Comments, corrections, and related references welcomed, as always!

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THE FOUR SORTS OF LENGTH-TWO MODULES

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Abstract. We consider modules M of length 2 over an arbitrary ring R. These can be classified as to whether M has one or more than one length-1 submodules, and also as to whether the length-1 subfactors of M belong to one or two isomorphism classes, giving four sorts of length-2 modules altogether.

We give several characterizations of each of these four classes, and note examples and properties of such modules. (Among modules over commutative rings, only three of these four sorts occur.)

We pose the question of how to enumerate the analogous classes for modules of larger finite lengths.

1. Main results

Conventions. *Rings* are associative and unital; modules will mean *right* modules; a *simple* module is a nonzero module with no proper nonzero submodules.

A module M is said to have *finite length* if it has a finite composition series (a chain $\{0\} = M_0 < M_1 < \cdots < M_d = M$ with all factor-modules M_i/M_{i-1} simple). By the Jordan–Hölder theorem [2, Theorem 3.2.1], [5], for each finite-length module M, the number d of steps is the same in all composition series. Their common value is called the *length* of M.

The automorphism group and the endomorphism ring of a module M will be denoted $\mathrm{Aut}(M)$ and $\mathrm{End}(M)$ respectively.

Clearly,

(1) A module N is simple if and only if it has length 1.

We note that

(2) The endomorphism ring of a simple module N is a division ring.

Indeed, N being simple implies that every nonzero endomorphism has all of N as image and has zero kernel, hence is invertible.

The four sorts of length-2 modules are characterized in the following four propositions.

Proposition 1. *If M is an R-module of length 2, the following conditions are equivalent:*

(i) *M is the direct sum of two isomorphic length-1 submodules.*
(ii) *M has two distinct length-1 submodules which are isomorphic to one another.*
(iii) *M has more than two length-1 submodules.*
(iv) *$\mathrm{End}(M)$ is isomorphic to the matrix ring $\mathrm{M}_2(D)$ for some division ring D.*
(v) *$\mathrm{End}(M)$ has at least one noncentral idempotent.*

Proof. Clearly, (i) $\Longrightarrow$ (ii).

To get (ii) $\Longrightarrow$ (iii), let $i : N_0 \to N_1$ be an isomorphism between distinct length-1 submodules of M. Since N_0 and N_1 are distinct and of length 1, they have intersection $\{0\}$, hence as M has length 2, $M = N_0 \oplus N_1$, and the graph of i in that direct sum is a length-1 submodule distinct from both summands, giving (iii).

To get (iii) $\Longrightarrow$ (i), let N_0, N_1, N_2 be distinct length-1 submodules of M. As above, the length hypotheses imply that $M = N_0 \oplus N_1$. It is easy to deduce that $N_2 < N_0 \oplus N_1$ must be the graph of an isomorphism between N_0 and N_1, giving (i). So (i)-(iii) are equivalent.

In view of (2), (i) $\Longrightarrow$ (iv), and clearly (iv) $\Longrightarrow$ (v). We shall complete our proof by showing (v) $\Longrightarrow$ (ii).

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Assuming (v) let $e \in \mathrm{End}(M)$ be a noncentral idempotent. Since e can be neither the identity nor the zero endomorphism, it must be the projection of M onto a summand N_0 along a summand N_1, both of which, being nonzero proper submodules of M, must have length 1. Let $u \in \mathrm{End}(M)$ fail to commute with e. Then u must fail to carry either N_0 or N_1 into itself. In the former case, $u(N_0)$, being a nonzero image of the length-1 module N_0, must be isomorphic thereto, and since it is distinct from N_0, this proves (ii); in the latter case, the corresponding observation holds for $u(N_1)$. □

Proposition 2. *If M is an R-module of length 2, the following conditions are equivalent:*

(i) *M is the direct sum of two nonisomorphic length-1 submodules.*
(ii) *M has two nonisomorphic length-1 submodules.*
(iii) *M has two and only two length-1 submodules.*
(iv) *$\mathrm{End}(M)$ is the direct product $D_0 \times D_1$ of two division rings.*
(v) *$\mathrm{End}(M)$ has a central idempotent e other than 0 and 1.*

Proof. As noted in the proof of Proposition 1, if a length-2 module M has two distinct length-1 submodules, it will be their direct sum, and other length-1 submodules of M, if any, will be the graphs of isomorphisms between them. From this, the equivalence of (i)-(iii) is clear.

Since there can be no nonzero homomorphisms between nonisomorphic length-1 modules, we see that in view of (2), (i) $\Longrightarrow$ (iv); and clearly (iv) $\Longrightarrow$ (v). Given e as in (v), we get a direct sum decomposition $M = eM \oplus (1-e)M$, with both summands necessarily of length 1. If, in contradiction to (iii), M had another length-1 submodule N, then N would be the graph of an isomorphism from eM to $(1-e)M$, which would induce an endomorphism of M carrying the former submodule to the latter; and such a homomorphism would not commute with e, contradicting (v). So (v) $\Longrightarrow$ (iii), completing the proof. □

Proposition 3. *If M is an R-module of length 2, the following conditions are equivalent:*

(i) *M has a unique length-1 submodule N, and $M/N \cong N$.*
(ii) *$\mathrm{End}(M)$ has no idempotents other than 0 and 1, but does have a nonzero nilpotent element.*
(iii) *$\mathrm{End}(M)$ has no idempotents other than 0 and 1, but does have a nonzero noninvertible element.*

Proof. Given the length-2 hypothesis, the first assertion of (i) says that M has only one proper nonzero submodule, hence it cannot be decomposed as a direct sum of two such modules, giving the part of (ii) concerning idempotents. The composite of the maps $M \twoheadrightarrow M/N \xrightarrow{\sim} N \hookrightarrow M$ gives the nonzero nilpotent element of (ii), so (i) $\Longrightarrow$ (ii). Clearly, (ii) $\Longrightarrow$ (iii).

Note that since M has length 2, it must have a length-1 submodule. If it had more than one, then in view of the length hypotheses, it would be the direct sum of any two of them, and the projection onto one of these direct factors would be an idempotent endomorphism. Hence the "no idempotents" condition of (iii) implies the first assertion of (i). Given that assertion, and letting $u \in \mathrm{End}(M)$ be a nonzero noninvertible element as in (iii), both the kernel and the image of u must be the N of (i), so

$$M/N \;=\; M/\ker(u) \;\cong\; u(M) \;=\; N, \tag{3}$$

giving the second assertion of (i). □

We now come to the last case, which in some ways is the most peculiar (we will see in the next section that there are no such examples if R is commutative), but which nevertheless has the most elegant equivalent condition (ii):

Proposition 4. *If M is an R-module of length 2, the following conditions are equivalent:*

(i) *M has a unique length-1 submodule N, and $M/N \ncong N$.*
(ii) *$\mathrm{End}(M)$ is a division ring.*
(iii) *$\mathrm{End}(M)$ has no idempotents other than 0 and 1, nor any nonzero nilpotents.*

Proof. Given the first condition of (i), any nonzero non-one-to-one endomorphism of M would have to have N as both its kernel and its image, but this would contradict the second condition of (i); so every nonzero endomorphism of M is one-to-one, which, in view of the finite length of M, means it must be invertible, giving (ii). Clearly, (ii) $\Longrightarrow$ (iii).

Assuming (iii) let us now prove (i). As in the proof of Proposition 3, the first condition of (iii) implies the first condition of (i). If the second condition of (i) failed, we would thus be in the situation of Proposition 3(i),

hence by condition (ii) of that result, M would have a nonzero nilpotent endomorphism, contradicting (iii). So (iii) indeed implies (i). □

From the statements (i) of the above four propositions, we see that

(4) Every length-2 module M over a ring R satisfies the equivalent conditions of one and only one of Propositions 1-4.

2. Further observations.

Properties of length-1 modules

On the one hand, let us observe that,

(5) If R is a *commutative* ring, or more generally, a ring in which every right ideal is 2-sided (e.g., a division ring), then for every length-1 R-module N, the group $\mathrm{Aut}(N)$ is transitive on nonzero elements of N. Equivalently, for all $x \in N \setminus \{0\}$, the map $\mathrm{End}(N) \to N$ given by $\alpha \mapsto \alpha(x)$ is bijective.

Indeed, under the hypothesis on R, if N is a length-1 module, then given any $x \in N \setminus \{0\}$, i.e., any generator of N, consider any other element of $N \setminus \{0\}$, which we may write xu for some $u \in R$. The annihilator of x in R is a right ideal, hence by hypothesis, a 2-sided ideal, hence closed under *left* multiplication by u, hence every element of R annihilating x also annihilates xu; so we get an endomorphism of $N = xR$ taking x to xu, which, since xu was chosen to be nonzero and N has length 1, is an automorphism, giving the first assertion. Since $\mathrm{End}(N) = \mathrm{Aut}(N) \cup \{0\}$, the above result is equivalent to the surjectivity of the map $\mathrm{End}(N) \to N$ described. That map is also one-to-one, since an endomorphism of N is determined by where it takes the generator x.

On the other hand, for general noncommutative R the conclusions of (5) do not hold. For example, if for some $n > 1$ (e.g., $n = 2$) R is the ring $\mathrm{M}_n(D)$ of $n \times n$ matrices over a division ring (e.g., a field) D, and N is the right R-module D^n of length-n rows, which is simple as an R-module, then the annihilator in R of the element having a 1 in i-th position and 0's elsewhere clearly depends on i; so $\mathrm{Aut}(N)$ is far from transitive on nonzero elements. In fact, in this case $\mathrm{End}(N) \cong D$, where each $a \in D$ acts by multiplication of all coordinates by a on the left.

Back to length-2 modules.

The classification of length-2 modules developed in section 1 concerned their structure in terms of submodules, factor-modules, and endomorphisms, but not in terms of elements. If we brought in the latter, we would get further subdivisions.

In particular, the situation of Proposition 1, where M is the direct sum of two copies of a length-1 module N, could be subdivided, as in the above subsection, according to whether $\mathrm{Aut}(N)$ is transitive on nonzero elements of N. If it is not transitive, then $M \cong N \oplus N$ can be generated by a single element, namely any element a whose components in those two summands are in different orbits of $\mathrm{Aut}(N)$, since these will have different annihilators in R, so on multiplying a by an element annihilating one component but not the other, we get an element au with only one nonzero component, and we see that a and au generate M, hence so does a. On the other hand, if $\mathrm{Aut}(N)$ is transitive on nonzero elements, it is easy to see that $N \oplus N$ cannot be generated by a single element.

Proposition 1 also ignores the possibility of subdividing the class of modules M considered there in terms of *how many* length-1 submodules M has. Namely,

(6) For M as in Proposition 1, the number of length-1 submodules of M is $1 + \mathrm{card}(\mathrm{End}(N)) = 2 + \mathrm{card}(\mathrm{Aut}(N))$.

This can be seen by writing $M \cong N \oplus N$, and noting that its length-1 submodules other than $\{0\} \oplus N$ are precisely the graphs of endomorphisms of N.

Rings over which modules as in Proposition 4 do not / do exist.

We note

(7) If (as in (5) above) R is commutative, or, more generally, every right ideal of R is 2-sided, then R has no modules M satisfying the equivalent conditions of Proposition 4.

Indeed, if M is a length-2 module having a unique length-1 submodule N, let x be an element of $M \setminus N$. Looking at the set of submodules of M, we see that $xR = M$. For some $u \in R$ we will have

$$N \;=\; x\,u\,R, \tag{8}$$

and the "right ideal is 2-sided" condition of (7) implies that the annihilator in R of x is closed under left multiplication by members of R, hence annihilates xu, hence that $N = xuR$ is a homomorphic image of $xR = M$. But the kernel of a surjective homomorphism $M \to N$ must be a proper nonzero submodule of M, hence must be N, so $N \cong M/N$, i.e., Proposition 4(i) fails.

On the other hand, an easy class of examples of rings R *not* satisfying the hypothesis of (7) which *do* have modules M as in Proposition 4 was pointed out to me by both Ken Brown and Dolors Herbera: Let R be the ring of upper triangular 2×2 matrices over any division ring D, and M the right R-module of length 2 row vectors over D. It is easy to see that the elements that generate M are those with nonzero first component; hence that the elements with first component zero form the unique proper nonzero submodule. Calling that submodule N, we see that the matrix $e_{11} \in R$ acts as the identity on M/N but annihilates N; so M satisfies Proposition 4(i).

Is the structure of a length-2 module of a given type determined by the structures of its length-1 subfactors?

Given a ring R and a module M satisfying the equivalent conditions of Proposition 1, the structure of M is clearly determined by the isomorphism class to which all its length-1 submodules belong. Similarly, in the situation of Proposition 2, the structure of M is determined by the isomorphism classes of its two length-1 submodules.

On the other hand, in the situation of Proposition 3, the structure of M is *not* necessarily determined by the common isomorphism class of N and M/N. For example, let k be any field, and R the k-subalgebra of the matrix algebra $\mathrm{M}_4(k)$ spanned by the identity matrix I and the two elements e_{12} and e_{34}. Let M be the R-module of length-4 row vectors over k of the form $(*, *, 0, 0)$, and M' the R-module of vectors of the form $(0, 0, *, *)$. These are easily seen to have unique nontrivial proper submodules, N and N', consisting respectively of vectors of the forms $(0, *, 0, 0)$, and $(0, 0, 0, *)$. The modules N, M/N, N', and M/N' are all isomorphic, all being 1-dimensional vector spaces over $R/(k\,e_{12} + k\,e_{34}) \cong k$; but M is not isomorphic to M', since on the former, e_{12} acts nontrivially and e_{34} trivially, while in the latter the reverse is true.

Similarly, given a ring R and a module M as in Proposition 4, the isomorphism class of M need not be determined by the (distinct) isomorphism classes of M/N and N. Indeed, if one takes R to be the k-subalgebra of $\mathrm{M}_4(k)$ spanned by the four elements $e_{11} + e_{33}$, $e_{22} + e_{44}$, e_{12}, and e_{34}, and defines M, N, M' and N' exactly as in the preceding paragraph, one gets the desired example.

(In both the above examples, k can, more generally, be a division ring.)

What got me interested in this topic.

The inspiration for this note was a question posed by Friedrich Wehrung (personal communication): If R is a noncommutative right and left principal ideal domain, and an element $z \in R$ has a factorization $z = pq$, where p and q are irreducible elements that are not similar i.e., such that $R/pR \ncong R/qR$, must it also have a factorization $z = q'p'$ where $R/p'R \cong R/pR$ and $R/q'R \cong R/qR$? This led me to the observation that if it failed to have such a factorization, then R/pqR would be a "peculiar" length-2 R-module, namely it would satisfy the condition that is now Proposition 4(i); and this eventually led me to the classification of length-2 modules in the preceding section. (We did find examples of noncommutative principal ideal domains with elements of the desired sort; cf. [7, Proposition 4.7] and the sentence that follows it.)

A question.

It is natural to ask,

(9) For $n > 2$, if we classify modules of length n in terms of their behavior with respect to submodules and factor modules, in the way analogous to what is done for $n = 2$ in Propositions 1-4 above, is there an elegant way of enumerating the resulting system of classes?

Related literature.

Some concepts in the literature related to those considered here are those of a uniserial module (a module whose submodules form a chain), a serial module (a direct sum of uniserial modules), and the Loewy length of a module M (which, when M has finite length, is the least integer n such that M has a chain of submodules $\{0\} = M_0 < \cdots < M_n = M$ such that each factor module M_i/M_{i-1} is a *sum* of length-1 submodules).

Looking on MathSciNet, some articles concerning modules of length 2 are [1], [3], [4] and [6].

3. Acknowledgements

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