Comments, corrections, and related references welcomed, as always!

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GROUPS AND FLIP-SETS

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ABSTRACT. Call a sequence $(a_i)_{i\in\mathbb{Z}}$ of elements of a group G a *trajectory* if for some $g, h \in G$, $a_i = g\,h^i$, equivalently, if $a_{i+1} = a_i\,a_{i-1}^{-1} a_i$ for all i. The latter formulation suggests the study of the derived operation $g \natural h = g\,h^{-1} g$ on groups, intuitively, "flipping h past g".

We determine the identities satisfied by this operation on the underlying sets of all groups, and define a *flip-set* to be a set with a binary operation $\natural$ that satisfies those identities. Not every flip-set is isomorphic to, or embeddable in, the underlying flip-set of a group, but we describe a way of mapping any flip-set into the flip-set of a group which is often one-to-one or close to one-to-one.

We look at the relation between additional identities on groups and on their $\natural$-structures. In particular, the groups satisfying all the $\natural$-identities satisfied by abelian groups are the groups of nilpotency class ≤ 2, i.e., the groups in which all commutators are central.

Note added 20 Nov., 2019: *Yves de Cornulier and Ualbai Umirbaev have pointed me to the concept of* quandle, *noting that what I here call the flip-structure of a group G is called its* core quandle, $\mathrm{Core}(G)$. (*If you are not familiar with the concepts see the Wikipedia page* Racks and quandles, *and the German Wikipedia page* Quandle, *which includes the example* $\mathrm{Core}(G)$, *and/or search for the terms on MathSciNet.*) *I will contact some people in the area, and ask them what parts of what I have written up are new, and if these seem to merit it, I will produce a new version of this note.*

1. Basics

We note

Lemma 1. *Let G be a group, and $(a_i)_{i\in\mathbb{Z}}$ a sequence of elements of G. Then the following conditions are equivalent.*

(i) *For some g, h in G, we have $a_i = g\,h^i$ for all $i \in \mathbb{Z}$.*
(i′) *For some g, h in G, we have $a_i = h^i g$ for all $i \in \mathbb{Z}$.*
(i″) *For some g, g', h in G, we have $a_i = g\,h^i g'$ for all $i \in \mathbb{Z}$.*
(ii) $a_{i+1} = a_i\,a_{i-1}^{-1}\,a_i$ *for all $i \in \mathbb{Z}$.*
(ii′) $a_{i-1} = a_i\,a_{i+1}^{-1}\,a_i$ *for all $i \in \mathbb{Z}$.*

Proof. Trivially, (i) $\implies$ (i″). Conversely, the equation of (i″) can be written $a_i = (gg')(g'^{-1}h\,g')^i$, an instance of (i); so (i) and (i″) are equivalent. Similarly, (i′) and (i″) are equivalent.

The equivalence of (ii) with (ii′) can be seen by solving the former relation for a_{i-1}.

It is straightforward to check that (i) implies (ii). Conversely, if (ii), and hence (ii′), hold, we claim that (i) holds with $g = a_0$, $h = a_0^{-1}a_1$. Indeed, for these g and h, the formula of (i) gives a sequence which agrees with (a_i) at $i = 0$, 1, and satisfies (ii) and (ii′). Hence by upward and downward induction on i, it agrees everywhere with (a_i). □

Definition 2. *A sequence $(a_i)_{i\in\mathbb{Z}}$ of elements of a group G satisfying the equivalent conditions of Lemma 1 will be called a* trajectory *in G.*

(The term "progression" has been used for the related concept of a subset of a group of the form $\{g\,h^i \mid 0 \leq i < n\}$; e.g., see [2].)

We note that

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Lemma 3. *If G is a group, the class of trajectories in G is closed under*

(i) *left and right translations:* $(a_i) \mapsto (f a_i)$ *and* $(a_i) \mapsto (a_i f)$ *for* f *in* G,

(ii) *termwise inversion:* $(a_i) \mapsto (a_i^{-1})$,

(iii) *shifts:* $(a_i) \mapsto (a_{i+n})$ *for* $n \in \mathbb{Z}$,

(iv) *the operations* $(a_i) \mapsto (a_{ni})$ *for* $n \in \mathbb{Z}$. *In particular, taking* $n = -1$, *the class of trajectories is closed under the reversal operation* $(a_i) \mapsto (a_{-i})$. □

On the other hand, if (a_i) and (b_i) are trajectories, the sequence $(a_i b_i)$ need not be. For instance, if elements x and y of a group G do not commute, and (a_i), (b_i) are the trajectories (x^i) and (y^i) respectively, we find that $(c_i) = (x^i y^i)$ does not satisfy condition (ii) of Lemma 1 for any i. Likewise, the termwise square of the trajectory $(x\, y^i)$ will not be a trajectory unless y and $x\, y\, x^{-1}$ commute.

I came upon the concept of trajectory in thinking about conditions on a group related to one-sided orderability. For instance, what is called "locally invariant orderability" of a group [3] is equivalent to the existence of a total ordering under which each trajectory is either monotone increasing, monotone decreasing, or decreasing up to a certain term and increasing thereafter. This condition is implied by the existence of a one-sided ordering; an intermediate condition is the existence of an ordering under which every trajectory is everywhere monotone increasing or decreasing.

I was not able to determine whether either of the above implications is reversible, and will not further discuss orderability questions here. However, the binary operation $(g,h) \mapsto g\, h^{-1} g$ suggested by Lemma 1 (ii) and (ii′) turned out to have some interesting properties, examined below. I find the concept of a trajectory sometimes useful for picturing the subject, but the binary operation simpler to work with. So let us make

Definition 4. *If G is a group, we will denote by* $\natural$ *the derived binary operation on the underlying set of G given by*

(1) $$g \natural h \;=\; g\, h^{-1} g,$$

and call this the flip operation *of G. (The idea is that* $g \natural h$ *represents the result of "flipping* h *past* g*".)*

The flip operation satisfies identities loosely paralleling the three identities defining groups:

Lemma 5. *The identities satisfied by the derived operation* $\natural$ *of all groups are the consequences of the three identities*

(2) $$g \natural g \;=\; g,$$

(3) $$g \natural (g \natural h) \;=\; h,$$

(4) $$(f \natural g) \natural h \;=\; f \natural (g \natural (f \natural h)).$$

Any word in a set X of symbols and the operation-symbol $\natural$ *can be reduced, using these identities, to a unique expression with parentheses clustered on the right,*

(5) $$x_0 \natural (x_1 \natural (\ldots \natural (x_{n-1} \natural x_n) \ldots)),$$ *where all* $x_i \in X$, *and no two successive terms* x_i, x_{i+1} *are the same.*

Proof. The verification of the identities (2)-(4) for the operation (1) is immediate. Postponing the claim that (2)-(4) imply all identities of that operation, we note that given any $\natural$-word in symbols from X, (4) can be used recursively to reduce it to one in which parentheses are clustered to the right, (3) can then be used recursively to eliminate cases where $x_{i-1} = x_i$ for some $i < n$, and, finally, (2) can be used recursively to eliminate cases where $x_{n-1} = x_n$, giving a word of the form (5).

To show uniqueness, note that in a group G, the formula in (5) represents the element

(6) $$x_0\, x_1^{-1} \ldots x_{n-1}^{\mp 1}\, x_n^{\pm 1}\, x_{n-1}^{\mp 1} \ldots x_1^{-1}\, x_0.$$

Hence if we take for G the free group on the elements of X, then by the restriction in (5) that no two successive x_i be equal, (6) is a reduced word in that free group, whose value in that group determines $x_0, \ldots, x_n$. So starting with an arbitrary $\natural$-word in the elements of X, any two expressions as in (5) obtainable from it using (2)-(4) must be the same, as desired.

Returning to the claim whose verification we postponed, suppose $u = v$ is an identity satisfied by $\natural$ in all groups. Applying (2)-(4) as above, we can reduce the two sides to words of the form (5). Since we have assumed the original expressions identically equal in groups, the above reduced expressions will have the same value in the free group on X, so they must be the same. So our reduction using (2)-(4) has indeed proved the given identity. $\square$

2. Abstract flip-sets

In what follows, for the sake of precision, we shall understand a group G to be a 4-tuple $(|G|, \cdot, ^{-1}, 1)$, where $|G|$ is the underlying set, and $\cdot$, $^{-1}$, 1 are the multiplication and inverse operations and the identity element. (However, we shall continue to write $g\,h$ for $g \cdot h$, and to call an element $g \in |G|$ "an element of G".) For notational simplicity as we develop these general results, all groups will be written multiplicatively. Groups presented by generators and relations will be written using the symbols $\langle \dots \mid \dots \rangle$.

Motivated by Lemma 5, we make

Definition 6. *A* flip-set *will mean an ordered pair* $A = (|A|, \natural)$, *where* $|A|$ *is a set, and* $\natural : |A|^2 \to |A|$ *a binary operation such that for all* $f, g, h \in |A|$,

(7) $\quad g \natural g \ = \ g,$

(8) $\quad g \natural (g \natural h) \ = \ h,$

(9) $\quad (f \natural g) \natural h \ = \ f \natural (g \natural (f \natural h)).$

(*Cf.* (2), (3), (4).)

If G *is a group, the pair* $(|G|, \natural)$, *with* $\natural$ *defined by* (1), *will be called the* underlying flip-set *of* G, *and denoted* $\mathrm{Flip}(G)$.

A degenerate but instructive class of examples is noted in

Lemma 7. *If* G *is a group, then the identity*

(10) $\quad x \natural y \ = \ y$

is satisfied by $\mathrm{Flip}(G)$ *if and only if* G *satisfies the identity* $g^2 = 1$, *i.e., has exponent* 2.

Proof. The identity (10) translates to the group-theoretic identity $x\,y^{-1}x = y$, equivalently, $(x\,y^{-1})^2 = 1$, which clearly holds for all $x, y \in |G|$ if and only if $g^2 = 1$ for all $g \in |G|$. $\square$

For G a group of exponent 2, we see from (10) that every subset of $\mathrm{Flip}(G)$ will be closed under $\natural$, hence will form a sub-flip-set. Taking such a set of finite cardinality not a power of 2, we see that the resulting flip-set cannot itself be $\mathrm{Flip}(G)$ for any group G; so not every flip-set has that form.

Is every flip-set at least embeddable in one of the form $\mathrm{Flip}(G)$? No.

Lemma 8. *For elements* x, y *of a group* G, *we have, in* $\mathrm{Flip}(G)$,

(11) $\quad x \natural y = y \iff y \natural x = x.$

On the other hand, if A *is the factor-flip-set of* $\mathrm{Flip}(\langle g \mid g^4 = 1 \rangle)$ *by the equivalence relation which identifies* g *with* g^3, *but leaves* 1 *and* g^2 *distinct, and we let* x *and* x' *be the respective images of* 1 *and* g^2, *and* y *the common image of* g *and* g^3, *then* $x \natural y = y$, *but* $y \natural x = x'$. *Hence that* 3*-element flip-set cannot be embedded in* $\mathrm{Flip}(G)$ *for any group* G.

Proof. Group-theoretically, the left-hand equality of (11) says $x\,y^{-1}\,x = y$, i.e., $(x\,y^{-1})^2 = 1$. Inverting this gives $(y\,x^{-1})^2 = 1$, the right-hand equality of (11).

It is straightforward to verify that the equivalence relation on $|\mathrm{Flip}(\langle g \mid g^4 = 1 \rangle)|$ described in the second paragraph of the lemma respects the flip-structure, and so indeed leads to a factor-structure, which we see has the asserted properties. $\square$

Nevertheless, given a flip-set A, there is a natural homomorphism to one of the form $\mathrm{Flip}(G)$ which often does a good job of separating elements.

Lemma 9. *Let A be any flip-set, and* $\mathrm{Sym}(|A|)$ *the group of all permutations of the set* $|A|$. *For* $x \in |A|$, *define* $\alpha(x) \in |\mathrm{Sym}(|A|)|$ *by*

$$(12) \qquad \alpha(x)(a) \;=\; x \natural a \qquad (a \in |A|).$$

Then α is a homomorphism of flip-sets, $A \to \mathrm{Flip}(\mathrm{Sym}(|A|))$.

If A above has the form $\mathrm{Flip}(G)$ *for a group G, then elements* $x,\, x' \in |A| = |G|$ *fall together under α if and only if they belong to the same coset of the group of elements of exponent 2 in the center of G.*

Proof. From (8) we see that for $x \in |A|$, $\alpha(x)^2 = 1$ (the identity permutation of $|A|$), hence $\alpha(x) \in |\mathrm{Sym}(|A|)|$. Let us check that α is a $\natural$-homomorphism. Given $x, y \in |A|$, we see (using (9) at the second step, and the fact that $\alpha(x)$ has exponent 2 in the fourth) that for all $z \in |A|$,

$$(13) \qquad \begin{aligned} &\alpha(x \natural y)(z) \;=\; (x \natural y) \natural z \;=\; x \natural (y \natural (x \natural z)) \;=\\ &\qquad \alpha(x)\,\alpha(y)\,\alpha(x)(z) \;=\; \alpha(x)\,\alpha(y)^{-1}\alpha(x)(z) \;=\; (\alpha(x) \natural \alpha(y))(z), \end{aligned}$$

as required.

To get the last assertion of the lemma, note that for elements $x,\, x' \in |G|$, we have $\alpha(x) = \alpha(x')$ if and only if all $y \in |G|$ satisfy $x\,y^{-1}x = x'y^{-1}x'$. Multiplying on the left by x'^{-1} and on the right by x^{-1}, this becomes

$$(14) \qquad x'^{-1}x\,y^{-1} \;=\; y^{-1}x'\,x^{-1}.$$

Taking $y = 1$ in (14) gives

$$(15) \qquad x'^{-1}x \;=\; x'x^{-1}.$$

Hence (14) implies that the common value of the two sides of (15) is central in G. In particular, the right-hand side of (15) is unaffected by conjugation by x; but the result of that conjugation is the inverse of the left-hand side, so the common value of the two sides has exponent 2, giving the "only if" direction of the desired statement. The "if" direction is straightforward. □

3. Notes on the arithmetic of $\natural$, and trajectories in flip-sets

We have observed that the identity (9) allows one to recursively bring any $\natural$-word in a set of symbols to a form with parentheses clustered to the right. It is helpful to note a consequence of that identity (of which the identity itself is the $n = 2$ case), which describes how such a right-clustered expression acts by $\natural$.

$$(16) \qquad \begin{aligned} &f_1 \natural (f_2 \natural (\dots \natural (f_{n-1} \natural f_n) \dots)) \natural g \;=\\ &\qquad f_1 \natural (f_2 \natural (\dots \natural (f_{n-1}(\natural f_n \natural (f_{n-1} \natural (\dots \natural (f_2 \natural (f_1 \natural g)) \dots)))) \dots)). \end{aligned}$$

This is straightforward to check when our flip-set has the form $\mathrm{Flip}(G)$ (cf. (6)). For a general flip-set A, the same result holds by fact proved in Lemma 5, that every identity holding in flip-sets $\mathrm{Flip}(G)$ holds in arbitrary flip-sets. Alternatively, one can prove (16) inductively from (9).

It is, of course, natural to make

Definition 10. (*Cf. Definition 2.*) *If A is a flip-set, then a sequence* $(a_i)_{i\in\mathbb{Z}}$ *of elements of A will be called a* trajectory *in A if it satisfies*

$$(17) \qquad a_{i+1} \;=\; a_i \natural a_{i-1} \quad \textit{for all } i \in \mathbb{Z},$$

equivalently,

$$(18) \qquad a_{i-1} \;=\; a_i \natural a_{i+1} \quad \textit{for all } i \in \mathbb{Z}.$$

(*Cf. Lemma 1,* (ii), (ii′).)

As noted in the proof of Lemma 1, if (a_i) is a trajectory in a group, we can write $a_n = g\,h^n$ where $g = a_0$, $h = a_0^{-1}a_1$. Writing $f = a_1$, this becomes

$$(19) \qquad a_n \;=\; g\,(g^{-1}\,f)^n.$$

In an abstract flip-set A, the description of the elements of a trajectory in terms of $g = a_0$ and $f = a_1$ is not quite as simple. Rather than writing a formula, I will illustrate the forms of a_{-3} to a_4, from which the pattern can be seen.

$$
\begin{array}{ll}
 & \ldots \\
 & a_{-3} = g \natural (f \natural (g \natural f)) \\
 & a_{-2} = g \natural (f \natural g) \\
 & a_{-1} = g \natural f \\
 & a_0 \;\, = g \\
(20) & a_1 \;\, = f \\
 & a_2 \;\, = f \natural g \\
 & a_3 \;\, = f \natural (g \natural f) \\
 & a_4 \;\, = f \natural (g \natural (f \natural g)) \\
 & \ldots
\end{array}
$$

Again, this family of formulas can be proved either by establishing them in flip-sets $\mathrm{Flip}(G)$ (where they are translations of the corresponding cases of (19)), or by computations using the identities defining flip-sets (in which case (16) is helpful).

Let us note

Lemma 11. *Given elements x and y of a group G, and an integer n, one can determine from the $\natural$-structure of $\mathrm{Flip}(G)$ whether $x^{-1}y$ is an n-th power in the group G. Namely, this will hold if and only if there exists a trajectory (a_i) in $\mathrm{Flip}(G)$ with $a_0 = x$ and $a_n = y$; equivalently, if and only if, writing $g = x$, there exists $f \in |G|$ such that y is given by the formula for a_n as in (20).* ☐

It follows in turn that we can tell whether, $x^{-1}\,y$ is, say, a product of squares, namely, by asking whether there is a sequence of elements $x = x_0,\, x_1,\, \ldots,\, x_n = y$ with each $x_{n-1}^{-1}\,x_n$ a square. This shows in turn that the structure of $\mathrm{Flip}(G)$ determines whether a property such as "every product of squares is a square", or "every product of two distinct squares in G has cube the identity" holds in a given group G.

On the other hand, we cannot, in general, tell from the $\natural$-structure of G whether different trajectories have the same factor $g\,f^{-1}$ between successive terms. Hence we cannot, so far as I can see, determine whether, say, two elements differ by a factor that can be written $a\,b\,c\,a\,c\,b$ for $a,\, b,\, c \in |G|$.

(However, we leave it to the interested reader to verify that we can, with a little ingenuity, determine whether a given two elements differ by a factor of the form $f^i\, g^j\, f^k$ for given integers $i,\, j,\, k$; or by a factor of the form $(f^i\, g^j)^k$; or by one of the form $a\,b\,c\,d\,a\,c\,b$. In the opposite direction, Lemma 16 will show that $\mathrm{Flip}(G)$ does *not* determine whether G is commutative; hence we cannot tell from the $\natural$-structure whether a given two elements of G differ by a factor of the form $f^{-1}g^{-1}f\,g$.)

The elements of a trajectory (20) in a flip-set comprise the sub-flip-set generated by f and g, and a flip-set that is not embeddable in a group may have trajectories with behavior that we do not see in groups. A trajectory in a group, if it involves any repetition, is periodic of some period n, with a repeating sequence of n distinct elements; while in the flip-set A of the last paragraph of Lemma 8, the trajectory with $a_0 = x$, $a_1 = y$ has the repeating sequence $x,\, y,\, x',\, y,$ where y appears twice.

4. Components of flip-sets

We saw in Lemma 7 that for G a group of exponent 2, the flip-set $\mathrm{Flip}(G)$ has the uninteresting operation (10), with the consequence that every subset of $|G|$ is a sub-flip-set.

Note that if a flip-set A admits a homomorphism to a flip-set B satisfying (10), then the inverse image of every subset of B is a sub-flip-set of A. Moreover, every flip-set A has a universal homomorphic image satisfying (10), whose elements are the equivalence classes of elements of $|A|$ under the equivalence relation $\sim$ generated by relations

$$(21) \qquad x \natural y \;\sim\; y \qquad (x, y \in |A|).$$

With the help of (8) it is easy to show that this equivalence relation has the form

$$(22) \qquad a \sim b \iff (\exists\, x_1, \ldots, x_n \in |A|) \;\; b = x_1 \natural (\ldots \natural (x_n \natural a) \ldots).$$

Calling these equivalence classes the "components" of A, we see that the union of any family of components is a sub-flip-set of A.

However, in contrast to the case of Lemma 7, the structure of A, or of such a sub-flip-set of A, is not in general determined by the flip-structures of its components: although each map $a \natural -$ $(a \in |A|)$ takes every component A_0 of A into itself, if a is not in A_0, the involution $a \natural -$ on A_0 carries information not determined by the $\natural$-structure of A_0.

Instead of constructing a sub-flip-set B of A by letting each component of A either wholly belong to B or be wholly absent, can we put together a B by choosing sub-flip-sets of the various components of A more or less independently of one another? Specifically, suppose we start with a flip-set $\mathrm{Flip}(G)$ for G a group, and let N be the normal subgroup of G generated by the squares, so that G/N is the universal exponent-2 image of G. Can we get a sub-flip-set of $\mathrm{Flip}(G)$ whose intersections with the various cosets of N include cosets of distinct subgroups of N? For instance, can we do this when G is an infinite cyclic group $\langle g\rangle$, so that $N=\langle g^2\rangle$?

The answer turns out to be no in that case, but yes for some other G.

Indeed, when $A=\mathrm{Flip}(\langle g\rangle)$, it is not hard to see that every nonempty sub-flip-set A' of A consists of the elements of a single trajectory. (Idea: If A' has more than one element, choose distinct $g^i,\, g^j\in|A'|$ so as to minimize $|i-j|$, and show that the existence of an element g^k not in the trajectory they generate would contradict that minimality.) For such a trajectory, $\{i \mid g^i\in|A'|\}$ either consists entirely of even integers, or consists entirely of odd integers, or the sets of even and of odd elements are cosets of a common subgroup of $\langle g^2\rangle$.

But for an example where something more interesting can happen, let G be the infinite dihedral group $\langle g,\, h \mid h^2=1,\ h^{-1}\,g\,h=g^{-1}\rangle$. It is easy to check that each coset of the subgroup $\langle g\rangle\subseteq G$ has trivial $\natural$-action on the other:

$$(23)\qquad g^i \natural (g^j h) \;=\; g^j h;\ \text{equivalently,}\ (g^j h)\natural g^i \;=\; g^i \qquad (i,j\in\mathbb{Z}).$$

From this it follows that the union of any $\natural$-closed subset of one of these cosets with any $\natural$-closed subset of the other gives a sub-flip-set of $\mathrm{Flip}(G)$; and those $\natural$-closed subsets can, independently, each be a nontrivial trajectory, a singleton, or empty.

(The cosets of $\langle g\rangle$ are not actually the components of $\mathrm{Flip}(G)$; each is the union of two such components. But each of those cosets is, as a flip-set, a single trajectory, so, as discussed earlier, the intersections of a sub-flip-set of G with its components has much less freedom.)

5. More on mapping flip-sets into groups

Given a flip-set A, we have seen that the map $\alpha: A\to \mathrm{Flip}(\mathrm{Sym}(|A|))$ of Lemma 9 is not, in general, one-to-one. Some of the elements that fall together under that map must do so under any homomorphism to the flip-set of a group, as seen in Lemma 8, while others need not: the second paragraph of Lemma 9 characterizes elements of flip-sets $\mathrm{Flip}(G)$ that fall together under the above map, but obviously do not under the identity map $\mathrm{Flip}(G)\to\mathrm{Flip}(G)$.

To avoid "unnecessary falling-together", one can try to embed A in a larger flip-set A', such that even if elements $x\neq x'$ satisfy $x\natural y = x'\natural y$ for all $y\in|A|$, this equality fails for some $y\in|A'|$, so that Lemma 9 yields a representation of A' that distinguishes them. If A has the form $\mathrm{Flip}(G)$ for some group G, this will always work: construct G' by adjoining to G one new generator g and no relations. Then nonidentity elements of G will not centralize g, so the cases of elements of G falling together under our map to $\mathrm{Sym}(|G'|)$ as described in Lemma 9 become trivial.

Given an arbitrary flip-set A, there will similarly exist a universal flip-set A' generated by an image of A and one additional generator g. If we could find a normal form for elements of this A' in terms of A, we could use it to tell which pairs of elements $x,\, x'$ fall together under all maps into flip-sets of groups. (Namely, if and only if $x\natural g = x'\natural g$.) But I do not see how to get such a normal form. Obviously, we can reduce any element of A' to an expression (5) in elements of $|A|\cup\{g\}$. But the identities of flip-sets will imply further equalities between such expressions. For instance, when we have an expression $\dots\natural(a_i\natural(a_{i+1}\natural(\dots)))\dots$ with a_i and a_{i+1} both coming from $|A|$, we can use (8) in reverse, to insert two terms a_i after a_{i+1}, getting an expression $\dots\natural(a_i\natural(a_{i+1}\natural(a_i\natural(a_i\natural(\dots)))))\dots$, then apply (9) to the first three of the terms shown to get $\dots\natural(b\natural(a_i\natural(\dots)))\dots$, where $b=a_i\natural a_{i+1}$. (Or we can do the mirror image, inserting two instances of a_{i+1} to the left of a_i and then applying (9).) For another example, if five successive terms $a_i,\dots,a_{i+4}$ all come from $|A|$ and satisfy $a_i=a_{i+2}=a_{i+4}$, then we can apply (9) either to $a_i,\, a_{i+1},\, a_{i+2}$, or to $a_{i+2},\, a_{i+3},\, a_{i+4}$, getting different reductions of our expression.

Contrast this with the case of the *group* gotten by adjoining a new generator g to an arbitrary group G. This has a normal form consisting of all alternating strings of nonidentity elements of G and nonzero powers of g, from which one quickly sees that no nonidentity element of G is central in the new group.

The reader might find it interesting to examine the case where A is the 3-element flip-set with elements x, x', y referred to in the last paragraph of Lemma 8, and see how the axioms for a flip-set force $x \natural g = x' \natural g$. (Outline: In $x \natural g$, substitute $y \natural x'$ for x, and expand the result using (9). Write the last y in the resulting expression as $x' \natural y$ and again expand by (9). Then apply (8) twice.)

By general nonsense (see [1, Exercise 9.9:8, or better, Theorem 10.4:3]) one can associate to any flip-set A a group $\mathrm{Group}(A)$ with a universal flip-set homomorphism $A \to \mathrm{Flip}(\mathrm{Group}(A))$. The pairs x, x' of elements of $|A|$ that fall together under this homomorphism will be those that fall together under all $\natural$-homomorphisms to groups. But, as in the approach of adjoining a universal g to A as a flip-set, it is not clear how to systematically detect such pairs.

Incidentally, let us note that the above universal map $A \to |\mathrm{Group}(A)|$ can never be *onto*. To see this, take any nontrivial group G and any $g \in |G| - \{1\}$. Then a $\natural$-homomorphism $c_g : A \to \mathrm{Flip}(G)$ is given by the constant map $c_g(a) = g \ \ (a \in |A|)$, and by the universal property of $\mathrm{Group}(A)$, c_g must factor $|A| \to |\mathrm{Group}(A)| \to |G|$, via a group homomorphism $\mathrm{Group}(A) \to G$. Since c_g takes no element of A to $1 \in |G|$, our map $|A| \to |\mathrm{Group}(A)|$ cannot take any element of $|A|$ to 1, and so cannot be surjective.

Here is another property of flip-sets arising from groups.

Lemma 12. *If G is a group, then the automorphism group of* $\mathrm{Flip}(G)$ *is transitive on the set* $|\mathrm{Flip}(G)| = |G|$.

Proof. By Lemma 3, the left translations of the group G (and likewise the right translations) are automorphisms of $\mathrm{Flip}(G)$, and they are, of course, transitive on $|G|$. □

The 3-element flip-set of the second paragraph of Lemma 8 is an example of a flip-set A whose automorphism group is not transitive: since the element y is fixed under all the operations $a \natural -$, while the elements x and x' are not, no automorphism of A can carry y to x or x'.

The above example is a homomorphic image of flip-set of the form $\mathrm{Flip}(G)$. A flip-set A which is, rather, embeddable in one of the form $\mathrm{Flip}(G)$, but again does not have transitive automorphism group, is the case of the example in the paragraph containing (23) where, as subsets of the cosets $\langle g \rangle$ and $\langle g \rangle h$ we use all of one, and a singleton subset of the other. Then A consists of an infinite trajectory together with a lone element which belongs only to trajectories of ≤ 2 elements; so no automorphism can carry that element to any other.

6. Identities of groups and flip-sets

If a group G satisfies nontrivial identities (identities not implied by the identities defining groups), this can lead to nontrivial identities on $\mathrm{Flip}(G)$.

(In (24) below, e, like f, g and h, denotes a general element of $|G|$; recall that identity elements are written 1 in this note.)

Lemma 13. *The identities satisfied by* $\mathrm{Flip}(G)$ *for all* abelian *groups G are the consequences* (*given the defining identities* (7)-(9) *for flip-sets*) *of the identity*

$$(24) \qquad e \natural (f \natural (g \natural h)) \ = \ g \natural (f \natural (e \natural h)).$$

Proof. That (24) holds in $\mathrm{Flip}(G)$ when G is abelian is immediate (cf. (6)).

To see that the only identities holding in all such flip-sets are the consequences of (24), first note that given an expression $x_0 \natural (x_1 \natural (\dots \natural (x_{n-1} \natural x_n) \dots))$, where the x_i are symbols in a set X, we can, assuming (24), rearrange in any way the x_i having *even* subscript $i < n$, and likewise rearrange the x_i having *odd* subscript $i < n$. In particular, if some $x \in X$ occurs in both even and odd positions, we can rearrange the terms so that these occurrences of x appear in adjacent positions, and then use (7) or (8) to shorten the word. (We use (7) if one of these occurrences of x is x_n, so that it was the other occurrence that had to be moved to become adjacent with it; (8) if neither occurrence is x_n, so that one, the other, or both can be moved till they become adjacent.)

Now suppose that $u = v$ is an identity in symbols from X satisfied by $\mathrm{Flip}(G)$ for all abelian groups G. Using (9) we can assume without loss of generality that in both u and v, parentheses are clustered to the right, while using (24), (7), and (8) as above, we can assume that in each of these words, no member of X occurs in both even-subscripted and odd-subscripted positions. Let us now evaluate u and v in the free abelian group G on X (which we write multiplicatively). For $x \in X$, an occurrence of x as the i-indexed term of the expression u or v will contribute $2\,(-1)^i$ to the exponent of x in the resulting element of G, unless the term in question is the final term (namely, x_n if our expression is $x_0 \natural (\dots \natural x_n) \dots)$ in which case it will contribute just $(-1)^i$. Since no term occurs in both even and odd positions in u or in v, we can conclude from the structure of free abelian groups that u and v must have the same length, the same number of occurrences of each element of X in nonfinal even position, the same number of occurrences of each element of X in nonfinal odd position, and the same final term. Hence u can be transformed into v by applications of (24); hence the identity $u = v$ is indeed a consequence of (24) (given (7)-(9)). □

What about the other direction? I.e., for which groups G will $\mathrm{Flip}(G)$ satisfy (24)?

Lemma 14. *If G is a group, then $\mathrm{Flip}(G)$ satisfies (24) if and only if G is nilpotent of nilpotency class ≤ 2, i.e., if and only if in G,*

(25) *every commutator* $[f, g] \;=\; f^{-1} g^{-1} f\, g$ *is central in G.*

Proof. In $\mathrm{Flip}(G)$, (24) translates to the group-theoretic identity

(26) $$e\, f^{-1}\, g\, h^{-1}\, g\, f^{-1}\, e \;=\; g\, f^{-1}\, e\, h^{-1}\, e\, f^{-1}\, g.$$

Let us start with the case where $f = 1$, write $h^{-1} = x$, and multiply the resulting equation both on the left and on the right by $g^{-1}\, e^{-1}$. Then we get

(27) $$x\, g\, e\, g^{-1} e^{-1} \;=\; g^{-1} e^{-1} g\, e\, x.$$

Taking $x = 1$, this tells us that $g\, e\, g^{-1}\, e^{-1}$ equals $g^{-1}\, e^{-1}\, g\, e$, i.e., $[g,\, e]$. So the general case of (27) tells us that $x\, [g,\, e] = [g,\, e]\, x$; so indeed, every commutator $[g,\, e]$ in G is central.

Conversely, suppose that in G every commutator is central. Now the string of symbols $e\, f^{-1}\, g$ can be turned into the string $g\, f^{-1}\, e$ by three transpositions of terms; and each of these transpositions corresponds group-theoretically to the insertion of a certain commutator at a certain point in the product in question. Hence the desired identity $e\, f^{-1}\, g\, h^{-1}\, g\, f^{-1}\, e \,=\, g\, f^{-1}\, e\, h^{-1}\, e\, f^{-1}\, g$ can be thought of as gotten from the trivial relation $g\, f^{-1}\, e\, h^{-1}\, g\, f^{-1}\, e = g\, f^{-1}\, e\, h^{-1}\, g\, f^{-1}\, e$ by inserting three commutators among the first three terms of the left-hand side, and the same three among last three the terms of the right-hand side. Since commutators are central in G, inserting them at different points yields the same value; so the two sides of (26) are in fact equal. □

If a flip-set $\mathrm{Flip}(G)$ satisfies (24), can it also be written $\mathrm{Flip}(G')$ for an abelian group G? In general, no.

Lemma 15. *For any group G, the following conditions are equivalent:*

(i) *In G, every product of squares is a square.*

(ii) *In $\mathrm{Flip}(G)$, for all elements x, y, z there exists an element w such that*

(28) $$x \natural (y \natural z) \;=\; w \natural z.$$

In particular, every abelian group satisfies these conditions. On the other hand, the group G free on two generators in the variety determined by (25)*, equivalently, the group of upper triangular 3×3 matrices over $\mathbb{Z}$ with 1's on the diagonal, does not. Hence the flip-structure of the latter group, though it satisfies* (24)*, is not isomorphic to the flip-structure of an abelian group.*

Proof. In the paragraph following Lemma 11, we noted that condition (i) above could be expressed in terms of the structure of $\mathrm{Flip}(G)$; condition (ii) is the explicit form that that condition takes (though we will not need to know what that form is to get the final result of this lemma).

Clearly, every abelian group satisfies (i). To see that the free group G of nilpotency class ≤ 2 on generators f, g does not, let us write the general element thereof as $f^i\, g^j\, [g, f]^k$ $(i,\, j,\, k \in \mathbb{Z})$, and note that the group operation is given by

(29) $$(f^i\, g^j\, [g, f]^k)\; (f^{i'}\, g^{j'}\, [g, f]^{k'}) \;=\; f^{i+i'}\, g^{j+j'}\, [g, f]^{k+k'+ji'}.$$

(Rough idea: in bringing the product on the left-hand side to normal form, each time we push one of the j occurrences of g in the first factor past one of the i' occurrences of f in the second, a $[g, f]$ is created.)

Now if, in (28), we take $x = f$, $y = g^{-1}$, $z = 1$, the left-hand side of that relation becomes

(30) $$f\, g\, 1\, g\, f \;=\; f^2\, g^2\, [g, f]^2.$$

For the right-hand side of (28) to equal this, w has to have the form $f\,g\,[g,f]^k$; but we see that this gives

(31) $\quad w \natural z = w\,w = f^2\,g^2\,[g,f]^{2k+1},$

which cannot agree with (30). □

On the other hand, if we adjoin to the noncommutative group of the above lemma a central square root of $[g,f]$, the above problem goes away:

Lemma 16. *Let G_1 be the free abelian group on three generators $f,\,g,\,h$, and G_2 the group obtained by adjoining to the free group of nilpotency class ≤ 2 on generators $f,\,g$ a central square root of the element $[g,f]$, which we shall write $[g,f]^{1/2}$, so that the general element of G_2 can be written in the normal form $f^i\,g^j\,[g,f]^{k/2}$ with $i,\,j,\,k\in\mathbb{Z}$, and the group multiplication is given by* (29)*, with k and k' everywhere replaced by $k/2$ and $k'/2$.*

Then $\mathrm{Flip}(G_2)\cong\mathrm{Flip}(G_1)$*, by the map*

(32) $\quad f^i\,g^j\,[g,f]^{k/2} \ \mapsto\ f^i\,g^j\,h^{k-ij}.$

Proof. The map (32) is clearly a bijection. Computation shows that it respects $\natural$. □

(The computation of the exponent of h in the image of the $\natural$-product of two elements of G_2 is very messy; I wish I could offer a nicer verification. For a bit of intuition on (32), observe that $f^i\,g^j\,[g,f]^{k/2}$ can also be written $g^j\,f^i\,[g,f]^{(k/2)-ij}$, and that $(k/2)+((k/2)-ij)=k-ij$ balances the asymmetry implicit in each of these two normal forms for G_2.)

Returning to the result in Lemma 15 that the underlying flip-sets of the free abelian group of rank 3 and the free group of nilpotency class ≤ 2 on two generators are not themselves isomorphic, we remark that each can nonetheless be embedded in the other. In one direction, restricting (32) to the case where $[g,f]$ has integer exponent, we have an embedding of the free nilpotent group in the free abelian group,

(33) $\quad f^i\,g^j\,[g,f]^k \ \mapsto\ f^i\,g^j\,h^{2k-ij}.$

For the other direction, note that the cases of the right-hand side of (33) with j even comprise the elements of the free abelian group on $f,\ g^2,\ h^2$, so renaming these elements as $f,\ g,\ h$ (but not changing our notation in the free nilpotent group), and turning the map around, we get the embedding

(34) $\quad f^i\,g^j\,h^k \ \mapsto\ f^i\,g^{2j}\,[g,f]^{k+ij}.$

Turning back to the identity (24), here is another way to look at that condition.

Lemma 17. *Let A be a flip-set, and z any element of A. Then A satisfies* (24) *if and only if* (*in the notation of Lemma 13*) *the elements of the set $\{\alpha(f)\,\alpha(z)\mid f\in|A|\}\subseteq|\mathrm{Sym}(|A|)|$ all commute with one another; in other words, if and only if the map*

(35) $\quad f \ \mapsto\ \alpha(f)\,\alpha(z),$

which is a $\natural$-homomorphism $A\to\mathrm{Flip}(\mathrm{Sym}(|A|))$, has image in an abelian subgroup of $\mathrm{Sym}(|A|)$.

Hence if the above equivalent conditions hold, and if, moreover, the map $f\mapsto\alpha(f)$ is one-to-one, then A is embeddable in $\mathrm{Flip}(G)$ *for an abelian group G. In particular, for every group H of nilpotency class ≤ 2 having no central elements of order* 2, $\mathrm{Flip}(H)$ *is embeddable in* $\mathrm{Flip}(G)$ *for an abelian group G.*

Proof. Suppose first that for some $z\in|A|$, the elements $\alpha(f)\,\alpha(z)$ $(f\in|A|)$ all lie in an abelian subgroup of $\mathrm{Sym}(|A|)$. Since $\alpha(z)^2=1$, these elements can be written $\alpha(f)\,\alpha(z)^{-1}$, hence for any $f,\,g\in|A|$, that abelian subgroup contains $(\alpha(f)\,\alpha(z)^{-1})\,(\alpha(g)\,\alpha(z)^{-1})^{-1}=\alpha(f)\,\alpha(g)^{-1}$; so our hypothesis is equivalent to the statement that all elements of $\mathrm{Sym}(|A|)$ the form $\alpha(f)\,\alpha(g)^{-1}$ commute. Again noting that the exponent $^{-1}$ makes no difference, we see in particular that for all $e,\,f,\,g\in|A|$, we have $(\alpha(e)\,\alpha(f))\,(\alpha(g)\,\alpha(f))=(\alpha(g)\,\alpha(f))\,(\alpha(e)\,\alpha(f))$, which, cancelling the $\alpha(f)$'s on the right, gives $\alpha(e)\,\alpha(f)\,\alpha(g)=\alpha(g)\,\alpha(f)\,\alpha(e)$. Applying this element of $\mathrm{Sym}(|A|)$ to elements $h\in|A|$, we get (24).

The reverse implication works essentially the same way.

That (35) is a $\natural$-homomorphism follows from the fact that $f\mapsto\alpha(f)$ is a $\natural$-homomorphism, and that, by the proof of Lemma 12, right translation by $\alpha(z)$ is a $\natural$-automorphism of $\mathrm{Sym}(|A|)$.

The first sentence of the second paragraph follows immediately. The final sentence follows in view of Lemma 9. □

We end this section by looking briefly at the other very simple sort of identity a group can satisfy, saying that all its elements have exponent n for some fixed n. Lemma 11 shows us that for each n, the groups satisfying this identity can be characterized by a $\natural$-identity on their flip-structures, namely,

Lemma 18. *Let n be a positive integer. Then a group G satisfies the identity $g^n = 1$ if and only if* $\mathrm{Flip}(G)$ *satisfies the identity equating the formulas for a_0 and a_n in* (20). ☐

The above "if and only if" shows that in this case we don't have the complication that we had for commutativity, where the effect of our $\natural$-identity was weaker than the group-identity we started with. But we have a different sort of complication. For each positive integer n we can ask

Question 19. *Does the $\natural$-identity described in Lemma 18 which characterizes groups of exponent n imply, for general flip-sets,* all *$\natural$-identities satisfied by groups of exponent n?*

Equivalently, is the free flip-set on any finite set of generators, subject to that identity, embeddable in the flip-set of a group?

It seems likely that the answer will depend on n. A positive answer is vacuously true for $n = 1$, and for $n = 2$ is easily deduced from Lemma 7. I have not investigated the question for higher n. We were able to obtain a positive result for the analogous question concerning the commutativity identity in Lemma 13 with the help of the explicit description of free abelian groups, and to get a result of the same sort for the class of all groups in Lemma 5, using the normal form for free groups; but I doubt that normal forms are known for free groups of exponent n, except for a few small values of n.

I have not examined the consequences for $\mathrm{Flip}(G)$ of any other identities on a group G.

7. Counting generators and relations

It is natural to ask

Question 20. *Under what conditions on a group G is* $\mathrm{Flip}(G)$ *finitely generated? Finitely presented?*

In such cases, what can be said about the number of generators or relations required?

In particular, what is true in the case where G is the free group on two generators?

Here is what I know about generator-counts.

Lemma 21. *For G a group, let* $\mathrm{gen}(G)$ *denote the minimum number of elements needed to generate G as a group, and likewise, for A a flip-set, let* $\mathrm{gen}(A)$ *denote the minimum number of elements needed to generate it as a flip-set.*

Then for every finitely generated group G, if we write N for the subgroup of G generated by the squares (so that G/N is the universal exponent-2 homomorphic image of G), we have

(36) $\quad \mathrm{gen}(\mathrm{Flip}(G)) \;\geq\; \max(\mathrm{gen}(G){+}1,\ \mathrm{card}(|G/N|)).$

If G is abelian, we in fact have equality in (36).

Proof. In view of Lemma 7, the homomorphic image $\mathrm{Flip}(G/N)$ of $\mathrm{Flip}(G)$ cannot be generated by any proper subset of $\mathrm{Flip}(G/N)$, hence requires $\mathrm{card}(|G/N|)$ generators; hence $\mathrm{Flip}(G)$ itself requires at least that many; so to prove (36) it remains to prove that $\mathrm{Flip}(G)$ also requires more than $\mathrm{gen}(G)$ generators.

Suppose $\mathrm{Flip}(G)$ is generated by a set S. Since $\mathrm{Flip}(G)$ is nonempty, S must be nonempty; choose $g \in S$. Since left translations under the group operation are $\natural$-automorphisms of $\mathrm{Flip}(G)$, $\mathrm{Flip}(G)$ is also generated by $g^{-1}S$; hence (since the $\natural$-operation of $\mathrm{Flip}(G)$ is a derived operation of G), the group G is generated by $g^{-1}S$. But $1 \in g^{-1}S$; so $S - \{1\}$ also generates G, so $\mathrm{Flip}(G)$ indeed requires at least one more generator than G.

To get the final sentence of the lemma, let us first note that if G is any *abelian* group, and X any subset of $|G|$ containing 1, then an element $g \in |G|$ will belong to the flip-subset generated by X if and only if

(37) g can be written as a product of powers of elements of $X - \{1\}$, in which the exponents of all but at most one of those elements are even.

Indeed, if we take an expression (6) with the x_i allowed to be arbitrary members of X, drop factors with $x_i = 1$, and combine occurrences of each x_i to the left and to the right of x_n, we get a product of the form described, where the only member of X that can appear with odd exponent is x_n if that is not 1. (If all terms are 1, we regard the result as the empty product, which we understand to have value 1.)

Now suppose G is a finitely generated abelian group, say with $\mathrm{gen}(G)=n$. Then G is the direct product of n cyclic subgroups $\langle g_i \mid g_i^{d_i}=1\rangle$ $(i=1,\dots,n)$ where each d_i is either 0 or >1. Without loss of generality, assume that $d_1,\dots,d_m$ are even, and $d_{m+1},\dots,d_n$ are odd. Thus, the universal exponent-2 homomorphic image G/N has order 2^m.

We now want to construct a generating set X for $\mathrm{Flip}(G)$, of the cardinality shown on the right-hand side of (36).

The key to insuring that the set we will describe generates $\mathrm{Flip}(G)$ will be to set things up so that the closure of X under $\natural$ contains, on the one hand, all 2^m products of subsets of $\{g_i,\dots,g_m\}$ (including the empty product 1), and, on the other hand, all the elements $g_{m+1},\dots,g_n$. We will then be able to express an arbitrary $g\in G$ in the form (37) by letting the product of those g_i with $i=1,\dots,m$ that occur with odd exponent in g be the one term occurring with (arbitrary) odd exponent in that product, bring in the elements from $\{g_1,\dots,g_m\}$ with appropriate even exponents to achieve the desired powers of those elements, and finally note that each g_i with $m<i\le n$ has odd order, hence the subgroup it generates is also generated by its square, i.e., can be regarded as consisting of even powers of g_i, so those g_i can also be brought into our product (37) with even exponent to achieve the desired value.

To get such a generating set X, let us first form the set X_0 of all products of finite subsets of $\{g_1,\dots,g_m\}$. What we do next depends on which of $n+1$ and 2^m (i.e, $\mathrm{gen}(G)+1$ and $\mathrm{card}(|G/N|)$) is larger. In either case, we keep unchanged the members of X_0 that are products of 0 or 1 of $g_i,\dots,g_m$. If $n+1\le 2^m$ (equivalently, $n-m\le 2^m-(m+1)$), we replace $n-m$ of the $2^m-(m+1)$ elements $y\in X_0$ that are products of two or more of $g_1,\dots,g_m$ with elements $g_i\,y$ for distinct $i=m+1,\dots,n$, and take for X the resulting modified version of X_0, which we note still has 2^m elements. If $n+1\ge 2^m$, we replace all 2^m-m-1 of the elements $y\in X_0$ that are products of two or more of $g_1,\dots,g_m$ with elements $g_i\,y$ as above; this leaves $(n-m)-(2^m-m-1)=n-2^m+1$ of $g_{m+1},\dots,g_n$ unused, and we make these into additional members of X, giving X a total cardinality of $2^m+(n-2^m+1)=n+1$. (If $n+1=2^m$, these two constructions agree.) Thus, in each case, we get a set X of cardinality $\max(n+1,2^m)$.

To show that this X is a generating set for $\mathrm{Flip}(G)$, it will suffice, by our earlier remarks, to show that for each element $g_i\,y$ that we introduced, the flip-set generated by X contains both g_i and y.

To recover g_i, let us first form an expression (37) in the elements of X in which $g_i\,y$ occurs with exponent 2, and each of the g_j whose product gives y occurs with exponent -2. (These g_j are products singleton subsets of $\{g_i,\dots,g_m\}$, which we left unchanged in constructing X from X_0.) The result is g_i^2, and as we have noted, since g_i has odd order, some power of g_i^2 is g_i, which thus lies in the flip-set generated by X. Now taking an expression (37) with $g_i\,y$ having exponent 1, and g_i (obtained above) having an even exponent that gives a value equal to g_i^{-1}, we recover y, as required. □

I have not investigated Question 20 beyond this.

8. Comparison with heaps

A derived operation on groups related to the flip-operation is the ternary operation

$$(38)\qquad \tau(x,\,y,\,z)\;=\;x\,y^{-1}\,z,$$

which satisfies the identities

$$(39)\qquad \tau(\tau(v,\,w,\,x),\,y,\,z)\;=\;\tau(v,\,\tau(y,\,x,\,w),\,z)\;=\;\tau(v,\,w,\,\tau(x,\,y,\,z)),$$

$$(40)\qquad \tau(x,\,x,\,y)\;=\;y\;=\tau(y,\,x,\,x).$$

A set with an operation τ satisfying (39) and (40) is called a *heap*. (Cf. [1, Exercises 9.6:10-11] for some background and references.) For G a group, let us write $\mathrm{Heap}(G)$ for the heap with underlying set G and operation defined by (38).

As with flip-set structures, the heap structure on $\mathrm{Heap}(G)$ does not determine the group structure: again, every right or left translation operation of the group structure gives an automorphism of the heap. However, in contrast to the case of flip-structures, every heap structure on a nonempty set does arise as above from a group structure, which is unique up to isomorphism, and which becomes unique when one chooses an element e to be the identity element. The group structure is then given by

$$(41)\qquad x\,y\;=\;\tau(x,\,e,\,y),\quad x^{-1}\;=\;\tau(e,\,x,\,e).$$

Because of this very close relationship with groups, heaps are not much studied for their own sake, though one sometimes calls on the concept in situations where a natural heap structure exists but not a natural group structure; namely, given two isomorphic objects C and D of a category, the set of isomorphisms $C \to D$ has a natural structure of heap, given by the same formula (38), but not a natural group structure.

The flip-structure on the underlying set of a group is, clearly, a further weakening of the heap structure, given by

$$(42) \qquad x \natural y \ = \ \tau(x,\, y,\, x).$$

This loses much more information than the heap structure. In particular, as we have seen, not every such structure is embeddable in one coming from a group.

9. Final thoughts

I do not know whether the ideas of this paper will prove useful in improving our understanding of groups or other structures, or are merely a curious side-trip. The results we have obtained have all been lemmas – no deep results.

In this development, unexpected behavior has repeatedly involved the exponent 2 in groups (e.g., Lemmas 7 and 21, and last paragraphs of Lemmas 8 and 9.) A generalization of the subject (if one wants to move further into areas that might or might not be of use), in which more exponents can be expected to show such behavior, would be to study, for any $n > 1$, the binary operator $\natural_n$ on underlying sets of groups defined to carry the terms a_1 and a_0 of a trajectory to a_n; in other words,

$$(43) \qquad g \natural_n h \ = \ g\,(h^{-1}g)^{n-1}.$$

(So the operation we have called $\natural$ is in this notation $\natural_2$.)

If (a_i) is a trajectory in a group, and S any subset of $\mathbb{Z}$, then it is not hard to show that the set of a_i generated under $\natural_n$ by $\{a_i \mid i \in S\}$ will have the property that each of its members has the form a_j for some j which is both congruent modulo n to some member of S and congruent modulo $n-1$ to some (possibly different) member of S. Note also that the expression (43) has value g if and only if $(h^{-1}g)^{n-1} = 1$, value h if and only if $(h^{-1}g)^n = 1$. So it seems that the $\natural_n$-analogs of flip-sets should show interesting behavior involving exponents dividing n or $n-1$.

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