Comments, corrections, and related references welcomed, as always!

TEXed February 3, 2023

AN ELEMENTARY RESULT ON INFINITE AND FINITE DIRECT SUMS OF MODULES

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Abstract. Let R be a ring, and consider an R-module M given with two (generally infinite) direct sum decompositions, $A \oplus (\bigoplus_{i\in I} C_i) = M = B \oplus (\bigoplus_{j\in J} D_j)$, such that the submodules A and B, and the D_j, are all finitely generated. We show that there then exist *finite* subsets $I_0 \subseteq I$, $J_0 \subseteq J$, and a direct summand $Y \subseteq \bigoplus_{i\in I_0} C_i$, such that $A \oplus Y \ = \ B \oplus (\bigoplus_{j\in J_0} D_j)$.

We then note some ways that this result can and cannot be generalized, and pose some related questions.

1. The result

Throughout this note, R will be an associative unital ring, and 'module' will mean unital left R-module. An easy observation is

Lemma 1.1 ($\approx$[4] Lemma 2.1, p.33)**.** *Given a module with a direct sum decomposition $S \oplus T$, and a submodule P thereof containing S, one has $P = S \oplus (T \cap P)$.* $\square$

The next proposition will be key to the proof of our main result, Theorem 1.3. I am grateful to Pace Nielsen and Mauricio Medina-Bárcenas for this version of the proposition, stronger than the one in an earlier draft of this note, and leading to a stronger version of Theorem 1.3, which answers a question I had previously posed.

Proposition 1.2 (P. Nielsen, M. Medina-Bárcenas)**.** *Suppose we are given a module with a direct sum decomposition $B \oplus V$, and a submodule which admits a direct sum decomposition $A \oplus X$, such that*

(1.1) $\quad A \ \subseteq \ B \ \subseteq \ A \oplus X \ \subseteq \ B \oplus V$

(*with no assumption on whether $X \subseteq V$*).

Then we have the direct sum decomposition

(1.2) $\quad B \ = \ A \oplus (X \cap B),$

in which the summand $X \cap B$ is also a direct summand in X; precisely:

(1.3) $\quad X \ = \ (X \cap B) \ \oplus \ (X \cap (A \oplus V)).$

Proof. We apply Lemma 1.1 on the one hand to the first three terms of (1.1), and on the other hand to the last three terms, in each case taking the leftmost term for S, the direct summand attached to it in the rightmost term for T, and the middle term for P. The application to the first three terms of (1.1) gives (1.2), while the application to the last three gives

(1.4) $\quad A \oplus X \ = \ B \oplus (V \cap (A \oplus X)).$

Substituting (1.2) into the right-hand side of (1.4) gives

(1.5) $\quad A \oplus X \ = \ A \oplus (X \cap B) \oplus (V \cap (A \oplus X)).$

We now make a third application of Lemma 1.1, this time taking for S the middle summand $X \cap B$ on the right-hand side of (1.5), for T the sum of the other two terms on that side, and for P the module X, which, in view of the left-hand side of (1.5), is contained in the right-hand side, and which clearly contains our chosen S. The result is

(1.6) $\quad X \ = \ (X \cap B) \ \oplus \ ((A \oplus (V \cap (A \oplus X))) \cap X).$

2020 *Mathematics Subject Classification.* Primary: 16D70. Secondary: 08B25, 18A30.

Key words and phrases. isomorphisms among direct sums of modules.

Archived at `http://arxiv.org/abs/2208.06511`; also readable at `http://math.berkeley.edu/~gbergman/papers/`. The latter version may be revised more frequently than the former. After publication, any updates, errata, related references, etc. found will be noted at `http://math.berkeley.edu/~gbergman/papers/abstracts`.

To get (1.3), it will suffice to show that the terms at which (1.3) and (1.6) differ are the same; i.e., that:

(1.7) $\quad X \cap (A \oplus V) \ = \ (A \oplus (V \cap (A \oplus X))) \cap X.$

Here $\supseteq$ is immediate, since the right-hand side can, essentially, be obtained from the left by shrinking V to $V \cap (A \oplus X)$. To get the reverse inclusion, consider any element of the left-hand side, and let us write it

(1.8) $\quad a + v \ = \ x$, where $a \in A,\ v \in V,\ x \in X$.

If we rewrite the above equation as

(1.9) $\quad v \ = \ -a + x$,

we see that v belongs to $V \cap (A \oplus X)$, hence $a + v$ belongs to $A \oplus (V \cap (A \oplus X))$, and since $a + v = x$, this element in fact belongs to $(A \oplus (V \cap (A \oplus X))) \cap X$, as required. □

We can now prove

Theorem 1.3. *Suppose M is a module having two (generally infinite) direct sum decompositions,*

(1.10) $\quad A \oplus (\bigoplus_{i \in I} C_i) \ = \ M \ = \ B \oplus (\bigoplus_{j \in J} D_j)$,

where the submodules A and B and each of the D_j are finitely generated.

Then there exist finite subsets $I_0 \subseteq I$ and $J_0 \subseteq J$, and a direct summand Y in the module $\bigoplus_{i \in I_0} C_i$, such that

(1.11) $\quad A \oplus Y \ = \ B \oplus (\bigoplus_{j \in J_0} D_j)$.

Proof. Since A is finitely generated, it is contained in the sum of a finite subset of the modules on the right-hand side of (1.10), which we may take to include B. That is,

(1.12) $\quad A \ \subseteq \ B \oplus (\bigoplus_{j \in J_0} D_j)$, where $J_0 \subseteq J$ is finite.

Since B and the D_j are all finitely generated, we can now, again in view of (1.10), choose finitely many of the C_i such that

(1.13) $\quad B \oplus (\bigoplus_{j \in J_0} D_j) \ \subseteq \ A \oplus (\bigoplus_{i \in I_0} C_i)$, where $I_0 \subseteq I$ is finite.

Bringing together the inclusions (1.12) and (1.13), and putting $\subseteq B \oplus (\bigoplus_{j \in J} D_j)$ on the far right (in view of (1.10)), we have a chain like (1.1), with $B \oplus (\bigoplus_{j \in J_0} D_j)$ in the role of the B of (1.1), $\bigoplus_{i \in I_0} C_i$ in the role of X, and $\bigoplus_{j \in J \setminus J_0} D_j$ in the role of V; so we can apply Proposition 1.2. The conclusion (1.2) of that proposition (with sides reversed) gives us a decomposition (1.11), with $Y = (\bigoplus_{i \in I_0} C_i) \cap (B \oplus (\bigoplus_{j \in J_0} D_j))$, and the line after (1.2) tells us that this Y is indeed a direct summand in $X = \bigoplus_{i \in I_0} C_i$ (with a complement that can be described using (1.3), though I haven't included that description in the statement of the Theorem). □

Note that if, rather than being given a module M with two internal direct-sum decompositions, we are given an isomorphism i between two external direct sums of families of modules,

(1.14) $\quad A \oplus (\bigoplus_{i \in I} C_i) \ \cong \ B \oplus (\bigoplus_{j \in J} D_j)$,

with A, B and the D_j again all finitely generated, then we can apply the above theorem, taking for M the left-hand side of (1.14), and regarding the inverse images in M of the summands on the right-hand side of that isomorphism as submodules of M which give a second internal direct sum decomposition of M. Theorem 1.3, applied to that pair of decompositions shows that given an isomorphism (1.14), there exist finite subsets $I_0 \subseteq I$ and $J_0 \subseteq J$, and a direct summand Y in the module $\bigoplus_{i \in I_0} C_i$ such that

(1.15) $\quad A \oplus Y \ \cong \ B \oplus (\bigoplus_{j \in J_0} D_j)$.

2. Remarks

Theorem 1.3 was motivated by a question posed to the author by Z. Nazemian; roughly: if Q and P are finitely generated projective modules over a ring R, such that every direct summand of a module P^a for a natural number a is isomorphic to P^b for a natural number b, and if $Q^m \oplus (\bigoplus_{i=1}^{\infty} P) \cong Q^n \oplus (\bigoplus_{i=1}^{\infty} P)$ for some natural numbers m and n, then must $Q^m \oplus P^k \cong Q^n \oplus P^l$ for some natural numbers k and l? I obtained an affirmative answer, and eventually generalized the proof to get a result which, with help from Pace Nielsen, became Theorem 1.3. (Nazemian's question had some further hypotheses which I don't state above. If the answer had been a counterexample, then, of course, it would have been important to find one that satisfied those restrictions; but since a positive result was found, the fewer hypotheses the better.)

I have no cases of Theorem 1.3 in mind in which the C_i are not finitely generated; they are not so assumed because the proof does not need such an assumption.

There are several papers in the literature proving relationships among expressions of a module as finite and/or infinite direct sums; cf. [1], [4], [5], [6]. All of these, however, have strong assumptions either on the base-ring (e.g., in [1], that it is what the authors call a *separative exchange ring*), or on the summand modules (e.g., in [5], that they have semilocal endomorphism rings). I am not aware of any known results with hypotheses as weak as those of Theorem 1.3.

The phrase "An elementary result" in the title of this note is half tongue-in-cheek. Theorem 1.3 is indeed elementary in that it does not call on any deep definitions or results, and its statement and proof are not long. But finding these involved much frustrating trial and error, and I still don't have good intuition about them. In the next section we will note, inter alia, variants of that theorem involving some not-so-elementary hypotheses.

Mauricio Medina-Bárcenas redevelops in [7] the results of the preceding section via results on modular lattices with appropriate completeness conditions.

3. Some ways our result can and cannot be generalized, and some questions

In the finite relation (1.11) that we obtain in Theorem 1.3 from the generally infinite relation (1.10), the summands A, B and D_j $(j \in J_0)$ are modules occurring in that original relation, but Y is a *direct summand* in a finite sum of such modules. Can we get a similar result in which *all* the summands are (at least up to isomorphism) terms from the original relation?

An easy example shows that we cannot. Let R be a field k, let I and J both be countably infinite sets, let each C_i and each D_j be a k-vector-space of dimension 2, and let A and B be finite-dimensional k-vector-spaces, one of odd and the other of even dimension. Note that both sides of (1.14) are then k-vector spaces of countably infinite dimension, so (1.14) holds; and this isomorphism can, as noted, be turned into a relation (1.10). But if we form the direct sum of A with some finite sum of the C_i and D_j, and likewise of B with some finite sum of the C_i and D_j, one of these sums will be odd-dimensional and the other even-dimensional; so no variant of (1.11) or (1.15) without a term like Y can hold.

A couple of people have pointed out to me that in Theorem 1.3, "finitely generated" can be everywhere weakened to the condition called "dually slender" in [3], and "small" in [4, §2.9]. A module A is so called if

(3.1) Every homomorphism of A into an infinite direct sum of modules has image in a finite subsum,

equivalently, if

(3.2) A is not the union of any countable strictly increasing chain of proper submodules, $A_0 \subset A_1 \subset \cdots \subset A_i \subset \ldots$.

(To see this equivalence, first assume (3.1), and note that if we had $A_0 \subset A_1 \subset \ldots$ with union A, the induced map $A \to \bigoplus_{i\in\omega} A/A_i$ would give a contradiction to that assumption. Conversely, assuming (3.2), suppose A had a homomorphism into a direct sum $\bigoplus_I E_i$, which did not land in a finite subsum. Then the induced homomorphism into some countable subsum $\bigoplus_{n\in\omega} E_{i_n}$ $(i_0, i_1, \ldots$ distinct elements of $I)$ would have the same property; and we see that the kernels A_n of the induced homomorphisms $A \to \bigoplus_{i>n} E_i$ $(n \in \omega)$ would contradict (3.2).)

In another direction, it is not hard to see that one can, in the statement and proof of Theorem 1.3, replace modules with *torsion-free* abelian groups, and finitely generated modules with such groups having finite rank (maximal number of linearly independent elements). More generally, one has the obvious analogous result for torsion-free modules over any commutative integral domain.

One might try to embrace all of these versions in a statement about direct sums in an appropriate sort of additive category; note that this would not in general be an abelian category, since the category of torsion-free abelian groups does not have cokernels. Rather than trying, at this time, to give a "definitive" generalization of Theorem 1.3, I hope that readers will be able to call on the proof, or note an appropriate modification thereof, in cases they want to use.

In a different direction, I wonder about

Question 3.1. *Can one get a result like Theorem 1.3, but with direct product modules* $A \times \prod_{i\in I} C_i$ *and* $B \times \prod_{j\in J} D_j$ *in place of the direct sums* $A \oplus \bigoplus_{i\in I} C_i$ *and* $B \oplus \bigoplus_{j\in J} D_j$ *?*

(The replacement of the *binary* operator "$\oplus$" with "$\times$" above is just a formality. It is for the infinite families that direct sums and direct products differ.)

A difficulty is that, unlike the situation for infinite direct sums, a finitely generated submodule of an infinite direct product need not lie in a finite subproduct. (E.g., in an infinite product of 1-dimensional vector spaces over a field, consider the 1-dimensional subspace generated by an element with infinite support.)

A quasi-cure for that problem is to replace the condition of finite generation by that of finitely *cogeneration.* Recall that a module is said to be 'finitely cogenerated' if every family of submodules thereof having zero intersection has a finite subfamily with zero intersection [8]. If a finitely cogenerated module is embedded in a possibly infinite direct product of modules, we see that its projection onto some finite subproduct thereof will indeed be an embedding. Using this, given an isomorphism

$$(3.3) \qquad A \times (\textstyle\prod_{i\in I} C_i) \;=\; M \;\cong\; B \times (\textstyle\prod_{j\in J} D_j)$$

where A, B and the D_i are finitely cogenerated, we can obtain a chain of embeddings

$$(3.4) \qquad A \;\hookrightarrow\; B \times (\textstyle\prod_{j\in J_0} D_j) \;\hookrightarrow\; A \times (\textstyle\prod_{i\in I_0} C_i) \;\hookrightarrow\; B \times (\textstyle\prod_{j\in J} D_j),$$

where I_0 and J_0 are finite subsets of I and $J.$

But so far as I can see, we cannot say in (3.4), as we could in the proof of Theorem 1.3, that the composite of two successive embeddings is the inclusion associated with the indicated direct product decompositions. If we could, then since *finite* direct products of modules are the same as finite direct sums, we would be able to apply Proposition 1.2 as before.

Actually, a formal dual of the approach of Theorem 1.3 should involve a chain of *surjections,* going in the opposite direction to the embeddings of (3.4).

Mauricio Medina-Bárcenas [7] has formulated a result which dualizes in a different way the condition of being a direct sum; whether his result is relevant to Question 3.1 is not clear. In any case, I don't at present see a way to answer Question 3.1.

A final question, about which I have not thought hard, is

Question 3.2. *For what sorts of structures other than modules do analogs of Theorem 1.3 hold?*

One finds that the development of that theorem works for not-necessarily-abelian groups (with 'direct sum' taken to mean the subgroup of the direct product consisting of tuples with only finitely many non-identity terms). I don't know whether there are versions of the result that allow, say, semidirect products in place of direct products. Results are obtained in [2] on direct decompositions of more general algebraic structures, but, as in the other publications cited, a strong hypothesis (in this case called the *exchange property*) is assumed.

4. Acknowledgements

I am indebted to Brian Davey, Jonathan Farley, Laszlo Fuchs, Mauricio Medina-Bárcenas, Pace Nielsen, Kevin O'Meara, and J.-P. Serre for references to related literature, and for other helpful comments on this note.

References

[1] P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998) 105-137. (As of Jan. 26, 2023, on the Springer webpage for the Abstract of [1], all occurrences of the symbol $\succcurlyeq$ should be $\cong$.) MR1639739

[2] Peter Crawley and Bjarni Jónsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math. **14** (1964) 797-855. MR0169806

[3] Paul C. Eklof, K. R. Goodearl and Jan Trlifaj, *Dually slender modules and steady rings,* Forum Math. **9** (1997) 61–74. MR1426454

[4] Alberto Facchini, *Module theory. Endomorphism rings and direct sum decompositions in some classes of modules*, Progress in Mathematics, 167. Birkhäuser Verlag, Basel, 1998. xiv+285 pp. ISBN: 3-7643-5908-0. MR1634015. Reprinted 2012 under Modern Birkhäuser Classics. ISBN: 978-3-0348-0302-1; 978-3-0348-0303-8. MR3025306.

[5] Alberto Facchini and Lawrence S. Levy, *Infinite progenerator sums*, pp. 73–77 of *Algebras, rings and their representations,* World Sci. Publ., Hackensack, NJ, 2006. MR2234301

[6] Irving Kaplansky, *Projective modules*, Ann. of Math. (2) **68** (1958) 372-377. MR0100017

[7] Mauricio Medina-Bárcenas, personal communication.

[8] *Wikipedia,* `https://en.wikipedia.org/wiki/Finitely_generated_module#Equivalent_definitions_and_finitely_cogenerated_modules` .

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