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TeXed April 10, 2026

## ADJOINING UNIVERSAL INVERSES TO FAMILIES OF ELEMENTS OF FREE MONOIDS

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ABSTRACT. Let  $\langle X \rangle$  be the free monoid on a generating set  $X$ , and suppose one adjoins to  $\langle X \rangle$  universal 2-sided inverses to a finite set  $S$  of its elements. We note an elementary algorithm which yields a normal form for elements of the resulting monoid  $M$ .

In particular, either  $M$  will be the free group on  $X$ , or, more generally, the free product as monoids of the free group on a subset  $X_0 \subseteq X$  with the free monoid on  $X \setminus X_0$ , or  $M$  will contain 1-sided invertible elements that are not 2-sided invertible.

If  $S$  is allowed to be infinite, we show that the corresponding normal form still exists, though it cannot necessarily be computed algorithmically.

We note work by others on the related topic of “special monoids”, monoids presented by finitely many generators and finitely many relations of the form  $w = 1$ .

### 1. INTRODUCTION

It is well-known that in a monoid  $M$ , if a product  $xy$  is invertible (has a 2-sided inverse), the product  $yx$  need not be. The bicyclic monoid  $\langle x, y \mid xy = 1 \rangle$  is a standard example. That example likewise shows that if  $xyxy$  is invertible, none of  $yx$ ,  $xyx$ , or  $xyyx$  need be.

On the other hand, if a product  $xyx$  is invertible, we see that  $x$  has both a right and a left inverse, hence it is invertible, hence so is  $y$ , so the submonoid of  $M$  generated by  $x$ ,  $y$  and  $(xyx)^{-1}$  is a group.

What is going on here?

In this note, we study the structure of the monoid  $M$  obtained by starting with the free monoid on a set  $X$ , and for some subset  $S$  of its elements, adjoining universal inverses to the members of  $S$ . In §3 we develop a simple algorithm for obtaining a well-behaved normal form for  $M$  if  $S$  is finite, and in §4 make some further observations on this construction.

If  $S$  is not assumed finite,  $M$  will be the direct limit of the monoids arising in this way from finite subsets  $S' \subseteq S$ , and in §5 we show that  $M$  inherits most of the properties obtained in the finite case – the notable exception being the property that its structure can be computed algorithmically from the set  $S$ .

In §6 we briefly compare these results with results in the literature on monoids presented using families of relations of the form  $w = 1$ .

### 2. GENERAL CONVENTIONS AND OBSERVATIONS

We make here some straightforward conventions and observations, so that we will be able to call on them subsequently without digressing. First, a convention.

(2.1) The words “inverse” and “invertible”, unless modified by “left”, “right” or “one-sided”, will mean two-sided inverse and two-sided invertible.

Now some trivial observations. Let  $M$  be any monoid.

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2020 *Mathematics Subject Classification*. Primary: 20M05. Secondary: 03B25, 08A50.

*Key words and phrases*. free monoid, adjunction of universal inverses, word problem.

Archived at <http://arxiv.org/abs/2510.07617>. Readable at <http://math.berkeley.edu/~gbergman/papers/>. The latter version may be revised more frequently than the former. After publication, any updates, errata, related references, etc., found will be noted at <http://math.berkeley.edu/~gbergman/papers/abstracts>.

(2.2) If an element  $w \in M$  has at least one right inverse and at least one left inverse, then  $w$  is invertible in  $M$ . All one-sided inverses of  $w$  in  $M$  are then equal, their common value being the 2-sided inverse of  $w$ .

(2.3) If an element  $w \in M$  is right invertible in  $M$ , then all left divisors of  $w$  are also right invertible in  $M$ . If an element  $w \in M$  is left invertible in  $M$ , then all right divisors of  $w$  are left invertible in  $M$ .

On the other hand, a *right* divisor of a right invertible element need not, in general, be right invertible, and analogously for a left divisor of a left invertible element. E.g., in the monoid  $\langle x, y, z \mid xyz = 1 \rangle$ ,  $y$  is a right divisor of the right-invertible element  $xy$ , but is not right-invertible, and is a left divisor of the left-invertible element  $yz$  but not left-invertible. However, a special case, where those implications are easily seen to hold is:

(2.4) If a right invertible element  $w$  has a factorization  $w = uv$ , where  $u$  is invertible, then  $v$  is also right invertible. Likewise, if a left invertible element  $w$  has a factorization  $w = vu$  with  $u$  invertible, then  $v$  is also left invertible.

To see the right invertibility case, one conjugates an equation showing right invertibility,  $uvw = 1$ , by  $u$ , and analogously for the left invertibility case.

Another notational convention:

(2.5) If  $w$  is an element of a monoid  $M$ , and we universally adjoin to  $M$  an inverse to  $w$ , we will denote that inverse  $\mathbf{i}(w)$  and the resulting monoid  $M\langle\mathbf{i}(w)\rangle$ . Similarly, if we adjoin inverses to all elements in a subset  $S \subseteq M$ , we shall use the notation  $M\langle\mathbf{i}(S)\rangle$ , or, if  $S = \{w_1, \dots, w_n\}$ ,  $M\langle\mathbf{i}(w_1), \dots, \mathbf{i}(w_n)\rangle$ .

Hence to “universally adjoin an inverse to  $w$ ” means to adjoin a free generator  $z$ , and then impose the relations  $zw = 1$ ,  $wz = 1$ . (It would feel more natural to write  $w^{-1}$  rather than  $\mathbf{i}(w)$  for the adjoined inverse; but the notation  $w^{-1}$  applies to all cases where an element  $w$  has an inverse, and I want a notation specific to an inverse universally adjoined to an element not assumed to have one.)

(2.6) In the context of (2.5) we will generally use the same symbols for elements of  $M$  and for their images in  $M\langle\mathbf{i}(S)\rangle$  (even though elements that are distinct in  $M$  may fall together in  $M\langle\mathbf{i}(S)\rangle$ ).

For an example where such elements fall together, let  $M = \langle x, y \mid xy = 1 \rangle$ . Then  $yx$  and  $1$  are distinct in  $M$ , but fall together in  $M\langle\mathbf{i}(x)\rangle$ .

Next, some observations requiring a bit more thought.

**Lemma 2.7.** *Let  $M$  be a monoid, and  $S, S'$  subsets of  $M$ . Then the following conditions are equivalent:*

(i) *In  $M\langle\mathbf{i}(S)\rangle$ , all elements of  $S'$  become invertible, and in  $M\langle\mathbf{i}(S')\rangle$ , all elements of  $S$  become invertible.*

(ii) *There exist homomorphisms  $f : M\langle\mathbf{i}(S)\rangle \rightarrow M\langle\mathbf{i}(S')\rangle$  and  $g : M\langle\mathbf{i}(S')\rangle \rightarrow M\langle\mathbf{i}(S)\rangle$  respecting the natural homomorphisms of  $M$  into these monoids.*

*In this situation, the  $f$  and  $g$  of (ii) are in fact inverse to one another, and give the unique isomorphism between  $M\langle\mathbf{i}(S)\rangle$  and  $M\langle\mathbf{i}(S')\rangle$  respecting the homomorphisms of  $M$  into those monoids.*

*Proof.* (i) implies (ii) by the universal properties of  $M\langle\mathbf{i}(S)\rangle$  and  $M\langle\mathbf{i}(S')\rangle$ . The reverse implication is trivial. The final statement is easily deduced from the uniqueness of inverses of invertible elements of a monoid.  $\square$

**Definition 2.8.** *If  $M$  is a monoid, we shall call two subsets  $S$  and  $S'$  of  $M$  inverse-equivalent if they satisfy the equivalent conditions of Lemma 2.7.*

The following lemmas describe in the context of a general monoid  $M$  two tools that we will apply in the next section to free monoids.

**Lemma 2.9.** *Let  $M$  be a monoid and  $S$  a subset of  $M$ . If  $S$  contains two elements  $u$  and  $v$  and their product  $uv$ , then  $S$  is inverse-equivalent to the subset one gets by deleting from  $S$  any one of these three elements, as long as it is distinct from both of the others (though those two others need not be distinct from each other).*

*Proof.* It will suffice to show that for the given  $S$ , condition (i) of Lemma 2.7 holds for  $S'$  the set obtained by deleting from  $S$  the element in question. Clearly every element of  $S'$  will be invertible in  $M\langle\mathbf{i}(S)\rangle$ ; it remains to show that the element we have deleted from  $S$  in forming  $S'$  remains invertible in  $M\langle\mathbf{i}(S')\rangle$ . If that element is  $w$ , this is true because a product of invertible elements is invertible. If it is  $u$ , it will again be the product in  $M\langle\mathbf{i}(S')\rangle$  of invertible elements, namely  $u = (uv)\mathbf{i}(v)$ , and if it is  $v$ , it will likewise be the product  $v = \mathbf{i}(u)(uv)$ .  $\square$

**Lemma 2.10.** *Let  $M$  be a monoid,  $S$  a subset of  $M$ , and  $w$  any element of  $M$ . Then if, in  $M$ ,  $w$  is a left divisor of some  $u \in S$  and is also a right divisor of some  $v \in S$ , then  $S \cup \{w\}$  is inverse-equivalent to  $S$ .*

*Proof.* The one non-obvious property we need is that the image of  $w$  in  $M\langle\mathbf{i}(S)\rangle$  is invertible. By (2.3), that image is right and left invertible in  $M\langle\mathbf{i}(S)\rangle$ , so by (2.2), it is invertible there.  $\square$

### 3. MONOIDS OBTAINED BY ADJOINING TO A FREE MONOID INVERSES TO FINITELY MANY ELEMENTS

We turn, at last, to the case of extensions of free monoids.

The usual notation in the literature for the free monoid on a set  $X$  is  $X^*$ , and combining this with the notation  $M\langle\dots\rangle$  of the preceding section, we would get expressions  $X^*\langle\dots\rangle$ . But let us use a slightly simpler notation:

**Definition 3.1.** *For any set  $X$ , the free monoid on  $X$  will be denoted  $\langle X \rangle$ . The results of adjoining to  $\langle X \rangle$  universal inverses to a list  $w_1, \dots, w_n$ , respectively to a set  $S$ , of elements of  $\langle X \rangle$  will be written  $\langle X : \mathbf{i}(w_1), \dots, \mathbf{i}(w_n) \rangle$ , respectively  $\langle X : \mathbf{i}(S) \rangle$ .*

*We shall say that two nonidentity elements of  $\langle X \rangle$  overlap if one of them can be written  $ab$  and the other  $bc$ , where  $b \neq 1$ , and at most one of  $a$  and  $c$  equals 1. (Note that if  $a = 1$ , then  $ab = b$ , a proper left substring of  $bc$ , while if  $c = 1$ ,  $bc = b$ , a proper right substring of  $ab$ .) If neither  $a$  nor  $c$  equals 1, we shall say that  $ab$  and  $bc$  overlap noninclusively.*

*We shall call a subset  $T \subseteq \langle X \rangle \setminus \{1\}$  overlap-free if no pair of elements of  $T$  (distinct or not) overlap.*

(The point of the phrase “distinct or not” is to make clear that an overlap-free set is not allowed to contain self-overlapping elements, such as  $xyx$  or  $xx$ .)

**Proposition 3.2.** *Let  $X$  be a set, and  $S$  any finite subset of the free monoid  $\langle X \rangle$ . Then there is a finite overlap-free subset  $T$  of  $\langle X \rangle \setminus \{1\}$  inverse-equivalent to  $S$  (Definition 2.8).*

*Such a  $T$  can be constructed algorithmically from  $S$ .*

*Proof.* If  $1 \in S$ , then deleting that element clearly will not change the inverse-equivalence class of  $S$ , so we may assume without loss of generality that  $S \subseteq \langle X \rangle \setminus \{1\}$ .

We will define below a function associating to every finite subset  $U$  of  $\langle X \rangle \setminus \{1\}$  a nonnegative integer “weight”,  $\text{wt}(U)$ , and show that if  $U$  has two elements which overlap, we can obtain a set  $V$  inverse-equivalent to  $U$ , but having lower weight. Thus, starting with  $S$ , this process must terminate in a set  $T$  inverse-equivalent to  $S$  which is overlap-free.

For any  $w \in \langle X \rangle$ , let  $\text{lgth}(w)$  denote its length as a word in the elements of  $X$ , and let

$$(3.3) \quad \text{wt}(U) = \sum_{w \in U} (\text{lgth}(w) - 1).$$

Suppose first that  $U$  contains two distinct elements, one of which left-divides the other:  $u$  and  $uv$ . Then by Lemma 2.9,  $U \cup \{v\}$  and  $U$  are inverse-equivalent, while by another application of that Lemma,  $U \cup \{v\}$  is inverse-equivalent to  $U \cup \{v\} \setminus \{uv\}$ ; so  $U$  is inverse-equivalent to  $U \cup \{v\} \setminus \{uv\}$ . Now adjoining  $\{v\}$  increased the weight of  $U$  by at most  $\text{lgth}(v) - 1$  (“at most” because  $v$  might already have belonged to  $U$ , in which case bringing it in left  $U$  unchanged); and deleting  $uv$  decreased that weight by the larger value  $\text{lgth}(uv) - 1$ . So the combined changes decrease the weight.

Similarly, if  $U$  contains two distinct elements one of which right-divides the other, then the left-right dual of the above construction decreases its weight.

Finally, if  $U$  contains elements  $ab$  and  $bc$  (not necessarily distinct) which overlap noninclusively, then by Lemma 2.10,  $U \cup \{b\}$  is inverse-equivalent to  $U$ , and after adjoining  $b$  to  $U$ , we can then, as above, replace  $ab$  by  $a$ . These steps together show that  $U$  is inverse-equivalent to  $(U \setminus \{ab\}) \cup \{a, b\}$ ; and again, this decreases the weight of  $U$  by at least 1. (Why was the last step above just to replace  $ab$  by  $a$ , and

not simultaneously replace  $bc$  by  $c$ ? That works nicely if  $ab$  and  $bc$  are distinct, but if they are equal, it requires us to consider various subcases. Deleting just one of these products avoids this complication. If removing  $ab$  still leaves  $bc$  in our set, then at a subsequent step we can handle this as in the above “one left-divides the other” case.)

Since the weight function is nonnegative-integer valued, the above process must terminate after finitely many steps, giving an overlap-free  $T$  which is inverse-equivalent to  $S$ .  $\square$

The above construction involves repeated choices of which elements to add to and remove from  $U$ , and so does not show  $T$  to be uniquely determined by  $S$ . But we will be able to prove that uniqueness after getting some further results. These intermediate results, which will also be used in §5, will not be restricted to finite sets  $T$ .

The next result makes use of a tool for establishing normal forms, the Diamond Lemma [2]. Though the title of [2] refers to ring theory, §9.1 thereof notes the version of the result for monoids, there called “semigroups”. (A similar procedure for establishing normal form results, often used in semigroup and monoid theory, is called “Knuth-Bendix reduction” [4, §12.2]. As discussed at [2, p. 179], the term “Diamond Lemma” goes back to a graph-theoretic result of M. H. A. Newman [9].)

**Proposition 3.4.** *Let  $X$  be a set, and  $T$  an overlap-free subset of  $\langle X \rangle \setminus \{1\}$ . Then every element of  $\langle X : \mathbf{i}(T) \rangle$  has a unique expression as the product of a (possibly empty) string of elements of  $X \cup \mathbf{i}(T)$  in which no element  $\mathbf{i}(w)$  ( $w \in T$ ) is immediately preceded or immediately followed by the corresponding string  $w$ .*

*Proof.* That every element of  $\langle X : \mathbf{i}(T) \rangle$  can be written as such a product is immediate even without the assumption that  $T$  is overlap-free. Indeed, by definition of  $\langle X : \mathbf{i}(T) \rangle$ , every element thereof is a product of elements of  $X \cup \mathbf{i}(T)$ . If such a product contains an  $\mathbf{i}(w)$  immediately preceded or followed by the string  $w$ , then the product obtained by dropping the substring  $w\mathbf{i}(w)$  or  $\mathbf{i}(w)w$  represents the same element, and is shorter. Repeating such rewritings, we eventually get an expression of the desired sort.

To show that if  $T$  is overlap-free, such representations are unique, we use the Diamond Lemma [2]. Given  $X$  and  $T$ , the relations defining  $\langle X : \mathbf{i}(T) \rangle$  in terms of the generating set  $X \cup \mathbf{i}(T)$  can be written in the notation of [2] as reductions

$$(3.5) \quad (w\mathbf{i}(w), 1) \text{ and } (\mathbf{i}(w)w, 1) \text{ for all } w \in T.$$

Note that each generator of the form  $\mathbf{i}(w)$  ( $w \in T$ ) appears at the end of just one of the  $2 \text{ card}(T)$  reducible strings shown in (3.5), and at the beginning of just one such string, and nowhere else. Hence the only overlaps among such strings in which the overlapping substrings involve an  $\mathbf{i}(w)$  correspond to the words  $w\mathbf{i}(w)w$ , and it is clear that the two ways of reducing such a word give the same value,  $w$ . So in the language of [2], the ambiguities whose overlapping portions involve an  $\mathbf{i}(w)$  are *resolvable*.

What about overlaps involving the strings  $w$ ? By assumption, no two members of  $T$ , distinct or not, overlap; this leaves only the case where the right-hand  $w$  of a product  $\mathbf{i}(w)w$  is equal to the left-hand  $w$  of the product  $w\mathbf{i}(w)$ , and the overlapping subword is in the whole string  $w$ . In that case, the ambiguously reducible product is  $\mathbf{i}(w)w\mathbf{i}(w)$ , and both reductions yield  $\mathbf{i}(w)$ .

So all ambiguities of our reduction-system are resolvable, hence the result of [2] for monoids shows that each element of  $\langle X : \mathbf{i}(T) \rangle$  has a unique reduced expression under the set of reduction (3.5), i.e., a unique expression of the asserted form.  $\square$

We next want to prove that the *invertible* elements of  $\langle X : \mathbf{i}(T) \rangle$  are just the obvious ones, and so, in particular, obtain a simple description of which elements of the original monoid  $\langle X \rangle$  become invertible in  $\langle X : \mathbf{i}(T) \rangle$ . Curiously, the route to these results requires us to first determine which elements become one-sided invertible.

**Proposition 3.6.** *Let  $X$  be a set, and  $T$  an overlap-free subset of  $\langle X \rangle \setminus \{1\}$ . Then the right-invertible elements of  $\langle X : \mathbf{i}(T) \rangle$  are precisely the elements which, when written in the normal form of Proposition 3.4, consist of products in any order (including the empty product 1) of elements of  $\mathbf{i}(T)$  and nonempty initial substrings (not necessarily proper) of the expressions in  $\langle X \rangle$  for elements of  $T$ .*

*In particular, the elements of the free monoid  $\langle X \rangle$  that become right-invertible in  $\langle X : \mathbf{i}(T) \rangle$  are the products of nonempty initial substrings of the expressions in  $\langle X \rangle$  for members of  $T$ .*

The elements of  $\langle X : \mathbf{i}(T) \rangle$  and  $\langle X \rangle$  that are left-invertible in  $\langle X : \mathbf{i}(T) \rangle$  are described by the left-right duals of the above characterizations.

*Proof.* By symmetry it suffices to prove the characterization of right-invertible elements.

So suppose  $u, v \in \langle X : \mathbf{i}(T) \rangle$  satisfy  $uv = 1$ . Assume inductively that all right-invertible words shorter than  $u$  (i.e., having shorter normal forms in terms of the generating set  $X \cup \mathbf{i}(T)$ ) are products of the asserted form.

Write  $u$  and  $v$  in normal form. If they are not both 1, then the string  $uv$  must not be in normal form, so the reducibility of  $uv$  must result from some nonempty final substring of  $u$ , when adjoined to some nonempty initial substring of  $v$ , giving an expression  $w \mathbf{i}(w)$  or  $\mathbf{i}(w)w$  for some  $w \in T$ . (If both of these situations occurred simultaneously, i.e., if when the expressions for  $u$  and  $v$  were joined, the result contained a substring  $\mathbf{i}(w)w \mathbf{i}(w)$  or  $w \mathbf{i}(w)w$ , one could arbitrarily use one of the substrings  $\mathbf{i}(w)w$  or  $w \mathbf{i}(w)$  thereof in the argument to follow. But in fact, as we shall note after the proof, this cannot happen.)

If a substring  $w \mathbf{i}(w)$  arises in this way, that means the expressions in  $\langle X \rangle$  for  $u$  and  $v$  have the forms

$$(3.7) \quad u = u' w_1, \quad v = w_2 \mathbf{i}(w) v', \quad \text{where } w_1 w_2 = w \text{ and the string } w_1 \text{ is nonempty.}$$

Now  $u'$ , as a left divisor of the right-invertible element  $u$ , is right invertible (by (2.3)), so by our inductive hypothesis,  $u'$  is a product of the asserted form. Hence, since  $w_1$  is an initial substring of  $w \in T$ , we see that  $u = u' w_1$  is also a product of the asserted form.

On the other hand, if the substring that makes  $uv$  reducible has the form  $\mathbf{i}(w)w$ , then the normal forms for  $u$  and  $v$  must have the forms  $u' \mathbf{i}(w) w_1$  and  $w_2 v'$ , where again  $w_1 w_2 = w$ , making  $w_1$  an initial substring (possibly empty) of a member of  $T$ . Again,  $u'$ , as a left divisor of a right-invertible element, is right invertible, so again applying our inductive hypothesis to  $u'$ , we get the desired conclusion for  $u = u' \mathbf{i}(w) w_1$ .

The assertion beginning ‘‘In particular’’ clearly follows.  $\square$

We remarked parenthetically that the situation where two elements  $u, v$  with  $uv = 1$  have normal forms that, when put together, contain a string  $\mathbf{i}(w)w \mathbf{i}(w)$  or  $w \mathbf{i}(w)w$  cannot actually occur. Indeed, if  $u = u' \mathbf{i}(w) w_1$  and  $v = w_2 \mathbf{i}(w) v'$  where  $w_1 w_2 = w$ , then the product  $u' \mathbf{i}(w) w_1 w_2 \mathbf{i}(w) v'$  would reduce to  $u' \mathbf{i}(w) v'$ . For this to reduce to 1, it would in particular have to be reducible, so either  $u'$  would have to end in  $w$ , or  $v'$  would have to begin with  $w$ ; but these properties would contradict the assumption that  $u$ , respectively,  $v$ , was in normal form. The case where putting together the normal forms of  $u$  and  $v$  gave an expression containing  $w \mathbf{i}(w)w$  is still simpler: if the indicated occurrence of  $\mathbf{i}(w)$  came from the expression for  $u$ , then that expression would not have been in normal form, and dually if it came from the expression for  $v$ .

We remark that if an element of a monoid  $\langle X : \mathbf{i}(T) \rangle$  has a 1-sided inverse, that 1-sided inverse need not be unique. E.g., in  $\langle x, y, z : \mathbf{i}(xy), \mathbf{i}(xz) \rangle$ , the element  $x$  has distinct right inverses with normal forms  $y \mathbf{i}(xy)$  and  $z \mathbf{i}(xz)$ .

Back to general results. We can now characterize 2-sided invertible elements:

**Proposition 3.8.** *Let  $X$  be a set, and  $T$  an overlap-free subset of  $\langle X \rangle \setminus \{1\}$ . Then the invertible elements of  $\langle X : \mathbf{i}(T) \rangle$  are precisely the elements which can be written as products of members of  $T \cup \mathbf{i}(T)$ . The normal form (as in Proposition 3.4) of each such element can be written uniquely as such a product which contains no substring  $w \mathbf{i}(w)$  or  $\mathbf{i}(w)w$  ( $w \in T$ ).*

*In particular, the elements of  $\langle X \rangle$  that are invertible in  $\langle X : \mathbf{i}(T) \rangle$  are precisely the products of elements of  $T$ , and each has a unique expression as such a product.*

*Thus, the monoid of all elements of  $\langle X \rangle$  with inverses in  $\langle X : \mathbf{i}(T) \rangle$  is the free monoid on  $T$ , and the group of all invertible elements of  $\langle X : \mathbf{i}(T) \rangle$  is the free group on  $T$ .*

*Proof.* We shall first show that every invertible element has an expression as in the first paragraph above, then that such expressions are unique. These results clearly imply the remaining assertions.

Let  $u$  be an invertible element of  $\langle X : \mathbf{i}(T) \rangle$ . If  $u = 1$ , then  $u$  is the empty product of members of  $T \cup \mathbf{i}(T)$ , so let  $u \neq 1$ , and assume inductively that all invertible elements having shorter normal forms than that of  $u$  are products of elements of  $T \cup \mathbf{i}(T)$ .

Since  $u$  is right invertible, Proposition 3.6 says that it can be written  $u = u_1 \dots u_n$  where all  $u_i$  are either nonempty left substrings of elements of  $T$  (hence right invertible) or members of  $\mathbf{i}(T)$  (hence invertible); and since  $u$  is also left invertible, that result likewise tells us that it can be written  $u = u'_1 \dots u'_n$ , where all  $u'_i$  are either nonempty right substrings of elements of  $T$  or members of  $\mathbf{i}(T)$ .

Let us first consider the case where  $n = n' = 1$ . Then  $u$  is either a member of  $\mathbf{i}(T)$ , or is both a left substring of a member of  $T$  and a right substring of a member of  $T$ . The former case clearly satisfies our desired conclusion, while in the latter, the condition that  $T$  be overlap-free tells us that  $u \in T$ , so that conclusion again holds.

It remains to consider the case where  $n$  and  $n'$  are not both 1. By left-right symmetry, it suffices to assume  $n > 1$ , so that  $u$  is a product  $u_1 \dots u_n$  of  $n > 1$  elements  $u_i$ , each of which is either a member of  $\mathbf{i}(T)$ , or a nontrivial left factor of a member of  $T$ .

In this case, note that the final factor  $u_n$  is by assumption right invertible; but being a right factor of the invertible element  $u$ , it is also left invertible. Thus,  $u_n$  is invertible; hence since  $u$  is also invertible, both of the factors in the decomposition  $u = (u_1 \dots u_{n-1}) u_n$  are invertible elements of shorter normal form than  $u$ . Hence by our inductive assumption, they are both products of members of  $T \cup \mathbf{i}(T)$ ; hence so is  $u$ , as desired.

As for the uniqueness of the decomposition as a product of members of  $T \cup \mathbf{i}(T)$ , suppose the normal form for  $u$  has two such decompositions,  $u = u_1 \dots u_n = u'_1 \dots u'_{n'}$ , and assume inductively that for invertible elements with shorter normal forms, the desired uniqueness holds. If the normal form for  $u$  begins with an element of  $\mathbf{i}(T)$ , then each of our decompositions must have that as its first factor, and the result of deleting that factor gives  $u_2 \dots u_n = u'_2 \dots u'_{n'}$ , so our inductive assumption implies the desired conclusion. If the normal form for  $u$  does not begin with an element of  $\mathbf{i}(T)$ , then the initial substrings  $u_1$  and  $u'_1$  will both be members of  $T$ , one of which is a left substring of the other. But since  $T$  is assumed overlap-free, no member of  $T$  can be a proper left substring of another, so we again get  $u_1 = u'_1$ , and conclude as before that the decompositions are the same.  $\square$

We can now, at last, get the uniqueness of the  $T$  constructed from an arbitrary finite  $S$  in Proposition 3.2. Indeed, Proposition 3.8 clearly implies

**Proposition 3.9.** *Let  $X$  be a set, and  $T$  an overlap-free subset of  $\langle X \rangle \setminus \{1\}$ . Then  $T$  is precisely the set of nonidentity elements of  $\langle X \rangle$  which are invertible in  $\langle X : \mathbf{i}(T) \rangle$ , but have no factorization into two nonidentity elements of  $\langle X \rangle$  that are invertible in  $\langle X : \mathbf{i}(T) \rangle$ .*

*Hence if  $S$  is any finite subset of  $\langle X \rangle$ , and  $T$  is an overlap-free subset inverse-equivalent to  $S$  (shown to exist by Proposition 3.2), then  $T$  is uniquely determined by  $S$ : it is the set of nonidentity elements of  $\langle X \rangle$  that are invertible in  $\langle X : \mathbf{i}(S) \rangle$  but have no factorization into two nonidentity elements of  $\langle X \rangle$  invertible therein.  $\square$*

Note, incidentally, that although the meaning given to ‘‘overlap’’ in Definition 3.1 includes the case where one of the two words is an initial or final subword of the other, it does not include the case where one is merely an *internal* subword of the other. For instance, for  $x, y, z \in X$ ,  $xyz$  and  $y$  do not count as overlapping words; so  $\langle x, y, z : \mathbf{i}(xyz), \mathbf{i}(y) \rangle$  is a presentation  $\langle X : \mathbf{i}(T) \rangle$  with overlap-free  $T$ . Hence, in this monoid,  $z\mathbf{i}(xyz)x \neq \mathbf{i}(y)$ , though these two elements fall together if multiplied on the left by  $xy$ , or on the right by  $yz$ , or on application of the natural homomorphism to  $\langle x, y, z : \mathbf{i}(x), \mathbf{i}(y), \mathbf{i}(z) \rangle$ .

#### 4. SOME OBSERVATIONS

What does the monoid  $\langle X : \mathbf{i}(T) \rangle$  of Proposition 3.8 look like?

Clearly, if  $T = X$ , it will be the free group on  $X$ . If, more generally,  $T$  is a subset of  $X$ , then it is easy to see that  $\langle X : \mathbf{i}(T) \rangle$  will be the free product as monoids of the free group on  $T$  and the free monoid on  $X \setminus T$ .

On the other hand, if the overlap-free set  $T$  is not contained in  $X$ , i.e., if it has at least one member  $w$  of length  $> 1$ , then writing such an element as a product  $w = w_1 w_2$  of nonempty words, we see that  $w_1$  will be right invertible but not left invertible, with right inverse  $w_2 \mathbf{i}(w)$ . Writing  $x = w_1$  and  $y = w_2 \mathbf{i}(w)$ , these elements will generate a bicyclic monoid  $\langle x, y \mid xy = 1 \rangle$ , since the relation  $xy = 1$  allows us to reduce every member of the monoid generated by  $x$  and  $y$  to the form  $y^m x^n$  ( $m, n \geq 0$ ), and from Proposition 3.4 and our assumptions on  $w$ ,  $w_1$  and  $w_2$ , we can see that no two of these elements are equal.

Such monoids  $\langle X : \mathbf{i}(T) \rangle$  will contain nonidentity idempotent elements, since in a bicyclic monoid  $\langle x, y \mid xy = 1 \rangle$ ,  $yx$  is such an element. (However, I suspect that if  $w$  is any element of a monoid  $\langle X : \mathbf{i}(S) \rangle$  which is *not* idempotent, then all the nonnegative integer powers of  $w$  are distinct.)

The construction in Proposition 3.2 of an overlap-free set  $T$  inverse-equivalent to a given finite set  $S$  tends to *increase* the number of elements in the set. On the other hand, if one is not concerned with overlap-freeness, one can easily *decrease* that number to 1:

**Lemma 4.1.** *Let  $X$  be a set, and  $S$  a finite subset of  $\langle X \rangle$ . Then there is an element  $w \in \langle X \rangle$  such that the singleton  $\{w\}$  is inverse-equivalent to  $S$ .*

*Proof.* If  $S$  is empty, we can take  $w = 1$ , and if  $S$  is a singleton  $\{w_1\}$  we can take  $w = w_1$ , so assume  $S = \{w_1, \dots, w_n\}$  with  $n > 1$ , and let

$$(4.2) \quad w = w_1 \dots w_{n-1} w_n w_{n-1} \dots w_1.$$

We shall use a technique from the proof of Proposition 3.2 to show  $\{w\}$  inverse-equivalent to  $S$ .

Note that  $w$  overlaps itself noninclusively via the initial and final factors  $w_1$ . Hence as in the next-to-last paragraph of the proof of Proposition 3.2, we can get an inverse-equivalent set by replacing the singleton  $\{w\} = \{w_1 \dots w_{n-1} w_n w_{n-1} \dots w_1\}$  with the 2-element set  $\{w_1, w_2 \dots w_{n-1} w_n w_{n-1} \dots w_1\}$ , which we can in turn replace with  $\{w_1, w_2 \dots w_{n-1} w_n w_{n-1} \dots w_2\}$ . Iterating this process, we eventually get  $\{w_1, \dots, w_n\}$ , i.e.,  $S$ . (Since we did not assume  $S$  overlap-free, there may be overlaps among the  $w_i$ , which in the proof of Proposition 3.2 we would have gotten rid of by further steps. But our goal here isn't to get an overlap-free  $T$ , so we only carry out the steps indicated above.)  $\square$

I will end this section with a curious observation not directly related to the subject of this paper. It seems unlikely that it is not known, but I have not been able to find it in the literature.

**Lemma 4.3.** *If two elements  $x$  and  $y$  of a monoid  $M$  satisfy  $xy = 1$  but  $yx \neq 1$ , then*

$$(4.4) \quad \text{the map } M \rightarrow M \text{ given by } z \mapsto yzx$$

*is a one-to-one endomorphism of  $M$  as a semigroup, but is not a monoid homomorphism.*

*The corresponding statement holds for associative unital rings.*

*Proof.* It is immediate that (4.4) respects multiplication (and in the ring case, addition), but does not send 1 to 1. Moreover, if  $yzx = yz'x$ , then multiplying by  $x$  on the left and  $y$  on the right, we get  $z = z'$ , so the map is one-to-one.  $\square$

## 5. ADJOINING INVERSES TO NOT NECESSARILY FINITE FAMILIES OF ELEMENTS

We begin this section with a consequence of Proposition 3.8 in a context far more general than that of §3:

**Proposition 5.1.** *Let  $X$  be a set, and  $f$  a homomorphism from  $\langle X \rangle$  to an arbitrary monoid  $M$ . Then the set*

$$(5.2) \quad N = \{w \in \langle X \rangle \mid f(w) \text{ is invertible in } M\}$$

*is a free submonoid of  $\langle X \rangle$ , whose (unique) free generating set is an overlap-free set  $T \subseteq \langle X \rangle \setminus \{1\}$ .*

*Conversely, every overlap-free subset  $T \subseteq \langle X \rangle \setminus \{1\}$  arises in this way from such a homomorphism; e.g., by the first assertion of Proposition 3.9, from the natural map  $\langle X \rangle \rightarrow \langle X : \mathbf{i}(T) \rangle$ .*

*Proof.* Let us first observe that for any submonoid  $N$  of a free monoid  $\langle X \rangle$ , if we let

$$(5.3) \quad T = \{w \in N \setminus \{1\} \mid w \text{ is not a product of two nonidentity elements of } N\},$$

then  $T$  generates  $N$ . (Given  $w \in N \setminus \{1\}$ , assume inductively that all elements of  $N$  of shorter length than  $w$  belong to  $\langle T \rangle$ . Then if  $w$  cannot be factored in  $N$ , we have  $w \in T$ , while if it can, our inductive hypothesis implies that  $w \in \langle T \rangle$ .)

Now assume  $X$ ,  $f$ ,  $M$  and  $N$  as in the statement of the Proposition, and define  $T$  by (5.3). If two elements of  $T$  overlap, write them  $ab$  and  $bc$ , with  $b \neq 1$  and  $a$  and  $c$  not both equal to 1. The image of  $b$  in  $M$  is both right and left invertible, hence invertible, hence so are the images of  $a$  and  $c$ , so if  $a \neq 1$  we get a contradiction to the assumption that  $ab$  belongs to (5.3), and dually if  $c \neq 1$ ; so  $T$  is indeed overlap-free, i.e., it satisfies the first sentence of Proposition 3.8. Hence the middle paragraph of that proposition shows that the submonoid of  $\langle X \rangle$  generated by  $T$ , which we have seen is  $N$ , is free on  $T$ .

The final converse statement is clear.  $\square$

Note that despite the nice characterization of the submonoids  $N$  that arise as in (5.2), very little can be said about  $f(N)$  in such situations. The map  $f$  need not be one-to-one on  $T$ , and even if it is, it may not be one-to-one on  $N$ , and even if that holds, so that the submonoid  $f(N) \subseteq M$  is free on  $f(T)$ , the subgroup of  $M$  that it generates may not be free. For an example of the last sort, let  $f$  map the free monoid  $\langle x, y \rangle$  into the group  $G$  of maps  $t \mapsto at + b$  ( $a \in \mathbb{R} \setminus \{0\}$ ,  $b \in \mathbb{R}$ ) of the real line into itself, by sending  $x$  to  $t \mapsto t/2$  and  $y$  to  $t \mapsto (t+1)/2$ . Looking at the behavior of these maps on the unit interval  $[0, 1]$ , one can show that  $f$  is one-to-one; but since the commutator subgroup of  $G$  is abelian, the induced homomorphism from the free group  $\langle x, y : \mathbf{i}(x), \mathbf{i}(y) \rangle$  to  $G$  cannot be one-to-one.

But in the case of Proposition 5.1 where  $M$  has the form  $\langle X : \mathbf{i}(S) \rangle$ , we can recover essentially everything we proved for finite  $S$  in §3, except the assertion that  $T$  can be computed algorithmically from  $S$ :

**Proposition 5.4.** *Let  $X$  be a set, and  $S$  a subset of  $\langle X \rangle$ .*

*Then there is a unique overlap-free subset  $T \subseteq \langle X \rangle \setminus \{1\}$  which is inverse-equivalent to  $S$ . An element  $w \in \langle X \rangle \setminus \{1\}$  belongs to  $T$  if and only if it satisfies the following equivalent conditions:*

(i)  *$w$  is invertible in  $\langle X : \mathbf{i}(S) \rangle$ , but is not a product of two elements of  $\langle X \rangle \setminus \{1\}$  that are invertible in  $\langle X : \mathbf{i}(S) \rangle$ .*

(ii)  *$w$  is invertible in  $\langle X : \mathbf{i}(S) \rangle$ , but does not have a proper left divisor in  $\langle X \rangle \setminus \{1\}$  that is left invertible in  $\langle X : \mathbf{i}(S) \rangle$ .*

(ii')  *$w$  is invertible in  $\langle X : \mathbf{i}(S) \rangle$ , but does not have a proper right divisor in  $\langle X \rangle \setminus \{1\}$  that is right invertible in  $\langle X : \mathbf{i}(S) \rangle$ .*

(iii) *There exists a finite subset  $S'$  of  $S$  such that for all finite  $S''$  with  $S' \subseteq S'' \subseteq S$ ,  $w$  belongs to the overlap-free set  $T''$  inverse-equivalent to  $S''$  (whose existence and uniqueness were shown in Propositions 3.2 and 3.9).*

*Proof.* The first assertion of Proposition 5.1, applied to the case  $M = \langle X : \mathbf{i}(S) \rangle$ , shows that the elements  $w \in \langle X \rangle \setminus \{1\}$  that satisfy (i) comprise the unique overlap-free subset of  $\langle X \rangle \setminus \{1\}$  that is inverse-equivalent to  $S$ . It remains to show the equivalence of (i) with the other conditions listed. Since (i) and (iii) are left-right symmetric, while (ii') is the left-right dual of (ii), it suffices to show (i), (ii) and (iii) equivalent.

We get (i)  $\implies$  (ii) by contradiction: If an element  $w \in \langle X \rangle \setminus \{1\}$  is invertible in  $\langle X : \mathbf{i}(S) \rangle$  and has a factorization  $w = uv$  in  $\langle X \rangle \setminus \{1\}$  with  $u$  left invertible in  $\langle X : \mathbf{i}(S) \rangle$ , then  $u$  is also right invertible there, since it is a left divisor of the invertible element  $w$ ; hence being both left and right invertible, it is invertible there, hence  $v = u^{-1}w$  is also invertible there, so the factorization  $w = uv$  contradicts (i).

(ii)  $\implies$  (i) is immediate, since a failure of (i) is clearly a failure of (ii).

We shall prove (i) equivalent to (iii) using the observation that  $\langle X : \mathbf{i}(S) \rangle$  is the direct limit of the monoids  $\langle X : \mathbf{i}(S') \rangle$  as  $S'$  ranges over the finite subsets of  $S$ .

Assuming  $w$  satisfies (iii), choose  $S'$  as in (iii), and let  $T'$  be the overlap-free set inverse-equivalent to  $S'$  given by Proposition 3.2. Then  $w \in T'$  is invertible in  $\langle X : \mathbf{i}(T') \rangle \cong \langle X : \mathbf{i}(S') \rangle$ , hence also invertible in  $\langle X : \mathbf{i}(S) \rangle$ , proving the first condition of (i). Next, suppose by way of contradiction that  $w$  could be written as a product  $w = uv$  where  $u, v \in \langle X \rangle \setminus \{1\}$  both become invertible in  $\langle X : \mathbf{i}(S) \rangle$ . Then taking  $S'$  as above, and letting  $S_u$  and  $S_v$  be finite subsets of  $S$  such that  $u$  becomes invertible in  $\langle X : \mathbf{i}(S_u) \rangle$  and  $v$  in  $\langle X : \mathbf{i}(S_v) \rangle$ , we see on the one hand that for  $S'' = S' \cup S_u \cup S_v$ , (iii) tells us that  $w$  belongs to the overlap-free set  $T''$ . On the other hand it has the factorization  $w = uv$  with the factors  $u$  and  $v$  invertible in  $\langle X : \mathbf{i}(T'') \rangle$ , a contradiction. So (iii)  $\implies$  (i).

The reverse implication is quicker: Assuming (i),  $w$  is invertible in  $\langle X : \mathbf{i}(S) \rangle$  hence in  $\langle X : \mathbf{i}(S') \rangle$  for some finite  $S' \subseteq S$ ; and if the final part of (iii) failed, this would mean that  $w$  had a nontrivial factorization in  $\langle X \rangle$  into factors invertible in some  $\langle X : \mathbf{i}(S'') \rangle$ , hence invertible in  $\langle X : \mathbf{i}(S) \rangle$ , contradicting (i).  $\square$

We remark that though the assumption that  $S$  was finite was used in Proposition 3.2 and the second paragraph of Proposition 3.9, we never assumed the generating set  $X$  finite; and it would not have simplified things if we had, since for finite  $X$ ,  $\langle X \rangle - \{1\}$  can still contain an infinite overlap-free subset  $T$ . E.g., for  $X = \{x, y, z\}$ , the set  $T = \{xy^i z \mid i \geq 0\}$  is overlap-free. And in fact, if we take for  $T$  the subset of the latter set gotten by letting  $i$  range over an algorithmically undecidable subset of the natural numbers, we

get a non-algorithmically-decidable  $T$ . (With a little thought one can see that the same observations hold with  $X = \{x, y\}$ , and elements  $xy^iz$  replaced by  $x^iy^i$ .)

In another direction, I have not investigated either of

**Question 5.5.** *Suppose  $X$  is a set and  $S_1$  a subset of  $\langle X \rangle - \{1\}$ , and we then take a subset  $S_2 \subseteq \langle X : \mathbf{i}(S_1) \rangle$ , and adjoin to  $\langle X : \mathbf{i}(S_1) \rangle$  universal inverses to all elements of  $S_2$ , getting a monoid that we might denote  $\langle X : \mathbf{i}(S_1) : \mathbf{i}(S_2) \rangle$ .*

*Can we get, for such a monoid, structure results of a similar nature to those obtained above?*

*More generally, can we get such results for monoids  $\langle X : \mathbf{i}(S_1) : \dots : \mathbf{i}(S_n) \rangle$  obtained by iterating this construction in the obvious way?*

**Question 5.6.** *Suppose  $X$  is a set, and  $S_l, S_r$  are subsets of  $\langle X \rangle - \{1\}$ . Let  $\langle X : \mathbf{i}_l(S_l), \mathbf{i}_r(S_r) \rangle$  denote the monoid obtained by adjoining a universal left inverse to each element of  $S_l$ , and a universal right inverse to each element of  $S_r$ .*

*Or more generally, since 1-sided inverses are not in general unique, suppose  $c_l$  and  $c_r$  are cardinal-valued functions on  $S_l$  and  $S_r$ , and we let  $\langle X : \mathbf{i}_l(S_l, c_l), \mathbf{i}_r(S_r, c_r) \rangle$  be the monoid obtained by adjoining to each  $w \in S_l$  a universal  $c_l(w)$ -tuple of left inverses, and to each  $w \in S_r$  a universal  $c_r(w)$ -tuple of right inverses.*

*Can one obtain nice structure results for such monoids? (And, again, perhaps, for monoids arising by iteration of these constructions?)*

## 6. COMPARISON WITH SPECIAL MONOIDS

There has been considerable research on what are called “special monoids”: monoids presented by a finite set of generators and a finite family of relations of the form  $w_i = 1$ . A striking result of S. Adjan [1], summarized in [3, p.1876, statement (A1), and Theorem 1.1 and following paragraph] is that any special monoid presented by just *one* such relation has solvable word problem.

The class of such one-relator monoids includes all monoids of the form  $\langle X : \mathbf{i}(S) \rangle$  with  $X$  and  $S$  finite, for given a monoid of the latter sort, we have seen in Lemma 4.1 that there exists a word  $w \in \langle X \rangle$  such that  $\langle X : \mathbf{i}(w) \rangle \cong \langle X : \mathbf{i}(S) \rangle$ ; and adjoining one more generator  $z$  to  $X$ , the monoid  $\langle X : \mathbf{i}(w) \rangle$  has the one-relator special monoid presentation

$$(6.1) \quad \langle X \cup \{z\} \mid wzw = 1 \rangle.$$

Indeed, the relation  $wzw = 1$  makes  $w$  a left and right factor of an invertible element, hence invertible, hence  $z = w^{-2}$ , so  $z$  can be dropped from the generators if we include in the presentation the inverse  $\mathbf{i}(w)$  to  $w$ .

Thus, monoids  $\langle X : \mathbf{i}(S) \rangle$  with  $X$  and  $S$  finite are equivalent to one-relator special monoids where the relation has the form  $wzw = 1$  for a word  $w$  not involving the generator  $z$ . In particular, the solvability of the word problem for one-relator special monoids gives an alternative proof of the solvability of the word problem for monoids  $\langle X : \mathbf{i}(S) \rangle$  obtained in §3.

The referee has pointed out that the final statement of Proposition 3.8, that the group of invertible elements of  $\langle X : \mathbf{i}(T) \rangle$  is free, can likewise be deduced from results on one-relator monoids. Namely, if we apply the result from [1] summarized in [3, Theorem 4.1] to the monoid (6.1), it tells us that there is a factorization of  $wzw$  into a product, possibly with repetitions, of subwords  $u_1, \dots, u_k$ , which generate the group of invertible elements of the monoid, and such that, writing  $wzw = u_{i_1} \dots u_{i_n}$ , that group is presented by those  $k$  generators and the one relation

$$(6.2) \quad u_{i_1} \dots u_{i_n} = 1.$$

But since a  $u_i$  involving  $z$  can occur only once in the decomposition of  $wzw$ , the relation (6.2) can be looked at as expressing that generator  $u_i$  as a monoid word in the inverses of the other generators, from which one deduces that the group in question is free on those other generators. The referee also notes that the algorithm used in the proof of Proposition 3.2 above to obtain the overlap-free set  $T$  from  $S$  is similar to what one would get on applying to (6.1) the algorithm introduced by Adjan [1] and described in [3, pp.1892-1893].

Special monoids defined using more than one relation  $w_i = 1$  are much less well-behaved. Indeed, it is proved in [8] that whether such a monoid is a group is undecidable. Moreover, if such a monoid *is* a group, it may have undecidable word problem, since every finitely presented group can be finitely presented as a special monoid, and there are known to be finitely presented groups  $G$  with undecidable word problem.

The former undecidability result answered a question I was wondering about. Suppose  $X$  is a set and  $\{w_1, \dots, w_n\}$  a finite subset of  $\langle X \rangle$ . In view of the natural homomorphism  $\langle X : \mathbf{i}(w_1), \dots, \mathbf{i}(w_n) \rangle \rightarrow \langle X \mid w_1 = 1, \dots, w_n = 1 \rangle$ , we see that

- (6.3) The set of elements of  $\langle X \rangle$  that become invertible in  $\langle X : \mathbf{i}(w_1), \dots, \mathbf{i}(w_n) \rangle$  is contained in the set that become invertible in  $\langle X \mid w_1 = 1, \dots, w_n = 1 \rangle$ .

Is that inclusion always an equality? The answer must be no, since we can see from the results of §3 that it is algorithmically decidable whether all elements of  $\langle X \rangle$  become invertible in  $\langle X : \mathbf{i}(w_1), \dots, \mathbf{i}(w_n) \rangle$ ; hence if the inclusion of (6.3) were an equality, it would be decidable whether all elements of  $\langle X \rangle$  become invertible in  $\langle X \mid w_1 = 1, \dots, w_n = 1 \rangle$ , i.e., whether that monoid is a group, contradicting the abovementioned result from [8]. (However, the referee has pointed out a much simpler example, not involving decidability, that likewise answers my original question. Simplifying and strengthening that example a bit, note that in  $\langle x, y, z : \mathbf{i}(x), \mathbf{i}(yz) \rangle$ , the last assertion of Proposition 3.2 shows that none of the elements  $yx^nz$  ( $n > 0$ ) are invertible; but clearly, in  $\langle x, y, z : x = 1, yz = 1 \rangle$  they all are.)

Some of the many works on special monoids are [1], [6], [7], [8], [12], [13], [14]. In particular, [14, §5] gives simplified proofs of results in the earlier literature, such as that the word problem for a finitely presented special monoid is decidable if and only if the word problem for its group of units is decidable. There is also much information about special monoids in [3], [5] and [10] (with some minor errata to [10] noted at [11]), though the main subject of [3] is *inverse monoids* (monoids  $M$  such that for every  $x \in M$  there is a unique  $x' \in M$  such that  $xx'x = x$  and  $x'xx' = x'$ , and where for all  $x$  and  $y$ , the elements  $xx'$  and  $yy'$  commute), and the main subject of [5] and [10] is monoids presented by an arbitrary single relation  $w_1 = w_2$ .

## 7. ACKNOWLEDGEMENTS

I am indebted to Benjamin Steinberg for informing me that it is undecidable whether a finitely presented special monoid is a group, to Robert Gray for further information on what is known about such monoids, and for other helpful comments on earlier drafts of this note, and to the referee for many useful suggestions.

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