to prove the final assertion; so suppose S and T are weakly cancellative semigroups such that $\mathbb{N}{+}k$ is a direct factor in $S \times T$. As in the preceding section, this property corresponds to the existence of maps $\pi : S \times T \to \mathbb{N}{+}k$ and $\nu : S \times T \to S \times T$ satisfying (3.9)-(3.11). As in (3.16), for $(s,t) \in S \times T$, we shall write $\nu(s,t) = (\nu_t(s),\, \nu_s(t))$.

We claim that under our present hypotheses, $\nu_t(s)$ must in fact be a function of s alone, and $\nu_s(t)$ a function of t alone. By symmetry, it suffices to prove the former statement.

So suppose, rather, that for some $s \in S$, $t,\, t' \in T$ we had $\nu_t(s) \neq \nu_{t'}(s)$. We now invoke weak cancellativity. By left-right symmetry, we may assume without loss of generality that there is some $s' \in S$ such that $\nu_t(s)\,s' \neq \nu_{t'}(s)\,s'$. But by (3.21), both sides equal $\nu(s\,s')$, a contradiction.

Hence the operators $\nu_t : S \to S$ $(t \in T)$ are all equal, and likewise the operators $\nu_s : T \to T$ $(s \in S)$. Writing the common values of each as $\nu : S \to S$ and $\nu : T \to T$, (3.16) becomes

(4.4) for all $s \in S$, $t \in T$, $\nu(s,t) = (\nu(s),\, \nu(t))$.

Now by Lemma 3.8, one of the functions π_S, π_T adds 1 whenever ν is applied to its argument; without loss of generality, let us assume that is π_S. Then by Corollary 3.9, π_S takes on all the values in $\mathbb{N}{+}k$ and π_T is identically 0.

From the fact that the maps π and ν on $S \times T$ satisfy the conditions (3.9)-(3.11) of Lemma 3.5 (with $S \times T$ in the role of S), we now see that $\pi_S : S \to \mathbb{N}+k$ and $\nu : S \to S$ satisfy those same conditions, hence, by that lemma, $\mathbb{N}+k$ is a direct factor in S. (If, rather, π_T is the function that adds 1 whenever ν is applied to its argument, then T has $\mathbb{N}+k$ as a direct factor.) □

Our other result is easier to prove, but applies only to $\mathbb{N}$, since for $k > 0$, $\mathbb{N}+k$ does not belong to the categories in question.

Theorem 4.3. *The object* $\mathbb{N}$ *is prime in the full subcategory of* **Semigp** *whose objects are the monoids, and, more generally, in the full subcategory of* **Semigp** *whose objects are the semigroups in which every element is a product.*

Proof. In the latter category, Lemma 3.7 shows that all the functions $\nu_t : S \to S$ are equal, and likewise the functions $\nu_s : T \to T$. This gives us (4.4), and we complete the proof as for Theorem 4.2. □

The monoid case of Theorem 4.3 is also implied by the "weakly cancellative" case of Theorem 4.2, since multiplication by the identity element is cancellable.

We can easily go from the above result about monoids as a full subcategory of **Semigp** to one about the category of monoids and monoid homomorphisms:

Proposition 4.4. *A monoid* S *which is prime in the full subcategory of* **Semigp** *whose objects are the monoids is also prime in the category of monoids* (*i.e., the subcategory thereof where morphisms are required to respect identity elements*)*. In particular,* $\mathbb{N}$ *is prime in that category.*

Sketch of proof. The direct product $S \times T$ of two objects in the category of monoids clearly has the same semigroup structure as their direct product in the category of semigroups. Also, if S is a monoid and T a semigroup such that $S \times T$ is a monoid, it is easy to verify that T must be a monoid. Hence a monoid S is a direct factor of a monoid A in the category of semigroups if and only if it is a direct factor of A in the category of monoids.

From these observations, it is easy to deduce the first assertion of this Proposition, and apply it to the monoid $\mathbb{N}$. □

Question 4.5. *Are any subsemigroups or submonoids of* $\mathbb{N}$ *other than those isomorphic to the ones treated in this section prime in some or all of the indicated categories? E.g., what about* $\{0\} \cup (\mathbb{N}+2)$ *?*

If A is an arbitrary subsemigroup or submonoid of $\mathbb{N}$, then to characterize semigroups or monoids S having A as a direct factor, I suppose that in place of the ν of Lemma 3.5, one would consider a family of bijections $\nu^{(a,b)} : \pi^{-1}(\{a\}) \to \pi^{-1}(\{b\})$ for $a, b \in A$, subject to appropriate identities.

Of course, it would be of interest to know what other sorts of semigroups are prime in these categories – what about the free semigroup or monoid on two generators, for instance?

In Example 6.2 below, we shall see that the group of integers is not prime (in the language of that section, not "Tarski-prime") in the category of groups, from which it easily follows that it is also non-prime in the categories of semigroups considered above.

5. Two other questions from [6].

The question answered in §2 above is the second of four questions listed in [6, middle of p. 263]. It was recently noted in [10] that the last of those questions, *"Is* $\mathbb{Z}$ *prime in the class of all Abelian groups?"*, has an easy affirmative answer. So far as I know, the other two questions remain open.

One of them falls under the general theme of this note. In it, an *idempotent* semigroup means one satisfying the identity $x\,x = x$. There is considerable literature on such semigroups, e.g. [2, §4.4] and [1].

Question 5.1. [6, p. 263] *Is there any prime in the class of all idempotent semigroups?*

The other question has a less obvious connection with this note. In it, an algebra "of type $<1,1>$" means an algebra having precisely two operations, both unary.

Question 5.2. [6, p. 263] *Is there any prime in the class of all finite algebras of type* $<1,1>$ (*or in any type that has more than one operation of arity* ≥ 1) *?*

An algebra of type $<1,1>$ can be thought of as a set given with an action on it of the free semigroup on two generators; so the question has a somewhat similar flavor to those considered above.

6. Some results and examples for groups, comparing Tarski-primeness and the two sorts of Rhodes-primeness.

As mentioned in §1, two concepts called primeness, somewhat related to but distinct from Tarski's, are studied by semigroup theorists. In this section, we examine the relationships among these three sorts of primeness when applied to groups. We give them distinct names in (iii)-(v) below.

This section can be read independent of the preceding material (ignoring a couple of brief comments about the contrast between Example 6.2 below and Theorems 4.2 and 4.3 above, and, much later, the third-from-last paragraph before Question 6.12, which likewise refers to a result in an earlier section).

Convention 6.1. *In this section, for objects G, H, etc. of the category of all groups, or the category of finite groups,*

(i) *we shall call H a* direct factor *of G if G is isomorphic to a direct product $H \times H'$,*

(ii) *we shall call H a* subquotient *of G if H is isomorphic to a homomorphic image of a subgroup of G,*

(iii) *we shall call G* Tarski-prime *if, whenever G is a direct factor of a group in the category in question, and the latter group can also be written as a direct product $G_0 \times G_1$, then G is a direct factor of one of G_0 or G_1,*

(iv) *we shall call G* Rhodes-prime with respect to direct products *if, whenever G is a* subquotient *of a group in the category in question, and the latter group can be written as a direct product $G_0 \times G_1$, then G is a subquotient of one of G_0 or G_1 (condition introduced at* [8, p. 482, line 6]), *and similarly,*

(v) *we shall call G* Rhodes-prime with respect to semidirect products *if, whenever G is a subquotient of a group in the category in question, and the latter group can be written as a semidirect product $G_0 \rtimes G_1$, then G is a subquotient of one of G_0 or G_1 (condition introduced at* [8, p. 228, paragraph before Lemma 4.1.31].)

(What we name a *subquotient* of G in (ii) above was earlier called a *factor* of G, e.g., [7, p.1 et seq.], is called a *divisor* of G in [8, Def. 1.2.25, p.32], and is now often called a *section* of G by group theorists.)

We will also follow the common conventions of calling a group G *nontrivial* if it has more than one element, and in general writing e for the identity elements of groups, though for particular groups we may use notation appropriate for them, e.g., 0 for the identity element of the additive group $\mathbb{Z}$.

In view of Theorems 4.2 and 4.3 of the preceding section, one might expect the additive group $\mathbb{Z}$ to be Tarski-prime in the category of all groups. But a key tool in proving those results was the fact that if two members of $\mathbb{N}$ satisfy $m + n = 1$ (3.26), one of them has to be 0. The failure of that statement in $\mathbb{Z}$ underlies the difference in its behavior.

Example 6.2. *In the category of groups, $\mathbb{Z}$ is not Tarski-prime, but* is *Rhodes-prime with respect to semidirect and direct products.*

Proof. To construct a counterexample to Tarski-primeness, let us choose any

(6.1) prime numbers p and q, and nonunit divisors $a \,|\, p-1$ and $b \,|\, q-1$, such that a and b are relatively prime.

(E.g., $p = 3$, $q = 7$, $a = 2$, $b = 3$. Or one could take both p and q to be 7, again with $a = 2$ and $b = 3$.)

Recalling that the multiplicative groups of the rings $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are cyclic of orders $p-1$ and $q-1$, we can

(6.2) let $G = (\mathbb{Z}_p \rtimes \mathbb{Z}) \times (\mathbb{Z}_q \rtimes \mathbb{Z})$ where, in the first factor, a generator of $\mathbb{Z}$ acts on $\mathbb{Z}_p$ by multiplication by a unit of the ring $\mathbb{Z}_p$ having multiplicative order a, and in the second, it acts on $\mathbb{Z}_q$ by multiplication by a unit of the ring $\mathbb{Z}_q$ having multiplicative order b.

Let us rewrite this group as $G = (\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes (\mathbb{Z} \times \mathbb{Z})$, using the action of $\mathbb{Z} \times \mathbb{Z}$ under which the first factor $\mathbb{Z}$ affects only the first factor of $\mathbb{Z}_p \times \mathbb{Z}_q$, and the second only affects the second.

Now since a and b are relatively prime, the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element (a, b) is a direct factor. (Explicitly, we can find integers c and d such that $bc - ad = 1$, and use the invertible matrix $\left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right)$ to convert the given direct product decomposition of $\mathbb{Z} \times \mathbb{Z}$ into one in which (a, b) generates the second factor. That matrix or its inverse necessarily has some negative entries, which can be thought of as the way the fact that $\mathbb{Z}$ includes negative integers comes in.) By (6.2), that second factor acts trivially on $\mathbb{Z}_p \times \mathbb{Z}_q$.

Thus, G can be written as a semidirect product $G = (\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes (\mathbb{Z} \times \mathbb{Z})$ where the second $\mathbb{Z}$ acts trivially, and all the action comes from the first. Hence we can rewrite this group as a direct product $G = ((\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}) \times \mathbb{Z}$. So the group $\mathbb{Z}$ is isomorphic to a direct factor of the direct product (6.2), though it is not a direct factor in either of the given factors. So $\mathbb{Z}$ is not Tarski-prime.

As to the Rhodes-primeness conditions, it is clear that $\mathbb{Z}$ is a subquotient of a group G if and only if G has an element x of infinite order. It is also easy to see that a direct or semidirect product $G_0 \times G_1$ or $G_0 \rtimes G_1$ has such an element x if and only if G_0 or G_1 does; so $\mathbb{Z}$ indeed satisfies both versions of Rhodes-primeness. □

For finite groups, the relationship between these conditions is quite different. A key result we shall use is the *Krull-Schmidt Theorem* [3, Theorem 3.8, p. 86], which says that every group with ACC and DCC on normal subgroups has a factorization as a direct product of finitely many nontrivial groups which are not themselves nontrivial direct products, and that the factors in such a decomposition are *unique* up to rearrangement and isomorphism. Thus, every group with those chain conditions which is *not* a nontrivial direct product is Tarski-prime in the category of such groups. In particular, every *finite* group which is not a nontrivial direct product is Tarski-prime in the category of finite groups.

We can now prove

Lemma 6.3. *In the category of finite groups, for G a nontrivial group we have the implications*

$$\begin{aligned} & G \text{ is Rhodes-prime with respect to semidirect products} \\ \Longrightarrow\ & G \text{ is Rhodes-prime with respect to direct products} \\ \Longrightarrow\ & G \text{ has a unique minimal nontrivial normal subgroup} \\ \Longrightarrow\ & G \text{ is Tarski-prime.} \end{aligned} \tag{6.3}$$

Proof. The first implication is clear from the definitions, since direct products are a special case of semidirect products.

We shall prove the remaining two implications by contradiction.

For the first of these, suppose G has minimal nontrivial normal subgroups $N \neq N'$. Then $N \cap N' = \{e\}$, so the natural map $G \to (G/N) \times (G/N')$ is an embedding. Thus G is isomorphic to a subgroup of the direct product $(G/N) \times (G/N')$, making it a subquotient thereof; but it is not a subquotient of either G/N or G/N', since these have smaller orders than G. So G is not Rhodes-prime with respect to direct products.

For the final implication, note that by the Krull-Schmidt Theorem, every non-Tarski-prime nontrivial group G is a direct product $G_0 \times G_1$ of nontrivial groups, and if we choose minimal nontrivial normal subgroups $M_0 \subseteq G_0$ and $M_1 \subseteq G_1$, then $M_0 \times \{e\}$ and $\{e\} \times M_1$ are distinct minimal nontrivial normal subgroups of G. □

Now for some positive results on when finite groups are Rhodes-prime, in each of the two senses; although our first result below is not, in fact, limited to finite groups:

Proposition 6.4. *Every nontrivial* simple *group G (commutative or noncommutative) is Rhodes-prime with respect to semidirect products in the category of all groups.*

Hence every nontrivial finite simple group G is Rhodes-prime with respect to semidirect products in the category of finite groups.

Proof. Let G be a nontrivial simple group, and suppose G is a subquotient of a semidirect product $G_0 \rtimes G_1$. That is, suppose that that we can write it as $f(H)$ for some subgroup $H < G_0 \rtimes G_1$, and some surjective homomorphism $f : H \to G$.

Since $G_0 \rtimes \{e\}$ is a normal subgroup of $G_0 \rtimes G_1$, $H \cap (G_0 \rtimes \{e\})$ will be a normal subgroup of H, so its image in G under f will be normal; so since G is simple, that image will either be G or $\{e\}$.

If $f(H \cap (G_0 \rtimes \{e\})) = G$, that makes G a homomorphic image of a subgroup of G_0, i.e., a subquotient of G_0.

On the other hand, if $f(H \cap (G_0 \rtimes \{e\})) = \{e\}$, we see that the image in G of an element $(g_0, g_1) \in H$ depends only on g_1, and easily deduce that G is a homomorphic image of the subgroup of G_1 consisting elements occurring as second components of members of H; so G is a subquotient of G_1.

This proves the Rhodes-primeness of G with respect to semidirect products. If G is finite, having that property in the category of all groups clearly implies that it has the same property in the category of finite groups. □

From Example 6.2 we know that in the category of all groups, the converse to Proposition 6.4 fails: the non-simple group $\mathbb{Z}$ is Rhodes-prime with respect to semidirect products. But Alexander Olshanskiy (personal communication) has provided a proof of that converse in the category of finite groups. Here is a simplified version of his proof.

Proposition 6.5 (A. Olshanskiy)**.** *If a nontrivial finite group G is Rhodes-prime with respect to semidirect products in the category of finite groups, then G is simple.*

Proof. Let N be a minimal nontrivial normal subgroup of G. (By Lemma 6.3 there is a unique such N, but we will not need to call on that fact.)

Being an extension of N by G/N, the group G can be written as a subgroup of the wreath product $N \wr G/N$, that is, of the semidirect product $N^{|G/N|} \rtimes G/N$, where $|G/N|$ denotes the underlying set of G/N, and G/N acts on that direct power of N by permutation of the factors (Theorem of Kaloujnine and Krasner, [7, Theorem 22.21, p.46]). Hence by Rhodes-primeness with respect to semidirect products, G must be a subquotient of G/N or of $N^{|G/N|}$. It can't be subquotient of G/N because that group has smaller order, so it must be a subquotient of $N^{|G/N|}$.

Though N is a minimal nontrivial normal subgroup of G, it need not be simple, but it will be *characteristically simple,* i.e., it will have no proper nontrivial subgroups invariant under all its automorphisms (since these include conjugation by members of G). Hence by [9, 3.3.15, p.87, and sentence at end of proof, p.88], N is a direct product of (mutually isomorphic) simple groups. Hence $N^{|G/N|}$ is a direct product of those same simple groups, with possibly more repetitions.

Thus G, being Rhodes-prime with respect to semidirect products, and hence in particular, with respect to direct products, must be a homomorphic image of a subgroup one of those simple groups, S. (Our definition of that Rhodes-primeness condition only refers to pairwise products, but by induction, it extends to arbitrary finite products.) But since S is a direct factor of the subgroup N of G, its order is less than or equal to that of G, so to be a homomorphic image of a subgroup of S, the group G must, in fact, be isomorphic to S, hence, as desired, simple. ☐

The next result concerns conditions for G to be Rhodes-prime with respect to *direct* products. In view of the second implication of (6.3), G must have a unique minimal nontrivial normal subgroup M. Given this, we find that some quite varied additional conditions imply the desired Rhodes-primeness.

Proposition 6.6. *Let G be a finite group which has a unique minimal nontrivial normal subgroup M.*

Then G is Rhodes-prime with respect to direct products in the category of finite groups if any of the following conditions holds:

(i) *M is noncommutative, or*
(ii) *G is a semidirect product $M \rtimes K$ of M with a subgroup $K < G$, or*
(iii) *G is a cyclic group of prime-power order, $\mathbb{Z}_{p^n}$.*

Proof. Given a finite group G that is *not* Rhodes-prime with respect to direct products, we shall show that neither (i) nor (ii) can hold. The proof that (iii) implies Rhodes-primeness will be more straightforward.

Assuming non-Rhodes-primeness with respect to direct products, let us write G as a homomorphic image $f(H)$ of a subgroup H of a direct product $G_0 \times G_1$ of finite groups, such that G is not a subquotient of G_0 or of G_1. We can clearly replace G_0 and G_1 by the subgroups given by the projections of H onto those two factors, so assume that those projections are surjective; i.e., that H is a subdirect product of G_0 and G_1. Let us further assume that, for the given group G, the groups G_0, G_1, and H are chosen so as to minimize the order of H.

Now let

(6.4) $$H_0 = \{h \in G_0 \mid (h, e) \in H\}, \quad \text{and} \quad H_1 = \{h \in G_1 \mid (e, h) \in H\}.$$

If H_0 were trivial, then H would be the graph of a homomorphism $G_1 \to G_0$, hence isomorphic to G_1, so the surjection $f : H \to G$ would factor through G_1, contradicting the assumption that G was not a subquotient of G_1. So H_0 is nontrivial; and similarly H_1. From our assumption that H projects surjectively to each of G_0, G_1, it is also easy to see that H_0 is normal in G_0, and H_1 in G_1. (E.g., given $h \in H_0$ and $g_0 \in G_0$, we can find $g_1 \in G_1$ such that $(g_0, g_1) \in H$; and conjugating (h, e) by (g_0, g_1), we effectively conjugate h by g_0.) Now if $\ker(f)$ had nontrivial intersection with the subgroup $H_0 \times \{e\}$ of H, then since H_0 and $\ker(f)$ are both normal in H, their intersection would be normal, and by dividing G_0

by the projection of this intersection, we could decrease the order of H. So $\ker(f)$ does not meet $H_0 \times \{e\}$; and similarly it does not meet $\{e\} \times H_1$.

Let $M_0 \times \{e\}$ be any subgroup of $H_0 \times \{e\}$ minimal for being normal in H. Because it lies in $H_0 \times \{e\}$, it maps one-to-one into G, hence its image is a minimal normal subgroup of G; hence that image is M. Combining with the corresponding observation about H_1, we get

(6.5) $H_0 \times \{e\}$ has a unique minimal normal subgroup $M_0 \times \{e\}$, and $\{e\} \times H_1$ has a unique minimal normal subgroup $\{e\} \times M_1$, and both of these map isomorphically to M under f.

But $M_0 \times \{e\}$ and $\{e\} \times M_1$ centralize one another in H, so the subgroup M must be self-centralizing in G, i.e., commutative, giving the desired contradiction to (i).

Next, suppose as in (ii) that G is a semidirect product $M \rtimes K$ for some $K < G$. Since M is invariant under conjugation by members of K, the same must be true of the centralizer of M in K, hence that centralizer is invariant under conjugation by all members of $KM = G$, i.e., it is normal in G; and as a subgroup of K it has trivial intersection with M. Hence if it were nontrivial, any minimal nontrivial G-normal subgroup of it would contradict the uniqueness of the minimal normal subgroup M. So the centralizer of M in K is trivial.

Returning to what we proved earlier about H, note that if $f^{-1}(K) < H$ had nontrivial intersection with H_0, then this would centralize H_1, hence its image in G would be a nontrivial subgroup of K that centralized the image of H_1, which we saw contains M, contradicting the conclusion of the preceding paragraph. So $f^{-1}(K)$ has trivial intersection with H_0; and similarly with H_1. In view of (6.4), this forces $f^{-1}(K)$ to be the graph of an isomorphism between subgroups $K_0 < G_0$ and $K_1 < G_1$, each isomorphic to K, and we see that these act on M_0 and M_1 as K acts on M. Hence the subgroup $K_0M_0 < G_0$ (and likewise $K_1M_1 < G_1$) is isomorphic to $KM = G$, again contradicting our assumption that G is not a subquotient of G_0 or G_1. This gives the desired contradiction to (ii) assuming G not Rhodes-prime with respect to direct products.

To prove Rhodes-primeness with respect to direct products in case (iii), suppose a subgroup H of a finite group $G_0 \times G_1$ maps surjectively to $G = \mathbb{Z}_{p^n}$. Then an element mapping to a generator of G must have order divisible by p^n. Since the order of an element of $G_0 \times G_1$ is the least common multiple of the orders of its components, one of those components must have order divisible by p^n, hence some power of that element will have order exactly p^n, hence the subgroup of G_0 or G_1 that it generates will have that order, making $G = \mathbb{Z}_{p^n}$ a subquotient of G_0 or G_1. □

(I obtained case (ii) of the above result in 2014, answering a question posed by John Rhodes (personal correspondence). That result is given in weakened form in [5, Theorem 4.20, p. 1275]. For the meaning of "G is ji" in that statement, see the last paragraph of [5, p. 1252], in particular, the display.)

We can now give some examples of the distinction between Rhodes-primeness with respect to semidirect and direct products:

Example 6.7. *In the category of finite groups, the following groups are Rhodes-prime with respect to direct products but, not being simple, are not Rhodes-prime with respect to semidirect products:*

(i) *The permutation groups* S_n *for all* $n \geq 3$.
(ii) *All semidirect products* $\mathbb{Z}_p \rtimes A$ *where* p *is a prime and* A *a nontrivial subgroup of* $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$.
(iii) *All groups* $\mathbb{Z}_{p^n}$ *for* p *prime and* $n > 1$.

Proof. To show S_n Rhodes-prime with respect to direct products, we use Proposition 6.6(ii). For all cases except $n = 4$, the subgroup $A_n < S_n$ is the unique minimal normal subgroup, and S_n is a semidirect product of A_n with the order-2 subgroup generated by any transposition. For $n = 4$, there is a different unique minimal normal subgroup, the Klein four-group V, consisting of the even permutations of exponent 2, and if we write S_3 for the subgroup of elements of S_4 fixing some one of the four elements on which S_4 acts, we find that S_4 is a semidirect product $V \rtimes S_3$, as required. (For $n \geq 5$, Rhodes-primeness of S_n with respect to direct products is also an instance of Proposition 6.6(i).)

In case (ii), we again have a semidirect product decomposition. To see that $\mathbb{Z}_p$ is the unique nontrivial normal subgroup of $\mathbb{Z}_p \rtimes A$, note that any $g \in \mathbb{Z}_p \rtimes A$ that is not in $\mathbb{Z}_p$ acts nontrivially on $\mathbb{Z}_p$, hence a commutator of g with a nonidentity member of $\mathbb{Z}_p$ is a nonidentity element of $\mathbb{Z}_p$, so no such g can belong to a normal subgroup not containing $\mathbb{Z}_p$.

In case (iii), Rhodes-primeness with respect to direct products is Proposition 6.6(iii). □

Each of the above examples shows that the first implication of Lemma 6.3 is not reversible. I was unsure whether the middle implication of that lemma was reversible, but Alexander Olshanskiy sent me Example 6.8 below, showing that for every prime p there is a group of order p^5 for which the reverse of this implication fails, and Example 6.9, showing that for $p=2$, there are also two such examples of order p^3.

Example 6.8 (A. Olshanskiy)**.** *Let p be any prime, and H (of order p^3) the group of upper triangular 3×3 matrices over $\mathbb{Z}_p$ with diagonal I. Within H, let $a = I + e_{23}$, $b = I + e_{12}$, $c = I + e_{13}$.*

Let G (of order p^5) be the quotient of $H \times H$ by the subgroup generated by the central element (c, c^{-1}). Then G has a unique minimal normal subgroup, the subgroup M generated by the image of (c, I), equivalently, by the image of (I, c). By construction, G is a quotient, hence a subquotient, of $H \times H$; but G is not a subquotient of either of the factors, since these have smaller orders. Hence G not Rhodes-prime with respect to direct products.

Proof. In H, we find that $a^p = b^p = c^p = I$, that c is central, and that $ba = abc$. This allows us to write every element in the form $a^i\, b^j\, c^k$ $(0 \leq i, j, k < p)$, and since the total number of such expressions is p^3, the order of H, this expression for each element must be unique. It is easy to verify that the subgroup generated by c is the commutator subgroup of H, the center of H, and the unique minimal nontrivial normal subgroup of H.

In G, the subgroup M described in the second paragraph of the statement is central, hence normal, and has order p, hence is minimal. To see that it is the unique minimal normal subgroup, note that any $g \in G$ that is not in M is the image of an element $(h_0, h_1) \in H \times H$ such that one of h_0 or h_1 involves a nonidentity power of a or of b. Assuming without loss of generality that h_0 involves a nonidentity power of a, we find that the commutator of (h_0, h_1) with (b, I) is a nonidentity power of (c, I). Hence in G, the commutator of g with the image of (b, I) is a generator of M. This shows that every normal subgroup of G not contained in M contains M, so that M is indeed the unique minimal nontrivial normal subgroup of G, completing the proof. □

Example 6.9 (A. Olshanskiy)**.** *Let us write Q_8 for the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, and D_4 for the dihedral group (the symmetry group of the square, $<p,\, q \mid p^4 = e,\ q^2 = e,\ q\,p\,q^{-1} = p^{-1}>$), both of order 8.*

Then Q_8 is a subquotient of $D_4 \times D_4$, and D_4 is a subquotient of $Q_8 \times Q_8$, but since neither Q_8 nor D_4 is a subquotient of the other, neither group is Rhodes-prime with respect to direct products.

However, each has a unique minimal normal subgroup: $\{\pm 1\}$ in Q_8, and $\{e, p^2\}$ in D_4.

Proof. We shall show that the 16-element group

(6.6) $\qquad H \,=\, <x,\, y \mid x^4 = e,\ y^4 = e,\ y\,x\,y^{-1} = x^{-1}>$

(a semidirect product $<x \mid x^4 = e> \rtimes <y \mid y^4 = e>$) is isomorphic both to a subgroup of $Q_8 \times Q_8$ and to a subgroup of $D_4 \times D_4$, and has both Q_8 and D_4 as homomorphic images, from which the above subquotient assertions follow.

Within $Q_8 \times Q_8$, let $x = (i, 1)$ and $y = (j, j)$. It is easy to check by looking at first coordinates, and then at second coordinates, that x and y satisfy the relations of (6.6), hence the group they generate is a homomorphic image of H. That group admits a homomorphism onto Q_8, given by projection to the first component; but $x^2\, y^2 = (1, -1)$ is in the kernel of that homomorphism, hence the group generated by x and y must have larger order than Q_8; hence it can't be a *proper* homomorphic image of H, so it must be isomorphic thereto.

Likewise, within $D_4 \times D_4$, consider the subgroup generated by $x = (p, 1)$ and $y = (q, p)$. As in the preceding paragraph, we verify that this group is a homomorphic image of H, and that the projection onto the first component maps it surjectively to D_4. However, $y^2 = (e, p^2)$ is a nonidentity element of the kernel of this homomorphism, so again the whole group is of larger order than Q_8, and so must be isomorphic to H.

The unique nontrivial normal subgroup assertion is easily verified in each case. □

(We remark that D_4 is a semidirect product, $<p \mid p^4 = e> \rtimes <q \mid q^2 = e>$. However, $<p \mid p^4 = e>$ is not the *least* nontrivial normal subgroup of D_4 – that is a subgroup thereof – so the above non-Rhodes-primeness result does not contradict Proposition 6.6(ii).)

Finally, it is easy to give examples showing nonreversibility of the last implication of Lemma 6.3. Here's a quick one.

Example 6.10. *Let p be any prime, and q_0, q_1 primes (possibly equal to one another) that are both $\equiv 1 \pmod{p}$, so that the automorphism groups of $\mathbb{Z}_{q_0}$ and $\mathbb{Z}_{q_1}$ both have orders divisible by p. Thus, $\mathbb{Z}_p$ has faithful actions on both these groups; let $G = (\mathbb{Z}_{q_0} \times \mathbb{Z}_{q_1}) \rtimes \mathbb{Z}_p$, defined using these actions.*

This group is Tarski-prime, but has at least two minimal normal subgroups, $\mathbb{Z}_{q_0}$ and $\mathbb{Z}_{q_1}$. (More if $q_0 = q_1$.)

Proof. To show Tarski-primeness, note that the order of G, $p\, q_0\, q_1$, is a product of just three primes, hence if G were a nontrivial direct product, one of the factors would have prime order, hence be commutative, and since it centralizes the other factor, it would be central in G. But G has no nonidentity central elements, so it is not such a product, so by the Krull-Schmidt Theorem, G is Tarski-prime.

But as noted, G has at least two minimal normal subgroups. (And if $q_0 = q_1$, then $\mathbb{Z}_{q_0} \times \mathbb{Z}_{q_1}$ is a 2-dimensional $\mathbb{Z}_{q_0}$-vector space, and has q_0+1 one-dimensional subspaces, all of which are minimal normal subgroups of G.) □

The examples and proofs given above made much use of subgroups, and direct and semidirect products, but less use of homomorphic images. This led me to wonder: Suppose we call an object *modified* Rhodes-prime with respect to direct products if it satisfies condition (iv) of Convention 6.1 with "subquotient of" replaced by "subgroup of"; and likewise define modified Rhodes-primes with respect to semidirect products by the corresponding variant of condition (v). How do these conditions compare with the unmodified conditions? It turned out that these questions are easily answered for finite groups G.

The modified Rhodes-primeness condition with respect to direct products for such G is precisely equivalent to the condition that G have a unique minimal normal subgroup M. Indeed, if G has such a subgroup M, then given any embedding $G \to G_1 \times G_2$, the kernels of the induced maps $G \to G_1$, $G \to G_2$ have trivial intersection, hence cannot both contain M, so one of them must be trivial, giving an embedding of G in G_1 or G_2, thus proving modified Rhodes-primeness. Conversely, let us assume G modified Rhodes-prime. By finiteness of the lattice of normal subgroups, it suffices to prove that any two nontrivial normal subgroups N_1, N_2 have nontrivial intersection. And indeed, looking at cardinalities, we see that G cannot be embedded in G/N_1 or G/N_2, so by our assumption of modified Rhodes-primeness, the natural map $G \to G/N_1 \times G/N_2$ cannot be an embedding, so its kernel $N_1 \cap N_2$ must indeed be nontrivial.

On the other hand, we can see from the proofs of Propositions 6.4 and 6.5 that modified Rhodes-primeness of finite groups with respect to *semidirect* products is equivalent to simplicity, hence equivalent to ordinary Rhodes-primeness with respect to semidirect products.

Going back to modified Rhodes-primeness with respect to direct products, if we drop the finiteness assumption on our groups, we see that the existence of a least nontrivial normal subgroup still implies the condition in question; as does the weaker condition (equivalent to that one in the finite case) that every two nontrivial normal subgroups have nontrivial intersection (satisfied, for example, by the additive group of integers); but the converse is not true. For example, every infinite-dimensional vector space over a field $\mathbb{Z}_p$, regarded as an abelian group G, satisfies the modified Rhodes-primeness condition: If $G \to G_1 \times G_2$ is an embedding, then the images of G in G/N_1 and G/N_2 will also be $\mathbb{Z}_p$-vector spaces, at least one of which must have the same dimension as G.

But perhaps positive results can still be proved under assumptions on G weaker than finiteness, such as ACC and/or DCC on normal subgroups.

A question I have not thought hard about is

Question 6.11. *Let G be a* finite *group.*

It is clear that if, in the category of all *groups, G is Rhodes-prime with respect to direct products, respectively Tarski-prime, then it also has that property in the category of finite groups. Is the converse to either or both of these implications true?*

Propositions 6.4 and 6.5 show that the answer to the analog of this question for semidirect products is "yes".

Further remarks:

The analog of the Krull-Schmidt theorem is not true for finite *semigroups.* To see this, first note that in the proof of Lemma 2.5 above, the case of finite κ uses only finite semigroups, and so shows that $\mathrm{Null}(\kappa)$

is not Tarski-prime among finite semigroups. But taking κ such that $\kappa{+}1$ is a prime number, $\mathrm{Null}(\kappa)$ also cannot be a nontrivial direct product. For another example, see [6, Exercise 4, p. 265].

On the other hand, the Jónsson-Tarski Theorem ([4], [6, p. 290]) proves unique factorization, and hence Tarski-primeness of all nontrivial non-factoring objects, for finite algebras in a large class of varieties, including the variety of monoids. (The identity elements of monoids give what are there called "zero" elements [6, paragraph beginning at bottom of p. 282]. Note that in that definition of "zero element", the operation "$+$" is not required to be commutative.)

These thoughts lead to our last question. (One can see from the Jónsson-Tarski Theorem mentioned above that a finite group will be Tarski-prime as a monoid if and only if it is Tarski-prime as a group, which is why (6.9) below shows fewer possibly distinct conditions than the other two displays.)

Question 6.12. *Understanding the two sorts of Rhodes-primeness to be defined for semigroups and monoids as we define them in Convention 6.1 for groups* (*they are so defined in* [8])*, and likewise for the condition of Tarski-primeness* (*as in* [6])*, the following five implications are clear for any* finite group G. *Are some or all of the converse implications true?*

(6.7)
G *is Rhodes-prime as a finite* semigroup *with respect to semidirect products*
$\Longrightarrow$ G *is Rhodes-prime as a finite* monoid *with respect to semidirect products*
$\Longrightarrow$ G *is Rhodes-prime as a finite* group *with respect to semidirect products.*

(6.8)
G *is Rhodes-prime as a finite* semigroup *with respect to direct products*
$\Longrightarrow$ G *is Rhodes-prime as a finite* monoid *with respect to direct products*
$\Longrightarrow$ G *is Rhodes-prime as a finite* group *with respect to direct products.*

(6.9)
G *is Tarski-prime as a finite* semigroup
$\Longrightarrow$ G *is Tarski-prime as a finite* monoid, *equivalently, as a finite* group.

Likewise, for an arbitrary group G, are some or all of the corresponding implications with "finite" everywhere deleted true?

Incidentally, because a finite semigroup or monoid is not in general isomorphic to its opposite, there are actually two versions of the concept of semidirect product for these objects, based on "action on the right" and "action on the left". But since the opposite of every semigroup or monoid is still a semigroup or monoid, general results about each of these constructions imply the same results about the other. Cf. [8, p. 24, first sentence of next-to-last paragraph].

7. An observation on Rhodes-primeness with respect to direct products in arbitrary varieties of algebras.

The concept of Rhodes-primeness with respect to direct products (Convention 6.1(iv) above) makes sense in any variety of algebras. (Rhodes-primeness with respect to semidirect products does not – there is no concept of "subdirect product" in a general variety of algebras, because, to start with, there is no concept of one object of such a variety acting on another.) We end this note with an observation on this Rhodes-primeness condition for finite objects in arbitrary varieties of algebras (possibly having infinitely many operations, and/or operations of infinite arities – the finiteness of the algebras makes those features unimportant).

Given an object or family of objects X in a variety V, I will write $\mathrm{Var}(X)$ for the subvariety of V generated by X. Abandoning the convention we made for semigroups at the beginning of this note, we will understand any variety with no zeroary operations to have an empty algebra. This will be a subquotient of every object, hence will be considered Rhodes-prime with respect to direct products.

Recall that an element x of a lattice is called *join prime* if $x \leq y \vee z$ implies $x \leq y$ or $x \leq z$.

Proposition 7.1. *Let V be any variety of algebras* (*in the sense of universal algebra*) *and X a finite object of V. Then the following conditions are equivalent:*

(i) *X is Rhodes-prime with respect to direct products in the category of finite objects of V.*

(ii) *If Y_0 and Y_1 are finite nonempty objects of V such that $X \in \mathrm{Var}(\{Y_0, Y_1\})$, then X is a subquotient either of Y_0 or of Y_1.*

(iii) $\mathrm{Var}(X)$ *is join-prime in the join-semilattice of subvarieties of* V *generated by finite families of finite algebras,* and *whenever* Y *is a finite nonempty algebra in* V *such that* $\mathrm{Var}(X) \subseteq \mathrm{Var}(Y)$, X *is a subquotient of* Y.

Proof. First assume (i), and suppose we are given Y_0, Y_1 as in the hypothesis of (ii). For any n, the free algebra on n generators in $\mathrm{Var}(\{Y_0, Y_1\})$ is a subalgebra of $Y_0^{\mathrm{card}(Y_0)^n} \times Y_1^{\mathrm{card}(Y_1)^n}$. Taking n such that X is generated by n elements, and thus, by the assumption $X \in \mathrm{Var}(\{Y_0, Y_1\})$, is a homomorphic image of the free algebra on n generators in that variety, this makes X a subquotient of a finite direct product of copies of Y_0 and Y_1. Applying the Rhodes-primeness condition (i) inductively, we conclude that that X is a subquotient of Y_0 or of Y_1, proving (ii).

Next assume (ii). Since any subvariety of V generated by finitely many finite algebras is generated by one finite algebra (the direct product of the nonempty algebras in the family), the first assertion of (iii) just says that if $\mathrm{Var}(X) \subseteq \mathrm{Var}(\{Y_0, Y_1\})$, then $\mathrm{Var}(X) \subseteq \mathrm{Var}(Y_0)$, or $\mathrm{Var}(X) \subseteq \mathrm{Var}(Y_1)$, and this conclusion is trivial if one of Y_0, Y_1 is empty, while assuming them nonempty, it a weakening of the subquotient conclusion of (ii). The second assertion of (iii) is obtained from (ii) by taking $Y_0 = Y_1 = Y$.

Finally, assuming (iii), suppose X is a subquotient of $Y_0 \times Y_1$. Then in particular, it belongs to the variety generated by Y_0 and Y_1, so by the first condition of (iii), X belongs to the variety generated by one of these, which we will call Y. Then assuming Y nonempty, the second condition of (iii) shows that X is a subquotient of Y, establishing (i). On the other hand, if Y is empty, then so is X, and since the empty algebra (if it exists in V) is prime, we again get (i). □

To see that neither of the two parts of (iii) is alone equivalent to (i), note that in the variety of abelian groups, the first condition is satisfied if X is any finite abelian group of prime-power exponent, e.g., $\mathbb{Z}_2 \times \mathbb{Z}_2$, while the second is satisfied by any finite cyclic abelian group, e.g., $\mathbb{Z}_6$, but neither of those two examples is Rhodes-prime with respect to direct products.

8. Acknowledgements.

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