Comments, corrections, and related references welcomed, as always!

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SOME RESULTS ON COUNTING LINEARIZATIONS OF POSETS

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ABSTRACT. In §1 we consider a 3-tuple $S = (|S|, \lessdot, E)$ where $|S|$ is a finite set, $\lessdot$ a partial ordering on $|S|$, and $E$ a set of unordered pairs of distinct members of $|S|$, and study, as a function of $n \geq 0$, the number of maps $\varphi: |S| \to \{1, \ldots, n\}$ which are both isotone with respect to the ordering $\lessdot$, and have the property that $\varphi(x) \neq \varphi(y)$ whenever $\{x, y\} \in E$. We obtain results on this function, which we use in an appendix to give a combinatorial proof of a ring-theoretic identity of G. P. Hochschild.

In §2 we generalize a result of R. Stanley on the sign-imbalance of posets in which the lengths of all maximal chains have the same parity.

In §3-§5 we study the linearization-count and sign-imbalance of a lexicographic sum of posets. In particular, we obtain results on these numbers for a lexicographic sum of chains over a fixed poset, as functions of the lengths of those chains.

This material is far from my areas of expertise, and I am not sure which, if any, of the results obtained are worth publishing. Advice would be welcome!

1. Denominators of order-chromatic polynomials

In this section, for $S$ as in the first paragraph of the abstract, we show that the function described there is a polynomial over the rational numbers, and obtain a bound on the primes dividing its denominator. That bound will be used in an appendix, §6 (which depends only on this section) to recover a ring-theoretic result of Gerhard Hochschild.

Some background first. Recall that the chromatic polynomial of a finite graph $G$ is the function $f$ associating to every positive integer $n$ the number of colorings of the vertices of $G$ with $n$ colors (not all of which need be used), so that adjacent vertices get different colors. (We recall below why this function is given by a polynomial.)

Richard Stanley notes in [4] that if $P = (|P|, \lessdot)$ is a finite poset (partially ordered set), then the function $e$ associating to every positive integer $n$ the number of isotone maps $\varphi: P \to \{1, \ldots, n\}$ (i.e., maps $\varphi: |P| \to \{1, \ldots, n\}$ such that $x \lessdot y \implies \varphi(x) \leq \varphi(y)$) is also given by a polynomial, conceptually similar to the chromatic polynomial of a graph, and that the same is true of the function $\tau$ associating to $n$ the number of strictly isotone maps $P \to \{1, \ldots, n\}$ (maps such that for distinct $x, y$, one has $x \lessdot y \implies \varphi(x) < \varphi(y)$).

To see the above statement for the function $e$, let us break the process of choosing an isotone map $\varphi: P \to \{1, \ldots, n\}$ into three steps. First, decide which elements will fall together under $\varphi$. Calling the quotient poset arising from these identifications $P'$, choose, next, the linear order which the embedded image of $|P'|$ in $\{1, \ldots, n\}$ is to have. Finally, choose a way of embedding the resulting linearly ordered set into $\{1, \ldots, n\}$. Now if we write

\begin{equation}
(1) \quad m = \text{card}(P),
\end{equation}

then the number of elements of $P'$ will be some $m' \leq m$, and once a linear ordering of $|P'|$ has been chosen, the number of embeddings of the resulting ordered set in $\{1, \ldots, n\}$ will be the number of ways of choosing $m'$ out of $n$ elements, which is the binomial coefficient $\binom{n}{m'} = n(n - 1) \ldots (n - m' + 1)/m'!$. This is a polynomial of degree $m'$ in $n$ over the rational numbers, and summing the polynomials arising in this way from all linearly ordered sets obtained as in the first two steps above, we get a polynomial of degree $m$.
Clearly, the coefficients of this polynomial can be written to the common denominator \( m! \), so every prime dividing the least common denominator of those coefficients is \( \leq m \). But we can get a better bound on such primes if the Hasse diagram of \( P \) has more than one connected component.

To do so, let us modify the procedure described. The first step, choosing which elements of \(|P|\) fall together, remains the same; again let \( P' \) denote the set so obtained, given with the weakest partial ordering making the map \( P \to P' \) isotone. (Here we are assuming that \(|P'|\) is a set-theoretic image of \(|P|\) which admits a partial ordering making the map \( |P| \to |P'| \) isotone. The weakest such partial ordering is then the intersection of all such partial orderings.) At the second step, rather than strengthening that ordering to a linear ordering on \(|P'|\), let us strengthen it to an ordering which is linear on each connected component of the Hasse diagram of \( P' \), but which leaves elements of distinct connected components incomparable. Finally, we choose a one-to-one isotone map of the resulting poset into \( \{1, \ldots, n\} \).

How many choices are there at that last step? If, under the ordering chosen at the second step, the connected components of \( P' \) are chains \( P'_1, \ldots, P'_k \), of cardinalities \( m'_1, \ldots, m'_k \) respectively, then there are \( \binom{n}{m'_1} \) isotone embeddings of \( P'_1 \) in \( \{1, \ldots, n\} \), then \( \binom{n-m'_1}{m'_2} \) such embeddings of \( P'_2 \) in the remaining elements of \( \{1, \ldots, n\} \), and so on. So the total number of isotone embeddings is \( \binom{n}{m'_1} \binom{n-m'_1}{m'_2} \cdots \binom{n-m'_1-\ldots-m'_{k-1}}{m'_k} \), which, writing \( m'_1 + \cdots + m'_k = m' = \text{card}(|P'|) \), is the multinomial coefficient

\[
\binom{n(n-1)\ldots(n-m'+1)/m!\ldots m_k!}{m'!}.
\]

This is a polynomial in \( n \) whose denominator is divisible only by primes that are less than or equal to one of the \( m'_i \), so we need to know how large the \( m'_i \) the cardinalities of the components of \( P' \), can be.

Suppose the original poset \( P \) had \( c \) connected components. Each identification of two elements of \( P \) reduces the number of connected components by at most one, hence if \( P' \) has \( n' \) elements, and so has undergone \( m - m' \) identifications, it still has at least \( c - (m - m') \) components. The largest that such a component can be is \( m' \) minus the number of other components, so each component has at most \( m' - (c - (m - m') - 1) = m - c + 1 \) elements.

Thus, \( m - c + 1 \) is an upper bound on the primes that can occur in the denominator of (2). Summing over all choices of our image \( P' \) of \( P' \), and all componentwise linearizations thereof, we conclude that the polynomial giving the total number of isotone maps \( P \to \{1, \ldots, n\} \) has denominator divisible only by primes less than or equal to \( m - c + 1 \).

As noted in the abstract, we actually want to count the smaller set of such maps \( \varphi : P \to \{1, \ldots, n\} \) that satisfy an additional set of restrictions on which pairs of elements of \( P \) can fall together under \( \varphi \). But such restrictions do not affect the above argument; they limit the set of images \( P' \) that we enumerate at the first step, while our bound on primes in the denominator comes from the last step. So we have

**Theorem 1.** Let \( S \) be a 3-tuple \((|S|, \preceq, E)\), where \(|S| \) is a finite set, \( \preceq \) is a partial ordering on \(|S|\), and \( E \) is a set of unordered pairs of distinct members of \(|S|\). Let \( m \) be the cardinality of \(|S|\), and \( c \) the number of connected components of the Hasse diagram of the poset \((|S|, \preceq)\), and for each \( n \geq 0 \), let \( C(S, n) \) be the number of maps \( \varphi : |S| \to \{1, \ldots, n\} \) which are both isotone with respect to the ordering \( \preceq \), and have the property that \( \varphi(x) \neq \varphi(y) \) whenever \( \{x, y\} \in E \).

Then \( C(S, n) \) is a polynomial in \( n \) with rational coefficients, such that all primes dividing the denominators of those coefficients are \( \leq \text{card}(|S|) - c + 1 \). \( \square \)

Note that in the case of the above result where \( \preceq \) is the trivial ordering, \( C(S, n) \) is the chromatic polynomial of the graph \(|S|, E)\). In that case \( c = \text{card}(|S|) \), so the theorem says \( C(S, n) \) is a polynomial with integer coefficients; which is indeed true of chromatic polynomials [7, section IX.2].

Theorem 1 actually remains true if we allow arbitrary Boolean conditions on which elements fall together. For instance, for particular \( x_1, x_2, y_1, y_2 \in |S| \), the condition, “if \( x_1 \) falls together with \( y_1 \), then \( x_2 \) falls together with \( y_2 \)” would cause no difficulties with the proof. I have assumed the restrictions to have the simple form stated in the theorem because such conditions seem to occur naturally; in particular, the problem that motivated this result, discussed in §6, has conditions of that form. The result for arbitrary Boolean expressions in such conditions is easily deduced from Theorem 1 by inclusion-exclusion considerations.

On the other hand, Theorem 1 does not remain true if we allow restrictions involving the order relations among elements \( \varphi(x) \); for instance, if for some particular \( x, y \in |S| \) we consider only maps \( \varphi \) such that \( \varphi(x) \leq \varphi(y) \). Imposing that is equivalent to replacing \( P \) by a poset with its order relation \( \preceq \) strengthened;
and this can fuse distinct connected components without reducing the number of elements. So, for instance, if \((|S|, \preceq)\) is an \(m\)-element antichain, then an obvious set of conditions of the above form restricts us to maps \(|S| \to \{1, \ldots, n\}\) which are isotone with respect to a particular total ordering \(\preceq'\) of \(|S|\), of which there are \(\binom{n+m-1}{m}\). This is a polynomial with denominator \(m!\), despite the fact that the original ordering \(\preceq\) satisfies \(m - c + 1 = 1\).

The following consequence of Theorem 1 will be used in §6.

**Corollary 2.** Let \(S = (|S|, \preceq, E)\) be as in Theorem 1, and \(p\) be any prime greater than \(\text{card}(|S|) - c + 1\).

Then for every positive integer \(n\) whose residue modulo \(p\) is less than the chromatic number of the graph \((|S|, E)\), the integer \(C(S, n)\) is divisible by \(p\). In particular, if \(|S|\) is nonempty, then \(C(S, n)\) is divisible by \(p\) whenever \(n\) is divisible by \(p\).

**Proof.** By Theorem 1 we may write \(C(S, n) = f(n)/r\), where \(f\) is a polynomial with integer coefficients, and \(r\) an integer not divisible by \(p\). The residue of \(f(n)\) modulo \(p\) depends only on the residue of \(n\) modulo \(p\), hence as \(r\) is invertible modulo \(p\), the same is true of the residue of \(C(S, n) = f(n)/r\). Now for positive \(n\) less than the chromatic number of \((|S|, E)\), there can be no maps of \(|S|\) into \(\{1, \ldots, n\}\) which distinguish \(x\) and \(y\) whenever \(\{x, y\} \in E\), in particular, no isotone maps with that property; hence \(C(S, n) = 0\) for such \(n\). Hence for \(n\) congruent to a value \(n_0\) in that range, we have \(C(S, n) \equiv (\text{mod } p) C(S, n_0) = 0\).

The final sentence follows, since a nonempty graph has chromatic number greater than zero. \(\square\)

The remaining sections of this paper, other than the appendix §6, are related to this one only in that they involve counting linear images of posets, and use somewhat similar methods.

A belated remark on notation: At the end of the second paragraph of this section, I spoke of the condition that “for distinct \(x, y\), one has \(x \preceq y \implies \varphi(x) < \varphi(y)\),” where it would be more natural to say that for all \(x\) and \(y\) one has \(x \prec y \implies \varphi(x) < \varphi(y)\). But I am avoiding the use of “\(\prec\)” for strict \(\preceq\)-inequality because many authors use \(\prec\) for the condition that \(y\) covers \(x\) with respect to a partial order \(\preceq\), i.e., that \(x < y\) and there is no element \(z\) with \(x < z < y\). At a few places below, we will indeed consider the covering relation with respect to a partial ordering \(\preceq\). Though we will not introduce any notation for it, I felt it best to avoid confusion with a notation commonly used by others.

## 2. **The Sign-Imbalance of a Bicolorable Poset**

If \(X\) is a set of \(n < \infty\) elements, then there are precisely \(n!\) bijections \(X \to \{1, \ldots, n\}\), and given one such bijection \(b\), we can write any other as \(b \pi\) for a unique permutation \(\pi\) of \(\{1, \ldots, n\}\). In this situation, one calls a bijection \(X \to \{1, \ldots, n\}\) “even” or “odd” (relative to \(b\)), or equivalently, of sign \(+1\) or \(-1\), according to whether \(\pi\) is an even or an odd permutation. Clearly, replacing \(b\) with another bijection \(X \to \{1, \ldots, n\}\) either preserves or reverses this classification of all such bijections, depending on whether \(b\) is changed by an even or an odd permutation.

If \(P = (|P|, \preceq)\) is an \(n\)-element poset, then the isotone bijections \(|P| \to \{1, \ldots, n\}\) correspond naturally to linearizations of \(|P|\) (total orderings refining \(\preceq\)), and one can ask how many of these are odd and how many are even. The number of linearizations that are even minus the number that are odd is called the sign-imbalance of the poset \(P\) (see [5] and papers referenced there). If that number is zero, \(P\) is called sign-balanced. The sign-imbalance is, of course, unique only up to sign unless a particular ordering \(b\) relative to which it is calculated has been specified.

In studying the sign-imbalance of a finite poset \(P\) in this section, our method will again be to break the choice of a linearization of \(|P|\) into steps, such that for each partial ordering achieved at the next-to-last step, the options available at the final step are easy to study.

Since only one-to-one images of \(|P|\) are considered, we have no use for a graph structure \(E\) on \(|P|\) restricting elements that fall together. We will, however, make use of the graph structure of the Hasse diagram of \(P\). Let us define a bicoloring of \(P = (|P|, \preceq)\) to be a coloring of its Hasse graph with two colors; equivalently, a partition of \(|P|\) into two subsets such that whenever one element of \(|P|\) covers another, the elements have different colors. (Thus, \(P\) admits a bicoloring if and only if every cycle in its Hasse diagram has an even number of vertices.)

If \(P = (|P|, \preceq)\) is a finite poset on which we have a bicoloring, and \(\preceq\) is some total orderings refining the given partial ordering \(\preceq\), let us break \(|P|\) into the “blocks” of elements of the same color that are consecutive under \(\preceq\). That is, let \(|P|_1\) be the subset of \(|P|\) consisting of the least element under \(\preceq\), and
all elements of the same color (if any) that follow it under $\leq$ without an intermediate element of the opposite color; let $|P|_2$ consist of the first element of the other color, and those that similarly follow it, $|P|_3$ the next block (which has the same color as $|P|_1$), and so on. Note that within each $|P|_i$, all elements are incomparable under $\preceq$, since by the definition of a bicoloring, any two comparable elements of the same color have an element of the opposite color between them under $\preceq$, and hence under $\leq$.

In the above situation, we can define an intermediate partial ordering $\leq_{\text{pre}}$ on $|P|$, under which members of each $|P|_i$ remain incomparable, while whenever $x \in |P|_i$ and $y \in |P|_j$ with $i < j$, we let $x \leq_{\text{pre}} y$. We shall call the partial ordering $\leq_{\text{pre}}$ the “prelinearization” of $\preceq$ determined by the linearization $\leq$.

Let us call linearizations $\leq$ and $\leq'$ of $\preceq$ “block-equivalent” if the prelinearizations $\leq_{\text{pre}}$ and $\leq'_{\text{pre}}$ are the same. Given a prelinearization of $P$, with block decomposition $|P| = |P|_1 \cup |P|_2 \cup \cdots \cup |P|_m$, the block-equivalence-class of linearizations of $\preceq$ that contains it will have exactly $\text{card}(|P|_1)! \text{card}(|P|_2)! \cdots \text{card}(|P|_m)!$ elements, every such linearization being obtained by choosing an arbitrary linearization of each $|P|_i$. The sign of such a linearization will be, up to a constant factor $\pm 1$, the product of the signs of the linearizations of these subsets.

Now if any $|P|_i$ has cardinality greater than 1, then the $\text{card}(|P|_i)!$ choices for the ordering of that block will be equally split between odd and even. Hence in the summation of $+1$’s and $-1$’s that gives the sign-imbalance of $P$, it suffices to look at linearizations in which each same-color block is a singleton; in other words, linearizations for which the given coloring of $P$ is still a bicoloring. Let us call such a linearization compatible with the given bicoloring. Then the above discussion yields

**Theorem 3.** If $P = (|P|, \preceq)$ is a finite bicolored poset, then the sign-imbalance of the set of all linearizations of $P$ (relative to any fixed indexing of $|P|$) is equal to the sign-imbalance (relative to the same indexing) of the set of linearizations compatible with the given bicoloring.

In particular, if $P$ has no linearizations compatible with the given bicoloring, then it is sign-balanced. \(\square\)

Easy examples of bicolored posets $P$ with no compatible linearizations are those in which the difference between the numbers of elements of the two colors is greater than 1. If a poset $P$ has only one connected component, then a bicoloring of $P$, if one exists, is unique up to interchange of colors; an example in which this bicoloring has the property just mentioned is $\bigwedge_n$, with three elements of one color and just one of the other. So that poset is sign-balanced.

If a bicolorable poset has more than one connected component, we have a larger choice of bicolorings. For instance, if $P$ consists of two connected components, each of which is a chain with an odd number of elements, and we bicolor those chains so that their least elements have the same color, then we find that the difference between the total numbers of elements of the two colors is 2: so this poset, too, is sign-balanced. However, if we color the two chains so that their least elements have opposite colors, then the same number of elements have each color, so though we have just seen that $P$ is sign-balanced, this second bicoloring does not give a proof of that fact.

Examples of non-sign-balanced posets are $\bullet \downarrow$ and $\downarrow \downarrow$, as one can quickly check by counting.

Theorem 3 allows us to give an alternative proof of the following result of R. Stanley. (The formulation of that result in [5] refers to the length of chains, i.e., the number of steps, which is one less than the cardinality of the chain, in terms of which I state it here.)

**Lemma 4 (R. Stanley [5, Corollary 2.2]).** If a finite poset $P = (|P|, \preceq)$ has the property that the cardinalities of its maximal chains all have the same parity, and this is the opposite of the parity of the cardinality of $P$, then $P$ is sign-balanced.

**Proof.** The fact that the cardinalities of all maximal chains of $P$ have the same parity implies in particular that for each $x \in |P|$, all maximal chains in the downset generated by $x$ have a common parity. (Otherwise, by combining two such chains of different parities with a maximal chain in the upset generated by $x$, one would get two maximal chains in $P$ of different parities.) Classifying elements $x$ according to whether maximal chains below them all have odd or even cardinality, we get a bicoloring of $P$, under which all minimal elements have a common color. From this we see that if the common parity of the cardinalities of all maximal chains of $P$ is odd, the maximal elements of $P$ will all be of the same color as the minimal elements, while if it is even, they will all have the opposite color.
Suppose now that \( P \) has a linearization compatible with its bicoloring. Because the parity of \( P \) is the opposite of that of its maximal chains, the relationship between the colors of the elements at the bottom and top of this linearization (same-color versus different-color) will be the opposite of what is the case for maximal chains of \( P \).

On the other hand, the bottom element of the linearization must be the least element of one of those maximal chains, and hence must be of the color which all those bottom elements have, and the top element must similarly be the color of the top elements of all the maximal chains, so, in contradiction to the preceding paragraph, the relationship between the colors of elements at the top and bottom of this linearization must be the same as for maximal chains of \( P \). This contradiction shows that no compatible linearization of \( P \) exists; so by Theorem 3, \( P \) is sign-balanced.

An easy example to which the above lemma applies is that in which \( P \) has two connected components, each a chain of odd cardinality (for which we got the same conclusion directly from Theorem 3). Two more easy examples are the posets \( \oplus \) and \( \bigotimes \).

3. Bringing sign-imbalance and linearization-count together

If \( P \) is a finite poset, let \( L_0(P) \) denote the number of even linearizations of \( P \), and \( L_1(P) \) the number of odd linearizations (relative to some fixed linearization). Then it is natural to associate to \( P \) the element \( L(P) = L_0(P) + L_1(P) \zeta \) of the group ring \( \mathbb{Z}\{1, \zeta\} \), where \( \{1, \zeta\} \) is the 2-element group, written multiplicatively.

The ring \( \mathbb{Z}\{1, \zeta\} \) admits two homomorphisms to \( \mathbb{Z} \), taking \( \zeta \) to \(+1\) and \(-1\) respectively; thus, the former carries \( L(P) \) to \( L_0(P) + L_1(P) \), the number of linearizations of \( P \), which we shall denote \( L_+(P) \), while the latter carries it to \( L_0(P) - L_1(P) \), the sign-imbalance, which we shall denote \( L_-(P) \). These homomorphisms together yield an embedding of \( \mathbb{Z}\{1, \zeta\} \) in \( \mathbb{Z} \times \mathbb{Z} \), whose image is \( \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod{2}\} \). This embedding carries \( L(P) = L_0(P) + L_1(P) \zeta \) to \( (L_+(P), L_-(P)) \), which we shall denote \( L_{\pm}(P) \).

For any finite poset \( P \), \( L(P) \) lies in the subsemiring \( \mathbb{N}\{1, \zeta\} \) of \( \mathbb{Z}\{1, \zeta\} \) consisting of elements in which the coefficients of \( 1 \) and \( \zeta \) are both nonnegative. The image of this semiring under the above ring embedding is

\[
\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod{2} \text{ and } |b| \leq a\}.
\]

It will follow from Theorem 6 below that for various finite posets \( P \) and \( P' \), we can say that the number of linearizations \( L_+(P) \) is a multiple of \( L_+(P') \). (For instance, this is true if \( P' \) is any of the connected components of \( P \).) In such situations, it is not surprising to find that the sign-imbalance \( L_-(P) \) is likewise a multiple of \( L_-(P') \). But in fact, one typically has the stronger statement that \( L(P) \) is a multiple of \( L(P') \) in \( \mathbb{N}\{1, \zeta\} \), equivalently, that the ordered pair \( L_{\pm}(P) \) is a multiple of \( L_{\pm}(P') \) in the semiring (3).

So, for instance, if \( L_{\pm}(P') = (4, 2) \), then \( L_{\pm}(P) \) cannot be \((12, 0)\) or \((8, 2)\) or \((8, 8)\). Separate consideration of the numbers \( L_+(P) \) and \( L_-(P) \) would not exclude these values; but the pairs of integers by which \( (4, 2) \) would have to be multiplied to get them would be \((3, 0), (2, 1)\) and \((2, 4)\), of which the first two don’t have coordinates of the same parity, while the last fails to satisfy the final inequality of (3). (The element \( (2, 4) \in \mathbb{Z} \times \mathbb{Z} \) corresponds to \( 3 - \zeta \notin \mathbb{N}\{1, \zeta\} \).)

We will, in the next three sections, consider \( L_+ \) and \( L_- \) together. We will encounter both parallelisms and differences in their behaviors.

We remark that the element \( L(P) \in \mathbb{N}\{1, \zeta\} \) defined above is the image, under the homomorphism taking \( q \) to \( \zeta \), of the polynomial \( I_{P, \omega}(q) \) defined in [5, p.881, display (1)]. That polynomial depends in a much stronger way than \( L(P) \) on the reference linearization (expressed there by an indexing \( \omega \) of \( |P| \)), and, as Stanley notes, does not seem easy to understand.

4. The linearization-count and sign-imbalance of a lexicographic sum

4.1. Review of lexicographic sums. Suppose \( P_0 = (|P_0|, \leq_0) \) is a poset, and for each \( x \in |P_0| \), we are given a poset \( P_x = (|P_x|, \leq_x) \). Then we can form the disjoint union of the sets \( |P_x| \), say constructed as \( \{(x, x') \mid x \in |P_0|, x' \in |P_x|\} \), and give this set a partial ordering under which, for each \( x \in |P_0| \), the copy \( \{(x, x') \mid x' \in |P_x|\} \) of \( |P_x| \) is made order-isomorphic to \( P_x \) via the correspondence \((x, x') \mapsto x'\), while for \( x \neq y \) in \( |P_0| \), the order-relation between elements \((x, x')\) and \((y, y')\) \((x' \in |P_x|, y' \in |P_y|)\) is determined solely by their first components: \((x, x') \leq (y, y')\) if and only if \( x \leq_0 y \).
In the case of finite posets, considered below, we shall find it convenient to assume \(|P_0|\) indexed as \{x_1, \ldots, x_{m_0}\}. We shall then abbreviate \(P_{x_i} = (|P_{x_i}|, \preceq_{x_i})\) to \(P_i = (|P_i|, \preceq_i)\), and assume that each \(|P_i|\) is indexed as \{x_{i,1}, \ldots, x_{i,m_i}\}, and that distinct symbols \(x_{i,j} (i \leq m_0, j \leq m_i)\) denote distinct elements. This makes the \(|P_i|\) disjoint, so we can forgo the construction of their disjoint union by ordered pairs, and simply use the union of the \(|P_i|\) as the underlying set of the lexicographic sum. Choosing a notation for such a lexicographic sum, and summarizing the above description of it, we have

\[
P_0 \ast (P_1, \ldots, P_{m_0}) = (|P|, \preceq), \text{ where}
\]
\[
|P| = \{x_{i,j} \mid 1 \leq i \leq m_0, 1 \leq j \leq m_i\}, \text{ and}
\]

(4) \(x_{i,j} \preceq x_{i',j'}\) if and only if either

\(i \neq i'\) and \(x_i \preceq_0 x_i'\) (in \(P_0\)), or

\(i = i'\) and \(x_{i,j} \preceq x_{i,j'}\) (in \(P_i\)).

Two easy classes of examples: A poset \(P\) decomposed into its connected components \(P_i\) can be regarded as the lexicographic sum of the \(P_i\) over an antichain \(P_0\); in particular, the next-to-last steps in the construction of \(\S 1\) were lexicographic sums of chains over antichains. On the other hand, the “prelinearizations” that occurred as next-to-last steps in \(\S 2\) were lexicographic sums of antichains over chains.

Let us make

Convention 5. In the context of (4), for the purpose of defining the parities of linearizations, we define on each \(|P_i|\) \((0 \leq i \leq m_0)\) the linear reference order \(x_{i,1} \preceq_{i,\text{ch}} \cdots \preceq_{i,\text{ch}} x_{i,m_i}\) (where \(ch\) is mnemonic for “chain”), and for each \(i\) we shall write \(P_{i,\text{ch}} = (|P_i|, \preceq_{i,\text{ch}})\). On \(|P_0 \ast (P_1, \ldots, P_{m_0})|\) we shall likewise use the reference order

(5) \(x_{1,1} \preceq_{\text{ch}} \cdots \preceq_{\text{ch}} x_{1,m_1} \preceq_{\text{ch}} \cdots \preceq_{\text{ch}} x_{m_0,1} \preceq_{\text{ch}} \cdots \preceq_{\text{ch}} x_{m_0,m_{m_0}}\).

We remark that in discussing lexicographic sums \(P_0 \ast (P_1, \ldots, P_{m_0})\), we allow some (or all) of the \(|P_i|\) to be empty. Occasionally, such cases will require special consideration.

4.2. Computing \(L\) of a lexicographic sum. An idea used in \(\S 1\) and \(\S 2\) generalizes to the following easy result.

Theorem 6. In the context of (4), we have

(6) \(L(P_0 \ast (P_1, \ldots, P_{m_0})) = L(P_0 \ast (P_{1,\text{ch}}, \ldots, P_{m_0,\text{ch}})) \cdot \prod_{1 \leq i \leq m_0} L(P_i)\).

(Hence, the same is true with \(L\) everywhere replaced by \(L\).)

Sketch of proof. One can obtain the general linearization of the poset \(P_0 \ast (P_1, \ldots, P_{m_0}) = (|P|, \preceq)\) in two steps: First, choose the linear order to be used on each \(|P_i|\); that can be any linearization of \(\preceq_i\). Then specify how the union of the linearly ordered sets is to be ordered. By the definition of lexicographic sum, the order-relation in \(P_0 \ast (P_1, \ldots, P_{m_0}) = (|P|, \preceq)\) between any element of \(P_i\) and any element of \(P_j\) where \(i \neq j\) depends only on \(i\) and \(j\); hence for each way of linearizing the separate \(P_i\), the ways of linearizing their union which are compatible with those linearizations and with the ordering of \(P_0 \ast (P_1, \ldots, P_{m_0}) = (|P|, \preceq)\) correspond to the ways of linearizing a lexicographic sum over \(P_0\) of a family of chains of the corresponding lengths, of which \(P_0 \ast (P_{1,\text{ch}}, \ldots, P_{m_0,\text{ch}})\) is one. It is also not hard to check that the parity of a linearization of \(P_0 \ast (P_1, \ldots, P_{m_0})\) (relative to the reference-ordering \(\preceq_{\text{ch}}\)) is the product of the parities of the chosen linearizations of the \(P_i\) (relative to the reference-orderings \(\preceq_{i,\text{ch}}\)) and that of the corresponding linearization of \(P_0 \ast (P_{1,\text{ch}}, \ldots, P_{m_0,\text{ch}})\) (again relative to \(\preceq_{\text{ch}}\)). The equality (6) follows.

The final assertion follows because the operator \(L\) is the composite of the operator \(L\) with a ring homomorphism \(Z\{1, \zeta\} \rightarrow Z \times Z\). \(\square\)

4.3. Partial results on \(L\) of a lexicographic sum of chains. For Theorem 6 to be useful, we need to be able to compute \(L(P)\) for \(P\) a lexicographic sum of chains. We shall obtain partial results in this direction below. Since in these results, the chains in question are not given as listings of the elements of other posets, we adjust our notation slightly.

Definition 7. Let \(P_0 = (|P_0|, \preceq_0)\) be a finite poset, with \(|P_0| = \{x_1, \ldots, x_{m_0}\}\). Then for nonnegative integers \(m_1, \ldots, m_{m_0}\), we define

(7) \(L(P_0; m_1, \ldots, m_{m_0}) = L(P_0 \ast (C_1, \ldots, C_{m_0}))\).
where for \( 1 \leq i \leq m_0 \), \( C_i = (|C_i|, \leq_i) \) is a chain of \( m_i \) elements \( x_{i,1} \leq_i \ldots \leq_i x_{i,m_i} \), these chains being taken pairwise disjoint, and the parities of linearizations of their lexicographic sum being taken relative to the ordering

\[
(8) \quad x_{1,1} \leq \ldots \leq x_{1,m_1} \leq \ldots \leq x_{m_0,1} \leq \ldots \leq x_{m_0,m_{m_0}}.
\]

We will also use the notation corresponding to \((7)\) with \( L_0 \), \( L_1 \), \( L_+ \), \( L_- \) and \( L_\pm \) in place of \( L \).

A trivial case is

**Proposition 8.** Suppose, in Definition 7, that \( P_0 \) is a chain, \( x_1 \leq_0 \ldots \leq_0 x_{m_0} \). Then for all \( m_1, \ldots, m_{m_0} \), \( L(P_0; m_1, \ldots, m_{m_0}) = 1 \). Equivalently, \( L_+(P_0; m_1, \ldots, m_{m_0}) = L_-(P_0; m_1, \ldots, m_{m_0}) = 1 \).

**Proof.** With \( P_0 \) and all of \( C_1, \ldots, C_{m_0} \) being chains, we see that \( P_0 \ast (C_1, \ldots, C_{m_0}) \) is a chain, so it has just one linearization. This is our reference ordering, so it is even, giving the asserted conclusions. \( \square \)

If \( P_0 \) is an antichain we can also get an exact result. In the proof of Proposition 9 below, and that of Theorem 10, which partly generalizes it, we shall, following [5], use the language of “dominoes”. Namely, to partly or completely cover a poset \( P \) with dominoes means to distinguish certain non-overlapping pairs of elements \((x, y)\) such that \( y \) covers \( x \) (i.e., \( x \) and \( y \) are distinct elements such that \( x \leq y \) and there are no elements strictly between these). If \((x, y)\) is a pair which we have so distinguished, we will speak of there being a domino lying over \( x \) and \( y \).

(Richard Stanley informs me that Proposition 9 is essentially known: Parts (i) and (ii) can be obtained by setting \( q = -1 \) in formula (1.68) of [6], and calling on (1.66) and (1.87) thereof.)

**Proposition 9.** Suppose, in the context of Definition 7, that \( P_0 \) is an antichain.

Then the linearization count \( L_+(P_0; m_1, \ldots, m_{m_0}) \) is the multinomial coefficient

\[
(9) \quad \left( \sum_{1 \leq i \leq m_0} m_i \right)! / \prod_{1 \leq i \leq m_0} m_i !.
\]

On the other hand, the sign-imbalance \( L_-(P_0; m_1, \ldots, m_{m_0}) \) is described as follows:

(i) If at most one of \( m_1, \ldots, m_{m_0} \) is odd, then \( L_-(P_0; m_1, \ldots, m_{m_0}) \) is the multinomial coefficient

\[
(10) \quad \left( \sum_{1 \leq i \leq m_0} [m_i/2] \right)! / \prod_{1 \leq i \leq m_0} [m_i/2] !.
\]

(ii) If more than one of \( m_1, \ldots, m_{m_0} \) are odd, then \( L_-(P_0; m_1, \ldots, m_{m_0}) = 0 \).

**Proof.** The number \( L_+(P_0; m_1, \ldots, m_{m_0}) \) of linearizations of our lexicographic sum of chains is the number of ways of partitioning the \( \sum_{1 \leq i \leq m_0} m_i \) slots in such a linearization into subsets of cardinalities \( m_1, \ldots, m_{m_0} \), and \((9)\) is a standard description of this number [6, p.20].

To get information on \( L_-(P_0; m_1, \ldots, m_{m_0}) \), let us cover as much as we can of each linearization of \( P_0 \ast (C_1, \ldots, C_{m_0}) \) with dominoes, starting from the bottom. Thus, if \( \sum m_i \) is even, each linearization will be entirely covered, while if it is odd, all but the top element of each linearization will be.

I claim that the set of linearizations which, when dominoes are so placed, have the property that at least one domino lies over two elements which are incomparable under \( \leq \) is sign-balanced. Indeed, let us pair off these linearizations as follows. Given such a linearization, find, among those dominoes that lie over \( \leq \)-incomparable pairs, the top one, and form a new linearization by reversing the positions of the pair of elements it lies over. Because those elements are \( \leq \)-incomparable, the result will again be a linearization of \( \leq \); and because exactly one pair has been interchanged, the new linear order has parity opposite to the old one. It is also clear that the above construction, applied to the new linearization, returns the given one, so we indeed have a pairing. Hence the contributions to the sign-imbalance of those linearizations sum to zero.

Thus, to determine the sign-imbalance of \( P_0 \ast (C_1, \ldots, C_{m_0}) \), it suffices to consider linearizations which, when dominoes are set down as above, have the property that every domino lies over a pair of comparable elements. Let us call such a linearization of \( \leq \) distinguished.

Since elements from different chains \( C_i \) are incomparable, a distinguished linearization will have the property that every domino lies over a pair of elements from the same \( C_i \). More precisely, we see by induction, working up from the bottom, that every domino lies over a pair of the form \((x_{i,2j-1}, x_{i,2j})\). From this it easily follows that any distinguished linearization can be turned into our reference ordering \((8)\) by repeatedly switching the places of two adjacent dominoes (hence, reordering a string \( w < x < y < z \) as \( y < z < w < x \)), or moving a domino past the lone dominoless element, if there is one (reordering a string \( x < y < z \) as \( z < x < y \)). A movement of the former sort can be done using \( 2 \cdot 2 \) transpositions of
elements; one of the latter sort using 2 transpositions; so in each case, the parity of the linearization does not change. Hence every distinguished linearization is even, so the sign-imbalance of \( P_0 \ast (C_1, \ldots, C_{m_0}) \) equals the number of distinguished linearizations.

From the above considerations, we can see that if there exists a distinguished linearization, at most one of the \( C_i \) can have odd cardinality (the \( C_i \) from which the “lone element”, if any, at the top of a distinguished linearization comes). This immediately gives statement (ii).

To get statement (i), suppose first that all the \( m_i \) are even. Then it is not hard to see that each distinguished linearization is determined by noting how the \( \sum m_i/2 \) dominoes are partitioned into \( m_1/2 \) dominoes lying over pairs of members of \( C_1 \), \( m_2/2 \) lying over pairs of members of \( C_2 \), etc.. By the same counting principle used in getting (9), the number of such partitions is \( \prod (m_i/2)! \), which agrees with (10) in this case.

Finally, suppose that exactly one of the \( m_i \), say \( m_{i_0} \), is odd. Then in a distinguished linearization, the one element not under any domino, namely, the top element of our linearization, is necessarily the largest element of the chain \( C_{i_0} \) with an odd number of elements. So a distinguished linearization will be determined by the arrangement of the dominoes covering the remaining elements, and as before, the number of possibilities is described by (10).

\[ \square \]

For general \( P_0 \), we do not have an exact formula for \( L_+(P_0; m_1, \ldots, m_{m_0}) \), nor does the analog of statement (ii) hold (see §4.4 below). But, curiously, we do have an analog of (i), though it is harder to prove.

**Theorem 10.** In the context of Definition 7, if no more than one of \( m_1, \ldots, m_{m_0} \) is odd, then

\[ \sum \frac{m_1}{2}, \ldots, \frac{m_{m_0}}{2} \]

\[ \text{(11)} \quad L_-(P_0; m_1, \ldots, m_{m_0}) = L_+(P_0; \lfloor m_1/2 \rfloor, \ldots, \lfloor m_{m_0}/2 \rfloor). \]

**Proof.** Consider first the case where all \( m_i \) are even.

In that case, as in the proof of Proposition 9, let us cover each linearization of \( P_0 \ast (C_1, \ldots, C_{m_0}) \) with dominoes, starting from the bottom, and pair off each linearization in which at least one domino lies over a pair of \( \prec \)-incomparable elements with the linearization that differs from it only in the reversal of the order of the positions of the two elements lying under the top such domino. Again this operation, repeated, returns the original linearization, and the linearizations so paired have opposite parity, and we conclude that to determine the sign-imbalance of our poset, it suffices to look at linearizations such that every domino lies over a comparable pair. Again, we call such linearizations distinguished.

It is no longer true, as it was in the proof of Proposition 9, that in \( P_0 \ast (C_1, \ldots, C_{m_0}) \), the only comparable elements are elements from the same \( C_i \); but I claim nonetheless that in a distinguished linearization, every domino lies over a pair of the form \( (x_{i,2j-1}, x_{i,2j}) \). For if a domino lay over a pair of elements coming from \( C_i \) and \( C_{i'} \) respectively, with \( i \neq i' \), we would have \( x_i \leq x_{i'} \), so the two elements over which our domino lies could only be, respectively, the top element of \( C_i \) and the bottom element of \( C_{i'} \). But, assuming inductively that every domino below the one in question does lie over a pair of elements from the same chain, the number of elements of \( C_i \) below the domino in question would be even; so, bringing in the top element, \( C_i \) would have an odd number of elements, contradicting our assumption.

It again follows that the every distinguished linearization can be determined by noting how the \( \sum m_i/2 \) dominoes are partitioned into \( m_1/2 \) lying over members of \( C_1 \), \( m_2/2 \) lying over members of \( C_2 \), etc.; and clearly, this information corresponds to a linearization of a lexicographic sum over \( P_0 \) of chains of lengths \( m_1/2, \ldots, m_{m_0}/2 \). As in the proof of Proposition 9, each of these linearizations has even parity, so

\[ L_-(P_0; m_1, \ldots, m_{m_0}) = L_+(P_0; m_1/2, \ldots, m_{m_0}/2), \]

giving (11) in this case.

Now suppose, instead, that exactly one of the \( m_i \), say \( m_{i_0} \), is odd. Given a linearization of \( P_0 \ast (C_1, \ldots, C_{m_0}) \), let us begin covering it with dominoes from the bottom, until we have either put down a domino that lies over \( x_{i_0,m_{i_0}} \) (the top element of our odd-sized chain), at which point we pause, or we can put down no more dominoes (if \( x_{i_0,m_{i_0}} \) is the top element of our linearization). In the former case, we then put down further dominoes starting from the top of our linearization and working downward as far as we can (i.e., until we are stopped by the condition that dominoes must be disjoint). Either way, we end up covering all but one element with dominoes.

We see that, as before, if a linearization has at least one domino which lies over a pair of incomparable elements, then looking at the highest such domino, the operation of interchanging the positions of the two elements it lies over pairs it with a linearization of opposite parity; hence, again calling a linearization
such that no domino lies over an incomparable pair “distinguished”, we can, as before, evaluate $L_{-}(P_{0} \ast (C_{1}, \ldots, C_{m_{0}}))$ by looking only at the distinguished linearizations.

The same reasoning used in the all-$m_{i}$-even case still shows that in a distinguished linearization, every domino below the one which contains $x_{i_{0}, m_{i_{0}}}$ (or every domino, if $x_{i_{0}, m_{i_{0}}}$ is at the top of the linearization) lies over a pair of elements from the same $C_{i}$; and a similar argument shows the same for the dominoes we have put down starting at the top, if any. From the first of these facts, we can deduce that if $x_{i_{0}, m_{i_{0}}}$ lies under a domino, it is the lower element under that domino. Indeed, if it were the higher element, then if the lower element came from another set $C_{i}$, it would have to be the highest element of $C_{i}$ (because it is comparable with $x_{i_{0}, m_{i_{0}}}$, and hence all members of $C_{i}$ are, like it, $\preceq x_{i_{0}, m_{i_{0}}}$), and this would force $C_{i}$ to have an odd number of elements, while if the lower element under that domino also came from $C_{i_{0}}$, that would force $C_{i_{0}}$ to have an even number! Since $x_{i_{0}, m_{i_{0}}}$ is thus the lower element under its domino, the element above $x_{i_{0}, m_{i_{0}}}$, which we know is comparable with it, must have the form $x_{i_{1}, 1}$ for some $i_{1}$ with $x_{i_{0}} \preceq x_{i_{1}}$.

What about the element above this in our linearization, the one not under any domino? I claim this must be $x_{i_{1}, 2}$. Indeed, if it belonged to any $C_{i_{2}}$ other than $C_{i_{1}}$, this would force $C_{i_{1}}$ to have an odd number of elements, as we see by looking at domino-covered pairs above that element.

This suggests that we pass to a slight modification of our domino-coverings of distinguished linearizations, gotten by moving the domino lying over $x_{i_{0}, m_{i_{0}}}$ and $x_{i_{1}, 1}$ so that it lies instead over $x_{i_{1}, 1}$ and $x_{i_{1}, 2}$. When we have done this, every domino lies over a pair elements in the same summand $C_{i}$, and the one element not under any domino is $x_{i_{0}, m_{i_{0}}}$. (This has been the case all along if $x_{i_{0}, m_{i_{0}}}$ was the top element of our linearization. Incidentally, the original domino-covering is essential to our proof – we used it in pairing off non-distinguished linearizations. The modified covering is introduced only after that has been done.)

Again associating to each domino the index $i$ such that the two elements lying under it come from $C_{i}$, we get a string of indices which, for each $i$ contains $[m_{i}/2]$ occurrences of the index $i$, and such that when $x_{i} \preceq x_{i'}$, all occurrences of $i$ precede all occurrences of $i'$. The number of such strings of indices is, of course, $L_{+}(P_{0}; [m_{1}/2], \ldots, [m_{m_{0}}/2])$. The above string of indices clearly determines the relative positions in our given linearization of $P_{0} \ast (C_{1}, \ldots, C_{m_{0}})$ of all the elements $x_{i,j}$ except, perhaps, for $x_{i_{0}, m_{i_{0}}}$. A little thought shows that the position of that element relative to the others is also uniquely determined. It will occur immediately below the lowest location of an element of some $C_{i_{1}}$ with $i_{1} \neq i_{0}$ such that $x_{i_{0}} \preceq x_{i_{1}}$ if there are any such elements, or at the top of the whole linearization if there are none. (The latter will be the case if $x_{i_{0}}$ is a maximal element of $C_{i_{0}}$, or, slightly more generally, if for all $i_{1}$ with $x_{i_{1}}$ above $x_{i_{0}}$ we have $m_{i_{1}} = 0$.) So the number of distinguished linearizations is $L_{+}(P_{0}; [m_{1}/2], \ldots, [m_{m_{0}}/2])$.

We see as before that every distinguished linearization is even, giving (11).

4.4. What if more than one of the $m_{i}$ are odd? Can we prove in the context of Theorem 10 the analog of Proposition 9(ii), namely, that $L_{-}(P_{0}; m_{1}, \ldots, m_{m_{0}}) = 0$ if more than one of $m_{1}, \ldots, m_{m_{0}}$ are odd?

No. For, instance, for any finite poset $P_{0}$ we have $L_{-}(P_{0}; 1, 1, \ldots, 1) = L_{-}(P_{0})$; so for any $P_{0}$ which is not sign-balanced and has $m_{0} \geq 2$, the counterexample $m_{1} = \cdots = m_{m_{0}} = 1$ is a counterexample to the suggested statement. Proposition 8 gives another class of counterexamples, using any $m_{0} \geq 2$ and any $m_{1}, \ldots, m_{m_{0}}$ with at least two of the $m_{i}$ odd.

The observations of §5.4 below will give us a larger class of posets $P_{0}$ for which we can completely determine the functions $L_{-}(P_{0}; m_{1}, \ldots, m_{m_{0}})$.

5. Further observations on $L_{\pm}$ of lexicographic sums

5.1. The case where $P_{0}$ is a 2-element antichain. It is interesting to look at the simplest nontrivial case of Proposition 9, where $m_{0} = 2$. Then $P_{0} \ast (C_{1}, C_{2})$ is the disconnected union of a chain of $m_{1}$ elements and a chain of $m_{2}$ elements, so its linearizations correspond to the ways of partitioning a chain of $m_{1} + m_{2}$ elements into sets of $m_{1}$ and $m_{2}$ elements respectively. The number of these is the binomial coefficient $\binom{m_{1} + m_{2}}{m_{2}}$, so the values of $L_{\pm}(P_{0}; m_{1}, m_{2})$ are the entries of Pascal’s triangle. Let us display the first few rows of that triangle, and likewise, the values of $L_{-}(P_{0}; m_{1}, m_{2})$ given by parts (i)-(ii) of the same proposition. In each array of (12) below, the rows correspond to the values of $m_{1} + m_{2}$, the diagonals going downward to the left to the values of $m_{2}$, and the diagonals going downward to the right to the values of $m_{2}$.
The familiar rule for producing Pascal’s triangle, that each entry is the sum of the two above it in the preceding row, can be interpreted in terms of \( L_0(P; m, m) \). The linearizations of \( P \) can be classified according to whether the top element belongs to \( C_1 \) or \( C_2 \). In the former case, the ordering of the remaining elements constitutes a linearization of the union of a chain of \( m-1 \) elements and a chain of \( m \) elements; in the latter case, a linearization of a union of a chain of \( m \) elements and a chain of \( m-1 \) elements.

The sign-imbalances \( L_-(P; m, m) \) in fact satisfy a similar law. Every entry that lies an even number of steps from the sloping left edge of the array is, as in Pascal’s triangle, the sum of the two entries above it, while if an entry lies an odd number of steps from that edge, it is the difference of those two entries (the one to the left minus the one to the right). The reader can justify this rule by classifying linearizations of our union of chains as in the preceding paragraph, and examining how the parity of a linearization compares with the parity of the linearization of the one-element-smaller poset that we get on dropping the element at the top of that linearization.

(In fact, I discovered the formulas of Proposition 9(i)-(ii) by first looking at the case \( m_0 = 2 \), deducing as suggested above the rule for calculating \( L_-(P; m, m) \), noticing the way values from Pascal’s triangle occurred in the resulting picture, and thinking about how to justify and generalize that pattern.)

5.2. **\( L_\pm(P; m_1, \ldots, m_m) \) as a function of its separate variables.** In general, \( L_\pm(P; m_1, \ldots, m_m) \) is not given by a polynomial in \( m_1, \ldots, m_m \). For instance, in the case just examined, note that \( L_+(P; m,m) = \binom{2m}{m} \geq 2^m \). Nevertheless, if in the function \( L_+(P; m_1, m_2) = \binom{m_1+m_2}{m_1} \) we hold \( m_2 \) constant, the resulting function is a polynomial in \( m_1 \) (of degree \( m_2 \)), while if we hold \( m_1 \) constant, it is a polynomial in \( m_2 \) (of degree \( m_1 \)). A similar result holds for a general finite poset \( P \), together with a somewhat more complicated statement for the sign-imbalance function:

**Theorem 11.** Let \( P_0 = (|P_0|, \leq_{P_0}) \) be a finite poset, with \(|P_0| = \{x_1, \ldots, x_{m_0}\}\), let \( i_0 \in \{1, \ldots, m_0\} \), and for all \( i \neq i_0 \) in \( \{1, \ldots, m_0\} \), let us fix values for the \( m_i \). Then, regarding \( L_\pm(P_0; m_1, \ldots, m_m) \) as a function of \( m_i \), we have

(a) \( L_+(P_0; m_1, \ldots, m_m) \) is a polynomial in \( m_i \), whose degree is the sum of those values of \( m_i \) (\( i \neq i_0 \)) such that \( x_i \) is \( \leq_{P_0} \)-incomparable with \( x_{i_0} \).

(b) The values of \( L_-(P_0; m_1, \ldots, m_m) \) are given by two polynomials in \( m_i \), one for \( m_i \) even and the other for \( m_i \) odd. Each has degree less than or equal to \( \lceil\sum m_i/2\rceil \), where the summation is again over those \( i \neq i_0 \) such that \( x_i \) is \( \leq_{P_0} \)-incomparable with \( x_{i_0} \).

**Proof.** Again let \( C_1, \ldots, C_m \) be disjoint chains of lengths \( m_1, \ldots, m_m \). Then a linearization of \( P = P_0 \ast (C_1, \ldots, C_m) \) can be determined by first choosing an arbitrary linearization of the subposet with underlying set \(|P| - |C_{i_0}|\), then specifying where to insert the elements of \( |C_{i_0}| \).

To see where those elements can go, let us partition \( \{1, \ldots, m_0\} \setminus \{i_0\} \) into three subsets: the subset \( I_< \) consisting of those \( i \) such that \( x_i \leq_{P_0} x_{i_0} \), the subset \( I_> \) of those \( i \) such that \( x_{i_0} \leq_{P_0} x_i \), and the subset \( I_\sim \) of those \( i \) such that \( x_i \) is \( \leq_{P_0} \)-incomparable with \( x_{i_0} \). Then given any linearization \( \preceq \) of \( \leq_{P_0} \) on \(|P| - |C_{i_0}|\), the elements of \( |C_{i_0}| \) can be inserted within the range bounded below by the highest location, under that linearization, of an element of \( \bigcup_{i \in I_<} |C_i| \), and above by the lowest location of an element of \( \bigcup_{i \in I_>} |C_i| \); where we understand the former restriction to be vacuous if \( \bigcup_{i \in I_<} |C_i| \) is empty, and the latter if \( \bigcup_{i \in I_>} |C_i| \) is empty.
In our linearization of $|P| - |C_{i_0}|$, the interval we have described will in general contain some subset of $\bigcup_{i \in I} |C_i|$. If $d$ is the number of elements of that set in that interval, then clearly
\begin{equation}
0 \leq d \leq \sum_{i \in I} m_i,
\end{equation}
and we see that the number of ways the $m_{i_0}$ elements of the chain $C_{i_0}$ can be interspersed among those of that subset of $\bigcup_{i \in I} |C_i|$ is $\binom{m_{i_0} + d}{d}$, which, as a function of $m_{i_0}$, is a polynomial of degree $d$. The maximum value allowed by (13), $d = \sum_{i \in I} m_i$, does in fact occur, since we can linearize $|P| - |C_{i_0}|$ so that all members of $\bigcup_{i \in I} |C_i|$ precede all members of $\bigcup_{i \in I > i} |C_i|$, and these precede all members of $\bigcup_{i \in I < i} |C_i|$. Summing the polynomials in $m_{i_0}$ obtained from all our linearizations of $|P| - |C_{i_0}|$, we get a polynomial $p(m_{i_0})$ describing $L_+(P_0; m_1, \ldots, m_d)$. Since the leading coefficients of the polynomials we have summed are all positive, the terms of degree $\sum_{i \in I} m_i$ cannot cancel, so that is the degree of $p(m_{i_0})$, completing the proof of (a).

If, instead, we look at the sign-imbalance, then the polynomials $\binom{m_{i_0} + d}{d}$ are replaced by the functions described in parts (i) and (ii) of Proposition 9 (and illustrated by the diagonals of the right-hand array of (12)), multiplied by $\pm 1$ depending on the details of our linearization of $|P| - |C_{i_0}|$. Each of these is given by one polynomial of degree $\leq d/2$ on odd inputs, and a generally different polynomial of degree $\leq d/2$ on even inputs, yielding (b). (Because of the varying signs, we cannot say in this case that the leading terms of the highest-degree polynomials will not cancel, hence we cannot exactly specify the degree of the polynomial we get.)

In view of Proposition 9, we might wonder whether for the two polynomials referred to in (b) above, the degree of the one that gives the sign-imbalance when $m_{i_0}$ is odd must be less than or equal to the degree of the one that does this when $m_{i_0}$ is even. This is not the case. For instance, suppose $P_0$ is the poset $x_1 \uparrow x_2 \uparrow x_3$, let $i_0 = 2$, and take $m_1$ and $m_3$ odd. Then for any value of $m_2$, the union of the posets $C_2$ and $C_3$ will be a chain of length $m_2 + m_3$, which has the opposite of the parity of $m_2$, whence we see from Proposition 9 that for $m_2$ even, $L_+(P_0; m_1, m_2, m_3)$ will be zero, while for $m_2$ odd, it will be given by a polynomial of degree $\lfloor m_1/2 \rfloor$.

5.3. Chains in $P_0$. At the beginning of the preceding section, we noted that for $P_0$ a 2-element antichain, the function $L_+(P_0; m_1, m_2)$ could not be a polynomial in its two variables. More generally, if $x_{i_1}$ and $x_{i_2}$ are incomparable elements of a finite poset $P_0$, and we fix values for all of $m_1, \ldots, m_{m_0}$ other than $m_{i_1}$ and $m_{i_2}$, we find that as a function of those two variables, $L_+(P_0; m_1, \ldots, m_{m_0})$ cannot be a polynomial.

The idea is that one can strengthen the ordering of $P_0$ to get an “almost-chain” $P_0'$, in which the only pair of incomparable elements is $\{x_{i_1}, x_{i_2}\}$, and note that $L_+(P_0'; m_1, \ldots, m_{m_0}) \geq L_+(P_0; m_1, \ldots, m_{m_0}) = \binom{m_1 + m_2}{d}$. In particular, when $m_{i_1} = m_{i_2} = m$, we have $\binom{m_1 + m_2}{d} = \binom{2m}{d}$, which grows too rapidly for a polynomial function. More generally, this shows that $L_+(P_0'; m_1, \ldots, m_{m_0})$ cannot be a polynomial function in any subset of its arguments two of which correspond to incomparable elements of $P_0$.

So can $L_+(P_0; m_1, \ldots, m_{m_0})$ ever be a polynomial function of more than one of its arguments? Yes – in precisely the cases not excluded by the above conclusion.

Theorem 12. Let $P_0 = (|P_0|, \leq_{\alpha_0})$ be a finite poset, with $|P_0| = \{x_1, \ldots, x_{m_0}\}$, and let $\{i_1, \ldots, i_r\}$ be any subset of $\{1, \ldots, m_0\}$. Suppose that in $L_+(P_0; m_1, \ldots, m_{m_0})$ we fix nonnegative values for all the $m_i$ other than $m_{i_1}, \ldots, m_{i_r}$. Then (regardless of the values so chosen), the resulting function of $m_{i_1}, \ldots, m_{i_r}$ is given by a polynomial if and only if $\{x_{i_1}, \ldots, x_{i_r}\}$ is a chain in $P_0$.

Sketch of proof. We have just seen “only if”; I shall sketch the proof of “if”. We may assume without loss of generality that $x_{i_1} \leq_{\alpha_0} \ldots \leq_{\alpha_0} x_{i_r}$.

Using the chosen values for all $m_i$ other than $m_{i_1}, \ldots, m_{i_r}$, and taking arbitrary values for those $r$ arguments, let $C_1, \ldots, C_{m_0}$ denote chains with $m_{i_1}, \ldots, m_{i_r}$ elements respectively. Now let $Q$ be the lexicographic sum over $P_0$ in which the $i$-th summand for each $x_i \notin \{m_{i_1}, \ldots, m_{i_r}\}$ is $C_i$, while the summands for $i \in \{m_{i_1}, \ldots, m_{i_r}\}$ are empty. On the other hand, let $C$ be the lexicographic sum over $P_0$ formed with $C_1, \ldots, C_{m_0}$ in positions $i_{1, \ldots, i_r}$, and empty posets in all other positions. Thus, $C$ is a chain of $m_{i_1} + \cdots + m_{i_r}$ elements. To specify a linearization of $P_0 \ast (C_1, \ldots, C_{m_0})$, we can first specify a linearization $Q'$ of $Q$, then specify how the elements of the chain $C$ are to be inserted between and/or around those of $Q'$.
To complete the proof of the theorem, it will suffice to show that for each linearization $Q'$ of $Q$, the number of ways of so placing members of $C$ is given by a polynomial in $m_1, \ldots, m_r$.

We shall, in fact, first show that there is a polynomial in $m_1, \ldots, m_r$ which gives this number whenever \textit{positive} integer values are assigned to all its arguments $m_1, \ldots, m_r$, then show that any such polynomial must continue to have that property when some of its arguments are allowed to be zero.

The complication in counting ways that members of $C$ can be inserted in and around those of $Q'$ is that the regions of $Q'$ where the elements of the various $C_i$ can be inserted are possibly overlapping intervals, so that information on where members of one of those subchains are distributed in its allowed interval may or may not restrict how far other subchains can be distributed through theirs.

But suppose now that $m_1, \ldots, m_r$ are all greater than 0, so that the $C_i$ are nonempty. In this case, let us further classify linearizations of $Q' \cup C$ according to the positions, relative to the elements of $Q'$, of the \textit{greatest} elements of $C_i, \ldots, C_r$; i.e., according to which successive pair of the finitely many elements of $Q'$ each of these greatest elements lies between, or whether it lies before or after all such elements. Once such a set of positions has been specified, the interval of $Q'$ in which the \textit{remaining} elements of each $C_i$ can be placed is independent of where the remaining elements of the other $C_i$ go. Namely, it will be the intersection of the interval determined by the restrictions arising from the order-relations of $P_0$, and the interval between the position of the highest member of $C_{i-1}$ (if $j > 1$) and that of the highest member of $C_i$. If that intersection contains $d_j$ elements of $Q'$ (where $d_j$ may be 0 if, for instance, the top members of $C_{i-1}$ and $C_i$ lie between the same pair of elements of $Q'$), then there are $(m_i-1)^{d_j}$ ways to populate it with the $m_i - 1$ nonmaximal members of $C_i$. This is a polynomial in $m_i$ (of degree $d_j$), hence the total number of linearizations of $Q' \cup C$ with highest elements of all the $C_i$ in the chosen positions will be

\begin{equation}
\prod_{1 \leq j \leq r} \binom{m_i - 1 + d_j}{d_j},
\end{equation}

a polynomial in $m_1, \ldots, m_r$. Summing over all linearizations $Q'$ of $Q$, and all ways of positioning the highest members of the chains $C_i$, we get a polynomial $p(m_1, \ldots, m_r)$ which yields the value of $L_+(P_0; m_1, \ldots, m_r)$ – provided that, as assumed above, none of $m_1, \ldots, m_r$ is zero.

Does the same polynomial work if one or more of $m_1, \ldots, m_r$ is zero? The key to proving that it does is Theorem 11(a), which says that if we fix all but one of the $m_i$, then $L_+(P_0; m_1, \ldots, m_r)$ is a polynomial in that variable. Hence, if we choose for all of the $m_i$ other than some particular $m_{i_0}$ nonzero values, then the resulting polynomial function in the one variable $m_{i_0}$ will necessarily agree with the polynomial obtained in the preceding paragraph at all values of $m_{i_0}$ except possibly 0. But two 1-variable polynomials which agree at infinitely many values are equal – so in fact they will also agree when $m_{i_0} = 0$. This shows that the result of the preceding paragraph extends to the case where at most one of the $m_i$ is zero. An obvious induction on the number of zero arguments similarly gives the general case.

I have not studied the analogous question for $L_-(P_0; m_1, \ldots, m_r)$.

We remark that the final argument in the proof of Theorem 12 combines two similar results: Theorem 11(a), which only looks at one variable, but does not require its value to be nonzero, and the preceding step in the proof of Theorem 12, which allows several variables, but needs them to have positive values, to insure that each chain $C_{i-1}$ is nonempty, and so has a greatest element. But if one looks closely, one sees that we can drop the positivity condition on the \textit{last} variable, $m_r$; there is no need for the last subchain, $C_r$, to be nonempty. Using this fact, one can modify the above proof and eliminate its dependence on Theorem 11(a) (though it will still involve an induction). However, I felt that the development given above was easier to follow, if a little less intrinsically elegant.

Theorem 11(a) specified the degree of the one-variable polynomial it referred to. We can similarly determine the degree of the $r$-variable polynomial of Theorem 12, with respect to all its variables, or any subset of them. To get this result we will need the following curious lemma, which says that one can strengthen the ordering of a poset so as to get rid of one class of incomparability conditions, while preserving “enough of” another related class of such conditions.

\textbf{Lemma 13.} Let $P = (|P|, \preceq)$ be a finite poset, and $C$ a chain in $P$. Then the ordering $\preceq$ of $P$ can be strengthened to an ordering $\preceq'$ such that the complement of $C$ in $P$ also becomes a chain under $\preceq'$, while every element of $P$ that is incomparable with at least one element of $C$ under $\preceq$ remains incomparable with at least one element of $C$ under $\preceq'$.
Proof. Given any pair of incomparable elements \( x, y \in |P| - |C| \), consider the strengthenings \( \preceq_{x,y} \) and \( \preceq_{y,x} \) of \( \prec \) obtained by imposing the relation \( x \preceq_{x,y} y \), respectively \( y \preceq_{y,x} x \). We shall show that

At least one of \( \preceq_{x,y} \), \( \preceq_{y,x} \) has the property that every element of \( P \) that is incomparable under \( \preceq \) with at least one element of \( C \) remains incomparable under that strengthened ordering with at least one element of \( C \).

Repeatedly strengthening our partial order in this way, we eventually get an ordering under which \( |P| - |C| \) has no incomparable elements, i.e., is a chain, but still has the desired condition on elements incomparable with elements of \( C \).

Recall (reference?) that the relation \( \preceq_{x,y} \) can be characterized by the condition that for all \( w, z \in |P| \),

\[
\begin{align*}
\preceq_{x,y} z & \text{ if and only if either } w \preceq z \text{ or } w \preceq x \text{ and } y \preceq z. 
\end{align*}
\]

Thus, if any \( u \in |P| \) satisfies a relation under \( \preceq_{x,y} \) that it does not satisfy under \( \prec \), we must have \( u \preceq x \) or \( y \preceq u \).

Suppose now that we have an element \( u \in |P| \) such that

\[
\begin{align*}
\text{ (17) } u \text{ is } \prec\text{-incomparable with at least one element of } C, \text{ but is } \preceq_{x,y}\text{-comparable with all elements of } C. 
\end{align*}
\]

Of the two possibilities noted at the end of the sentence following (16), let us begin by assuming

\[
\begin{align*}
\text{ (18) } u \preceq x. 
\end{align*}
\]

Let us write \( \downarrow u \) for \( \{ z \in |P| \mid z \preceq u \} \) and \( \downarrow_{x,y} u \) for \( \{ z \in |P| \mid z \preceq_{x,y} u \} \), and make the obvious corresponding definitions of \( \uparrow u \) and \( \uparrow_{x,y} u \). The statement that \( u \) is \( \preceq_{x,y}\text{-comparable with all elements of } C \) thus says that

\[
\begin{align*}
\text{ (19) } (\downarrow_{x,y} u \cap |C|) \cup (\uparrow_{x,y} u \cap |C|) = |C|. 
\end{align*}
\]

Now applying (16) with \( u \) in the role of \( z \) and \( w \) ranging over \( P \), we see (in view of (18)) that \( \downarrow_{x,y} u = \downarrow u \), while applying (16) with \( u \) in the role of \( w \) and \( z \) ranging over \( P \) (and again using (18)), we get \( \uparrow_{x,y} u = \uparrow u \cup \uparrow y \). Since \( C \) is a chain, \( (\uparrow u \cup \uparrow y) \cap |C| \) must be either \( \uparrow u \cap |C| \) or \( \uparrow y \cap |C| \). If it were \( \uparrow u \cap |C| \), then in view of these descriptions of \( \downarrow_{x,y} u \) and \( \uparrow_{x,y} u \), (19) would say that \( u \) was \( \prec\text{-comparable with all elements of } C \), contrary to the first condition of (17). Hence it is \( \uparrow y \cap |C| \), and (19) instead says

\[
\begin{align*}
\text{ (20) } (\downarrow u \cap |C|) \cup (\uparrow y \cap |C|) = |C|. 
\end{align*}
\]

In view of (18), \( \downarrow u \subseteq \downarrow x \), so (20) implies

\[
\begin{align*}
\text{ (21) } (\downarrow x \cap |C|) \cup (\uparrow y \cap |C|) = |C|. 
\end{align*}
\]

If, rather than (18) we are in the other case, \( y \preceq u \), we get the variant of (21) with order-relations reversed and the roles of \( x \) and \( y \) interchanged – which is again (21). So (21) holds in either case.

Now suppose that in addition to an element \( u \) satisfying (17), there is also an element \( v \in |P| \) such that

\[
\begin{align*}
\text{ (22) } v \text{ is } \prec\text{-incomparable with at least one element of } C, \text{ but is } \preceq_{y,x}\text{-comparable with all elements of } C. 
\end{align*}
\]

Then we get the variant of (21) with only the roles of \( x \) and \( y \) interchanged:

\[
\begin{align*}
\text{ (23) } (\downarrow y \cap |C|) \cup (\uparrow x \cap |C|) = |C|. 
\end{align*}
\]

From (21) and (23), it is not hard to deduce that \( \downarrow x \cap |C| = \downarrow y \cap |C| \) and \( \uparrow x \cap |C| = \uparrow y \cap |C| \), and that these are complementary subsets of \( C \); i.e., that both \( x \) and \( y \) are \( \prec\text{-comparable with all elements of } C \), and that the order relation of each element of \( C \) with \( x \) is the same as its order relation with \( y \). (Quick Venn diagram proof: Draw a square, to represent properties of an element \( c \in |C| \). Divide it by vertical lines into three regions according to the \( \prec\text{-relation of } c \) to \( x \): smaller, incomparable, or greater; and similarly by horizontal lines according to its \( \prec\text{-relation to } y \). Interpret each of (21) and (23) as saying that every \( c \in |C| \) lies in a certain region of this diagram. Shade those regions, and note their intersection.)

It follows from this and (16) that passing from \( \prec \) to \( \preceq_{x,y} \) does not affect the order-relation or lack of it between any element of \( P \) and any element of \( C \) (and similarly for \( \preceq_{y,x} \)). This contradicts our assumption (17) (respectively (22)). Thus, our assumption that there existed both \( u \) satisfying (17) and \( v \) satisfying (22) has led to a contradiction, proving (15), and completing the proof of the lemma. \( \square \)
We can now get the result we are aiming for. Below, the “total degree” of a polynomial $p$ in a given subset of its indeterminates will mean the maximum, over the monomials occurring in $p$, of the sum of the exponents of those indeterminates.

**Corollary 14** (to proof of Theorem 12). Let $P_0$ be a finite poset, and $C_0$ a chain in $P_0$, with $|C_0| = \{x_1, \ldots, x_{m_0}\}$, and as in Theorem 12, consider the function $L_+ (P_0; m_1, \ldots, m_{m_0})$ with fixed values chosen for all the $m_i$ other than $m_{i_1}, \ldots, m_{i_r}$, as a function of $m_{i_1}, \ldots, m_{i_r}$. Let $p(m_{i_1}, \ldots, m_{i_r})$ be the polynomial which, by that theorem, gives this function.

Then for each subchain $C_0'$ of $C_0$, the total degree of $p(m_{i_1}, \ldots, m_{i_r})$ in the variables $m_{i_j}$ corresponding to the elements $x_{i_j}$ of $C_0'$ is equal to the number of elements of $P_0$ that are incomparable with at least one element of $C_0'$.

**Sketch of proof.** The degree we are looking for will be the maximum of the total degrees, in the variables corresponding to the elements of $C_0'$, of the leading terms of the polynomials (14) that are summed to get $p(m_{i_1}, \ldots, m_{i_r})$. (A general multivariable polynomial does not have a well-defined “leading term”; but if we look at polynomials (14) that not only maximize the total degree of their leading term in the variables we are interested in, but also, among those that do so, maximize the total degree in the remaining variables, it is clear that their leading terms cannot be cancelled by any term of any polynomial in our set.) Now for each such polynomial, its total degree in the variables we are interested in is the sum over $x_{i_j} \in |C_0'|$ of the number called $d_j$ in the proof of Theorem 12; i.e., the number of elements of $P_0$ lying in the range into which the non-maximal members of the subchain $C_{i_j}$ can be placed. Every element in one of those ranges must belong to a $C_i$ such that $x_i$ is incomparable with at least one member of $C_0'$; so the sum of the cardinalities of those chains $C_i$ (i.e. the sum of the corresponding $m_{i_j}$) is certainly an upper bound for the desired degree.

To get a linearization $Q'$ of the $Q$ of that proof, with the help of which we can realize that upper bound, we apply Lemma 13, taking for the $P_i$ of that lemma the lexicographic sum over $P_0$ having a chain of length $m_i$ in the $i$-th position for all $i \notin \{i_1, \ldots, i_r\}$, and a singleton (which by abuse of notation we denote $x_{i_j}$), in the $i_j$-th position for $j = 1, \ldots, r$. Identifying the set of these singletons with $C_0$, we take for the $C_i$ of that lemma our subchain $C_0'$ of $C_0$. The ordering $\precsim'$ given by Lemma 13 makes $|P| - |C_0'|$ a chain, hence it does the same for its subset $|P| - |C_0| = |Q|$, and we take this linearization of $Q$ to be our $Q'$. It remains to choose for each $i_j$ the position of the largest element of $C_{i_j}$ relative to the elements of $Q'$. We place each of these as high as we can consistent with the order $\precsim'$; i.e., we place it just below the least element of $Q'$ that is above $x_{i_j}$ under that ordering if there is one; if not, we place it above all elements of $Q'$. One then finds that every element of $Q'$ incomparable under $\precsim'$ with at least one member of $C_0'$ is in the range which, in the construction of Theorem 12, can be populated by elements of $C_{i_j}$ for some $x_{i_j} \in C_0'$; so the asserted total degree is achieved. \[\square\]

### 5.4. Associativity of the lexicographic sum, and its consequences.

Suppose as in (4) that $P_0 = (|P_0|, \precsim_0)$ is a poset with $|P_0| = \{x_1, \ldots, x_{m_0}\}$, and that for each $i \in \{1, \ldots, m_0\}$ we are given a poset $P_i = (|P_i|, \precsim_i)$ with $|P_i| = \{x_{i_1}, \ldots, x_{i_{m_i}}\}$, where $x_{i,j}$ and $x_{i',j'}$ are distinct unless $(i, j) = (i', j')$.

Now suppose further that for each pair $(i, j)$ with $1 \leq i \leq m_0$ and $1 \leq j \leq m_i$, we are given a poset $P_{i,j} = (|P_{i,j}|, \precsim_{i,j})$ with $|P_{i,j}| = \{x_{i,j,1}, \ldots, x_{i,j,m_{i,j}}\}$, such that $x_{i,j,k}$ and $x_{i',j',k'}$ are distinct unless $(i, j, k) = (i', j', k')$. Then we can define a partial ordering on

\[|P| = \{x_{i,j,k} | 1 \leq i \leq m_0, 1 \leq j \leq m_i, 1 \leq k \leq m_{i,j}\}\]

by letting

\[x_{i,j,k} \precsim x_{i',j',k'}\] if and only if either

\[i \neq i'\] and $x_i \precsim_0 x_{i'}$, or

\[i = i'\] but $j \neq j'$, and $x_{i,j} \precsim_i x_{i,j'}$, or

\[i = i', j = j',\] and $x_{i,j,k} \precsim_{i,j} x_{i,j,k'}$.

It is not hard to see that the resulting poset $P$ can be looked at both as the lexicographic sum over $P_0$ of the posets $P_i \ast (P_{i,1}, \ldots, P_{i,m_i})$, and as the lexicographic sum over $P_0 \ast (P_{i,1}, \ldots, P_{m_0})$ of the posets $P_{i,j}$. \(\square\)
Denoting their common value $P_0 \ast (P_1)_{1 \leq i \leq m_0} \ast (P_{i,j})_{1 \leq i \leq m_0, 1 \leq j \leq m_1}$, we thus have

$$P_0 \ast ((P_1) \ast (P_{i,j}))_{1 \leq j \leq m_1})_{1 \leq i \leq m_0} = (P_0 \ast (P_1)_{1 \leq i \leq m_0}) \ast (P_{i,j})_{1 \leq i \leq m_0, 1 \leq j \leq m_1} = (P_0 \ast (P_1)_{1 \leq i \leq m_0}) \ast (P_{i,j})_{1 \leq i \leq m_0, 1 \leq j \leq m_1}. $$

The equality between the first and last lines of (26) constitutes an associative law for lexicographic sums. (Of course, if one is given families of posets essentially as above, but without the disjointness assumptions on their underlying sets, or the indexing using natural numbers, one can construct lexicographic sums using ordered tuples, as at the beginning of §4.1, and one gets natural isomorphisms rather than equalities in (26).)

Now let us suppose each of the posets $P_{i,j}$ is a chain $C_{i,j}$ of $m_{i,j}$ elements, but make no such assumption on $P_0$ or the $P_i$. Then if we apply the function $L_\pm$ to (26), the final term is the function $L_\pm(P_0 \ast (P_1)_{1 \leq i \leq m_0}; m_{1,1}, \ldots, m_{m_0,m_0})$, while the initial term can be computed using Theorem 6 from the functions $L_\pm(P_0)$ and $L_\pm(P_i; m_{i,1}, \ldots, m_{i,m_i})$ ($i = 1, \ldots, m_0$).

Thus, if we know the functions $L_\pm(P; m_{1,1}, \ldots)$ for some family of finite posets $P$, we can get the corresponding function for any poset constructed as a lexicographic sum of members of that family over a member of that family; and, more generally, for any poset obtained in that way by iterated lexicographic sums.

In particular, since Propositions 8 and 9 give formulas for $L_\pm(P; m_{1,1}, \ldots)$ when $P$ is a finite chain or antichain, we can use the above technique to get such formulas for all posets constructed from chains and antichains by iterated lexicographic sums.

For example, consider the poset $P = \bigsqcup$. If we name the bottom two elements $x_{1,1}$ and $x_{1,2}$, and the top two $x_{2,1}$ and $x_{2,2}$, then our poset is the lexicographic sum of two antichains with underlying sets $\{x_{1,1}, x_{1,2}\}$ and $\{x_{2,1}, x_{2,2}\}$, over a chain whose element-set we may label $\{x_1, x_2\}$. With the help of Propositions 8 and 9, we find that $L_+(P; m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2}) = (m_{1,1} + m_{1,2})(m_{2,1} + m_{2,2})$, while $L_-(P; m_{1,1}, m_{1,2}, m_{2,1}, m_{2,2})$ is zero if either both $m_{1,1}$ and $m_{1,2}$ are odd, or both $m_{2,1}$ and $m_{2,2}$ are odd, while it is $\frac{1}{2}(m_{1,1} + m_{1,2})(m_{2,1} + m_{2,2})$ otherwise. Using iterated lexicographic sums, one can build up from chains and antichains arbitrarily complicated posets for which these functions can similarly be computed. These are called “series-parallel” posets in [2, Chapter 9, Exercise 6].

But “most” finite posets are not series-parallel; the simplest example is $P = \bigsqcup$. In fact, it is shown in [2, Chapter 9, Exercises 6–7] that a finite poset is series-parallel if and only if it does not contain a copy of that 4-element poset. (For further results on the characterization of classes of posets by “forbidden” subposets, see [3].) I have not studied the function $L_\pm(P; m_{1,1}, m_{2,1}, m_{3,1}, m_{4,1})$ determined by that 4-element poset; it would be interesting to describe it.

6. Appendix: A Formula of G. Hochschild

The results of §1 were motivated by a question Arthur Ogus asked me, on how one might understand, computationally, a formula of Gerhard Hochschild. In this appendix, which assumes only that section, we recover that formula.

Our development is far lengthier than Hochschild’s, so its interest (if any) lies in its different approach to the result, and in the possibility that the method may be applicable to questions not as easy to answer by other means.

Hochschild’s result (in which I have changed almost all the notation — but the translation between his and mine is straightforward) concerns an associative ring $R$ of prime characteristic $p$, a commutative subring $A$ of $R$, and an element $r \in R$ such that the commutator map $a \mapsto ad_r(a) = ra - ar$ carries $A$ into itself. What he shows is that for all $a \in A$,

$$ (ar)^p = a^p r^p + ad_r^{p-1}(a) r. $$

Since every term of (27) ends with a factor $r$, Ogus suggested that the corresponding identity with those factors removed should hold, namely

$$ (ar)^{p-1} a = a^p r^{p-1} + ad_r^{p-1}(a). $$

We shall see that this is true. Precisely, dropping the assumption that $R$ has characteristic $p$, we shall prove
Lemma 15 (after Hochschild [1, Lemma 1]). Let $R$ be an associative ring, $r$ an element of $R$, $A$ a commutative subring of $R$ such that the operation $ad_r : a \mapsto ra - ar$ carries $A$ into itself, and $p$ a prime number. Then the function taking every $a \in A$ to the element

$$\tag{29}(ar)^{p-1}a - a^p r^{p-1} - ad_r^{p-1}(a)$$

of $R$ can be written as a noncommutative polynomial in $r$ and the elements $a, \ ad_r(a), \ ad_r^2(a), \ldots$, in which the coefficients of all monomials are divisible by $p$.

Proof. Let us consider the term $(ar)^{p-1}a$ of (29), and repeatedly use the formula

$$\tag{30}rx = xr + ad_r(x) \quad \text{for} \quad x \in A$$

to eliminate occurrences of $r$ preceding elements of $A$. We begin with the rightmost occurrence of $r$, which precedes the final $a$; an application of (30) to that pair of factors turns $(ar)^{p-1}a$ into a sum of two monomials, in one of which that $r$ has jumped to the end, while in the other, it has been absorbed in the process of turning the final $a$ to $ad_r(a)$. We then apply (30) to the $r$ that was originally second from the right. This can either be absorbed in the $a$ immediately to its right, turning that into $ad_r(a)$, or jump past it. In the latter case it can, in turn, either be absorbed in the next factor (which is $a$ or $ad_r(a)$ depending on which output of the first step we are looking at, and which is turned into $ad_r(a)$ or $ad_r^2(a)$ respectively), or jump past that factor, becoming an (additional) $r$ at the far right. We proceed similarly with the third $r$ from the right, and so on. Since there are two possibilities for the fate of the rightmost $r$, three for the next, etc., we get $p!$ terms.

The choices leading to one of these $p!$ terms can be represented visually by taking the given string

$$\tag{31}ara \ldots ra \quad \text{with} \quad p \ a's \text{ and } p-1 \ r's$$

and drawing, from each member of some subset of the occurrences of $r$, an arrow to some occurrence of $a$ to its right. In the resulting expression, each $a$ that receives $m$ arrows becomes $ad_r^m(a)$ (with those receiving none remaining as $a$), and the $r$'s at the other ends of those arrows are deleted, while those $r$'s from which arrows were not drawn move to the far right.

Note that the output of this process has exactly one term in which none of the $p-1$ $r$'s acts on any of the $a$'s, and that term, $a^p r^{p-1}$, is cancelled by the $-a^p r^{p-1}$ of (29).

At the opposite extreme are the $(p-1)!$ terms in which every occurrence of $r$ acts on an element of $A$ (either an $a$, or the image of $a$ under previous actions of other occurrences of $r$). I claim that the sum of these is precisely $ad_r^{p-1}(a)$, and thus cancels the $-ad_r^{p-1}(a)$ of (29). To see this, note first that for any $x \in A$, we have $ad_r(x) = arx - xar = arx - axr = a ad_r(x)$, where the middle equality holds because $A$ is commutative. Hence $ad_r^{p-1}(a)$ can be written as

$$\tag{32} (a ad_r(\ldots(a ad_r(a ad_r(a)))\ldots))$$

Here the rightmost (better: innermost) occurrence of $ad_r$ acts on its argument $a$. The next occurrence acts on the product $a ad_r(a)$, so since $ad_r$ is a derivation, i.e., satisfies

$$\tag{33} ad_r(xy) = ad_r(x)y + x ad_r(y) \quad \text{for} \quad x,y \in A,$$

it turns $a ad_r(a)$ into a sum of two terms, in one of which it acts on the first factor and in the other on the second. The action of the next $ad_r$, on the product of $a$ with each of these two-factor terms, gives a sum of three terms; and so on. The resulting terms can be classified by writing out the string

$$\tag{34} a \ ad_r \ldots a \ ad_r \ a \ ad_r \ a \quad \text{with} \quad p \ a's \text{ and } p-1 \ ad_r's$$

and drawing an arrow from each $ad_r$ to an arbitrary $a$ to the right of it, on which it acts. The results are clearly the same as the expressions in our expansion of $(ar)^{p-1}a$ in which an arrow comes out of every occurrence of $r$, so, as desired, these terms cancel the $-ad_r^{p-1}(a)$ in (29).

What remains is to show that in the expansion of $(ar)^{p-1}a$, each monomial

$$\tag{35} ad_r^{m_1}(a) \ ad_r^{m_2}(a) \ ad_r^{m_k}(a) \ r^{p-1-m_1-\cdots-m_k}$$

that is not of either of the above two sorts, i.e., which satisfies

$$\tag{36} m_1 + \cdots + m_k \text{ is greater than } 0, \text{ but less than } p-1,$$
occurs with coefficient divisible by \( p \). Since \( A \) is commutative, we are regarding monomials \( (35) \) as the same if they differ by a permutation of the string of exponents \( m_1, \ldots, m_k \). Thus, we may make the notational assumption that the nonzero exponents in \( (35) \) form an initial substring, say \( m_1, \ldots, m_t \).

To determine the coefficient of a monomial \( (35) \), we need to count the ways of attaching arrows to \( (31) \) that lead to it. I claim such a system of arrows will be determined by an appropriately indexed family of elements of \( \{1, \ldots, p\} \) (corresponding to the positions of the \( r \)'s from which arrows begin, and the \( a \)'s at which they end) subject to certain inequalities – and that the set of these indexed families is the sort of object whose cardinality was studied in \( \S 1 \).

Indeed, consider any monomial \( (35) \), and let \( P \) be a partially ordered set consisting of \( \ell \) components, namely, for each \( i \leq \ell \), let the component \( P_i \) be a chain of \( m_i + 1 \) elements.

Let us, to begin with, assume for simplicity that the nonzero exponents in \( (35) \), \( m_1, \ldots, m_t \), are distinct. Then to each diagram of arrows on the string of symbols \( (31) \) that yields the monomial \( (35) \), we associate the map \( \varphi : P \rightarrow \{1, \ldots, p\} \) such that for \( i = 1, \ldots, \ell \), if the factor \( a \) \( m_i(a) \) in \( (35) \) arises by actions of the \( k_1 \)-th, \( k_2 \)-nd, through \( k_{m_i} \)-th occurrences of \( r \) on the \( k_{m_i+1} \)-st occurrence of \( a \), then \( \varphi \) maps the successive terms of the chain \( P_i \) to the integers \( k_1 < k_2 < \cdots < k_{m_i} < k_{m_i+1} \).

Which maps \( P \rightarrow \{1, \ldots, p\} \) can arise in this way from arrow-diagrams giving the monomial \( (35) \)? It is not hard to see that they will be precisely those isotone maps such that no two elements of \( P \) fall together, except that the maximal element of a component \( P_i \) is permitted to fall together with a nonmaximal element of a component \( P_j \) if \( j \neq i \). (The images of non-maximal elements of \( P \) all have to be distinct because they represent the sources of distinct arrows, and the images of maximal elements must be distinct because they represent the recipients of distinct families of arrows. Finally, the images of a non-maximal element and the maximal element in the same component must be distinct because the \( k \)-th occurrence of \( r \) can’t have an arrow to the \( k \)-th occurrence of \( a \), since the latter precedes it. On the other hand, there is no contradiction if for some \( k \), the \( k \)-th \( a \) is the recipient of some family of arrows, and the \( k \)-th \( r \) is the source of an arrow with a different destination; these are the cases where elements of \( P \) are allowed to fall together.) So letting \( S = ([P], \leq, E) \), where \( ([P], \leq) \) is the poset \( P \) described above, and \( E \) consists of all two-element subsets of \( |P| \) other than those whose members are the maximal element of one component of \( P \) and a nonmaximal element of a different component, we see that the coefficient of \( (35) \) in \( (ar)^{p-1}a \) is \( C(S, p) \), as defined in Theorem 1.

The partially ordered set \( P \) has \( (m_1 + 1) + \cdots + (m_\ell + 1) = m_1 + \cdots + m_\ell + \ell \) elements, and \( \ell \) connected components, so the term written \( |S| - c + 1 \) in Corollary 2 is here \( m_1 + \cdots + m_\ell + 1 \). By the second condition of \( (36) \), \( m_1 + \cdots + m_\ell \) is strictly less than \( p - 1 \), hence \( m_1 + \cdots + m_\ell + 1 \) is strictly less than \( p \), so \( p \) satisfies the condition in the first sentence of Corollary 2. By the first condition of \( (36) \), \( P \) is nonempty, so we can apply the final statement of that corollary to conclude that \( C(S, p) \) is divisible by \( p \), as desired.

What if the \( m_i \) are not all distinct? If a given value \( m \) occurs as \( h_m \) different \( m_i \)’s (i.e., if after collecting like factors in \( (35) \), \( a \) \( m_i(a) \) appears with exponent \( h_m \)), then the poset \( P \) constructed as above will have \( h_m \) \( m+1 \)-element components. In this situation, for each way the \( h_m \) factors \( a \) \( m_i(a) \) can arise from an arrow-diagram, we can choose, arbitrarily, which of those \( h_m \) components of \( P \) is mapped to which family of \( m \) arrows. This gives \( h_m ! \) possibilities. We see from this that the coefficient of our monomial \( (35) \) in the expansion of \( (ar)^{p-1}a \) will now be \( C(S, n) / \prod_m h_m ! \). But this creates no problem, since each of the \( h_m \) is less than \( p \). (Indeed, by \( (36) \), \( p - 1 > m_\ell \geq h_m m \geq h_m \) for each \( m \).) So since \( C(S, n) \) is divisible by \( p \), \( C(S, n) / \prod_m h_m ! \) is also, as required.

We remark that in the last paragraph of the above proof, we could have avoided dividing by \( \prod_m h_m ! \) if we had strengthened the ordering of our poset \( P \) by imposing, for each \( m \), an arbitrary total ordering on the maximal elements of the \( h_m \) chains of length \( m + 1 \). However, this would have decreased the number \( c \) of connected components of our poset, and we could no longer have asserted that \( p > \text{card}(|S|) - c + 1 \). (For instance, if \( p = 5 \), \( \ell = 3 \), and \( m_1 = m_2 = m_3 = 1 \), so that \( h_1 = 3 \), we would have had \( \text{card}(|S|) - c + 1 = 6 - 1 + 1 = 6 > p \).) Writing \( S' \) for the \( ([P], \leq, E) \) of the proof of Lemma 15, and \( S' \) for the variant with order-relation strengthened as described, we have \( C(S', n) = C(S, n) / \prod_m h_m ! \), and since we have shown that for \( n = p \), the latter value is divisible by \( p \), the same is true of the former. But I don’t see any natural strengthening of Theorem 1 or Corollary 2 that would give this fact directly.
7. Appendix: Notes on Lemma 13

I don’t know whether Lemma 13 has uses other than as a tool for proving Corollary 14; but it has piqued my curiosity, and I give below several related observations.

First, some quick examples. For a case of (15) in which one, but not the other of \( \preceq_{x,y} \) and \( \preceq_{y,x} \) has the property asserted there, let \( P \) be the 4-element poset \( \mathcal{N} \), let \( C \) be the 2-element chain in the middle of that picture, and let \( x \) and \( y \) be (necessarily) the two elements of \( P - C \). The reader can easily check the details.

For a case of the lemma itself in which the set of elements of \( C \) incomparable with one or more elements of \( P \) must always decrease when \( P - C \) is made a chain (in contrast to the assertion of the lemma, about elements of \( P \) incomparable with elements of \( C \)), let \( P \) be the disconnected union of a 3-element chain and a singleton, and take for \( C \) the subchain consisting of the top and bottom elements of the 3-element component.

Finally, for an example showing that the statement of the lemma fails if we drop the assumption that \( C \) is a chain, let \( P \) be the disconnected union of two 2-element chains, and let \( C \) consist of the two minimal elements of \( P \). Then under any strengthening of the ordering of \( P \) that makes \( P - C \) a chain, the larger element of \( P - C \) will lie above, and in particular, be comparable with, both elements of \( C \), though it was incomparable with one of them under the original ordering. (Note that under the full hypotheses of the lemma, the conclusion about elements of \( P \) incomparable with at least one element of \( C \) is equivalent to the same statement about elements of \( P - C \) incomparable with at least one element of \( C \). Without the hypothesis that \( C \) is a chain, the latter version is a priori weaker than the statement for arbitrary elements of \( P \); but the example shows that even that weaker version fails.)

It is not obvious from the proof we gave of Lemma 13 how to tell, given elements \( x \) and \( y \), which of the orderings \( \preceq_{x,y} \) and \( \preceq_{y,x} \) has the property asserted in (15), or whether both do. The equivalence \( (a) \iff (c) \) of the next corollary, and its variant with the roles of \( x \) and \( y \) reversed, give a simple criterion for one or the other of those possibilities to be excluded.

Corollary 16 (to proof of Lemma 13). Let \( P = (|P|, \preceq) \) be a finite poset and \( C \) a chain in \( P \), let \( x \) and \( y \) be two \( \preceq \)-incomparable elements of \( P - C \), and let \( \preceq_{x,y} \) be the strengthened partial ordering on \( P \) gotten by imposing the relation \( x \preceq_{x,y} y \) (described by (16)). Then the following three conditions are equivalent.

(a) There is some \( u \in |P| \) that is incomparable under \( \preceq \) with at least one element of \( C \), but is comparable under \( \preceq_{x,y} \) with every element of \( C \).

(b) One of \( x \) or \( y \) has the property that it is incomparable under \( \preceq \) with at least one element of \( C \), but is comparable under \( \preceq_{x,y} \) with every element of \( C \).

(c) \( (\downarrow x \cap |C|) \cup (\uparrow y \cap |C|) = |C| \), but \( (\downarrow y \cap |C|) \cup (\uparrow x \cap |C|) \neq |C| \). (Cf. (21), (23)).

Proof. We shall prove \((a) \implies (c) \implies (b) \implies (a)\).

Given (a), the proof of Lemma 13 gives us the equality (21) with which (c) begins. On the other hand, if alongside (21) we have equality instead of inequality in the second condition of (c) (i.e., if (23) also holds), then the two paragraphs of that proof following (23) show that \( \preceq_{x,y} \) does not introduce any relations between elements of \( P - C \) and elements of \( C \) that don’t hold under \( \preceq \), contradicting (a). So we must also have that inequality; so (c) indeed holds.

Next, assume (c). Note that in the equality \( (\downarrow x \cap |C|) \cup (\uparrow y \cap |C|) = |C| \), the sets \( \downarrow x \cap |C| \) and \( \uparrow y \cap |C| \) must be disjoint, since if their intersection contained an element \( z \), we would have \( y \preceq z \preceq x \), contradicting our general assumption that \( x \) and \( y \) are \( \preceq \)-incomparable. The inequality \( (\downarrow y \cap |C|) \cup (\uparrow x \cap |C|) \neq |C| \) says we can find a \( w \in |C| \) in neither \( \downarrow y \) nor \( \uparrow x \). By the preceding observation, \( w \) lies either in \( \downarrow x \) or in \( \uparrow y \), but not in both. If it is in \( \downarrow x \) but not \( \uparrow y \), then it is in neither \( \uparrow y \) nor \( \downarrow y \), showing that \( y \) is incomparable with at least one element of \( C \). Now every \( c \in |C| \) incomparable with \( y \) must, by the relation \( (\downarrow x \cap |C|) \cup (\uparrow y \cap |C|) = |C| \), lie in \( \downarrow x \), hence \( c \preceq_{x,y} x \preceq_{x,y} y \). So \( y \), though \( \preceq \)-incomparable with some elements of \( |C| \), is \( \preceq_{x,y} \)-comparable with all such elements, which is the “\( x \)” case of (b). If, on the other hand, our element \( w \) lies in \( \uparrow y \) rather than \( \downarrow x \), the corresponding considerations give the “\( y \)” case of (b). Thus, we have proved \((c) \implies (b)\).

The implication \((b) \implies (a)\) is trivial. \(\square\)

The proof of Lemma 13 shows how to build up all linearizations of \( |P| - |C| \) which extend to orderings of \( |P| \) of the sort asserted in the lemma, by making successive choices of order on unordered pairs of elements of
$|P| - |C|$, taken in any order. The above corollary tells us at each such step which choices are available. We end this section with a more systematic construction of orderings as in that lemma, based on a suggestion of Stefan Felsner (personal correspondence).

**Sketch of an alternative proof of Lemma 13, after S. Felsner.** Listing the elements of $C$ as $c_1 \preceq \ldots \preceq c_r$, let us partition $|P| - |C|$ into disjoint subsets

$$\text{(37)} \quad |P| - |C| = |S_1| \cup |S_2| \cup \ldots \cup |S_{2r+1}|$$

as follows. If $x \in |P| - |C|$ is already $\preceq$-comparable with all elements of $C$, say with $c_i \preceq x \preceq c_{i+1}$, we assign $x$ to $S_{2i+1}$, with the obvious modifications in the end-cases, namely, when $x \preceq c_1$ we assign it to $S_1$, and when $c_r \preceq x$ we assign $x$ to $S_{2r+1}$. On the other hand, if $x$ is incomparable with at least one element of $C$, let $c_i$ be the largest such element, and assign $x$ to $S_{2i}$.

It is not hard to check that for $1 \leq i < j \leq 2r + 1$, no element of $S_i$ is $\preceq$ any element of $S_j$. Hence we can strengthen the ordering $\preceq$ on $|P| - |C|$ to make all elements of $S_i$ precede all elements of $S_j$ whenever $i < j$, keeping the relative order of elements within each $S_i$. We can then go further and linearize each $S_i$, getting a total order $\preceq'$ on $|P| - |C|$.

On $|C|$, on the other hand, we let $\preceq'$ agree with $\preceq$, since $\preceq$ is already a total order there.

It remains to specify how $\preceq'$ should relate elements of $|P| - |C|$ and elements of $|C|$. If $x \in |P| - |C|$ belongs to a set $S_{2i+1}$, there is no choice: under $\preceq$, $x$ lies above all $c_j$ with $j \leq i$ and below all $c_j$ with $j \geq i + 1$, so we give it these same relations under $\preceq'$.

If $x \in S_{2i}$, we must again let $x$ lie below all $c_j$ with $j \geq i + 1$. In this case, there may or may not be choices as to how it should relate to lower members of $C$; but we make a choice that will always work: let $x$ be incomparable with $c_i$, and above all $c_j$ with $j < i$.

It is routine, though tedious, to verify that the relation $\preceq'$ so defined is a partial ordering on $|P|$. By construction, it is a strengthening of $\preceq$ which makes $|P| - |C|$ a chain, and has the property that every element of $|P| - |C|$ that was $\preceq$-incomparable with at least one element of $|C|$ (i.e., which belongs to some $|S_2|$), remains incomparable with some element of $|C|$ (namely, $c_i$). This completes the proof of the lemma.

\[ \square \]

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