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ON VAUGHAN PRATT'S CROSSWORD PROBLEM

GEORGE M. BERGMAN AND PACE P. NIELSEN

Abstract. Vaughan Pratt has introduced objects consisting of pairs (A, W) where A is a set and W a set of subsets of A, such that (i) W contains $\emptyset$ and A, (ii) if C is a subset of $A \times A$ such that for every $a \in A$, both $\{b \mid (a, b) \in C\}$ and $\{b \mid (b, a) \in C\}$ are members of W (a "crossword" with all "rows" and "columns" in W), then $\{b \mid (b, b) \in C\}$ (the "diagonal word") also belongs to W, and (iii) for all distinct $a, b \in A$, the set W has an element which contains a but not b. He has asked whether for every A, the only such W is the set of *all* subsets of A.

We answer that question in the negative. We also obtain several positive results, in particular, a positive answer to the above question if W is closed under complementation. We obtain partial results on whether there can exist counterexamples to Pratt's question with W countable.

1. Definitions and conventions

We begin by defining the type of structures we will be considering. These are called "$\mathbf{chu}_2$ comonoids" by Vaughan Pratt; we shall call them *Pratt comonoids*. The category-theoretic background of Pratt's terminology is not a prerequisite for reading this note; we sketch that background in an appendix, §10.

Definition 1. By a *Pratt comonoid* we shall mean a pair (A, W), where A is a set, and W a set of subsets of A such that

(i) $\emptyset$ and A are members of W, and

(ii) whenever C is a subset of $A \times A$ such that for every $a \in A$, both $\{b \mid (a, b) \in C\}$ and $\{b \mid (b, a) \in C\}$ are members of W, we also have $\{b \mid (b, b) \in C\} \in W$.

In this situation, we will call A the *base-set* of the Pratt comonoid (A, W), and W the *Pratt comonoid structure* on A.

A set W of subsets of a set A will be called T_1 if it satisfies

(iii) for all $a, b \in A$ with $a \neq b$, the set W has an element which contains a but not b.

A Pratt comonoid (A, W) will be called T_1 if W is T_1 as a set of subsets of A. It will be called *discrete* if $W = 2^A$, the full power set of A.

We shall generally identify subsets of A or $A \times A$ with $\{0, 1\}$-valued functions on those sets. In particular, we may call subsets of A "words" on A, and a subset $C \subseteq A \times A$ satisfying the hypotheses of (ii) a "crossword" over W, since its rows and columns are words lying in W. We will use the notation $x \vee y$ and $x \wedge y$ for the union and intersection (or from the $\{0, 1\}$ point of view, pointwise sup and pointwise inf) of words x and y, and likewise $\leq$ and $\geq$ for inclusion, and $<$ and $>$ for strict inclusion between words.

For $C \subseteq A \times A$ and $a \in A$, we shall follow the matrix-theoretic convention of calling $\{b \mid (a, b) \in C\}$ the a-th row, and $\{b \mid (b, a) \in C\}$ the a-th column (rather than the convention of the cartesian plane, where the first member of an ordered pair is the horizontal and the second the vertical coordinate).

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This note was inspired by the following question, which will be answered in the negative in §6.

(1) (V. Pratt [9], [7, pp. 27–28], [6]) Is every T_1 Pratt comonoid discrete?

Though (1) concerns the T_1 case, many of the general results we prove will concern arbitrary Pratt comonoids, with the T_1 condition only brought in for the *coups de grace* of our main results. Likewise, since our only known example of a T_1 Pratt comonoid that is not discrete is quite complicated, examples showing the obstructions to one or another approach will in general be non-T_1.

We shall use set-theorists' notation for ordinals; in particular, the set of natural numbers (nonnegative integers) will be denoted ω. We shall write relative complements of sets as $x - y = \{a \in x \mid a \notin y\}$. When subsets of a given set A are under consideration, we shall often write $\neg x$ for the relative complement $A - x$ of a subset x of A.

The next two sections mainly summarize known material.

2. Some quick examples and immediate results

The easiest examples of Pratt comonoids other than discrete ones are gotten by taking a preorder $\preccurlyeq$ on a set A, and defining W to be the set of all *down-sets* of A, that is, sets x such that $a \preccurlyeq b \in x \implies a \in x$ [8, Proposition 2.1]. The reader can easily verify that such pairs (A, W) satisfy the definition.

For an example that deviates slightly from this form, let A consist of the set ω of natural numbers together with one additional element ∞, greater than every natural number; and let

(2) $W \;=\; \{\,\text{down-sets of } A \text{ other than } \omega\,\}.$

In other words, W consists of those down-sets which, if they contain all natural numbers, also contain ∞. Since by the preceding paragraph, the set of *all* down-sets of A yields a Pratt comonoid, to show that the W we have just described also determines one, we just have to show that any crossword over W whose diagonal contains all pairs (n, n) $(n \in \omega)$ must contain (∞, ∞). Now moving upward or to the left from the diagonal in such a crossword C (i.e., decreasing one or the other coordinate), we see that every pair (m, n) with $m, n \in \omega$ belongs to C. Hence the word in W given by each natural-number-indexed row contains all natural numbers, hence, by definition of W, also contains ∞. This says the element of W corresponding to the *column* indexed by ∞ contains all natural numbers, hence, again, contains ∞; so $(\infty, \infty) \in C$, as required.

(The above example, essentially [8, Proposition 2.4], is an instance of the more general result [8, Proposition 2.2], which says that what is called the *Scott topology* on a directed-complete partial order yields a Pratt comonoid.)

Here are some easy ways of getting new Pratt comonoids from old.

Lemma 2. (i) *If* (A, W) *is a Pratt comonoid, then so is* $(A, W^{\neg})$, *where* $W^{\neg} = \{\neg w \mid w \in W\}$, *the set of complements in* A *of members of* W. *If* (A, W) *is* T_1, *then so is* $(A, W^{\neg})$.

(ii) *If* $\{(A, W_i) \mid i \in I\}$ *is a (finite or infinite) set of Pratt comonoids with the same base-set* A, *then* $(A, \bigcap_{i \in I} W_i)$ *is a Pratt comonoid.*

(iii) *If* $f : A \to A'$ *is a set map and* (A, W) *a Pratt comonoid, and we let* $W' = \{w \subseteq A' \mid f^{-1}(w) \in W\}$, *then* (A', W') *is a Pratt comonoid.*

(iv) *If* (A, W) *is a Pratt comonoid, and* $u \leq v$ *are elements of* W, *then the pair* $(A_{u,v}, W_{u,v})$, *where* $A_{u,v} = v - u \subseteq A$, *and* $W_{u,v} = \{x - u \mid x \in W \text{ with } u \leq x \leq v\}$, *is also a Pratt comonoid.*

Sketch of proof. We get (i) by interchanging 0's and 1's in the conditions defining a Pratt comonoid, and likewise in the T_1 condition.

Statement (ii) holds because the condition for (A, W) to be a Pratt comonoid is a closure condition on W, and for any closure operator, an intersection of closed subsets is closed.

To see (iii), note that given any crossword C' over W', its inverse image under $f \times f$ will be a crossword C over W, hence the diagonal thereof lies in W, and that diagonal is the inverse image of the diagonal of C', which therefore lies in W'.

Finally, to see (iv), note that if $C \subseteq A_{u,v} \times A_{u,v}$ is a crossword over $W_{u,v}$, then $(u \times v) \vee (v \times u) \vee C$ will be a crossword over W, hence its diagonal belongs to W, which translates to say that the diagonal of C belongs to $W_{u,v}$, as required. □

Part (i) of the above lemma shows that when we prove a result about Pratt comonoids, we can immediately get a dual statement by applying that result to complements of words. Likewise, (iv) allows us to "relativize" any general result about Pratt comonoids to yield a result about the set of words $x \in W$ such that $u \leq x \leq v$ for given $u \leq v$ in W.

Next, we note two easy ways of getting new words from old within a given Pratt comonoid.

Lemma 3 ([8, Proposition 2.5])**.** *Let* (A, W) *be a Pratt comonoid. Then* W *is closed under* (*pairwise, hence finite*) *meets and joins. That is, if* $x, y \in W$, *then* $x \wedge y$ *and* $x \vee y$ *also belong to* W.

Proof. Given $x, y \in W$, we find that $x \times y$ is a crossword on W (every row gives either the word $\emptyset$ or the word y; every column gives either $\emptyset$ or x), and its diagonal gives the word $x \wedge y$, so W is closed under intersections.

That it is also closed under unions follows by dualization, i.e., applying the above result to the comonoid $(A, W^{\neg})$ constructed as in Lemma 2(i). Alternatively, one can verify directly that $(x \times A) \vee (A \times y)$ is a crossword on W having $x \vee y$ as diagonal. □

Some observations on the above results:

Though we have seen that any intersection of Pratt comonoid structures on a set A is again a Pratt comonoid structure, the same is not true of unions, even pairwise unions. For instance, if $A = \{0, 1, 2\}$, and we let $W_{\leq}$ be the set of all down-subsets of A, and $W_{\geq}$ the set of up-subsets of A, then each of these is a Pratt comonoid structure, but their union is not, since it contains both $\{0, 1\}$ and $\{1, 2\}$, but not $\{0, 1\} \cap \{1, 2\}$.

Of course, since the condition of being a Pratt comonoid structure on A is a closure condition, there is a least such structure containing the union of two given structures. But the resulting closure operation can expand the given union enormously. For instance, suppose we let $A = \mathbb{Z}$, the set of integers, again let $W_{\leq}$ and $W_{\geq}$ be the systems of down-subsets and up-subsets of A, and let W be the least Pratt comonoid structure containing $W_{\leq} \cup W_{\geq}$. By taking intersections, we see that W contains all singletons $\{i\}$ $(i \in \mathbb{Z})$. Hence for any subset $x \subseteq \mathbb{Z}$, the set $\{(i, i) \mid i \in x\} \subseteq \mathbb{Z} \times \mathbb{Z}$ is a crossword over W, since every row or column is either empty or a singleton. So W consists of all subsets $x \subseteq \mathbb{Z}$. Thus, closing under taking diagonals of crosswords has carried the countable set $W_{\leq} \cup W_{\geq}$ to a set of continuum cardinality.

We saw in Lemma 3 that each Pratt comonoid structure on a set A is closed under pairwise unions and intersections. However, such structures need not be closed under infinite unions and intersections, as may be seen from the example (2) above, where W contains all the finite down-subsets of ω, but not their union, ω itself.

Though we do not in general get all infinite unions and intersections, everything we do get can be expressed in terms of such operations:

Lemma 4. *Let* A *be any set, and* $C : A \times A \to 2$ *any map. Then the subset* $z \subseteq A$ *corresponding to the diagonal of* C *can be written as a* (*possibly infinite*) *union of* (*possibly infinite*) *intersections of subsets of* A *corresponding to rows and columns of* C.

Proof. For each $a \in A$ that occurs as a member of at least one row or column of C, let $x_a \subseteq A$ be the intersection of all the rows and columns of C that contain a. Clearly, $a \in x_a$. We claim that, in fact, $z = \bigvee_{a \in z} x_a$. This will clearly imply the desired conclusion.

Indeed, for every $a \in z$, the element a lies in the above union, namely, in the joinand indexed by a. Conversely, if b is in that union, this says it lies in x_a for some $a \in z$. Looking at the a-th row of C we conclude from the definition of x_a that since $C(a, a) = 1$, we have $C(a, b) = 1$; and looking at the b-th column, $C(a, b) = 1$ similarly implies $C(b, b) = 1$. So $b \in z$ as required. □

The following easily verified observation (which does not refer to diagonal words) will also be useful.

Lemma 5. *If* A *and* A' *are sets, and* C *a subset of* $A \times A'$ *which has precisely* κ *distinct rows* (*resp. columns*) *then it has at most* 2^{κ} *distinct columns* (*resp. rows*).

In particular, if C *has only finitely many distinct rows* (*columns*), *it has only finitely many distinct columns* (*rows*). □

3. The case where A is countable

We sketch below the proof of the known result that every T_1 Pratt comonoid (A, W) with countable base-set A is discrete. First, a general observation.

Lemma 6. *If (A, W) is a T_1 Pratt comonoid and A_0 a finite subset of A, then for any subset y of A_0 there exists an $x \in W$ with $x \cap A_0 = y$.*

Proof. For all $a, b \in A_0$ with $a \neq b$, the T_1 condition allows us to choose an $x_{a,b} \in W$ containing a but not b. By Lemma 3, the set $x_a = \bigwedge_{b \in A_0 - \{a\}} x_{a,b}$ is a member of W containing a but no other member of A_0. We see that $x = \bigvee_{a \in y} x_a$ has the desired property. $\square$

We can now get:

Theorem 7 (V. Pratt [6, 2nd exercise on p. 28])**.** *If (A, W) is a T_1 Pratt comonoid, and A is countable, then (A, W) is discrete.*

Sketch of proof (after Mark G. Pleszkoch = "Mark Aujus" [9, Solution to puzzle 1.4]). Assume without loss of generality that $A = \omega$. Take any $z \subseteq A$, and assume inductively that for some n we have found $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1} \in W$ which are possible candidates for the first n rows and n columns of a crossword having z as diagonal; i.e., such that the partial crossword formed by using the x_m's as its first n rows and the partial crossword formed by using the y_m's as its first n columns agree on their $n \times n$ intersection, and the diagonal of that intersection yields the first n entries of z.

Now using the preceding lemma, with $A_0 = \{0, \ldots, n\}$, we can find $x_n, y_n \in W$ which extend our partial crossword; i.e., such that the first n entries of x_n are the entries of $y_0, \ldots, y_{n-1}$ in the position indexed by n, while its next entry is the entry of z needed in the corresponding position on the diagonal; and such that y_n has the symmetric property. This construction, continued recursively, leads to a full crossword over W with z as diagonal. $\square$

Now, on to new results.

4. The complement-closed case

We shall prove in this section that if (A, W) is a T_1 Pratt comonoid, and W is closed under complements, then (A, W) is discrete. We begin with some observations on not necessarily T_1 Pratt comonoids.

Though we have seen that for (A, W) a Pratt comonoid, W need not be closed under infinite unions, we claim that it is closed under unions of families of subsets that are pairwise disjoint. Here is a still more general statement.

Lemma 8. *Suppose (A, W) is a Pratt comonoid, and x_i $(i \in I)$ are elements of W such that for each $a \in A$, only finitely many of the x_i contain a. Then $\bigvee_{i \in I} x_i \in W$.*

Proof. Let $u = \bigvee_{i \in I} (x_i \times x_i) \subseteq A \times A$. For each $a \in A$, the a-th row (respectively, the a-th column) of u is the union of finitely many of the x_i, namely, those that contain a. Hence, since W is closed under finite unions, u is a crossword over W, hence its diagonal, $\bigvee_{i \in I} x_i$, indeed lies in W. $\square$

Corollary 9. *If (A, W) is a Pratt comonoid, then W is closed under forming unions of disjoint families (of arbitrary cardinality).* $\square$

Corollary 10. *If (A, W) is a Pratt comonoid such that W is closed under complementation, then W is closed under forming arbitrary unions.*

Hence by duality, W is also closed under forming arbitrary intersections.

Proof. By Lemma 3, W is closed under forming unions of finite families. Let κ be an infinite cardinal, assume inductively that W is closed under unions of families indexed by sets of cardinality $< \kappa$, and let x_β $(\beta \in \kappa)$ be a κ-indexed family of members of W.

For each $\beta \in \kappa$, our inductive hypothesis tells us that $\bigvee_{\gamma < \beta} x_\gamma \in W$, hence since W is closed under complementation, the set $y_\beta = x_\beta - \bigvee_{\gamma < \beta} x_\gamma$ belongs to W. The y_β are easily seen to be pairwise disjoint and to have union $\bigvee_{\beta \in \kappa} x_\beta$. (Namely, each $a \in \bigvee_{\beta \in \kappa} x_\beta$ belongs to y_β for β the least ordinal with $a \in x_\beta$.) So by Corollary 9, that union belongs to W. $\square$

We now get:

Theorem 11. *If* (A, W) *is a* T_1 *Pratt comonoid such that* W *is closed under forming complements, then* (A, W) *is discrete* (*i.e.,* W *is the set of all subsets of* A).

Proof. Because (A, W) is T_1, for each $a \in A$ the intersection of all members of W that contain a is $\{a\}$, so by the final assertion of Corollary 10, W contains every singleton. Since every subset of A is a union of singletons, another application of that corollary shows that W contains every subset of A. $\square$

5. An interesting non-T_1 Pratt comonoid structure on 2^ω

In the first paragraph of §2, we noted that easy examples of Pratt comonoids (A, W) in which W was not the set of all subsets of A could be gotten by starting with any partially ordered set A, and letting W be the set of all its down-sets; and in the next paragraph, we noted a case where such an A admitted a slightly smaller comonoid structure W, determined by a sort of "continuity" condition. Below, we construct another example of a sub-comonoid of the Pratt comonoid arising from a partially ordered set, which differs from it much more strikingly: The partially ordered set A we start with will be 2^ω, ordered by inclusion; the partially ordered set of *all* its down-sets can be shown to have cardinality $2^{2^{\aleph_0}}$, but our W will be countable. This example will be a key ingredient in our construction, in the next section, of a non-discrete T_1 example.

Actually, we will construct the Pratt comonoid of this section as a sub-comonoid of the comonoid of all *up-sets* of 2^ω, i.e., families of subsets closed under enlargement. The up-sets of any partially ordered set A form a Pratt comonoid structure on A for the same reason that the down-sets do, and since the partially ordered set 2^ω is isomorphic to its opposite, the two comonoids are isomorphic (cf. Lemma 2(i)). We will use up-sets because it will be conceptually simpler to take for the building blocks of our construction the up-sets e_n $(n \in \omega)$ consisting of all subsets of ω that contain n, rather than the down-sets given by their complements.

Recall that for every set A, the set 2^A of all subsets of A, regarded as a direct product of copies of the discrete topological space 2, is a compact Hausdorff space (compact by Tychonoff's Theorem). A subbasis for its open sets is given by the sets $\{x \in 2^A \mid a \in x\}$ for $a \in A$, and their complements, $\{x \in 2^A \mid a \notin x\}$. We shall call this topology the *natural topology* on 2^A.

In particular, both 2^ω and 2^{2^ω} have such topologies; note that the definition of the topology on the latter set uses only the set-structure of the former, and ignores its topology. (But a relation between the topologies of these two sets will be key to the proof of the final result of this section.)

We start with some observations on a general partially ordered set A. Recall that if $(A, \preccurlyeq)$ is a partially ordered set and $a \in A$, then $\uparrow(a) = \{b \in A \mid b \succcurlyeq a\}$ is called the *principal up-set* determined by a. Likewise $\downarrow(a) = \{b \in A \mid b \preccurlyeq a\}$ is called the *principal down-set* determined by a. (One normally calls these the principal up-set and principal down-set *generated* by a, but we will use "determined" to avoid confusion with comonoids generated by sets of subsets.)

Lemma 12. *Let* A *be a set given with a partial ordering* $\preccurlyeq$, *and let* $U_\preccurlyeq(A) \subseteq 2^A$ *denote the set of up-sets of* A, *with the topology induced by the natural topology on* 2^A. *Then the following conditions on an element* $x \in U_\preccurlyeq(A)$ *are equivalent.*

(i) x *is an isolated point of the topological space* $U_\preccurlyeq(A)$; *i.e.,* x *is not in the closure of* $U_\preccurlyeq(A) - \{x\}$.

(ii) x *is both the union of a finite* (*possibly empty*) *family of principal up-sets, and the intersection of a finite* (*possibly empty*) *family of complements of principal down-sets.*

Proof. To prove (ii) $\implies$ (i), note that if x is a union $\uparrow(a_0) \vee \cdots \vee \uparrow(a_{m-1})$, then it is the smallest (under inclusion) element of $U_\preccurlyeq(A)$ containing all of $a_0, \ldots, a_{m-1}$. Likewise, if it is an intersection $\neg\downarrow(b_0) \wedge \cdots \wedge \neg\downarrow(b_{n-1})$, then it is the largest element of $U_\preccurlyeq(A)$ not containing any of $b_0, \ldots, b_{n-1}$. These conditions together make it the *unique* element of $U_\preccurlyeq(A)$ containing each of the a_i and none of the b_j. Now the property of containing or not containing a specified element of A defines an open subset of 2^A, hence of $U_\preccurlyeq(A)$. Thus, intersecting the $m+n$ open sets arising from the above description, we get an open subset of $U_\preccurlyeq(A)$ having x as its only point; so x is isolated.

Conversely, since the subsets of $U_\preccurlyeq(A)$ defined by the conditions of containing or not containing a given element of A form a subbasis of its open sets, if x is isolated it must be the unique point in a finite intersection of such sets; i.e., the unique up-set that contains all members of a finite family of points

$a_0, \dots, a_{m-1}$ and no members of another finite family $b_0, \dots, b_{n-1}$. It is easy to see that there is a least up-set containing $a_0, \dots, a_{m-1}$ (namely, $\uparrow(a_0) \vee \dots \vee \uparrow(a_{m-1})$) and a greatest containing none of $b_0, \dots, b_{n-1}$ (namely, $\neg\downarrow(b_0) \wedge \dots \wedge \neg\downarrow(b_{n-1})$). Given that the families of up-sets defined by these two properties have nonempty intersection, if the least member of one family and the greatest member of the other did not coincide, then the intersection of the two families would not be a singleton. So they do coincide, giving a description of x as in (ii). □

(M. Erné has kindly pointed out to us that the above result can be deduced from the general theory of continuous lattices [4], [5]. Namely, in any lattice which is *bicontinuous* in the sense of [5, Chapter VII], the points that are isolated in the bi-Scott topology are those that are both *isolated from above* and *isolated from below* in the sense of [5, Chapter I]. A subclass of the bicontinuous lattices are the *superalgebraic* lattices [2] [3], which are, up to isomorphism, the up-set lattices of partially ordered sets A; and in these, the elements isolated from below are the finitely generated upsets, those isolated from above are the complements of finitely generated down-sets, and the bi-Scott topology agrees with the topology induced by the natural topology on 2^A, yielding the statement of the lemma.)

We now apply the above lemma to the case where A is the set 2^ω, partially ordered by inclusion. (We could allow any set in place of ω, but we shall see in §9 that this example can be generalized in other ways; so we will just consider here the case we are about to use.) Conditions (i) and (ii) below are as in the lemma; (iii) is what is new.

Corollary 13. *Let $A = 2^\omega$, partially ordered by inclusion, $\subseteq$, and $U_\subseteq(A) \subseteq 2^A$ its set of up-sets. For each natural number n, let $e_n \in U_\subseteq(A)$ denote the set of all subsets of ω containing n. Then the following conditions on an element $x \in U_\subseteq(A)$ are equivalent.*

(i) *x is an isolated point of $U_\subseteq(A)$ under the natural topology.*

(ii) *x is both the union of a finite* (*possibly empty*) *family of principal up-sets, and the intersection of a finite* (*possibly empty*) *family of complements of principal down-sets of A.*

(iii) *x lies in the lattice generated by $\{e_n \mid n \in \omega\} \cup \{\emptyset, A\}$; i.e., the closure of that set under pairwise unions and intersections.*

Proof. By Lemma 12, (i) $\iff$ (ii), so it will suffice to show that (iii) $\implies$ (ii), and that (i) $\wedge$ (ii) $\implies$ (iii).

Assume (iii). If $x = \emptyset$, it is both the union of the empty family of principal up-sets and the intersection of a 1-element family of complements of principal down-sets, namely $\{\neg\downarrow(A)\}$, so it satisfies (ii). The case $x = A$ is seen similarly.

If x is neither $\emptyset$ nor A, it can be written as a lattice-theoretic expression in the e_n, and using distributivity, we can express it both as a finite join of finite meets of these elements, and as a finite meet of finite joins thereof. Using the former expression, we note that each finite meet $e_{n_0} \wedge \dots \wedge e_{n_{i-1}}$ is the principal up-set determined by $\{n_0, \dots, n_{i-1}\}$, so we have the first condition of (ii).

On the other hand, when we express x as a finite meet of finite joins of the e_n, each of those finite joins can be looked at as the complement of a finite meet of complements of the e_n; and we see that such a meet $\neg e_{n_0} \wedge \dots \wedge \neg e_{n_{i-1}}$ is a principal down-set, the set of elements of A that are $\leq \neg\{n_0, \dots, n_{i-1}\}$. So x is a finite meet of complements of principal down-sets, giving the second condition of (ii).

Conversely, assume (i) $\wedge$ (ii). By (ii), x can be written as a finite join of principal up-sets, $x = \uparrow(s_0) \vee \dots \vee \uparrow(s_{n-1})$ $(s_0, \dots, s_{n-1} \subseteq \omega)$. Without loss of generality we may assume that none of the s_i contains any of the others. Suppose one of them, s_i, were infinite. Then $\uparrow(s_i)$ is the intersection of the downward-directed set of up-sets $\uparrow(s)$ as s ranges over the finite subsets of s_i, and we see that it will be the limit of those up-sets under the topology on $U_\subseteq(A)$. Now holding the other s_j in our expression for x fixed, and letting $s \to s_i$ as above, we get a family of up-sets approaching x. Moreover, if one of these up-sets by which we are approaching x coincided with x, say the one constructed from a finite subset $s \subseteq s_i$, then x would have s as a member, which it does not, since none of the other s_j's is contained in s_i. Thus x is a limit of points distinct from x, contradicting (i). So all s_i are finite, hence each $\uparrow(s_i)$ is a finite (possibly empty) meet of the e_n, so x is a finite (possibly empty) join of such finite meets, proving (iii). □

We can now prove:

Theorem 14. *Let $A = 2^\omega$; for each natural number n let $e_n \subseteq A$ be the set of subsets of ω containing n, and let $W \subseteq 2^A$ be the closure of $\{e_n \mid n \in \omega\} \cup \{\emptyset, A\}$ under pairwise unions and intersections. Then (A, W) is a Pratt comonoid.*

Proof. We first note that every $x \in W$, regarded as a function $A \to 2$, is continuous with respect to the product topology on $A = 2^\omega$ and the discrete topology on 2. Indeed, the generators e_n and their complements are the characteristic functions of the open-closed generating sets for the topology on A, hence are continuous, as are the constant functions $\emptyset$ and A; and the general member of W is obtained from the e_n, $\emptyset$, and A using the operations $\wedge$ and $\vee$ on 2, which are necessarily continuous in the discrete topology on 2.

Thus, if $C : A \times A \to 2$ is any crossword whose *rows* are given by elements of W, then each row represents a continuous map $A \to 2$. Hence the whole crossword, thought of as a map taking each element of A to the column it indexes, is a continuous function $A \to 2^A$.

Suppose now that C as above has infinitely many distinct columns. Let B be an infinite subset of A whose members index distinct columns of C. Since A is compact, we can find some limit-point $b \in A$ of B. By continuity, the column of C indexed by b is a limit-point of columns indexed by the elements of B. Hence, since elements of W are isolated points, the b-th column of C cannot be a member of W.

It follows that if all rows *and* columns of C belong to W, then C can have only finitely many distinct columns, and thus by Lemma 5, also only finitely many distinct rows. Hence we can apply Lemma 4 (with its qualifier "*possibly infinite*" irrelevant because of the above finiteness results), and the fact that W is closed under pairwise unions and intersections, to conclude that the diagonal of C is a member of W; proving that (A, W) is a Pratt comonoid. □

We will have further use for the technique of the last sentence of the above proof; so let us record what that argument gives us.

Corollary 15 (to Lemmas 3 and 4)**.** *Suppose A is a set and W a set of subsets of A which is closed under pairwise unions and intersections, and contains $\emptyset$ and A. Suppose, moreover, that no crossword $C : A \times A \to 2$ with all rows and columns in W has infinitely many distinct rows* (*equivalently, by Lemma 5, has infinitely many distinct columns*)*. Then (A, W) is a Pratt comonoid.* □

6. A non-discrete T_1 Pratt comonoid

We are now ready to construct a Pratt comonoid (A, W) that answers the question (1) in the negative.

Intuitively, the idea will be to make our base-set A the union of many "islands", and take W to be generated by an uncountable family of subsets $w_{n,\gamma}$ of A, such that each generator, when restricted to certain of the islands, looks like one of the generators in the example of Theorem 14 above, while everywhere else, it looks like one of a countable family of generators u_n of a discrete Pratt comonoid structure on A.

The fact that the $w_{n,\gamma}$ look "in most places" like the u_n will make our structure T_1. On the other hand, the system of "islands" will be set up so that given any *countable* family of the $w_{n,\gamma}$, there is some island on which that family acts precisely like our generating set for the construction of Theorem 14. We will use this property to show that, as in that theorem, no crossword formed from lattice expressions in our generators can have infinitely many distinct rows or columns; whence those lattice expressions will in fact form a Pratt comonoid structure on A, which we shall see is T_1 but not discrete.

We begin with an easy general observation.

Lemma 16. *If A is a set of continuum cardinality, then there is a countable T_1 family $\{u_0, u_1, \dots\}$ of subsets of A.*

Proof. It suffices to construct such a family for $A = 2^\omega$. To do this, let us define each even-indexed set u_{2n} to be the set e_n of subsets of ω which contain n, and each odd-indexed set u_{2n+1} to be the set $\neg e_n$ of subsets of ω which do not contain n. The T_1 property is immediate. □

Recalling that ω_1 denotes the first uncountable ordinal, we now define the base-set A of our example:

$$
\begin{aligned}
A &= A' \times A'', \text{ where} \\
A' &= \text{the set of all one-to-one maps } \omega \to \omega \times \omega_1, \\
A'' &= 2^\omega.
\end{aligned}
\tag{3}
$$

The typical element of A will be written (a', a''), with $a' \in A'$, $a'' \in A''$. The "islands" referred to in the above sketch will be the sets $\{a'\} \times A''$.

Observe that A has the cardinality of the continuum. Indeed, since A'' has that cardinality, it suffices to show that A' has at most that cardinality. Now A' is contained in the set of *all* maps $\omega \to \omega \times \omega_1$, and the cardinality of that set is bounded above by the result of replacing $\omega \times \omega_1$ in that description by 2^ω, i.e., by the cardinality of the set of maps $\omega \to 2^\omega$, which is $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$, as required.

Hence by Lemma 16, we can

(4) let $\{u_n \mid n \in \omega\}$ be a countable T_1 family of subsets of A.

We now define the family of subsets of A (which we will express as $\{0,1\}$-valued functions) that we will take as the generators of our Pratt comonoid. Namely, for each $(n,\gamma) \in \omega \times \omega_1$, we let $w_{n,\gamma} : A \to 2$ be defined by

(5) $$w_{n,\gamma}(a', a'') = \begin{cases} u_n(a', a'') & \text{if } (n,\gamma) \text{ is not among the elements } a'(i) \text{ for } i \in \omega \text{ (cf. (3))}, \\ a''(i) & \text{if } i \in \omega \text{ satisfies } a'(i) = (n,\gamma). \end{cases}$$

Thus, for each "island" $\{a'\} \times A''$, the coordinate a' specifies a countable sequence of ordered pairs (n,γ) such that the corresponding generators, $w_{n,\gamma}$, act on the A''-components of members of that island like the generating functions e_i of the example of Theorem 14; namely, if $(n,\gamma) = a'(i)$, then $w_{n,\gamma}$ selects the i-th coordinate of a''.

Now let

(6) $W =$ the lattice of subsets of A generated by $\{w_{n,\gamma} \mid n \in \omega,\ \gamma \in \omega_1\} \cup \{\emptyset, A\}$ under pairwise unions and intersections.

Most of our work will go into showing that (A, W) is a Pratt comonoid.

Turning back to the definition (3) of A, observe that each $a' \in A'$ has countable image; hence the set of second coordinates of elements of that image will be bounded within ω_1. Hence defining, for each $\beta < \omega_1$,

(7) $$A_\beta \;=\; \{(a', a'') \in A \mid \text{the second coordinates of all elements of the image of } a' \text{ are } < \beta\},$$

we have

(8) A is the union of the chain of subsets A_β $(\beta \in \omega_1)$,

and we see from the first line of (5) that

(9) for $\beta \in \omega_1$, every $w_{n,\gamma}$ with $\gamma \geq \beta$ acts on A_β by u_n.

We can now deduce that

(10) W is T_1 on A.

Namely, given $a_1,\ a_2 \in A$, the T_1 property of the u_n lets us choose an n such that $u_n(a_1) = 1$, $u_n(a_2) = 0$, and (8) allows us to choose a β such that $a_1,\ a_2 \in A_\beta$; so by (9), $w_{n,\beta}(a_1) = u_n(a_1) = 1$, $w_{n,\beta}(a_2) = u_n(a_2) = 0$, as required.

Let us show next that

(11) for every $\beta \in \omega_1$, the restrictions to $A_\beta \subseteq A$ of the elements of $W \subseteq 2^A$ are countable in number.

By (6) it suffices to show that the restrictions to A_β of the generating elements $w_{n,\gamma}$ are countable in number. By (9), those $w_{n,\gamma}$ with $\gamma \geq \beta$ have restrictions given by the countably many elements u_n. By definition of ω_1, there are only countably many $\gamma < \beta$, and hence only countably many $w_{n,\gamma}$ with such γ, completing the proof of (11).

We need, next, a combinatorial lemma (which we will apply to occurrences of the $w_{n,\gamma}$ appearing as arguments in a j-variable lattice term).

Lemma 17. *Let X be a set, j a positive integer, and S an infinite set of ordered j-tuples of elements of X, such that in each member of S, the j entries are distinct.*

Then there exist an infinite subset $S' \subseteq S$ and an $i < j$ such that, after we apply some permutation of the j coordinates to all our j-tuples, all members of S' begin with the same initial i-element string, while every element of X that occurs in one of the last $j - i$ positions of an element of S' occurs in no other member of S'.

Thus, cutting S' down to a countable set if it was uncountable, and writing $k = j - i > 0$, we can find distinct elements $x_\ell \in X$ $(\ell \in \omega)$ such that S' consists of the j-tuples

(12) $$(x_0, \ldots, x_{i-1},\, x_{i+hk}, \ldots, x_{i+(h+1)k-1}) \quad \textit{for } h \in \omega.$$

Proof. Let $i < k$ be the largest integer such that S contains infinitely many elements which agree in some common i-tuple of their coordinates; let us perform a permutation of indices that makes $0, \dots, i-1$ such a set of coordinates, and let us choose $x_0, \dots, x_{i-1}$ which appear in that order as the first i coordinates of infinitely many elements of S. Let $S_0 \subseteq S$ be the infinite set of those elements of S beginning with the string $x_0, \dots, x_{i-1}$. Note that the maximality of i implies that no element of X other than $x_0, \dots, x_{i-1}$ occurs in infinitely many members of S_0.

Now let $k = j - i$, and choose any $x_i, \dots, x_{i+k-1}$ such that $(x_0, \dots, x_{i-1}, x_i, \dots, x_{i+k-1}) \in S_0$. As just noted, each of $x_i, \dots, x_{i+k-1}$ appears as an entry in only finitely many members of S_0, so if we let S_1 be the set of elements of S_0 in which none of them appear, this will still be infinite, and we can pick $x_{i+k}, \dots, x_{i+2k-1}$ such that $(x_0, \dots, x_{i-1}, x_{i+k}, \dots, x_{i+2k-1}) \in S_1$. Letting S_2 be the infinite set of elements of S_1 involving none of $x_{i+k}, \dots, x_{i+2k-1}$, we can similarly pick $x_{i+2k}, \dots, x_{i+3k-1}$ to get an element of S_2; and so forth. Thus we get an infinite family S' of the desired form.

(We shall only need a countably infinite S'; but if we wished, we could get S' to have any regular cardinality $\kappa \leq \mathrm{card}(S)$, by replacing references to finite and infinite subsets of S in the above proof with subsets of cardinalities $< \kappa$ and $\geq \kappa$.) □

Using this lemma, let us prove:

Lemma 18. *For A and W defined as in* (3)-(6)*, every uncountable subset $W_0 \subseteq W$ contains a countable subset W_1 such that the set of distinct functions $W_1 \to 2$ obtained by evaluation at different elements of A has continuum cardinality.*

Proof. Each member of W can be written as a lattice expression $b(w_{n_0,\gamma_0}, \dots, w_{n_{j-1},\gamma_{j-1}})$ of some finite length j, whose arguments $w_{n_0,\gamma_0}, \dots, w_{n_{j-1},\gamma_{j-1}}$ are distinct, and which depends nontrivially on all j of its arguments (i.e., such that inserting all combinations of 0's and 1's in those j positions in b, the resulting function depends on each of its variables). Note that, ignoring the choice of variables $w_{n_0,\gamma_0}, \dots, w_{n_{j-1},\gamma_{j-1}}$, there are only countably many distinct finite lattice terms; hence the uncountability of W_0 implies that there is some lattice term b, say in j variables, from which one can get infinitely many of the members of W_0 by inserting appropriate elements $w_{n,\gamma}$ as its arguments. Let us fix such a b.

We can now apply Lemma 17 with $X = \{w_{n,\gamma}\}$, and S the set of j-tuples of elements of X which, when used as the argument-string of b, give an element of W_0. That lemma gives us a sequence of distinct elements w_{n_ℓ,γ_ℓ} $(\ell \in \omega)$ such that

(13) $\quad$ for each $h \in \omega$, $\quad b(w_{n_0,\gamma_0}, \dots, w_{n_{i-1},\gamma_{i-1}}, w_{n_{i+hk},\gamma_{i+hk}}, \dots, w_{n_{i+(h+1)k-1},\gamma_{i+(h+1)k-1}}) \in W_0.$

In particular, the set

(14) $\quad W_1 \ = \ \{\, b(w_{n_0,\gamma_0}, \dots, w_{n_{i-1},\gamma_{i-1}}, w_{n_{i+hk},\gamma_{i+hk}}, \dots, w_{n_{i+(h+1)k-1},\gamma_{i+(h+1)k-1}}) \mid h \in \omega\}$

will be a countably infinite subset of W_0. Having so chosen the n_ℓ and γ_ℓ, let us encode them as an element $a' \in A'$, setting

(15) $\quad a'(\ell) \ = \ (n_\ell, \gamma_\ell)$ for $\ell \in \omega$.

To complete the proof of the lemma, we shall construct for this a' a continuum-sized family of points of $\{a'\} \times 2^\omega$ such that restriction to distinct points of that family induces distinct $\{0,1\}$-valued functions on the countable set W_1.

To do this, recall that our lattice-expression b depends on all $j = i + k$ of its arguments. Combining this with the fact that lattice operations are isotone (order-respecting), we see that for some choice of values $c_0, \dots, c_{i-1} \in \{0,1\}$, we will have

(16) $\quad b(c_0, \dots, c_{i-1}, 0, \dots, 0) = 0, \quad b(c_0, \dots, c_{i-1}, 1, \dots, 1) = 1.$

Let us fix such $c_0, \dots, c_{i-1} \in \{0,1\}$. Within $\{a'\} \times 2^\omega$, let us now choose the 2^ω-tuple consisting of all elements whose A''-coordinates have the values $c_0, \dots, c_{i-1}$ at $0, \dots, i-1$, then have a common value, 0 or 1, at $i, \dots, i+k-1$, likewise a common value, 0 or 1, at $i+k, \dots, i+2k-1$, and, generally, for each $h \in \omega$, a common value, 0 or 1, at $i+hk, \dots, i+(h+1)k-1$. We see from (5) and (16) that on the ω-tuple of elements comprising W_1, this family of points of A will induce all 2^ω possible ω-tuples of values in $\{0,1\}$, yielding the assertion of the lemma. □

Contrasting (11) and Lemma 18, we get:

Corollary 19. *Let A and W be defined as in* (3)-(6)*, and C be any function $A \times A \to \{0,1\}$.*

Then if C has uncountably many rows which give distinct values in W, it has at least one column which is not in W. Likewise, if C has uncountably many columns giving distinct values in W, it has at least one row not in W.

Proof. If C has rows with uncountably many distinct values in W, let $W_0 \subseteq W$ be the set of values of these rows. Choose $W_1 \subseteq W_0$ as in Lemma 18, and choose $\beta \in \omega_1$ such that all of the elements of W_1 occur as rows indexed by members of A_β. (This can be done because W_1 is countable, while ω_1 has uncountable cofinality.) Then by choice of W_1 and β, the restrictions of the columns of C to A_β give continuum many distinct functions $A_\beta \to 2$. But by (11), only countably many of these restrictions can arise from members of W; so not all columns of C belong to W. The statement with rows and columns interchanged is seen in the same way. $\square$

Now assume that C is a crossword over W. The above corollary shows that C can have at most countably many distinct rows and at most countably many distinct columns. Hence the rows and columns of C are expressible as lattice-theoretic expressions in countably many of the $w_{n,\gamma}$; let w_{n_ℓ,γ_ℓ} $(\ell \in \omega)$ be a countably infinite family in terms of which they can be expressed. (If only finitely many are needed, choose the rest arbitrarily.) Let $a' \in A'$ be defined again as in (15), but now using this family of pairs. Then as noted in the paragraph following (5), the elements w_{n_ℓ,γ_ℓ} will behave on $\{a'\} \times 2^\omega \subseteq A$ like the generators e_ℓ in Theorem 14. Hence restricting C to a crossword on $\{a'\} \times 2^\omega \subseteq A$, and regarding $\{a'\} \times 2^\omega$ as a copy of 2^ω, this crossword will, by the proof of that theorem, have only finitely many distinct rows and columns.

Moreover, rows of the whole crossword C that represent distinct elements of W must be described by distinct elements of the free distributive lattice generated by the w_{n_ℓ,γ_ℓ}, hence their restrictions to their entries indexed by elements of $\{a'\} \times 2^\omega$ will also be distinct; so if C has infinitely many distinct rows, it must have infinitely many distinct columns indexed by members of $\{a'\} \times 2^\omega$; hence the induced crossword on that set will have infinitely many distinct columns, which we have just seen is impossible. So such a C can have only finitely many distinct rows; hence by Corollary 15, (A,W) is a Pratt comonoid.

Since a discrete Pratt comonoid allows all combinations of rows to appear in a crossword, results such as (11) or Corollary 19 show that (A,W) is not discrete. Let us also note that the cardinality of W is the cardinality of a free distributive lattice with $\aleph_1$ generators, which is $\aleph_1$. We have thus proved:

Theorem 20. *For A and W as described by* (3)-(6)*, (A,W) is a non-discrete T_1 Pratt comonoid, with* $\mathrm{card}(A) = 2^{\aleph_0}$ *and* $\mathrm{card}(W) = \aleph_1$. $\square$

7. Can a T_1 Pratt comonoid structure be countably infinite? First results

We saw in §3 that a T_1 Pratt comonoid (A,W) such that A is countable must be discrete. What if we restrict W rather than A? In the example of the preceding section, W was uncountable because of the index-set ω_1 occurring in the definition of its generators $w_{n,\gamma}$; and we needed this uncountability in proving T_1-ness, i.e., condition (10). (We used it again in proving Lemma 18, but we wouldn't have had to prove that lemma if in place of the ω_1 in our definitions we had been able to use, say, ω, since then W would have been countable.)

So let us pose:

Question 21. Does there exist an infinite T_1 Pratt comonoid (A,W) such that W is countable?

If not, does there exist a non-discrete infinite T_1 Pratt comonoid such that W is countably generated (under the closure operator of forming diagonals of crosswords)?

We have not been able to answer either of these questions.

Our first thought was that if we took an uncountable set A and a fairly "random" countable T_1 family of subsets, and closed it under finite unions and intersections, then the "randomness" might prevent the resulting set W from having crosswords with infinitely many distinct rows and columns, so by Corollary 15, W would be a Pratt comonoid structure on A.

But Corollary 9 already warns us that there will be difficulties: W must not have infinite pairwise disjoint families. In trying to find examples, the authors came up with some intricate ways of constructing T_1 lattices of subsets of a set A having no disjoint pairs of nonempty elements. For instance, suppose we let A be an antichain of subsets of ω (a family of elements none of which contains another), which can be taken to be uncountable, and for each natural number n let e_n be the set of members of A which contain n; and consider the Pratt comonoid structure W on A generated by these e_n. The condition that A be an antichain guarantees that the family $\{e_n\}$ is T_1; and it is not hard to get examples where the sublattice generated by $\{e_n \mid n \in \omega\} \cup \{\emptyset, A\}$ is a free distributive lattice with 0 and 1 on the e_n, and hence has no nontrivial pairs of disjoint elements. Yet when we examined examples of this sort, we found repeatedly that they admitted unexpected crosswords which brought more subsets into W, and ultimately led to discrete structures. We shall now show why something like this was inevitable.

Again, we begin with a result on not-necessarily T_1 structures.

Lemma 22. *Let (A, W) be a Pratt comonoid, $x_0 \geq x_1 \geq \dots$ an ω-indexed descending chain of elements of W, and $y_0 \leq y_1 \leq \dots$ an ω-indexed ascending chain of elements of W, such that either $\bigwedge_{n\in\omega} x_n = \emptyset$ or $\bigvee_{n\in\omega} y_n = A$.*

Then $\bigvee_{n\in\omega} x_n \wedge y_n \in W$.

Proof. In the case where $\bigwedge_{n\in\omega} x_n = \emptyset$, note that any $a \in A$ can belong to only finitely many of the x_n, hence, a fortiori, can belong to only finitely many of the sets $x_n \wedge y_n$. Hence the conclusion is an immediate consequence of Lemma 8.

Under the alternative hypothesis $\bigvee_{n\in\omega} y_n = A$, we get the dual of the preceding result, with the roles of the x_i and y_i interchanged; but this does not quite give the conclusion we want. Rather, it says that given $x_0 \geq x_1 \geq \dots$ and $y_0 \leq y_1 \leq \dots$ with $\bigvee_{n\in\omega} y_n = A$, we have $\bigwedge_{n\in\omega} x_n \vee y_n \in W$. Now since $\bigvee_{n\in\omega} y_n = A$, and the y_n form an increasing chain, every $a \in A$ lies in almost all the sets $x_n \vee y_n$. If in fact $a \in y_0$, it lies in all of them; if not, it lies in all of them if and only if it lies in all x_n *before* the point where it appears in y_n. These observations together show that $\bigwedge_{n\in\omega} x_n \vee y_n = y_0 \vee \bigvee_{n\in\omega} x_n \wedge y_{n+1}$; so our dualized result says that $y_0 \vee \bigvee_{n\in\omega} x_n \wedge y_{n+1} \in W$.

If we now apply this result with the sequence y_n replaced by the sequence that has $\emptyset$ in place of y_0, and y_{n-1} in place of y_n for $n > 0$, we get the desired conclusion. ☐

The next result is somewhat more complicated to state, so for simplicity we will only formulate the conclusion under one of the two dual hypotheses. But we remark that if both conditions $\bigwedge_{m\in\omega} x_m = \emptyset$ and $\bigvee_{n\in\omega} y_n = A$ hold, then the final hypothesis says that although the sequence of x's gets small, and the sequence of y's gets large, no y_n contains any x_m. The form shown is the modification of that condition needed when $\bigvee_{n\in\omega} y_n$ is not necessarily everything.

When we refer to W containing a complete sublattice, we mean a subset closed in 2^A under (possibly infinite) unions and intersections, but not necessarily containing $\emptyset$ or A.

Proposition 23. *Again let (A, W) be a Pratt comonoid, $x_0 \geq x_1 \geq \dots$ an ω-indexed descending chain of elements of W, and $y_0 \leq y_1 \leq \dots$ an ω-indexed ascending chain of elements of W, and suppose that $\bigwedge_{m\in\omega} x_m = \emptyset$. Suppose moreover that for no pair $(m, n) \in \omega \times \omega$ does y_n contain $x_m \wedge \bigvee_{i\in\omega} y_i$.*

Then W has at least continuum cardinality; in fact, it contains a complete sublattice isomorphic to the lattice of all subsets of ω.

Proof. Let us show, first, that we can construct sequences of integers $m(0) < m(1) < \dots$ and $n(0) < n(1) < \dots$ such that for all i,

$$(17) \qquad x_{m(i)} \wedge y_{n(i+1)} \;\not\leq\; (x_{m(i)} \wedge y_{n(i)}) \vee (x_{m(i+1)} \wedge y_{n(i+1)}).$$

We take $m(0)$ and $n(0)$ arbitrary. Suppose, recursively, that we have found $m(0), \dots, m(j)$ and $n(0), \dots, n(j)$ so that (17) holds for all $i < j$. By the final hypothesis of the proposition, $y_{n(j)} \not\geq x_{m(j)} \wedge \bigvee_{i\in\omega} y_i$, so there is some a in the latter set that is not in the former; i.e., which lies in $x_{m(j)}$ and some y_i but not in $y_{n(j)}$. In particular, we can find $n(j{+}1) > n(j)$ such that $a \in y_{n(j+1)}$; moreover, since $\bigwedge_{m\in\omega} x_m = \emptyset$, we can find $m(j{+}1) > m(j)$ such that $a \notin x_{m(j+1)}$. With a, $n(j{+}1)$, and $m(j{+}1)$ so chosen we see that (17) also holds for $i = j$. Thus we get sequences $m(i)$ and $n(i)$ satisfying (17) for all $i \in \omega$.

Now let

$$(18) \qquad z \;=\; \bigvee_{i\in\omega} x_{m(i)} \wedge y_{n(i)}.$$

Since $\{m(i)\}$ is a cofinal subsequence of ω, we have $\bigwedge_{i\in\omega} x_{m(i)} = \emptyset$, hence by Lemma 22, $z \in W$. We claim that if for all $i \in \omega$ we define

(19) $\quad z_i \;=\; z \vee (x_{m(i)} \wedge y_{n(i+1)})$

(again an element of W), then

(20) $\quad$ for all i, $z_i > z$,

but

(21) $\quad$ for $i \neq j$, $z_i \wedge z_j = z$.

To see (20), apply (17) to get an element $a \in x_{m(i)} \wedge y_{n(i+1)}$ that is not in $x_{m(i)} \wedge y_{n(i)}$ or $x_{m(i+1)} \wedge y_{n(i+1)}$. Thus, $a \in (x_{m(i)} - x_{m(i+1)}) \wedge (y_{n(i+1)} - y_{n(i)})$. The fact that a is in the first of these two difference-sets tells us that for $j > i$, $a \notin x_{m(j)}$, while the fact that it is in the second set tells us that for $j \leq i$, $a \notin y_{n(j)}$, so in each case, $a \notin x_{m(j)} \wedge y_{n(j)}$. Hence $a \in z_i$ does not lie in any of the joinands of (18), giving (20).

To get (21), note that "$\geq$" holds by (20), so it suffices to prove "$\leq$". Assume without loss of generality that $i < j$. Then

$$
\begin{aligned}
z_i \wedge z_j \;&=\; z \vee ((x_{m(i)} \wedge y_{n(i+1)}) \wedge (x_{m(j)} \wedge y_{n(j+1)})) && \text{(by (19) and distributivity)}\\
&=\; z \vee (x_{m(j)} \wedge y_{n(i+1)}) && \text{(because } x_{m(j)} \leq x_{m(i)} \text{ and } y_{n(i+1)} \leq y_{n(j+1)})\\
&\leq\; z \vee (x_{m(j)} \wedge y_{n(j)}) && \text{(because } i{+}1 \leq j, \text{ so } y_{n(i+1)} \leq y_{n(j)})\\
&=\; z && \text{(by (18), in particular, the term indexed by } j).
\end{aligned} \tag{22}
$$

Statements (20) and (21) together say that the z_i are sets properly containing z, which on removing z give pairwise disjoint sets. By Lemma 2(iv), the results of removing z from all elements of W which contain it form a Pratt comonoid on $A - z$; so the z_i yield pairwise disjoint nonempty elements of that Pratt comonoid, hence by Corollary 9, the unions of arbitrary subsets of these sets are members of that Pratt comonoid. The corresponding elements of W give a complete lattice of subsets of the asserted form. $\square$

A problem with finding applications of the above proposition is that it is often hard to find natural hypotheses on a Pratt comonoid that lead to a countable descending chain with empty intersection, or, if we have such a chain, that lead to an ascending chain which does not eventually "swallow up" the descending chain. However in the next result we get both of these, using the T_1 assumption together with the condition that (A, W) be generated by the sort of family discussed following Question 21 above.

When we speak of (A, W) being "generated by" a family S of subsets of A, we mean, of course, that W is the least Pratt comonoid structure on A which contains S.

Theorem 24. *Suppose (A, W) is a T_1 Pratt comonoid which can be generated by a countably infinite family S of subsets of A that satisfy no nontrivial distributive lattice relations* (*i.e., which form a set of free generators of a free distributive lattice*). *Then W has at least continuum cardinality, and in fact contains a complete sublattice isomorphic to the lattice of subsets of ω.*

Proof. If A is countable, Theorem 7 gives the desired conclusion, so assume A uncountable. Now if we associate to each $a \in A$ the set of members of S which contain it, then to distinct elements of A we will associate distinct subsets of S (because W is T_1, so S must also be). Hence only countably many elements of A can yield cofinite subsets of S, so let $a \in A$ be an element such that the associated set is non-cofinite in S. Let us now write S as a disjoint union of two infinite sets: a set S_1 which *properly* contains the set of elements of S containing a, but is still non-cofinite; and its complement, $S_2 = S - S_1$.

Since S_1 properly contains the set of elements of S containing a, it includes some element not containing a, so the intersection of the members of S_1 does not contain a. On the other hand, for any $b \neq a$, the T_1 condition tells us that some $s \in S$ which contains a (and hence belongs to S_1) does not contain b; so the intersection of the members of S_1 is empty.

Let us now index each of S_1 and S_2 by ω, and for each $i \in \omega$, let x_i be the intersection of the first i elements of S_1, and y_i the union of the first i elements of S_2. We claim that for all m and n we have $y_n \not\geq x_m \wedge (\bigvee_{i\in\omega} y_i)$. For if $y_n \geq x_m \wedge (\bigvee_{i\in\omega} y_i)$, then in particular, $y_n \geq x_m \wedge y_{n+1}$, which is a nontrivial lattice relation among the first m elements of S_1 and the first $n{+}1$ elements of S_2, contradicting the assumption on S. So no such relation holds, hence Proposition 23 gives the conclusion of the theorem. $\square$

Did the above proof use the full strength of our hypothesis that the elements of S satisfy no nontrivial distributive lattice relations? If we write S_1 as $\{s_{1,i} \mid i \in \omega\}$ and S_2 as $\{s_{2,i} \mid i \in \omega\}$, then we find that the condition used at the end of the proof reduces to the statement $s_{2,0} \vee \cdots \vee s_{2,n-1} \not\geq s_{1,0} \wedge \cdots \wedge s_{1,m-1} \wedge s_{2,n}$. Now the condition that the elements of S satisfy no nontrivial distributive lattice relations is indeed equivalent to saying that for no two disjoint finite families of elements of S does the join of one majorize the meet of the other. (To see this, note that any lattice word in a set of elements of a distributive lattice can be reduced to a join of meets, and also, dually, to a meet of joins. Hence every relation on a family of elements of such a lattice is equivalent to the statement that some meet of joins majorize some join of meets. But a meet majorizes an element if and only if each meetand majorizes it; and a join is majorized by an element if and only if each joinand is majorized by it; so every such relation reduces to a family of relations each of which says that a join of elements of our given set majorizes a meet of elements of that set; hence freeness says that no such nontrivial relation, i.e., no such relations in which no generator appears as both a meetand and a joinand, holds.)

However, looking at the quantifications involved in the proof of the theorem, one finds that one can slightly restrict the set of relations that one needs to assume do not hold.

Corollary 25 (to the proof of Theorem 24)**.** *In the situation of Theorem 24, the condition that S generate a free distributive lattice can be weakened to say that there exists a partition of S into finitely many disjoint subsets, $S^{(0)}, \ldots, S^{(k-1)}$ such that no relation $t_0 \vee \cdots \vee t_{n-1} \geq s_0 \wedge \cdots \wedge s_{m-1}$ holding in W with every t_j distinct from every s_i has all but at most one of the s_i belonging to the same set $S^{(i)}$.*

Proof. Assume S has the above weakened property. In the proof of Theorem 24, after choosing S_1 to properly contain the set of elements of S containing a, and to have infinite complement, note that this complement must contain infinitely many members of one of our sets, say $S^{(i)}$. So rather than taking S_2 to be the full complement of S_1 in S, let us take it to consist of the members of that complement which belong to $S^{(i)}$. We can now complete the proof as before. At the last step, if we have a relation $s_{2,0} \vee \cdots \vee s_{2,n-1} \geq s_{1,0} \wedge \cdots \wedge s_{1,m-1} \wedge s_{2,n}$, we note that the meetands on the right satisfy $s_{1,0}, \ldots, s_{1,m-1} \in S^{(i)}$, with at most the last meetand $s_{2,n}$ not in $S^{(i)}$, contradicting the hypothesis on S. □

We do not know how one might make use of this weaker hypothesis in studying Question 21.

8. Further results on Pratt comonoids with W smaller than the continuum

The arguments in the preceding section used the fact that if (A, W) is a Pratt comonoid such that W has cardinality less than the continuum, then W cannot contain infinitely many pairwise disjoint sets (by Corollary 9). This fact restricts the structure of such comonoids in other ways as well.

The next lemma seems likely to be known, but since we do not know a reference, we will give the proof. We will be applying it to a lattice, the W of a Pratt comonoid (A, W), but since it only involves the operation $\wedge$, the natural context for stating it is that of a $\wedge$-semilattice. Recall that this means a set L with a single idempotent commutative associative operation $\wedge$; one then regards L as partially ordered by taking $x \leq y$ if $x \wedge y = x$.

Let us make:

Definition 26. If L is a $\wedge$-semilattice with least element 0, we shall call two elements of L *disjoint* if their meet is 0, and we will call a nonzero element $x \in L$ *strongly indecomposable* if there do not exist two disjoint nonzero elements $< x$ in L.

We will call two strongly indecomposable elements x, $y \in L$ *equivalent* if they are not disjoint.

That the above condition is indeed an equivalence relation on strongly indecomposable elements is immediate from the definitions.

Three examples: In the lattice of open-closed subsets of the Cantor set, regarded as a $\wedge$-semilattice, there are no strongly indecomposable elements. In the lattice whose elements are all the neighborhoods of a point p of a topological space, together with the empty set, which plays the role of 0, all nonzero elements are strongly indecomposable, and form a single equivalence class. In the lattice of all subsets of a set X, the singletons are the strongly indecomposable elements, and no two distinct singletons are equivalent.

Lemma 27. *If L is a $\wedge$-semilattice with least element 0 which has no infinite family of pairwise disjoint nonzero elements, then L has only finitely many equivalence classes of strongly indecomposable elements, and every nonzero element of L majorizes at least one strongly indecomposable element.*

Proof. There cannot be infinitely many equivalence classes of strongly indecomposable elements of L, because a family of representatives of such equivalence classes would form an infinite family of pairwise disjoint elements.

To prove that every nonzero $x \in L$ majorizes at least one strongly indecomposable element, suppose $x > 0$ does not. In particular, x is not itself strongly indecomposable, so there exist disjoint nonzero elements $x_0, y_0 < x$. By our assumption on x, the element x_0 is not strongly indecomposable, so it in turn majorizes disjoint elements $x_1, y_1 < x_0$. Continuing in this manner, we see that $y_0, y_1, \dots$ will form an infinite family of pairwise disjoint nonzero elements, contradicting the hypothesis on L. □

In the situation above, if $E_0, \dots, E_{n-1}$ are the equivalence classes of strongly indecomposable elements, and we map each $x \in L$ to the set of those E_i such that x majorizes a member of E_i, this yields a homomorphism of $\wedge$-semilattices from L to the $\wedge$-semilattice of subsets of $\{E_0, \dots, E_{n-1}\}$, which sends only 0 to the empty set.

By Corollary 9, if (A, W) is a Pratt comonoid such that W has less than continuum cardinality, the above lemma is applicable to W. More generally, for every w in such a W, that lemma is applicable to the lattice of members of W containing w by Lemma 2(iv); moreover these same statements apply to the dual lattice to W by Lemma 2(i).

Given elements $w < x$ in a $\wedge$-semilattice, let us call x strongly indecomposable *relative to* w if it is strongly indecomposable in the $\wedge$-semilattice $\{y \in W \mid y \geq w\}$ (with w regarded as 0-element). Likewise, in a $\vee$-semilattice L with greatest element 1, let us call an element x *dually strongly indecomposable* if it is strongly indecomposable in the dual $\wedge$-semilattice; and define dual strong indecomposability relative to an element $w > x$ in the obvious way. Then the above observations give:

Corollary 28. *Suppose (A, W) is a Pratt comonoid such that W has less than continuum cardinality. Then for every $w \in W$ (including $\emptyset$), the set of equivalence classes of elements strongly indecomposable relative to w is finite, and every element $> w$ majorizes at least one such element. Likewise, the set of equivalence classes of elements* dually *strongly indecomposable relative to w is finite, and every element $< w$ is majorized by at least one such element.* □

Not only elements of W, but also elements of A have obligatory relationships with equivalence classes of strongly indecomposable elements:

Lemma 29. *Suppose (A, W) is a Pratt comonoid such that W has less than continuum cardinality. Let us say that an element $a \in A$* dominates *an equivalence class E of strongly indecomposable elements if every member of W which contains the element a majorizes some member of E.*

Then every $a \in A$ dominates one or more equivalence classes of strongly indecomposable elements of W.

Moreover, if (A, W) is T_1 and a dominates an equivalence class E, then either $\bigwedge_{w \in E} w = \emptyset$, or $\bigwedge_{w \in E} w = \{a\}$. In the latter case, E is the set of all strongly indecomposable elements of W containing a, E is the only equivalence class dominated by a, and a is the only element dominating E.

Hence if (A, W) as above is T_1 and A is infinite, there exists at least one equivalence class E of strongly indecomposable elements such that $\bigwedge_{w \in E} w = \emptyset$.

Proof. Suppose some $a \in A$ dominated no equivalence class E of strongly indecomposable elements. Then for each such E we could find a $w_E \in W$ containing a but majorizing no member of E. The intersection of these finitely many elements would be a member of W containing a, hence nonempty, but majorizing no strongly indecomposable element, contradicting Lemma 27. This gives our first conclusion.

If (A, W) is T_1 and a dominates E, then for every $b \neq a$ we can find a member of W containing a but not b, and this will majorize an element of E; hence $\bigwedge_{w \in E} w$ can contain no $b \neq a$, and so must be $\emptyset$ or $\{a\}$. In the latter case, it is easy to verify that the set of strongly indecomposable elements of W containing a must coincide with E, giving the first two assertions under this hypothesis. Moreover, for any $b \neq a$, the T_1 property gives a member of W containing b but not a, giving the third assertion.

Since there are only finitely many E, there are only finitely many a dominating equivalence classes E with nonempty intersection; so if A is infinite, we can apply the first conclusion of the lemma to any a not of that sort, and get the final conclusion. □

Further observations: If (A, W) is a Pratt comonoid with W countably infinite, and E an equivalence class of strongly indecomposable elements of W having empty intersection, then we see that we can construct an ω-indexed descending chain $x_0 > x_1 > \dots$ downward cofinal in E, and so in particular, having empty intersection. Given the countability of W, Proposition 23 implies that for any ascending chain $y_0 < y_1 < \dots$ in W, there must be some m and n such that

(23) $\quad y_n \ > \ x_m \wedge \bigvee_{i\in\omega} y_i.$

But in fact, we had enough information to see that without calling on Proposition 23. On the one hand, (23) holds for all m and n if $x_0 \wedge \bigvee_{i\in\omega} y_i = \emptyset$. On the other hand, if this intersection is nonempty, that means some y_n has nonempty intersection with x_0. That intersection belongs to E, hence will majorize some x_m, and the relation $y_n > x_m$ then implies (23).

Though such increasing and decreasing chains do not have the properties that would allow us to call on Proposition 23 and get a contradiction, they do lead to the following result, which contrasts with the behavior of the examples of Theorems 14 and 20.

Proposition 30. *If (A, W) is a T_1 Pratt comonoid with W countably infinite, then there exist crosswords over W with infinitely many distinct rows and columns.*

Proof. As noted above, we can construct an infinite descending chain (x_m) with empty intersection, and by duality, an infinite ascending chain (y_n) with union A. Without loss of generality, let us take these chains to be strictly decreasing and strictly increasing respectively. Then $C = \bigvee_n x_n \times y_n$ will be the desired crossword. Indeed, given $a \in A$, let m be the greatest integer such that $a \in x_m$; then we see that the row of C indexed by a will be given by $y_m \in W$, and a dual statement applies to columns; so the rows and columns of C comprise precisely $\{x_m\} \cup \{y_n\}$, an infinite subset of W. $\square$

Here is another sort of chain that we can construct in any infinite T_1 Pratt comonoid with W countable, or more generally, with $\mathrm{card}(W)$ both less than the continuum and less than $\mathrm{card}(A)$. The relativized version of Lemma 29 shows that for every $w \in W$ there are only finitely many elements $a \notin w$ which dominate equivalence classes relative to w that have intersection strictly larger than w. Since $\mathrm{card}(W) < \mathrm{card}(A)$, there must be an a which, relative to every w not containing it, dominates no such equivalence classes. Now starting a recursion with $y_0 = \emptyset$, assume we have elements $y_0 < \dots < y_{i-1}$, each strongly indecomposable relative to the one before, with $a \notin y_{i-1}$. Then we can take an equivalence class E_i of elements strongly indecomposable relative to y_{i-1} which is dominated by a relative to y_{i-1}, and take a $y_i \in E_i$ which does not contain a. We thus get an infinite chain with these properties; but how this fact might be useful we again don't know.

If we assume the negation of the continuum hypothesis, then the example of Theorem 20 has W of less than continuum cardinality, hence Corollary 28, Lemma 29, and the observation of the last paragraph apply to that example. Thus, we cannot expect those results to yield contradictions without some stronger assumption, such as that W be countable.

A question we have not studied, which is also suggested by Theorem 20, is:

Question 31. Assuming the negation of the continuum hypothesis, can there exist a non-discrete T_1 Pratt comonoid (A, W) whose base-set A has less than continuum cardinality?

9. Generalizing the construction of §5

Let us pick up a loose end. In §5 we saw that the lattice of subsets of $A = 2^\omega$ generated by $\{e_n \mid n \in \omega\} \cup \{\emptyset, A\}$ formed a Pratt comonoid. We sketch here, as promised in the paragraph preceding Corollary 13, a generalization of that result.

First let us generalize Corollary 13. For brevity we will not, this time, include condition (ii) of Lemma 12 among the equivalent conditions in the statement, though the equivalence of conditions (i) and (ii) of that lemma will still be essential to the proof.

Lemma 32. *Let $(A_i)_{i\in I}$ be any family of* finite *partially ordered sets, such that each A_i has a least element 0_i and a greatest element 1_i, and let $A = \prod_{i\in I} A_i$, with $\preccurlyeq$ the componentwise partial ordering. Let S be the set of elements of A which have i-th coordinate 0_i for all but at most one i, and S' the set of elements having i-th coordinate 1_i for all but at most one i. Then the following conditions on an element $x \in U_{\subseteq}(A)$ are equivalent.*

(i) *x is an isolated point of $U_{\subseteq}(A)$ under the natural topology.*

(ii) *x is the inverse image under the projection of A onto a finite sub-product $A_{i_0} \times \cdots \times A_{i_{n-1}}$ (where $i_0, \ldots, i_{n-1}$ are distinct elements of I) of an up-set of that sub-product.*

(iii) *x lies in the lattice of up-sets of A generated by $\emptyset$ and the elements $\uparrow(s)$ for $s \in S$.*

(iv) *$\neg x$ lies in the lattice of down-sets of A generated by $\emptyset$ and the elements $\downarrow(s)$ for $s \in S'$.*

Proof. We shall show (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii), whence by symmetry also (ii) $\Longleftrightarrow$ (iv), and then that (iii)$\wedge$(iv) $\Longrightarrow$ (i).

The proof that (i) $\Longrightarrow$ (ii) follows the idea of Corollary 13 (i) $\wedge$ (ii) $\Longrightarrow$ (iii). Assuming (i), we can, by Lemma 12(ii), write x as a union $\uparrow(a_0) \vee \cdots \vee \uparrow(a_{m-1})$ with each $a_j \in A$, and we can assume without loss of generality that none of the a_j majorizes any of the others. If any of the a_j had infinitely many nonzero coordinates, then we could write $\uparrow(a_j)$ as an intersection of principal up-sets determined by elements $\uparrow(a)$ which take finitely many nonzero coordinates from a_j, and have zero-elements in all other coordinates. As in the proof of that corollary, this would make x a limit, in the natural topology on $U_{\subseteq}(A)$, of elements $\neq x$, contradicting (i). Hence x is a join of finitely many elements each constraining only finitely many coordinates of elements of A. If we write $\{i_0, \ldots, i_{n-1}\}$ for the full set of indices whose coordinates x constrains, we see that x is the inverse image of a subset of $A_{i_0} \times \cdots \times A_{i_{n-1}}$, which will be an up-set in that product, proving (ii).

To get (ii) $\Longrightarrow$ (iii), note that since $A_{i_0} \times \cdots \times A_{i_{n-1}}$ is a finite partially ordered set, every up-set of that set is a finite (possibly empty) union of principal up-sets. Writing such a principal up-set as $\uparrow(a_0, \ldots, a_{n-1})$, we see that it is the intersection over $j = 0, \ldots, n-1$ of the principal up-sets determined by the elements which have a_j in the i_j coordinate and zeroes in all other coordinates. Hence the inverse image of that up-set in A is the intersection of the principal up-sets of A having the same descriptions, but with "other coordinates" now ranging over I rather than just $\{i_0, \ldots, i_{n-1}\}$. This leads to the description of x as in (iii). The reverse implication is clear.

By symmetry, we likewise get (ii) $\Longleftrightarrow$ (iv).

Finally, (iii)$\wedge$(iv) immediately gives Lemma 12(ii), and hence (i). □

This leads to the following generalization of Theorem 14.

Theorem 33. *For A a partially ordered set constructed as in Lemma 32, the set W of all up-sets of A that satisfy the equivalent conditions of that lemma is a Pratt comonoid structure on A.*

Sketch of proof. Let us note how to adapt the proof of Theorem 14. Condition (iii) of Lemma 32, which is analogous to the hypothesis of that theorem, makes for the easiest translation of the proof. We replace occurrences of "2" in that proof, where they represent value-sets of coordinates of elements of A, by the appropriate finite partially ordered sets A_i, while where 2 occurs as the value-set of members of $W \subseteq 2^A$, it remains unchanged.

In Theorem 14, the generators $e_n : A \to 2$ of W were projections to the n-th coordinates. The corresponding generators $\uparrow(s)$ $(s \in S)$ of our present W can be regarded as composite maps $A \to A_i \to 2$, where the first arrow is the projection onto the i-th component, and the second is the characteristic function of the principal up-set determined by some element of A_i. Thus, as in the proof of Theorem 14, they are continuous maps. Where A was included in our list of lattice generators in Theorem 14, it does not require separate mention here, since it can be written $\uparrow(0)$, but $\emptyset$ is still needed, and indeed appears in Lemma 32(iii). □

10. Appendix: Background on the concept of Pratt comonoid

In [6], [7], [8] Vaughan Pratt studies, for Σ a set, the category $\mathbf{chu}_\Sigma$, whose objects, *Chu spaces*, are pairs (A, r, X), where A and X are sets, and $r : A \times X \to \Sigma$ a set map, and where a morphism $(A, r, X) \to (B, s, Y)$ is given by a pair of set-maps, $f : A \to B$ and $g : Y \to X$ such that $s(f(a), y) = r(a, g(y))$ for $a \in A$, $y \in Y$. These spaces are used to model various programming concepts. It is noted in [7, §1.6] that the definition is based on ideas from the Master's thesis of Po Hsiang Chu.

If one restricts attention to objects (A, r, X) such that distinct elements of X induce distinct maps on A, then one can regard X as a set of maps $x : A \to \Sigma$, and drop the map r from the description of these objects. If one also takes $\Sigma = 2 = \{0, 1\}$, then X, now a set of $\{0, 1\}$-valued functions on A, can be regarded as a set of distinguished subsets of A, and morphisms $(A, X) \to (B, Y)$ correspond to set-maps

$A \to B$ under which the inverse image of every distinguished subset of B is a distinguished subset of A. Pratt notes that various sorts of mathematical structures can be described as instances of Chu spaces; for instance, the category of topological spaces can be considered a subcategory of $\mathbf{chu}_2$, determined by the condition that the distinguished subsets are closed under arbitrary unions and finite intersections.

Given objects (A, X) and (B, Y) of $\mathbf{chu}_\Sigma$, Pratt defines $(A, X) \otimes (B, Y)$ to be the object whose first component is the set $A \times B$, and whose second component is the set of those maps $A \times B \to \Sigma$ which form "crosswords" with rows from Y and columns from X. He then defines a *comonoid* in $\mathbf{chu}_\Sigma$ to be an object (A, X) given with a map $(A, X) \to (A, X) \otimes (A, X)$ which makes certain diagrams commute, dual to the diagrams of set-maps that define the ordinary concept of monoid.

(The first author of this note, having worked with coalgebra objects as representing objects for algebra-valued functors [1], prefers to use the unmodified term "comonoid" for an object given with an appropriate sort of morphism into the *coproduct* of two copies of itself, and would call an object of the sort Pratt considers a "$\otimes$-comonoid".)

Pratt then shows that a morphism $(A, X) \to (A, X) \otimes (A, X)$ can satisfy this definition of comonoid if and only if it is determined by the diagonal map $A \to A \times A$; so such comonoids correspond to objects (A, X) with the property that their diagonal maps *are* morphisms; in other words, that every "crossword" over X determines, via its diagonal, an element of X.

In [8] and [9], Pratt focuses on the case $\Sigma = 2$, and, in view of the fact that these comonoids can be developed, for the nonspecialist, in language that does not require familiarity with the category $\mathbf{chu}_2$, defines them roughly as we have done here, emphasizing the "crossword" metaphor. We have deviated from his notation only in replacing X with W, as a mnemonic for "words".

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University of California, Berkeley, CA 94720-3840, USA
E-mail address: `gbergman@math.berkeley.edu`

Brigham Young University, Provo, UT 84602-1231, USA
E-mail address: `pace@math.byu.edu`