Ordering Coproducts of Groups and Semigroups

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In memory of Yitz Herstein

A. A. Vinogradov proved that a coproduct of orderable groups is orderable, and R. E. Johnson extended this result to semigroups. A simpler version of their proof is given, and results are obtained on orderability of group coproducts with amalgamation. In particular, it is shown that the coproduct of two or more copies of an orderable group H with amalgamation of a subgroup $T \subseteq H$ is orderable if and only if T is the difference kernel of a pair of homomorphisms from H into another orderable group, equivalently, if and only if T is convex under some *one-sided* group-ordering of H. For coproducts of families of nonisomorphic overgroups of a group T, partial results are obtained.

Some related results are proved for one-sided orderability.

1. MOTIVATION.

That the free group F on any number of generators is orderable (definition recalled below) is a useful, but nontrivial, fact. One proof of this result begins by embedding F in the multiplicative group of units of a ring of noncommuting formal power series $k \ll X \gg$. For instance if F is free on two generators f and g, one takes any ordered field k (e.g., the real numbers) and maps F to the units of $k \ll x, y \gg$ by

(1) $$f \mapsto 1 + x, \qquad g \mapsto 1 + y.$$

One then orders the units of $k \ll x, y \gg$ with constant term 1 by choosing for each n an ordering of the k-vector space of polynomials homogeneous of degree n, and comparing two elements by looking at the first degree in which they disagree, and seeing which has ''larger'' component in that degree. That this gives a group ordering is easy to verify; what is not so obvious is that the map determined by (1) is an embedding.

To establish this, it suffices to find *some* completed graded k-algebra containing elements ξ and η, homogeneous of degree 1, such that $1 + \xi$ and $1 + \eta$ generate a free group. Happily, an easy example

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exists. In the 2×2 matrix ring over the polynomial algebra in one indeterminate, $M_2(k[t])$, the group generated by the two invertible matrices $\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$ is free. (This is related to the uniqueness of the Euclidean algorithm on $k[t]$.) Hence these matrices still generate a free group in the completion of this ring, $M_2(k[[t]])$, allowing one to complete the proof sketched.

Observe that one can now bypass the use of the power series algebra $k\langle\langle x,y\rangle\rangle$; for the method used to order units with constant term 1 in that algebra can equally be applied to units of $M_2(k[[t]])$ with constant term the identity matrix. Hence the latter group is orderable, hence so is the free subgroup generated by our pair of matrices.

Below, we shall generalize the result that this pair of matrices generates a free group, by showing that certain sorts of pairs F, G of (semi)groups of matrices generate semigroups isomorphic to the coproduct of F and G, possibly with amalgamation of a common subgroup. From this we shall prove results on orderability of such coproducts.

(Actually, there *is* an easy direct proof that (1) gives an embedding, [**19**, proof of Theorem 5.6]. If I had realized that from the beginning, I would not have discovered the results of this paper.)

Vinogradov's and Johnson's proofs [**29**], [**15**] of the orderability of coproducts (without amalgamation) of orderable groups and semigroups respectively, are ‘‘in essence’’ the same as the corresponding cases of the proofs given here, but use representations by infinite matrices where we describe the same objects, more conceptually, as 2×2 matrices over formal power series rings.

Some observations and results related to the material of this paper, but which I felt not sufficiently definitive to include here, are given in [**2**].

2. Definitions.

We recall some definitions and basic facts.

An *ordered group* means a group G with a total ordering on its underlying set which satisfies, for all $a, b, c, d \in G$,

(2) $$a > b \;\Rightarrow\; ac > bc \text{ and } da > db.$$

A group G is called *orderable* if it admits such an ordering. The same definitions apply to semigroups; note that an orderable semigroup will be right and left cancellative.

We understand all semigroups to have neutral element 1. (Thus, we say ‘‘semigroup’’ where most authors now use ‘‘monoid’’.) We shall consider 1 to be the product of the *empty string* of elements of any group or semigroup; a product of a string of >0 elements will be called a *nonvacuous* product.

We note

(3) The direct product of an arbitrary family of orderable groups or semigroups is orderable ([**22**, 3.5], cf. [**3**, Theorem 2.1.1], [**17**, Proposition 1.2.2]).

Indeed, we may choose an ordering on each factor in the product, well-order the index-set over which the product is taken, and then order the product lexicographically. (Direct products are called ‘‘complete direct products’’ in [**17**] and ‘‘cartesian products’’ in [**3**].) We also note

(4) Orderability is a local property; that is, a group or semigroup is orderable if and only if every finitely generated sub(semi)group is orderable.

The proof of (4) for groups in [**3**, last sentence of Theorem 1.3.2], [**17**, Corollary 1 to Theorem II.1.3] depends on an explicit criterion for orderability of groups, not applicable to semigroups. But the original proof in [**23**] is based on a very general principle, which works for semigroups. (As noted in [**26**], this principle is a case of a general result of Model Theory; cf. [**20**, Theorem 7.2.1] [**5**, Corollary VI.2.9].)

We shall use the category-theoretic term ‘‘coproduct’’ and the notation $F \amalg G$ where the literature generally speaks of the ‘‘free product, $F * G$’’ of groups or semigroups. A ‘‘coproduct with amalgamation’’ (of a common subgroup S) will likewise mean what is often called a ‘‘free product with amalgamation’’.

We remark that though (2) is equivalent, for groups, to the corresponding condition with $>$ everywhere replaced by $\geq$, the latter condition is weaker for semigroups, allowing the possibility of noncancellativity. We are using the version with $>$ here because it is more amenable to our techniques, though the weaker condition is perhaps more natural. In [**9**], the weaker definition of orderability is used, and semigroups are not assumed to have 1, leading to some different results from those proved for semigroups here. (Incidentally, in [**15**], the case without 1 is ‘‘reduced’’ to the case with 1 by an argument that is in fact valid only if the semigroups without 1 have no idempotent elements.)

Rings will be associative with 1. An *ordered ring* means a nontrivial ring given with a total ordering which makes the additive group an ordered group, and such that the *positive* elements (elements > 0 under the ordering) form a multiplicative semigroup; the positive elements will then form an ordered semigroup under the given ordering. Note that an orderable ring has no zero-divisors. The analog of (3) is false for rings, since nontrivial direct product rings always have zero-divisors.

If R is a ring, $M_n(R)$ will denote the ring of $n \times n$ matrices over R, $R[t]$ will denote the ring of polynomials over R in one central indeterminate t, and $R[[t]]$ its completion, the ring of formal power series over R. The degree of an element $a \in R[t]$ means, as usual, the largest i such that t^i occurs with nonzero coefficient in a, or $-\infty$ if $a = 0$.

The relative complement of a set X in a set Y will often be written $Y - X$.

3. Alternating Products.

Our first generalization of the fact that the matrices $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ generate a free group is the following. (For convenience, we are using a common numbering system for displays and enunciations.)

Lemma 5. *Let R be a ring, and F and G multiplicative subgroups of $M_2(R[t])$. Suppose that in every element of $F - G$, the upper right-hand entry has strictly greater degree than the other three entries and has a non-zero-divisor for its highest degree coefficient, and similarly that in every element of $G - F$, the lower left-hand entry has strictly greater degree than the other three entries and has a non-zero-divisor for its highest degree coefficient.*

Then if $F \cap G = \{I\}$, the group of matrices generated by $F \cup G$ is the coproduct group $F \amalg G$. In general, this group is the coproduct with amalgamation, $F \amalg_{F \cap G} G$.

Proof. There is a natural homomorphism from the coproduct or coproduct with amalgamation described into the multiplicative structure of $M_2(R[t])$; what we need to show is that this map is one-to-one. We will deduce this from the computational fact

(6) Suppose one applies, on the left, to the column vector $\begin{pmatrix}1\\1\end{pmatrix}$, a nonvacuous product whose factors are *alternately* elements of $F-G$ and elements of $G-F$ (with no restriction on which kind of element this alternating product starts or ends with). Then if the last (leftmost) factor comes from F, the upper entry of the resulting vector has strictly greater degree than the lower entry, while if the last factor comes from G, the lower entry has strictly greater degree than the upper entry.

This is easily seen by induction. (In fact, we note for use in the next Lemma that it does not require that F and G be groups, only that they be sets satisfying the indicated condition on corner elements.)

Let us first prove the Lemma in the case $F\cap G=\{I\}$, even though this will be subsumed in the general case. In this case every nonidentity element of $F\amalg G$ can be written (uniquely) as a nonvacuous product whose factors are alternately nonidentity elements of F and of G. From (6) we see that the image in $M_2(R[t])$ of such a product will be distinct from the identity, hence the natural homomorphism from the group coproduct into the matrix group has trivial kernel, and we have an embedding, as claimed.

In the general case, the standard normal form for the coproduct with amalgamation is based on choosing representatives of the right cosets of $F\cap G$ in F and in G. However, we do not need this full result, but only the observation (which is straightforward) that every element of $F\amalg_{F\cap G} G$ is *either* an element of $F\cap G$, *or* can be written as a nonvacuous alternating product of elements of $F-G$ and elements of $G-F$. In the first case, if it is a nonidentity element then it clearly has nonidentity image in $M_2(R[t])$; in the second case the same conclusion follows from (6). ▌

If we were to start with semigroups F and G rather than groups, the same arguments would give results on when the map from the coproduct semigroup $F\amalg_{F\cap G} G$ to the matrix semigroup has *trivial kernel*. But, of course, this is much weaker than the desired statement that the map is one-to-one. To prove the latter, we will need stronger assumptions. We will also forgo extending this result to coproducts with amalgamation, which seems to get rather messy, though someone may wish to try it.

LEMMA 7. *Let R be a ring, and F, G multiplicative subsemigroups of $M_2(R[t])$. Suppose every element of F has the form $\begin{pmatrix}p&q\\0&1\end{pmatrix}$, where p is a non-zero-divisor, and that for every two distinct such elements, $\begin{pmatrix}p&q\\0&1\end{pmatrix}$ and $\begin{pmatrix}p'&q'\\0&1\end{pmatrix}$, one has*

(8) $\deg(q'-q)>\max(\deg(p),\deg(p'))$, *and the leading coefficient of $q'-q$ is a non-zero-divisor.*

Assume likewise that every element of G has the form $\begin{pmatrix}1&0\\q&p\end{pmatrix}$, again with p a non-zero-divisor and (8) *holding for distinct elements. Then the semigroup of matrices generated by $F\cup G$ is the coproduct semigroup $F\amalg G$.*

Proof. Let H denote the multiplicative semigroup generated by $F\cup G$. Every member of H can be written as a possibly vacuous alternating product of nonidentity members of F and of G; the desired result is equivalent to showing that two such product-expressions representing the same element of H must be the same expression.

Because elements of F and G are triangular with non-zero-divisors as diagonal entries, they are non-zero-divisors in the matrix ring, from which it follows that H is a cancellative semigroup. Hence it will suffice to prove the assertion for expressions which, if both nonvacuous, have distinct leftmost factors. By comparing their behavior on $\begin{pmatrix}1\\1\end{pmatrix}$ and applying (6), we see that if either is the vacuous product, then they both are, while if neither is, then they must have leftmost factors in the same set F or G. By symmetry (corresponding to interchanging the subscripts 1 and 2 in our matrices and vectors, and the names of F and G), we may assume without loss of generality that both have leftmost factor in F, i.e., that the given equality has the form

$$fs = f's',$$

where $f, f' \in F$, and s and s' are alternating products of nonidentity members of F and of G, which, if nonvacuous, have members of G as their leftmost factors. Writing f and f' as matrices, and the images of the vector $\begin{pmatrix}1\\1\end{pmatrix}$ under s and s' as columns, we get an equality of column vectors of the form

$$\begin{pmatrix}p & q\\0 & 1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}p' & q'\\0 & 1\end{pmatrix}\begin{pmatrix}u'\\v'\end{pmatrix},$$

where $\deg(v) \geq \deg(u)$ and $\deg(v') \geq \deg(u')$. (Equality of degrees is possible if s and/or s' is the vacuous product; in the contrary cases (6) gives strict inequality.) Now comparing the bottom entries of the two sides of this equation, we see that $v = v'$. Hence the equality of the top entries becomes

$$pu + qv = p'u' + q'v, \qquad \text{or}$$

$$pu - p'u' = (q' - q)v.$$

But since f and f' were assumed distinct, the hypothesis of our Lemma says that the first factor on the right-hand side of the last equation has larger degree than p or p' and a non-zero-divisor for leading coefficient, and we have already noted that the element $v = v'$ has larger degree than u or u'. This renders the above equality impossible, and completes the proof of the Lemma. ∎

We note in passing that either of our Lemmas easily yields our motivating observation, and a companion statement:

COROLLARY 9 (cf. [**4**]). *If R is a ring of characteristic 0, then the subgroup of $GL_2(R[t])$ generated by $\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\t & 1\end{pmatrix}$ is free on these generators. If R is of prime characteristic p, then the subgroup generated by these elements is the coproduct of two copies of the cyclic group of order p.* ∎

To construct more general situations to which Lemmas 5 and 7 are applicable, let R be a ring without zero-divisors, and consider any group F of units of R, or more generally, any multiplicative semigroup F of nonzero elements of R. Let us embed F in $M_2(R[t])$ by $f \mapsto \begin{pmatrix}f & 0\\0 & 1\end{pmatrix}$. To get an upper right hand entry of positive degree, let us now conjugate by $\begin{pmatrix}1 & t\\0 & 1\end{pmatrix}$, getting the homomorphism

(10) $$f \mapsto \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f & t(f-1) \\ 0 & 1 \end{pmatrix}.$$

If G is another subgroup or subsemigroup of R, let us similarly take a diagonal embedding of G in 2×2 matrices, and conjugate it by $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, getting an embedding of G in matrices

(11) $$g \mapsto \begin{pmatrix} 1 & 0 \\ t(g-1) & g \end{pmatrix}.$$

Applying Lemma 7 to these embeddings, we get

COROLLARY 12. *If R is a ring without zero divisors, and F, G multiplicative semigroups of nonzero elements of R, then the subsemigroup of $M_2(R[t])$ generated by the representation* (10) *of F and the representation* (11) *of G is isomorphic to the coproduct $F \amalg G$.* ▌

We remark that the case of the above Corollary where F and G are *groups* can be gotten using the easily proved first assertion of Lemma 5, rather than Lemma 7. (For results on embeddings of coproduct groups in matrices of various sizes over *fields*, see [**24**], [**30**, Theorem 4], [**28**], [**21**].)

Because the data describing coproduct groups with *amalgamation* is more complicated, we shall find it convenient to prove at this point a restricted case, and derive more general results from this one later, when we have passed from matrix groups to abstract orderable groups.

Suppose M is a group of units in a ring R, let $N = M\times M$, and let us write $\Delta M = \{(m, m) \mid m\in M\} \subseteq N$ for the diagonal image of M. Now N can be identified with a group of diagonal 2×2 matrices over R. (Terminological caveat: the ''diagonal'' subgroup $\Delta M \subseteq N$ consists of pairs with both components equal, but ''diagonal'' matrices need not have equal diagonal entries.) Conjugating this embedding by the same two matrices as before, we get a pair of embeddings of N in $M_2(R[t])$,

(13) $$(m, m') \mapsto \begin{pmatrix} m & t(m-m') \\ 0 & m' \end{pmatrix}, \qquad (m, m') \mapsto \begin{pmatrix} m & 0 \\ t(m'-m) & m' \end{pmatrix},$$

which intersect in the common image of the diagonal subgroup $\Delta M \subseteq N$. Lemma 5 now gives

COROLLARY 14. *Let M be a group of units in a ring R, let $N = M\times M$, and let $\Delta M = \{(m, m) \mid m\in M\} \subseteq N$. Then the multiplicative subgroup of $M_2(R[t])$ generated by the two representations* (13) *of N is isomorphic to $N \amalg_{\Delta M} N$.* ▌

4. CONSTRUCTING ORDERINGS.

We will state the next result in the generality of formal power series, though the power series arising in our application will be polynomials.

LEMMA 15. *Let R be an ordered ring, and $U \subseteq M_2(R[[t]])$ the multiplicative subsemigroup of elements whose constant terms are diagonal matrices, with* positive *diagonal entries. Then U is orderable.*

Proof. For each n, let us order the R-submodule $t^n M_2(R) \subseteq M_2(R[[t]])$ by choosing an arbitrary order among the four ''positions'' in a 2×2 matrix, and calling a nonzero element of this module positive if in the ''first'' position in which a nonzero coefficient occurs, the coefficient is in fact positive. (The orderings of the positions can be the same for all n, but need not – there is a lot of freedom here.) Note that the product, in either order, of a positive element of this module with a diagonal element of $M_2(R)$ having positive diagonal entries, is positive. Now given distinct elements $a, b\in U$, let $n\geq 0$ be the least integer such that t^n has nonzero coefficient in $a-b$, and let us write $a>b$ if and only if this coefficient is positive. This is clearly a total ordering on the set U, and from the above observations it follows that it will satisfy (2), making U an ordered semigroup. ∎

To combine this result with those of the preceding section and get results on orderability of coproducts of orderable groups and semigroups, we need ways of embedding orderable groups and semigroups in ordered rings. If k is any ordered ring and G an ordered semigroup, the semigroup ring kG can be made an ordered ring, by calling an element ''positive'' if the highest element of G having nonzero coefficient in k has positive coefficient. Now any two orderable semigroups F and G are embeddable in a common orderable semigroup, for instance $F\times G$. Hence F and G can be embedded in the multiplicative semigroup of positive elements of a common ordered ring. It now follows from Corollary 12 and Lemma 15 that a coproduct of two orderable semigroups is orderable. By induction, the same applies to a coproduct of finitely many orderable semigroups. Using (4), we deduce

THEOREM 16 (Johnson [**15**], Vinogradov [**29**]). *Any coproduct of orderable semigroups is orderable. In particular, any coproduct of orderable groups is orderable.* ∎

From Corollary 14 and Lemma 15 we similarly get the following result, the consequences of which we will develop in §6.

PROPOSITION 17. *Let* M *be an orderable group,* $N=M\times M$, *and* $\Delta M=\{(m,m)\mid m\in M\}\subseteq N$. *Then* $N\amalg_{\Delta M} N$ (*the coproduct of two copies of* N *with amalgamation of* ΔM) *is orderable.* ∎

5. ORDERING AND ORDERABILITY.

The above results were stated in terms of orderability, rather than in terms of specific orderings. In most cases, I consider mathematical statements to the effect that, say, two objects are isomorphic to be sloppy versions of statements that there is an isomorphism between them characterized by certain properties; but we seem to have a different situation here, the class of *orderable* (semi)groups being at least as natural as that of *ordered* (semi)groups. For instance, from the facts that the class of orderable semigroups is closed under passing to direct products and to subsemigroups, it follows that

(18) Every semigroup has a universal orderable homomorphic image, i.e., has a least congruence $\sim$ such that $G/{\sim}$ is orderable. (In the case of groups: a least normal subgroup N such that G/N is orderable.)

On the other hand, a group or semigroup will not in general have a universal *ordered* homomorphic image.

Another indication that specific orderings are ''not too important'' is given by

(19) If $f: F \to G$ is a homomorphism of orderable semigroups, then for every semigroup ordering on G, there is a semigroup ordering on F which makes f isotone ($\leq$-respecting).

To see this, note that the map $(f, \mathrm{id}_F): F \to G \times F$ is an embedding. Order its codomain lexicographically using the given ordering on G and an arbitrary semigroup ordering on F, and restrict this ordering to the embedded copy of F. The induced ordering on F will clearly have the desired property.

If in the above situation we are dealing with groups, and we let $N = \mathrm{Ker}(f)$ and examine how the new ordering on F is related to the given ordering on G and the old ordering on F, we find that it involves the latter only via its restriction to N, and that in this situation the construction can be generalized to the following easily verified statement:

(20) If F is a group and N a normal subgroup, then for every group ordering on F/N, and every group ordering on N which is invariant under the action of F by conjugation, there exists a unique group ordering on F making the natural maps $N \to F \to F/N$ isotone. (F. W. Levi [**18**]. Cf. [**22**, (2.33′), p.7 *ftn.*], [**17**, Theorem 2.2.4], [**3**, Lemma 1.3.5].)

Now given *ordered* semigroups F and G, once we know from Theorem 16 that their coproduct is *orderable*, we can apply (19) to the natural map $F \amalg G \to F \times G$, using a lexicographic order on the codomain, and conclude that the coproduct can be given a group ordering making this map isotone, and in particular, extending the given orderings on F and G. So we lost no information by failing to put these facts into the statement of Theorem 16.

In contrast to (19), however, if $f: F \to G$ is a homomorphism, or even an embedding, of orderable semigroups, and a semigroup ordering is given on F, it is *not* true that there must exist a semigroup ordering of G making f isotone. For instance, if F and G are groups and x and y are elements of F such that $f(x)$ and $f(y)$ are conjugate in G, then no ordering of F under which x and y have ‘‘opposite signs’’ (one > 1, the other < 1) is compatible with any group ordering of G. In particular, if F is the free group on x and y, G free on x and z, and f the embedding taking x to x and y to zxz^{-1}, then there are orderings of F under which x and y have opposite sign, but there is no group ordering on G making f isotone with respect to such an ordering on F.

Another obstruction to finding an ordering of G compatible with a given ordering of F occurs if f is not one-to-one: the kernel of f may not be convex under the latter ordering.

6. Ordering Coproducts with Amalgamation.

Proposition 17 showed that a particular sort of coproduct group with amalgamation was orderable. We shall now deduce a wider class of results from that case.

When we work with the direct product M^X of a family of copies of a group M indexed by a general set X, the diagonal image of M in this power-group will be denoted $\Delta^{(X)}M$.

Theorem 21. *Let H be an orderable group, and T a subgroup of H. Then the following conditions are equivalent:*

(i) *There exists an orderable overgroup M of H and a subset $X \subseteq M$ such that T is the centralizer* *of* X *in* H.

(ii) *There exists an orderable overgroup* M *of* H *and an element* $x \in M$ *such that* T *is the centralizer of* x *in* H.

(iii) *There exists an orderable group* M *and a pair of homomorphisms* $\alpha, \beta\colon H \to M$ *such that* T *is the equalizer* $\{h \in H \mid \alpha(h) = \beta(h)\}$. *Equivalently, there exists an orderable group* M *and a homomorphism* $(\alpha, \beta)\colon H \to M \times M$ *such that* $T = (\alpha, \beta)^{-1}(\Delta M)$.

(iv) $H \amalg_T H$ *is orderable.*

(v) *For every set* I, *the coproduct* $\amalg_T^{(I)} H$ *of a family of copies of* H *indexed by* I, *with amalgamation of* T, *is orderable.*

(vi) *The group gotten by universally adjoining to* H *a* T*-centralizing element* x *is orderable.*

Proof. We shall prove (i)-(v) equivalent, each to the next, and obtain the equivalence with (vi) as part of the step (iv)⇒(v). The easy directions in the chain of equivalences are (i)⇐(ii)⇒(iii)⇐(iv)⇐(v). The first and last of these implications are clear. For the second, we let α and β be the inclusion of H in M and the conjugate of this inclusion by the element x respectively. For the third, we let α and β be the two canonical injections of H into $H \amalg_T H$. Let us now prove the various reverse implications.

(i)⇒(ii). Given (i), we form the product group M^X, and let x be the element of this group given by the inclusion map $X \to M$. Then the centralizer of x in the diagonal subgroup $\Delta^{(X)} M \cong M$ is the copy, in that subgroup, of $C_M(X)$, so the centralizer of x in the image of H therein is the image of $C_M(X) \cap H = T$, establishing (ii).

(ii)⇐(iii). Suppose there exist α and β as in (iii). We may assume that their kernels are trivial, since if this condition does not hold, we can achieve it by replacing M by $M \times H$. Hence we may embed H in 2×2 matrices over the group ring on M, by the map $h \mapsto \begin{pmatrix} \alpha(h) & 0 \\ 0 & \beta(h) \end{pmatrix}$. The image lies in the ordered group U described in Lemma 15, and the centralizer, in this image, of the element $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in U$ is the image of T.

(iii)⇒(iv). Given a homomorphism as in the second statement of (iii), assume as above that it is an *embedding* of H in $N = M \times M$. By Proposition 17, $N \amalg_{\Delta M} N$ is orderable. The subgroup thereof generated by the images of H in the two copies of N will be isomorphic to $H \amalg_{(H \cap \Delta M)} H = H \amalg_T H$ [**27**, Proposition I.1.3.3], so this group is orderable.

(iv)⇒(v). Let $\langle x \rangle$ denote the infinite cyclic group on one generator x. If the inclusion $T \subseteq H$ satisfies (iv), then by our previous observations it satisfies (iii), from which one can easily show that the inclusion $T \subseteq H \times \langle x \rangle$ also satisfies (iii), hence (iv); thus the coproduct of two copies of $H \times \langle x \rangle$ with amalgamation of T is orderable. Now one finds that the group generated by the image of x in one of these copies and the image of H from the other is a universal extension of H by a T-centralizing element x. Within this extension, the subgroups $x^i H x^{-i}$ $(i \in \mathbf{Z})$ generate a countable coproduct of copies of H with amalgamation of T. This yields (v) for I countable, and the result for arbitrary I follows using (4). We see that we have also proved the equivalence with (vi). ▮

Remark: In [**6**], the analog of the implication (iii)⇒(iv) is proved for ordered division rings, and the result for ordered groups is deduced. This gives an alternative (though somewhat roundabout) way of proving this key step in our Theorem.

We shall obtain further conditions equivalent to those of the above Theorem in the next section. At this point, let us deduce some sufficient conditions for a coproduct of not necessarily isomorphic orderable groups with amalgamation of a common subgroup to be orderable.

THEOREM 22. *Let* S *be an orderable group, and let us be given a family of orderable overgroups* $G_i \supseteq S$ $(i \in I)$. *Suppose*

(i) *that the* G_i *can be embedded in a common orderable overgroup* H, *by embeddings agreeing on* S, *so that for some set* $X \subseteq H$, *the centralizer subgroup* $C_H(X) \subseteq H$ *intersects each* G_i *in precisely the subgroup* S; *or* equivalently

(ii) *that there exists an orderable overgroup* M *of* S, *and a system of embeddings* $(\alpha_i, \beta_i): G_i \to M \times M$ *respecting* S, *such that for each* i, *the inverse image of* ΔM (*in other words, the subgroup of* G_i *on which* α_i *and* β_i *agree*) *is precisely* S.

Then the coproduct group $\coprod_S G_i$ *is orderable.*

Proof. The equivalence of (i) and (ii) can be deduced from the equivalence of conditions (i) and (iii) of the preceding Theorem. That Theorem also shows that, for H as in condition (i), $\coprod^{(I)}_{C_H(X)} H$ will be orderable. From the fact that each G_i meets $C_H(X)$ in exactly S, it follows that the subgroup of $\coprod^{(I)}_{C_H(X)} H$ generated by the G_i's has the form $\coprod_S G_i$ [**27**, Proposition I.1.3.3]. So this group is orderable. ▮

Note that this result includes the group case of Theorem 16, i.e., the orderability of a coproduct group $\coprod G_i$ without amalgamation, for in this case we can establish condition (ii) above by taking M to be the direct product group $\prod G_i$, and letting the α_i be the natural injections, and the β_i the trivial maps.

7. DOMINIONS.

To view the class of subgroups defined by the equivalent conditions of Theorem 21 in a more general context, recall that Isbell [**13**] defines the *dominion* of a subalgebra A of an algebra H in a fairly general category of algebras to mean the algebra of all $h \in H$ such that every pair of homomorphisms $H \rightrightarrows K$ with a common codomain K which agree on the subalgebra A also agree at the element h. Thus, the dominion construction is a closure operator on subalgebras of H. If the category in question is closed under small products, the dominion can always be represented as the equalizer of a single pair of homomorphisms $H \rightrightarrows K$. Hence the subalgebras closed under the dominion operator could be referred to as the *equalizer* subalgebras of H, a terminology suggested by the definition on the first page of [**13**]. But I find it more natural to call them the *dominion* subalgebras, in view of the relation to this closure operator.

We now see from Theorem 21 (iii)$\Leftrightarrow$(iv), that for T a subgroup of the orderable group H, the coproduct group $H \coprod_T H$ is orderable if and only if T is a *dominion subgroup* of H in the category of orderable groups.

Is this a nonvacuous restriction? In the category of all groups, every subgroup $T \subseteq H$ is a dominion, since, by the standard description of $H \coprod_T H$ [**27**, Theorem I.1.2.1] [**19**, Theorem 4.4], the equalizer of the two injections of H into this group is just T. However not every subgroup of an orderable group will be a dominion in the subcategory of *orderable* groups. This is clear from condition (i) of the next Lemma; the subsequent points give still stronger restrictions.

LEMMA 23. *Suppose* T *is a dominion subgroup of* H *in the category of orderable groups. Then*

(i) *If* $h \in H$, *and* $h^n \in T$ *for some* $n > 0$, *then* $h \in T$.

More generally,

(ii) *If* $h \in H$, *and there exist* $t_1, \ldots, t_n \in T$ *such that* $(t_1^{-1} h t_1) \ldots (t_n^{-1} h t_n) \in T$, *then* $h \in T$.

(*In fact, condition* (i) *is satisfied by dominions in the larger category of groups in which the* n*th power map is one-one for all* n, *and* (ii) *in the category of what are called, in the language of* [**17**], *''groups* H *without H-torsion'' and in* [**3**], *''R*-groups''.*)

Still more generally, if for $h_1, \ldots, h_m \in H$ *we let* $\Sigma_T(h_1, \ldots, h_m)$ *denote the set of all nonvacuous products of conjugates* $t^{-1} h_i t$ $(t \in T,\ i \leq m)$, *then*

(iii) *For all* $h_1, \ldots, h_m \in H - T$, *there exist* $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$ *such that* $\Sigma_T(h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m})$ *is disjoint from* T.

Proof. The parenthetical statement following (ii) will not be used here; the first part is clear; those familiar with the condition referred to in the second will see that it is exactly what is needed to make that assertion hold. We will now prove the unparenthesized assertions without reference to these observations.

Clearly, (i) is a special case of (ii). The hypothesis of (ii) says that $\Sigma_T(h)$ has nonempty intersection with T; since T is closed under inverses, this is equivalent to saying that both $\Sigma_T(h)$ and $\Sigma_T(h^{-1})$ have nonempty intersection with T, so assuming (iii), this indeed implies $h \in T$. Hence (i) and (ii) will follow if we prove (iii).

Suppose $h_1, \ldots, h_m \in H - T$. Since T is a dominion, we can find homomorphisms α and β from H into an orderable group M which agree on T but disagree everywhere else. Let us choose an ordering of M. Since α and β disagree at each of $h_1, \ldots, h_m$, we may, for $i = 1, \ldots, m$, take ε_i to be $+1$ if $\alpha(h_i) < \beta(h_i)$, and to be -1 if $\alpha(h_i) > \beta(h_i)$. Then for $h \in \Sigma_T(h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m})$, we see that $\alpha(h) < \beta(h)$, so $h \notin T$. ∎

In [**3**], a subgroup $T \subseteq H$ closed under taking roots, as in point (i) above, is called *isolated*, while one such that a relation $(g_1^{-1} h g_1) \ldots (g_n^{-1} h g_n) \in T$, with all $g_i \in H$, implies $h \in T$ is called *strongly isolated*. I find no name for the intermediate condition used in (ii), with conjugation only by elements of T; probably one would say that T is ''T-isolated'' or ''self-isolated'' in H. A subgroup T satisfying condition (iii) might then be called ''poly-self-isolated''. Incidentally, we are using the symbol Σ where [**3**] and [**17**] use S (for ''semigroup generated by conjugates of'') because of our use of S for a subgroup.

Instead of expressing (ii) and (iii) in terms of products of *conjugates* of the given elements of H by elements of T, we could equivalently have used products of the given elements of H and arbitrary elements of T, and the condition that such a product (respectively no such product) equal 1. Our choice of the former formulation is for the sake of parallelism with other results in the theory of orderable groups; cf. [**3**, Theorems 1.3.1-2, 1.4.1] = [**17**, Theorems II.1.1-3, II.3.1].

From now on, we shall frequently use the term ''dominion'' to mean ''dominion in the category of orderable groups''. There is no chance of misunderstanding since, as we have noted, the dominion concept is trivial in the category of groups.

Note that if T is a subgroup of H *not* satisfying condition (i) or (ii) of the preceding Proposition, then that condition gives us a particular element h that is not in T but which must lie in the dominion of T. But condition (iii) is not as good in that way: if $h_1, \ldots, h_m$ fail to satisfy it, we can see that the dominion of T in H must contain at least *one* of the h_i, but we do not know which.

The argument can, however, be refined to show that if such a family is, in a sense we shall make precise, irredundant, then *all* of $h_1, \ldots, h_m$ lie in the dominion of T. Given $h_1, \ldots, h_m \in H$, let us define a *T-trap* for $\{h_1, \ldots, h_m\}$ to mean a family X such that (a) each member of X is an equation holding in H, whose left hand side is an element of T and whose right hand side is a nonvacuous product of elements $t^{-1} h_i^{\pm 1} t$ $(i \leq m,\ t \in T)$, and (b) for every choice of $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$, X contains at least one equation whose right hand side has the form of a member of $\Sigma_T(h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m})$. (I.e., each factor is a conjugate of one of the $h_i^{\varepsilon_i}$. Note that the expression constituting the right hand side of a member of X need not involve all of the h_i, and that if it does not, it will have the form of a member of more than one $\Sigma_T(h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m})$, differing in the exponents assigned to the absent h_i's.)

LEMMA 24. *Let H be an orderable group and T a subgroup. If $h_1, \ldots, h_m \in H$ $(m \geq 1)$, and if there exists a T-trap X for $\{h_1, \ldots, h_m\}$, no subset of which is a T-trap for a proper nonempty subset of $\{h_1, \ldots, h_m\}$, then $h_1, \ldots, h_m$ all belong to the dominion of T in H.*

Proof. Given H, T, $h_1, \ldots, h_m$ and X as in the statement, suppose α, β are homomorphisms from H into an ordered group M which agree on T. It will suffice to show that they agree on all of $h_1, \ldots, h_m$.

Consider the set of all assignments of $\varepsilon_1, \ldots, \varepsilon_m$ such that ε_i is $+1$ if $\alpha(h_i) < \beta(h_i)$, ε_i is -1 if $\alpha(h_i) > \beta(h_i)$, and either choice is allowed if $\alpha(h_i) = \beta(h_i)$. For each such assignment, choose a member of X which expresses an element $t \in T$ as a member of $\Sigma_T(h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m})$. By choice of $\varepsilon_1, \ldots, \varepsilon_m$, we see that for each such t,

$$\alpha(t) \leq \beta(t), \tag{25}$$

with strict inequality if our expression for t actually involves any of the h_i for which $\alpha(h_i) \neq \beta(h_i)$. Since $t \in T$, however, we must have equality in (25), hence the expression for t does not involve any of these h_i. From this it is not hard to verify that the subset of X we have chosen constitutes a T-trap for

$$\{h_i \mid \alpha(h_i) = \beta(h_i)\}. \tag{26}$$

Furthermore, the set (26) is nonempty, since each of our expressions is a *nonvacuous* product of conjugates of members of that set. Hence, by our minimality hypothesis on X, (26) cannot be a proper subset of $\{h_1, \ldots, h_m\}$; so it is the whole set, as claimed. ▮

Note that if a subgroup $T \subseteq H$ fails to satisfy condition (iii) of Lemma 23, then by going to a minimal counterexample to that condition we get a family of elements $h_1, \ldots, h_m \in H - T$ satisfying the condition of Lemma 24; and, of course, conversely the existence of such a family shows that T does not satisfy condition (iii) of Lemma 23. Hence, Lemma 23 (iii) and Lemma 24 express the same property of dominions; their respective advantages are that the first is a simpler statement, while the second is explicitly a closure condition. (The idea of the proof of equivalence between these conditions can be abstracted as a combinatorial result. I do this in [**2**, §1], and apply this result to a known criterion for a group to be orderable.)

Recall that a subgroup T of an orderable group H is called *relatively convex* if it is convex with respect to *some* group ordering of H [**3**] [**17**]. Every relatively convex *normal* subgroup T of an orderable group H is clearly a dominion, since T is the equalizer of the canonical map and the trivial map from H to the orderable group H/T. Translating this, by Theorem 21, to a statement about orderability of coproducts with amalgamation, we get a case of the only positive result on the subject that seems to have been known till now, the full statement of which is that $F \amalg_S G$ is orderable if F and G are orderable and S is normal and relatively convex in each of F and G, and admits an ordering invariant under conjugation by each of these overgroups [**3**, Proposition 2.3.3], [**17**, Proposition III.1.1]. This is easily proved directly from (20) and Vinogradov's Theorem (the group case of Theorem 16), applied to $(F/S) \amalg (G/S)$.

Kopytov has conjectured the converse statement, that a coproduct with amalgamation $F \amalg_S G$, with

S relatively convex in F and G, is never orderable *unless* S is normal in one or both of F and G ([**17**, Problem 8(c)] or, better stated, [**3**, p. 36, lines 6-7]). This was useful in focusing attention on the question of amalgamation of relatively convex subgroups, but as a conjecture it turns out to be quite false. We shall show now that the dominion subgroups T of an orderable group H include *all* the relatively convex subgroups of H, and in fact, constitute a natural generalization of this class.

To formulate this result, we recall that a *right ordering* on a group (or semigroup) G is an ordering on the underlying set which is required to satisfy only the first part of (2):

(27) $$a > b \;\Rightarrow\; ac > bc.$$

Left orderings are defined analogously (but do not have to be treated separately because whenever $\geq$ is a right ordering, the relation $\geq'$ defined by $a \geq' b \;\Leftrightarrow\; b^{-1} \geq a^{-1}$ is a left ordering, with the same positive cone as the right ordering $\geq$.)

Let us call a subgroup T of a right orderable group H *right relatively convex* if it is convex under some right ordering of H (equivalently, under some left ordering).

THEOREM 28. *Let H be an orderable group and T a subgroup of H. Then the equivalent conditions of Theorem 21 are also equivalent to each of the following.*

(vii) *There exists a right action of H by order-preserving maps on a totally ordered set B, such that T is the stabilizer subgroup in H of some element $x \in B$.* (*Equivalently, there exists such an action for which T is the pointwise stabilizer of some subset $X \subseteq B$.*)

(viii) *T is right relatively convex in H.*

(ix) *For all $h_1, \ldots, h_m \in H - T$, there exist $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$ such that $\Sigma_T(h_1^{\varepsilon_1}, \ldots, h_m^{\varepsilon_m})$ is disjoint from T* (*condition* (iii) *of Lemma 23*). (*Equivalently, if $h_1, \ldots, h_m \in H$, and there exists a T-trap for this family which does not contain a T-trap for a proper nonempty subfamily, then $h_1, \ldots, h_m \in T$.*)

Proof. The two versions of (vii) are equivalent by the same reasoning that shows the equivalence of (i) and (ii) in Theorem 21. To see that the conditions of Theorem 21 imply (vii), assume H embedded in an overgroup M so that T is the centralizer in H of an element $x \in M$, as in condition (i) of that Theorem. Then for any group ordering of M, the action of H on M by conjugation is order-preserving, and T is the stablizer of x with respect to that action. (Alternatively, given maps α, β: $H \to M$ with equalizer T as in Theorem 21 (iii), let H act on the set M so that $h \in H$ carries $m \in M$ to $\alpha(h)^{-1} m \beta(h)$. Then T is the stabilizer of $1 \in M$, and again the action preserves any group ordering on M.)

Conversely, suppose H acts on an ordered set B as in (vii). Let us choose any ordered ring k, and order the free k-module kB in the same way we order the group ring of an ordered group. Then the action of H on B induces an order-preserving k-linear action of H on kB. Let M denote the semidirect product, based on this action, of H with the additive group of kB. By (20) M is orderable, and we can see that the centralizer in H of $x \in kB \subseteq M$ is the stabilizer in H of $x \in B$, which by assumption is T, giving condition (ii) of Theorem 21.

(vii)$\Rightarrow$(viii): Again assume an action as in (vii) given. We may let H act on the set $B \times H$, using the given action on B and its action on itself by right multiplication, and we can order this product lexicographically using the given ordering on B and any group ordering on H. The result is an action without fixed points by order-preserving maps on an ordered set; hence a right ordering may be induced on H by choosing any element y of this set, and declaring $h \geq h'$ if and only if $yh \geq yh'$. Let us take $y = (x, 1)$. Then T can be described as the set of elements of H which carry y into the convex set

$\{x\} \times H$. So T is convex with respect to this right ordering.

(viii)⇒(vii): Assuming T is convex under a right ordering on H, the right coset space $T \backslash H$, on which H has a natural right action, has an induced ordering under which this action is order-preserving; and the stabilizer in H of the coset T itself is clearly T, giving (vii).

We noted in the discussion following Lemma 24 the equivalence of the two versions of (ix), and that these conditions hold for any dominion subgroup $T \subseteq H$, i.e., any subgroup satisfying the equivalent conditions of Theorem 21. Conversely, assume the first statement of (ix). It is easy to deduce from the Compactness Theorem of Model Theory that there exists a ''global'' function $\varepsilon: H - T \to \{\pm 1\}$ such that the set of all nonvacuous products of elements $t^{-1} h^{\varepsilon(h)} t$ $(h \in H - T,\ t \in T)$ does not meet T. It is easy to verify that the set $P = \{h \in H - T \mid \varepsilon(h) = +1\}$ must be closed under internal multiplication, and under left and right multiplication by elements of T, and that P, T, P^{-1} will partition H. From this it follows that $T \backslash H$ may be totally ordered by setting $Th > Th'$ if $h h'^{-1} \in P$. Again, the action of H on this set is order-preserving, and (vii) follows. ▮

The class of dominion subgroups, that is, *right* relatively convex subgroups, of an orderable group is in general strictly larger than the class of relatively convex subgroups. For a relatively convex subgroup T must be *infrainvariant*, that is, the class of its conjugates $h^{-1} T h$ $(h \in H)$ must be totally ordered by inclusion, while if M a group, the subgroup $\Delta M \subseteq M \times M$, which is the prototypical case of a dominion, is not infrainvariant unless M is commutative. Further, not even every infrainvariant right relatively convex subgroup is relatively convex. For every *normal* subgroup is trivially infrainvariant, but one can verify that a normal subgroup is relatively convex if and only it is the kernel of a homomorphism to an orderable group, and is right relatively convex if and only it is the kernel of a homomorphism to a right orderable group; and there exist right orderable groups that are not orderable.

In a general category of algebras, if not every subalgebra of an algebra is a dominion, it typically happens that a dominion subalgebra of a dominion subalgebra need not be a dominion in the larger algebra. (This is because there can be pairs of morphisms out of the subalgebra that do not extend to pairs of morphisms on the original algebra.) However, we can show from Theorem 28 that this does not happen in the category of orderable groups.

Corollary 29. *If H is an orderable group, D a dominion subgroup of H, and T a dominion subgroup of D, then T is a dominion subgroup of H.*

Equivalently, if T is any subgroup of the orderable group H, and D the dominion of T in H, then the inclusion of T in D is an epimorphism (*in the category-theoretic sense*) *in the category of orderable groups.*

Proof. Let us first verify the equivalence of the two statements in a general category of algebras. Assume the first statement. Then in the context of the second statement, the dominion of T in D is a dominion subalgebra of a dominion subalgebra, hence by assumption is a dominion subalgebra of H; but D is the least dominion subalgebra of H containing T, hence the dominion of T in D is all of D, which is the desired epimorphicity. Conversely, assuming the second statement we see that in the context of the first, the dominion of T in H will be a subalgebra of D in which T is epimorphically included. But since T is a dominion in D, the only such subalgebra of D is T itself, so the dominion of T in H is T, i.e., T is a dominion in H.

Hence it suffices to establish the second assertion for orderable groups. Given $T \subseteq H$, it follows from Theorem 28 and the comments following Lemma 24 that every element of D can be obtained, starting from T, by repeated adjunctions of families of elements $h_1, \ldots, h_m$ which satisfy the conditions of that Lemma relative to the subgroup obtained at the current stage. Such families will satisfy these conditions within D just as within H (for the hypotheses of that Lemma do not involve any elements of H other than those which, under its conclusion, belong to D). Hence if we now take the dominion of T in D, we get all these elements, i.e., the inclusion of T in D is an epimorphism. ▮

8. Coproducts with Amalgamation of Nonisomorphic Overgroups.

Let us return to Theorem 22, concerning orderability of a group $\coprod_S G_i$ where the G_i are not necessarily isomorphic overgroups of S. It may seem unfortunate that each of our sufficient conditions for orderability requires that we already know that the G_i are embeddable in some common orderable group, by maps agreeing on S. However, some assumption of the sort is needed, for as observed at the end of §5, an embedding of an orderable group S in an orderable group G may limit the class of group orderings on S which extend to group orderings of G. Consequently, embeddings into more than one such overgroup may impose incompatible restrictions.

For example, let S be the free group on a countable set of generators $\{y_i \mid i\in \mathbf{Z}\}$, and let F be the semidirect product of S with the infinite cyclic group $<x>$, where x acts by the automorphism $y_i \mapsto y_{i+1}$. Then F is the free group on x and y_0, hence orderable, and S is the kernel of the obvious map $F \to <x>$, hence is the equalizer of this map and the trivial map, hence is a dominion subgroup. Observe also that any group ordering of F must have the property that the sequence $\dots, y_{-1}, y_0, y_1, y_2, \dots$ is monotone, either increasing or decreasing (depending on whether $x y_0 x^{-1} = y_1$ is greater or less than y_0). Now if we let G be an overgroup of S constructed in the same way as F, but using some ''scrambled'' indexing of the y_i (one that is not monotone increasing or decreasing in the original indexing), we can see that no overgroup of S containing copies of both the overgroups F and G can be orderable, so in particular, $F \amalg_S G$ is not, though S is a dominion subalgebra of each of F and G. See also [**3**, Example 2.3.2] = [**17**, II.1.2°].

Are the sufficient conditions of Theorem 22 for the orderability of a coproduct with amalgamation of distinct overgroups of S perhaps (like those of Theorems 21 and 28 for the same-overgroup case) also necessary?

They are not. Note that those conditions imply that S is a dominion subgroup of each of the G_i. Here is an example of an orderable coproduct $F \amalg_S G$ where S is not a dominion in either F or G. Let H be the free group $<x, y>$, F and G the subgroups $<x, y^2>$ and $<x^2, y>$ respectively, and S their intersection, $<x^2, y^2>$. Then one can show (using the freeness of H) that H has the universal property of $F \amalg_S G$, hence this coproduct with amalgamation is orderable, though S is not closed under taking square roots in either F or G, and so is not a dominion subgroup of either.

However, the *same* element of S does not have both a square root in $F - S$ and a square root in $G - S$. It is not hard to show that this is a necessary condition for the orderability of a coproduct with amalgamation $F \amalg_S G$. More generally, a system of equations representing a ''minimal S-trap'' in the sense of Lemma 24 cannot have solutions in both F and G. We shall formalize this statement presently.

The observations made so far actually give conditions necessary for F and G to be embeddable in *some* common orderable group H by maps agreeing on S, so that the images of F and G intersect in S only. But in contrast to the situation described by Theorem 21, even the existence of such embeddings does not guarantee orderability of $F \amalg_S G$. For example, let H be the infinite cyclic group $<x>$, let F and G be the subgroups $<x^m>$ and $<x^n>$ for relatively prime integers $m, n > 1$, and let S be their intersection, $<x^{mn}>$. Clearly the above ''common embedding'' condition is satisfied. Now note that in the coproduct $F \amalg_S G$, the images of the generators x^m of F and x^n of G do not commute, though both commute with the generator x^{mn} of S. But in an orderable group, if an element x commutes with a nonzero power y^n of another element y, it must commute with y. Hence $F \amalg_S G$ is not orderable.

We shall see below that this, too, is a specific instance of a more general condition.

We shall now formalize the three sorts of necessary conditions for orderability of a coproduct with amalgamation arrived at in the above discussion. For the second condition, generalizing the observation that the same element of S cannot have an nth root in two different G_i, we will want the concept of a *formal S-trap*. This will mean a family X of sentences in group-theoretic symbols $x_1, \ldots, x_m$ and elements of S, each equating a nonvacuous product of conjugates $s^{-1} x_i^{\pm 1} s$ $(s \in S,\ i \leq m)$ to an element of S, such that for every choice of $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$, the family X involves at least one sentence in which each x_i, whenever it appears (if at all), has exponent ε_i. A *solution* to a formal S-trap in an overgroup of S means, of course, an m-tuple of elements of that overgroup satisfying this system of sentences.

PROPOSITION 30. *Let S be an orderable group, and G_i $(i \in I)$ a family of orderable overgroups of S.*
(i) *If the G_i can be mapped homomorphically into a common orderable overgroup H of S by homomorphisms respecting S, then the intersection over I of the set of group orderings on S arising as restrictions of group orderings on G_i is nonempty.*
(ii) *If the G_i can be* embedded *in a common orderable overgroup H of S by maps respecting S, so that the images of any two of these groups, G_i and G_j, intersect only in S, then for every pair of indices $i \neq j$, the diagonal image of S in $G_i \times G_j$ is a dominion subgroup thereof.*

(*Equivalently, if X is a formal S-trap in variables $x_1, \ldots, x_m$, no subset of which is a formal S-trap in a proper nonempty subset of these variables, and if this trap has solutions both in G_i and in G_j, then these solutions must coincide, i.e., must lie in S.*)
(iii) *If the G_i can be embedded by maps respecting S into a common orderable overgroup H so that no element of any $G_i - S$ commutes with any element of any $G_j - S$ $(i \neq j)$, then for every i, the centralizer in S of every element of $G_i - S$ must be a dominion subgroup* (*not merely in S, but*) *in each G_j $(i \neq j)$.*

(*For instance, if an element of $G_i - S$ centralizes an element $s \in S$, then s cannot be a power of an element of $G_j - S$ for $i \neq j$.*)

Hence, the conclusions of (i)-(iii) *are all necessary conditions for the group $\coprod_S G_i$ to be orderable.*

Proof. To see (i), choose an ordering on H. The restriction of this ordering to S is an ordering induced, for each i, by an ordering on the *image* of G_i in H. If this image is an embedding, we are done. If not, we use (20) to lift the ordering in question to an ordering of G_i.

To prove (ii), suppose $\alpha_i\colon G_i \to H$ $(i \in I)$ are embeddings as in the hypothesis of that statement, and suppose $(f, g) \in G_i \times G_j$ lies in the dominion of the diagonal image of S. Consider the two homomorphisms $G_i \times G_j \to H$ gotten by projecting onto G_i, respectively G_j, then applying the homomorphism α_i respectively α_j of this group into H. Since α_i and α_j respect S, the two composites agree on the diagonal image of S in $G_i \times G_j$. Hence they agree on the dominion of this diagonal image, and in particular at (f, g), i.e.,

$$\alpha_i(f) = \alpha_j(g).$$

But by hypothesis, the images of α_i and α_j are disjoint except for the common image of S, hence $(f, g) \in S \times S$, and since α_i and α_j respect the common subgroup S, the above displayed equation becomes $f = g$. Hence (f, g) lies in the diagonal image of S in $G_i \times G_j$, proving that subgroup a dominion. The parenthetical restatement follows from Theorem 28 applied to $G_i \times G_j$, and the observation that a solution to a formal S-trap in that group is equivalent to a solution in G_i and a solution in G_j.

(iii) is straightforward, using the equivalence of conditions (ii) and (iii) of Theorem 21. The parenthetical case follows from Lemma 23 (i). ▮

Actually, condition (iii) can be further generalized. Assume S and the G_i as in the hypothesis of the above Proposition. Then we have

(31) For every element $g \in G_i - S$ $(i \in I)$, let A_g be the partial endomorphism of S which acts as the inner automorphism of G_i induced by g on the subgroup of elements of S carried into S by that automorphism, and is undefined elsewhere. If $\coprod_S G_i$ is orderable, then for any sequence of elements $g_1, \ldots, g_m \in \bigcup G_i - S$, no two successive terms of which come from the same G_i, the fixed subgroup of the composite partial endomorphism $A_{g_1} \ldots A_{g_m}$ of S must be a dominion subgroup in every G_i other than the one containing g_m.

We leave the detailed verification to the interested reader. The idea is that if $m > 1$, this subgroup of S is the centralizer, in each of the indicated G_i, of $g_1 \ldots g_m \in \coprod_S G_i$.

Question 32. *Are the conclusions of Proposition 30* (i), (ii), *and of condition* (31) *above, sufficient to insure that* $\coprod_S G_i$ *is orderable?*

We remark that the converse to Proposition 30 (i) is false, as shown by the example of [**8**]. However, that example fails to satisfy the other two conditions mentioned in Question 32. (Indeed, the two overgroups in that example are constructed by adjoining square roots to an element of the subgroup ‘‘in different ways’’.) If such an example could be found which satisfied these conditions, it would give a counterexample to that Question.

We noted in (18) that every group has a universal orderable homomorphic image. In particular, given a family of orderable overgroups G_i of an orderable group S, the universal orderable homomorphic image of the group $\coprod_S G_i$ will have the universal property of the coproduct, in the category of orderable groups, of the G_i, with amalgamation of S. It may not, of course, be an ‘‘amalgamation’’ in the traditional sense, i.e., the G_i may not be faithfully embedded in this group. However, it might be of interest to look for general results on the structure of this group. Writing it $O\text{-}\coprod_S G_i$, we record one easy result.

Corollary 33 (to Theorem 22). *If* $S \subseteq H$ *are orderable groups,* T *the dominion of* S *in* H *as orderable groups, and* I *a set, then* $O\text{-}\coprod_S^{(I)} H \cong \coprod_T^{(I)} H$. ▮

9. Right-Ordering Coproducts.

In preceding sections we have been concerned with *two-sided* orderability of coproducts; in this last section we turn to one-sided orderability. The definition of a right or left orderable group or semigroup was recalled in (27). We remark that under this definition, a right orderable semigroup will be right cancellative, but not necessarily left cancellative.

It is easy to verify the following analog of (20):

(34) If F is a group and N a normal subgroup, then for any right ordering on F/N and any right ordering on N, there exists a unique right ordering on F making the natural maps $N \to F \to F/N$ isotone.

Note that (34) contains no requirement that the ordering on N be F-invariant. We can now deduce

THEOREM 35. *If S is a group and G_i $(i \in I)$ a family of overgroups of S, then the following conditions are equivalent.*

(i) *The G_i can be embedded in a common right orderable overgroup H by maps respecting S.*

(ii) *The coproduct group $\coprod_S G_i$ is right orderable.*

Proof. Given embeddings in a group H as in (i), we get a homomorphism $\coprod_S G_i \to H$. Let N be its kernel. Since the maps into H are embeddings, N has trivial intersection with each G_i, hence as it is normal, it has trivial intersection with every conjugate of every G_i. Hence N is a free group [**19**, Corollary 4.9.2], hence orderable, hence by (34), $\coprod_S G_i$ is right orderable. The converse is trivial. ▮

So, in contrast to what we found for two-sided orderability, we see that groups such as $\langle x\rangle \coprod_{\langle x^m\rangle} \langle x\rangle$ and $\langle x^m\rangle \coprod_{\langle x^{mn}\rangle} \langle x^n\rangle$ are right orderable.

COROLLARY 36 (See [**10**], [**3**, Theorem 7.3.2]). *A coproduct $\coprod G_i$ (without amalgamation) of right-orderable groups G_i is right-orderable.*

Proof. Apply Theorem 35 with $H = \prod G_i$. ▮

We saw earlier that the subgroups T of an *orderable* group H such that $H \coprod_T H$ was orderable could be characterized as those subgroups convex under some *right* ordering of H. What can we expect to be the corresponding condition on T for $H \coprod_T H$ to be *right* orderable? It is easy to see from Theorem 35 that it is vacuous!

COROLLARY 37. *For every subgroup T of a right orderable group H, $H \coprod_T H$ is right orderable.* (*More generally, every coproduct $\coprod_T^{(I)} H$ is right orderable.*)

Hence, every subgroup T of a right orderable group H is a dominion subgroup of H in the category of right orderable groups. ▮

On the other hand, condition (i) of Theorem 35 is nonvacuous; this is shown by the following example. Let G_1 be the semidirect product of an infinite cyclic group $\langle v\rangle$ with an infinite cyclic group $\langle x\rangle$, in which conjugation by x takes v to v^{-1}. G_1 is right orderable by (34). Let us write u for the element x^2, which is central in G_1, and let S be the subgroup of G_1 generated by u and v. Thus S is free abelian on these two generators, and is of index 2 in G_1.

Now for any integer m, $(xv^m)^2 = x^2$ in G_1. This implies that under any right ordering of that group, xv^m and x have the same ''sign''. Since the set of elements of a given sign is closed under multiplication, we can multiply on the left by x and deduce that uv^m and u have the same sign; in other words, in the ordered abelian subgroup S, we have $|v| << |u|$. (In an abelian group, a right ordering is a 2-sided ordering.)

Regarding G_1 as an extension of S by one generator x subject to the relations

$$x^2 = u, \qquad x^{-1} v x = v^{-1},$$

let us similarly form an extension G_2 of S by adjoining to S an element y satisfying

$$y^{-1}u\,y = u^{-1}, \qquad y^2 = v.$$

By symmetry, G_2 is right orderable, and every right ordering on it satisfies $|u| << |v|$. Hence if the group $G_1 \amalg_S G_2$ (or any common extension of G_1 and G_2 over S) admitted a right ordering, this ordering would satisfy $|u| << |v| << |u|$, a contradiction. (Cf. [**25**, Proposition 3.7.1] and references there. The example of [**11**, bottom of p. 557], though constructed for a slightly different purpose, has similar properties.)

Of the three obstructions that we found to putting a two-sided ordering on a general coproduct of ordered groups with amalgamation, Proposition 30 (i), Proposition 30 (ii), and (31), the last two involved dominions, and hence vanish in the case of right ordered groups. The only kind of obstruction that is apparent is thus that of compatibility of orderings of the subgroup S induced by orderings on the various G_i. This, however, is a more complicated topic in the one-sided case than in the two-sided case, as discussed in [**2**, §3].

It is interesting to compare the above observations with a known result about *locally indicable* groups. These are by definition groups all of whose nontrivial finitely generated subgroups admit nontrivial homomorphisms to the infinite cyclic group; they form a proper subclass of the right orderable groups [**1**]. It is known [**16**, Theorem 9] [**12**, Proposition 3.1] that a coproduct of two locally indicable groups with amalgamation of a *cyclic* subgroup is locally indicable. I do not know whether the corresponding statement is true for right orderable groups; some partial results in this direction are proved in [**2**, §4]. (A powerful generalization of the cited result on locally indicable groups is [**12**, Theorem 4.2], which proves the local indicability of any group obtained by imposing on a coproduct of locally indicable groups a single relation which involves more than one of the factor groups, and is not a proper power.)

Right orderings can also be studied by the ring-theoretic methods of our earlier sections. When applied to groups, this approach gives no improvement on Theorem 35 above. However, it is the only technique I know how to apply to right ordering coproducts of *semigroups*, so I will sketch it.

If k is an ordered ring and G a right-ordered semigroup, let us again define an ordering on the underlying k-module of the group ring kG by calling an element a positive if the coefficient of the greatest element of G having nonzero coefficient in a is positive. This will not in general be a ring ordering – the product of two positive elements of kG need not be positive. However, multiplication on the right by elements of G will still carry positive elements of kG to positive elements, and this is enough to give a version of Lemma 15:

LEMMA 38. *Let G be a right orderable semigroup, and k an ordered ring. Let $U \subseteq M_2(kG[[t]])$ be the multiplicative subsemigroup of elements whose constant terms are diagonal, with members of G as both diagonal entries. Then U is right orderable.* ▮

Combining this with Lemma 7, we get

THEOREM 39. *A coproduct of right orderable cancellative semigroups is right orderable.* ▮

Note added in proof. The example given after Corollary 37 appears in [**31**, Theorem 7].

REFERENCES

1. George M. Bergman, Right orderable groups that are not locally indicable, *to appear*.
2. George M. Bergman, Notes on orderable and right-orderable groups (addenda to ‘‘Ordering coproducts of groups and semigroups’’), unpublished note, available from the author.
3. Roberta Botto Mura and Akbar H. Rhemtulla, *Orderable Groups*, Dekker Lecture Notes in Pure and Applied Math., v. 27, 1977. MR **58** #10652.
4. Joël Lee Brenner, Quelques groupes libres de matrices, *C. R. Acad. Sci. Paris,* **241** (1955) 1689-1691. MR **17**, p. 824.
5. P. M. Cohn, *Universal Algebra*, 2nd ed., Reidel, Dordrecht, 1981. MR **82j**:08001.
6. P. M. Cohn, On copowers of an ordered skew field, *Order,* **1** (1985) 377-382. MR **86j**:16019.
7. P. M. Cohn, On coproducts of ordered systems, *Algebra and Order,* Proc. First Int. Symp. Ordered Algebraic Structures, Luminy-Marseilles 1984, S. Wolfenstein (ed.), Heldermann Verlag Berlin, 1986. MR **89d**:06030.
8. A. M. W. Glass, Dan Saracino and Carol Wood, Non-amalgamation of ordered groups, *Math. Proc. Camb. Phil. Soc.,* **95** (1984) 191-195. MR **86c**:06020.
9. G. Hetyei Jr., On ordering the free product of ordered semigroups, *Annales Univ. Sci. Budapest, R. Eötvös, Sect. Math.,* (to appear).
10. W. Charles Holland and E. B. Scrimger, Free products of lattice ordered groups, *Algebra Universalis,* **2** (1972) 247-254. MR **47** #8384.
11. James Howie, Aspherical and acyclic 2-complexes, *J. London Math. Soc.,* (2) **20** (1979) 549-558. MR **81e**:57004.
12. James Howie, On locally indicable groups, *Math. Zeitschr.,* **180** (1982) 445-451. MR **84b**:20036.
13. John R. Isbell, Epimorphisms and dominions, pp. 232-246 in *Proc. Conf. Categorical Algebra* (*La Jolla, CA, 1965*), Springer-Verlag, 1966. (The statement of the Zigzag Lemma for *rings* in this paper is incorrect; the correct version is given in [**14**].) MR **35** #105a.
14. John R. Isbell, Epimorphisms and dominions, IV, *J. London Math. Soc.,* (2) **1** (1969) 265-273. MR **41** #1774.
15. R. E. Johnson, Free products of ordered semigroups, *Proc. Amer. Math. Soc.,* **19** (1968) 697-700. MR **37** #2864.
16. Abraham Karrass and Donald Solitar, The subgroups of a free product of two groups with an amalgamated subgroup, *Trans. Amer. Math. Soc.,* **150** (1970) 227-255. MR **41** #5499.
17. Ali I. Kokorin and Valerij M. Kopytov, *Linearly Ordered Groups* (Russian), Nauka, Moscow, 1972. Translated as *Fully Ordered Groups,* Halsted Press, NY, 1974. MR **50** #12852 and **51** #306.
18. F. W. Levi, Contributions to the theory of ordered groups, *Proc. Indian Math. Soc.,* **17** (1943) 199-201. MR **5**, p. 58.
19. Wilhelm Magnus, Abraham Karrass and Donald Solitar, *Combinatorial Group Theory: Presentations of groups in terms of generators and relations,* Interscience, 1966. MR **34** #7617.
20. Anatolij I. Mal’cev, *Algebraic Systems* (Russian), Nauka, Moscow, 1972; (English), Springer, 1973. MR **44** #142, **50** #1878.
21. Zbigniew Marciniak, A note on free products of linear groups, *Proc. Amer. Math. Soc.,* **94** (1985) 46-48. MR **86k**:20037.
22. Berhnard H. Neumann, On ordered groups, *Amer. J. Math.,* **71** (1949) 1-18. MR **10**, p. 428.
23. Berhnard H. Neumann, An embedding theorem for algebraic systems, *Proc. London Math. Soc.,* (3) **4** (1954) 138-153. MR **17**, p. 448.
24. V. L. Nisnevič, On groups isomorphically representable by matrices over a commutative field, *Math. Sb.,* NS **8** (1940) 395-403. MR **2**, p. 309.
25. Donald S. Passman, *Infinite Crossed Products,* Academic Press, 1989.
26. A. Robinson, Note on an embedding theorem for algebraic systems, *J. London Math. Soc.,* **30** (1955) 249-252. MR **17**, p. 449.
27. Jean-Pierre Serre, *Arbres, Amalgames,* SL_2, Astérisque, No. 46, Soc. Math. France, 1977; translated as *Trees,* Springer, 1980. MR **57** #16426, **82c**:20083.
28. P. B. Shalen, Linear representations of amalgamated products, *J. Pure and Applied Algebra,* **15** (1979) 187-197. MR **80e**:20011.
29. A. A. Vinogradov, On the free product of ordered groups, *Mat. Sb.,* NS **25** (67) (1949) 163-168. MR **11**, p. 157.
30. B. A. F. Wehrfritz, Generalized free products of linear groups, *Proc. London Math. Soc.,* (3) **27** (1973) 402-424. MR **51** #3322.
31. N. Ya. Medvedev, On quasi varieties of *l*-groups and groups, *Siberian Math. J.,* **26** (1985), 111-117.