COMPLETENESS RESULTS FOR METRIZED RINGS AND LATTICES

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Dedicated to George Grätzer
and
to the memory of Jonathan Gleason

Abstract. The Boolean ring $B$ of measurable subsets of the unit interval, modulo subsets of measure zero, has primary ideals that are closed under the natural metric (e.g., $\{0\}$), but has no prime ideals closed under that metric; hence closed primary ideals are not, in general, intersections of closed prime ideals. Moreover, $B$ is known to be complete in its metric; together, these facts answer a question posed by J. Gleason. From this example, rings of arbitrary characteristic with the corresponding properties are also obtained.

The result that $B$ is complete is generalized to show that in a lattice $L$ given with a metric satisfying some natural inequalities, if every increasing Cauchy sequence converges and every decreasing Cauchy sequence converges, then every Cauchy sequence converges; i.e., $L$ is complete as a metric space.

1. Introduction: a ring-theoretic question leading to a lattice-theoretic result

A standard result of ring theory says that if $I$ is an ideal of a commutative ring $R$, then the nil radical of $I$ (the ideal of elements having some power in $I$) is the intersection of the prime ideals of $R$ containing $I$ [1, Proposition 10.2.9, p. 352].

Jonathan Gleason\(^1\) (personal communication) asked the present author about a possible generalization of that result. Namely, suppose $R$ is a topological commutative ring. For $x \in R$, denote by $(x)$ the closed ideal of $R$ generated by $x$; and for any ideal $I$ of $R$, let $\sqrt{I}$ denote the intersection of all closed ideals $J \supseteq I$ such that $J$ contains every element $x$ such that $(x)^n \subseteq J$ for some $n \geq 1$. Then must $\sqrt{I}$ be the intersection of all closed prime ideals containing $I$? If not in general, does this become true if $R$ is complete with respect to the given topology?

We shall see below that the answer is negative: If $R$ is the Boolean ring of measurable subsets of the unit interval modulo sets of measure zero, topologized using the metric given by the measure of the symmetric difference of two such sets, then $\{0\} = \sqrt{\{0\}}$ (as defined above), but $R$ has no closed prime ideals, so $\{0\}$ is not an intersection of such ideals, although $R$ is indeed complete in the given metric. We give the details in §2, and note in §3 how to get, from this characteristic-2 example, examples of arbitrary characteristic.

The one not-so-obvious property of our example is its completeness as a metric space. In §4 (which is independent of §§2-3) we note how this follows from a standard result of measure theory, then prove a general result on when a metrized lattice is complete, which yields an alternative proof of that statement.

2. The Boolean example

Most of the desired properties of the example sketched above are straightforward to verify:

Let $B_0$ be the set of measurable subsets of the unit interval $[0, 1]$, and for $S \in B_0$, let $\mu(S) \in [0, 1]$ be its measure. Then $B_0$ clearly forms a subring of the Boolean ring of subsets of $[0, 1]$; and, recalling that a sum $S + T$ in the Boolean ring of subsets of a set is the symmetric difference $S \cup T - (S \cap T)$, we see that for $S, T \in B_0$,

$$\mu(S \cup T) = \mu(S + T) + \mu(S \cap T) = \mu(S) + \mu(T) - \mu(S \cap T).$$

\(^1\)Jonathan Gleason, then a graduate student in mathematics at the University of California, Berkeley, raised this question shortly before his unexpected tragic death on January 16, 2018.
For $S, T \in B_0$, let
\[d_0(S, T) = \mu(S + T).\]
Then $d_0$ is a pseudometric, i.e., for all $S, T, U \in B_0$,
\[(2) \quad d_0(S, T) = \mu(S + T).\]
Then (3)-(8) clearly carry over to
\[B.\]
It follows easily that these operations are continuous in our metric.

Now let $B$ be the quotient ring
\[(9) \quad B = B_0 / \{S \mid \mu(S) = 0\}.\]
Let us write \([S] \in B\) for the residue class of $S \in B_0$, and define
\[(10) \quad d([S], [T]) = d_0(S, T) = \mu(S + T) \quad \text{for} \quad [S], [T] \in B.\]
Then (3)-(8) clearly carry over to $B$, with the “$\implies$” of (4) strengthened to “$\iff$”. In particular, $d$ is a metric, and

**Lemma 1.** The operations of the Boolean ring $B$ of measurable subsets of $[0,1]$ modulo sets of measure zero are continuous with respect to the metric $d$ of (10). \(\square\)

Since, as a Boolean ring, $B$ has no nonzero nilpotent elements, the ideal $\{0\}$ of $B$ certainly has the property of containing every element $x$ such that $(x)^n \subseteq \{0\}$ for some $n > 0$. Being a singleton, $\{0\}$ is closed in the metric topology, so $\sqrt{\{0\}} = \{0\}$ under the definition suggested by Gleason.

Is $\{0\}$ an intersection of closed prime ideals? A negative answer follows from

**Lemma 2.** Every prime ideal $P$ of the ring $B$ has for topological closure the whole ring $B$.

Hence $B$ has no closed prime ideals.

**Proof.** Given a prime ideal $P$, let us first show that $P$ has elements $[S]$ arbitrarily close to 1, i.e., such that the complement $1 + S$ of $S$ has measure arbitrarily close to 0. To do this, we shall show that for any $S \in P$, there exists $T \in P$ with $\mu(1 + T) = \mu(1 + S)/2$.

Indeed, let us write $1 + S$ as the union of two disjoint measurable subsets $X$ and $Y$, each of measure $\mu(1 + S)/2$. (E.g., one can take $X = (1 + S) \cap [0, t]$ and $Y = (1 + S) \cap (t, 1]$ for appropriate $t \in [0, 1]$; such a value of $t$ will exist by continuity of $\mu((1 + S) \cap [0, t])$ in $t$.) Since $[X][Y] = 0 \in P$, one of $[X], [Y]$ belongs to $P$; say $[X] \in P$. Then $[S] + [X] \in P$; but $\mu(1 + (S + X)) = \mu((1 + S) + X) = \mu(1 + S) - \mu(X)$ (since $X \subseteq 1 + S$), which is $\mu(1 + S) - \mu(1 + S)/2 = \mu(1 + S)/2$.

So $P$ has elements $[S]$ arbitrarily close to 1; and multiplying arbitrary $[U] \in B$ by such elements, we conclude that $P$ has elements arbitrarily close to every $[U]$; so the closure of $P$ contains every $[U] \in B$, i.e., is all of $B$, as claimed.

Hence no prime ideal $P$ is itself closed, giving the final assertion. \(\square\)

So $\{0\} = \sqrt{\{0\}}$ is not an intersection of closed prime ideals.

In view of the completeness result referred to in the Introduction (to be proved in two ways in §4), the Boolean ring $B$ answers the strongest form of Gleason’s question.

(To be precise, Gleason’s question concerns completeness of a topological ring $R$ in the uniform structure arising from additive translates of neighborhoods of 0. Our metrized rings have additive-translation-invariant metrics by (7), so completeness in that uniform structure is equivalent to completeness in the metric.)
3. Non-Boolean algebras

Is the behavior of above example limited to Boolean rings, or perhaps to rings of finite characteristic? No. We show below how to generalize the construction of the preceding section to algebras in the sense of General Algebra (a.k.a. Universal Algebra), and note that when the algebras in question are rings (of arbitrary characteristic), these give further examples of the properties asked for.

We start with the analog of the construction \( B_0 \).

**Definition 3.** For \( X \) a set, let \( X^{[0,1]} \) denote the set of all \( X \)-valued functions on the unit interval, and for each \( f \in X^{[0,1]} \) and \( x \in X \), let
\[
(11) \quad f_x = \{ t \in [0,1] \mid f(t) = x \}.
\]

Let \( X' \) denote the subset of \( X^{[0,1]} \) consisting of those \( f \) such that
\[
(12) \quad \text{for all } x \in X, \text{ the set } f_x \subseteq [0,1] \text{ is measurable, and}
\]
\[
(13) \quad \{ x \in X \mid f_x \neq \emptyset \} \text{ is countable (i.e., finite or countably infinite).}
\]

For \( f, g \in X' \), let
\[
(14) \quad d'(f, g) = \mu(\{ t \in [0,1] \mid f(t) \neq g(t) \}) = (\sum_{x \in X} d_0(f_x, g_x))/2,
\]
where \( d_0 \) is the pseudometric on measurable subsets of \([0,1]\) defined in (2).

The final equality of (14) is, intuitively, a consequence of the fact that every \( t \in [0,1] \) such that \( f(t) \neq g(t) \) contributes twice to the summation in the final term: via the summand with \( x = f(t) \) and the summand with \( x = g(t) \). This idea is easily formalized to show that that summation is indeed twice the middle term of (14).

**Lemma 4.** For any set \( X \), the function \( d' \) defined by (14) is a pseudometric on \( X' \).

For any finitary operation \( u : X^n \rightarrow X \), the induced pointwise operation \( u^{[0,1]} : (X^{[0,1]})^n \rightarrow X^{[0,1]} \) carries \((X')^n \) to \( X' \), and for \( f^{(0)}, \ldots, f^{(n-1)}, g^{(0)}, \ldots, g^{(n-1)} \in X' \) we have
\[
(15) \quad d'(u^{[0,1]}(f^{(0)}, \ldots, f^{(n-1)}), u^{[0,1]}(g^{(0)}, \ldots, g^{(n-1)})) \leq d'(f^{(0)}, g^{(0)}) + \cdots + d'(f^{(n-1)}, g^{(n-1)}).
\]

**Proof.** That \( d' \) is a pseudometric is straightforward. (The triangle inequality is easily verified using the right-hand side of (14), and the fact that \( d_0 \) is a pseudometric.)

We need to know next that given \( u : X^n \rightarrow X \), and \( f^{(0)}, \ldots, f^{(n-1)} \in X' \), we have \( u^{[0,1]}(f^{(0)}, \ldots, f^{(n-1)}) \in X' \). Note that for each \( x \in X \), \( u^{[0,1]}(f^{(0)}, \ldots, f^{(n-1)})_x \) will be the union, over all \( n \)-tuples \((x_0, \ldots, x_{n-1})\) satisfying \( u(x_0, \ldots, x_{n-1}) = x \), of the sets
\[
(16) \quad f_x^{(0)} \cap \cdots \cap f_x^{(n-1)}.
\]

Now for each \( i \in n \), only countably many values of \( x_i \) make \( f^{(i)}_x \) nonempty, so only countably many \( n \)-tuples \((x_0, \ldots, x_{n-1})\) can make (16) nonempty; and by (12), each of those intersections is measurable; so for each \( x \), \( u^{[0,1]}(f^{(0)}, \ldots, f^{(n-1)})_x \) is a countable union of measurable sets, hence measurable; i.e., \( u^{[0,1]}(f^{(0)}, \ldots, f^{(n-1)}) \) satisfies the condition of (12). It also satisfies the condition of (13), since the countably many cases where (16) is nonempty lead to only countably many possibilities for the element \( u(x_0, \ldots, x_{n-1}) \). So \( u^{[0,1]} \) indeed carries \((X')^n \) to \( X' \).

Finally, (15) follows from the fact that \( u(f^{(0)}, \ldots, f^{(n-1)}) \) and \( u(g^{(0)}, \ldots, g^{(n-1)}) \) can differ only at points \( t \in [0,1] \) where \( f^{(i)} \) and \( g^{(i)} \) differ for at least one \( i \).

We now want to deduce corresponding results, with the set of functions \( X' \) replaced by the set of equivalence classes of such functions under the relation of differing on a set of measure zero. We will need the following observation.

**Lemma 5.** As in §2, let \( B_0 \) denote the Boolean ring of measurable subsets of \([0,1]\).

Let \( S_0, S_1, \ldots \) be a finite or countably infinite family of elements of \( B_0 \) such that
\[
(17) \quad \mu(S_i \cap S_j) = 0 \quad \text{whenever } i \neq j,
\]
and
\[
(18) \quad \sum_i \mu(S_i) = 1.
\]

Then there exist \( T_0, T_1, \ldots \in B_0 \) such that
\[
(19) \quad d'(S_i, T_i) = 0.
\]
and the $T_i$ partition $[0,1]$, i.e.,

$$f(28) \quad f \quad \text{we define} \quad f \quad \text{and since a subset of } [0, d \quad x \quad (26).$$

$$S \quad \text{see that} \quad S \quad \text{shall write} \quad (22)$$

Proof. Let

$$T_i = S_i \setminus \bigcup_{0 \leq j < i} S_j \quad \text{for} \quad i > 0,$$

and

$$T_0 = [0,1] \setminus \bigcup_{i>0} T_i.$$

These sets are clearly measurable and partition $[0,1]$. Since the sets $S_j$ whose members are removed from $S_i$ in (22) have, by (17), only a set of measure zero in common with $S_i$, we see that for $i > 0$, $T_i$ differs from $S_i$ in a set of measure zero, giving (19) for such $i$. By (22), no $T_i$ with $i > 0$ contains elements of $S_0$, so by (23), $T_0 \supseteq S_0$: hence to prove the $i=0$ case of (19), it suffices to show that $\mu(T_0) = \mu(S_0)$. We can get this by noting that $\sum_{i \geq 0} \mu(T_i) = \mu([0,1]) = 1 = \sum_{i \geq 0} \mu(S_i)$ by (18), and subtracting from that relation the equations $\mu(T_i) = \mu(S_i)$ for $i > 0$, which follow from the cases of (19) already obtained.

Now – still assuming the completeness result to be proved in the next section – we can get

**Proposition 6.** For $X$ a set, let $X^*$ denote the quotient of $X'$ (defined in Definition 3) by the equivalence relation $d'(f, g) = 0$, and let $d^*$ be the metric on $X^*$ induced by $d'$.

Then $X^*$ is a complete metric space.

Proof. Consider any Cauchy sequence $[f^{(0)}], [f^{(1)}], \cdots \in X^*$, where $f^{(0)}, f^{(1)}, \cdots \in X'$. For each $n$, the set of elements $x \in X$ such that $f^{(n)}_x$ is nonempty is countable; hence there exists a countable (possibly finite) list of distinct such elements:

$$x_0, x_1, \ldots = \{x \in X \mid (\exists n) f^{(n)}_x \neq \emptyset\}.$$

Now (writing $d_0$ and $d$ as in the preceding section, for our pseudometric on $B_0$ and metric on $B$), we have for all $i, m, n$,

$$d([f^{(m)}_x], [f^{(n)}_x]) = d_0(f^{(m)}_x, f^{(n)}_x) \leq d'(f^{(m)}, f^{(n)}) = d^*([f^{(m)}], [f^{(n)}]);$$

hence the Cauchyness of the sequence of $[f^{(n)}]$ implies for each $x_i$ the Cauchyness of the sequence of $[f^{(n)}_{x_i}] \in B$. Hence by the completeness of $B$, for each $i$ the sequence $[f^{(0)}_{x_i}], [f^{(1)}_{x_i}], \ldots \; \text{converges to an element which we shall write} \; [S_i], \; \text{choosing an arbitrary representative} \; S_i \in B_0 \text{ of the limit in } B.

I claim that these sets $S_i$ satisfy (17) and (18). To get the first of these equations, note that for any $\varepsilon > 0$, one can choose $n$ such that $d'(f^{(n)}_{x_i}, S_i) < \varepsilon/2$ and $d'(f^{(n)}_{x_j}, S_j) < \varepsilon/2$. Since $f^{(n)}_{x_i}$ and $f^{(n)}_{x_j}$ are disjoint, we see that $S_i$ and $S_j$ intersect in a set of measure at most $\varepsilon$. Since this holds for all $\varepsilon > 0$, they in fact intersect in a set of measure 0.

To get (18), choose $m$ such that for all $n \geq m$ we have $d^*([f^{(m)}], [f^{(n)}]) < \varepsilon/2$, equivalently,

$$d'(f^{(m)}, f^{(n)}) < \varepsilon/2.$$

Also, since $x_0, x_1, \ldots$ partition $[0,1]$, we can choose $j$ such that

$$\mu(f^{(m)}_{x_0}) + \cdots + \mu(f^{(m)}_{x_j}) \geq 1 - \varepsilon/2.$$

For $n \geq m$, (26) guarantees that $d'(f^{(m)}_{x_0}, f^{(m)}_{x_j}) + \cdots + d'(f^{(m)}_{x_j}, f^{(n)}_{x_j}) < \varepsilon/2$, equivalently, $d([f^{(m)}_{x_0}], [f^{(m)}_{x_j}]) + \cdots + d([f^{(m)}_{x_j}], [f^{(n)}_{x_j}]) < \varepsilon/2$, hence passing to the limit as $n \to \infty$, $d([f^{(m)}_{x_0}], [S_0]) + \cdots + d([f^{(m)}_{x_j}], [S_j]) \leq \varepsilon/2$; and combining with (27) we get $\mu([S_0] + \cdots + [S_j]) \geq 1 - \varepsilon$. Since this holds for all $\varepsilon$, we have $\mu([S_0] + \cdots + [S_j]) \geq 1$, and since a subset of $[0,1]$ cannot have measure larger than 1, we get (18).

Lemma 5 now gives us a partition of $[0,1]$ into sets $T_i$ which differ from the $S_i$ by sets of measure zero. If we define $f \in X'$ by

$$f_{x_i} = T_i \quad \text{for all} \quad i,$$

then $[f]$ is a limit of the given Cauchy sequence $[f^{(0)}], [f^{(1)}], \ldots$, proving completeness. □
A note on condition (13): If we had allowed larger cardinalities, we would not have been able to use basic properties of measure, e.g., in concluding that the set in the middle term of (14) was measurable, or in proving in Lemma 4 that \( u^{[0,1]} \) carries \((X')^n\) to \(X'\). On the other hand if we had restricted the set of nonempty \( f_x \) to be finite, our \( X^* \) would not have been complete, except in the case where \( X \) was finite. So countability is the only choice that gives the construction \( X^* \) the desired properties.

We have not yet called on (15). Let us now use it to show that our construction behaves nicely with respect to algebra structures.

**Proposition 7.** Suppose \( A \) is an algebra in the sense of General Algebra, that is, a set given with a (finite or infinite) family of operations, each of finite arity.

Then for each operation \( u : A^n \to A \) of \( A \), the operation \( u^* \) of \( A^* \) defined by

\[
(29) \quad u^*([f_0], \ldots, [f_{n-1}]) = [u^{[0,1]}(f_0, \ldots, f_{n-1})]
\]

is uniformly continuous in the metric \( d^* \); indeed, it satisfies

\[
(30) \quad d^*(u^*([f_0], \ldots, [f_{n-1}]), u^*([g_0], \ldots, [g_{n-1}])) \leq d^*([f_0], [g_0]) + \cdots + d^*([f_{n-1}], [g_{n-1}]).
\]

The algebra \( A^* \) satisfies all identities satisfied by \( A \). In fact, every finitely generated subalgebra of \( A^* \) is contained in a subalgebra isomorphic to a countable direct product of copies of \( A \).

**Sketch of proof.** By the case of (15) where all \( d^*(f_i, g_i) \) are zero, the operations \( u^{[0,1]} \) of \( A' \), respect the equivalence relation used in defining \( A^* \), so (29) gives a well-defined operation. The general case of (15) then gives the inequality (30). Finally, given any finite family of elements \( [f_0], \ldots, [f_{N-1}] \) of \( A^* \), the countably many nonempty sets \( f_a^i \) \( (0 \leq i < N, a \in A) \) yield a decomposition of \([0,1]\) into countably many intersections as in (16), on each of which all of \( f_0, \ldots, f_{N-1} \) are constant. Dropping those intersections that have measure zero, and looking at members of \( A' \) that are constant on the remaining countably many subsets, we see that this has as its image in \( A^* \) a subalgebra isomorphic to a countable direct product of copies of \( A \), completing the proof. \( \square \)

Finally, some observations specific to rings:

**Proposition 8.** Let \( A \) be an associative ring. Then in the complete metrized ring \( A^* \) arising by the construction of Proposition 7, the closure of every prime ideal is all of \( A^* \); hence \( A^* \) has no closed prime ideals.

On the other hand, if \( A \) has no nonzero nilpotent elements, then the topological radical \( \sqrt{\{0\}} \), defined as in \( \S 1 \), is \( \{0\} \).

**Sketch of proof.** Let \( P \) be a prime ideal of \( A \). As in the proof of Lemma 2, let us first show that \( P \) has elements arbitrarily close to \( 1 \). Namely, given \( f \in P \), let us, as in that proof, partition \([0,1] \setminus f_1 \) (the set where \( f \) has value \( \neq 1 \)) into measurable subsets \( S \) and \( T \) of equal measure. Now let \( f' \) be defined to agree with \( f \) on \( f_1 \cup S \), and equal 1 on \( T \), and let \( f'' \) agree with \( f \) on \( f_1 \cup T \), and equal 1 on \( S \). Then \( f'f'' = f \), so one of these elements must belong to \( P \), but \( d(f', 1) = d(f'', 1) = d(f, 1)/2 \). So we get elements of \( P \) arbitrarily close to \( 1 \); and as in the proof of Lemma 2 we deduce that the closure of \( P \) is all of \( A^* \).

The final assertion is straightforward. \( \square \)

So the rings \( A^* \) for \( A \) a ring without nilpotents generalize the properties of the example \( B \) of the preceding section.

Remark: The development of the above results in terms of measurable \( X \)- and \( A \)-valued functions on \([0,1]\), modulo disagreement on sets of measure zero, feels artificial. Surely one should be able to perform our constructions abstractly in terms of the Boolean ring \( B \) and the real-valued function on \( B \) induced by the measure on \([0,1]\), and hence get our results for a general Boolean rings with appropriate real-valued functions on them.

If we were interested in functions to \( X \) and \( A \) assuming only finitely many values, these could be described as the continuous functions from the Stone space of \( B \) to the discrete space \( X \). But for functions allowed to assume countably many values, it is not clear to me what the best abstract formalization is; so I leave this to experts in Boolean rings to carry further.

In contrast, the result of the next section will be carried out in a satisfyingly general context.

4.Completeness

When I first suggested the Boolean ring \( B \) of measurable subsets of \([0,1]\), modulo sets of measure zero as an answer to J. Gleason’s question, the one property that was not clear to me was completeness in the natural metric, though it seemed intuitively likely.
One might naively hope to prove completeness by showing that every Cauchy sequence in \( B \) “converges almost everywhere” on \([0, 1]\); i.e., that almost every point either belongs to all but finitely many members of the sequence, or is absent from all but finitely many. But this is not so; a counterexample is the sequence whose first term is \([0, 1]\), whose next two are \([0, 1/2]\) and \([1/2, 1]\), whose next three are \([0, 1/3], [1/3, 2/3], [2/3, 1]\), and, generally, whose \(1 + 2 + \ldots + (n - 1) + i\)-th term for \(1 \leq i \leq n\) is \([(i - 1)/n, i/n]\). Since the measures of these sets approach zero, the sequence approaches 0 in our metric; but clearly every \(t \in [0, 1]\) occurs in infinitely many of these sets. Looking at this example, one might still hope that given a Cauchy sequence in \( B \), almost every \(t \in [0, 1]\) has the property that the set of members of the sequence which contain \(t\) is either “eventually scarce”, or has eventually scarce complement. But this, too, fails; to see this, take the above example, and “stretch it out” by repeating the \(m\)-th term \(2^m\) times successively, for each \(m\).

However, an online search turned up a proof of the desired completeness statement, in a set of exercises [4, see point 6 on p. 2], which I described in the first version of this note as the one reference for the result that I could find. David Handelman then pointed out that the desired statement followed immediately from the standard fact that \(L^1\) of the unit interval is complete in its natural metric ([3, Theorem VI.3.4, p.133], [2, Theorem 22.E, p.93]), on identifying measurable sets with their characteristic functions. And indeed, in [2, Exercise 40(1), p.169], the reader is asked to deduce the result for measurable sets from the result for \(L^1\).

In all these sources, the key to the proof of completeness is to pass from an arbitrary Cauchy sequence to a subsequence with the property that the distance between the \(i\)-th and \(i + 1\)-st terms is \(2^{-i}\). Rather magically, a sequence with this property does converge almost everywhere, giving a limit of the original Cauchy sequence.

And in fact, this trick can be abstracted from the context of measure theory to that of lattices (or even semilattices) as in the next theorem; from which we will recover the completeness statement for measurable sets modulo null sets in a final corollary.

Since we no longer need the notation “\(f_i\)” that we used in the preceding section for the point-set at which a function takes on the value \(x\), we will use subscripts below in the conventional way to index the terms of sequences.

We remark that the condition of completeness as a metric space, obtained in the theorem, is independent of the concept of completeness as a lattice, i.e., the existence of least upper bounds and greatest lower bounds of not necessarily finite subsets (though the condition that certain infinite least upper bounds and greatest lower bounds exist will be key to the argument). For instance, any discrete lattice, given with the metric that makes \(d(x, y) = 1\) whenever \(x \neq y\), is complete as a metric space, and, indeed, satisfies the hypotheses of the next theorem, but need not be complete as a lattice. Inversely, the totally ordered subset of the real numbers \([-1] \cup (0, 1) \cup \{2\}\) is complete as a lattice, but not as a metric space.

**Theorem 9.** Let \(L\) be a lattice, whose underlying set is given with a metric \(d\) which satisfies, identically, at least one of the inequalities

\[
\begin{align*}
(31) \quad d(x \lor y, x \lor z) & \leq d(y, z) \quad (x, y, z \in L), \\
(32) \quad d(x \land y, x \land z) & \leq d(y, z) \quad (x, y, z \in L).
\end{align*}
\]

(or, more generally, let \(L\) be an upper semilattice satisfying \((31)\), or a lower semilattice satisfying \((32)\)).

Suppose moreover that in \(L\)

\[
\begin{align*}
(33) \quad \text{every increasing Cauchy sequence } x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \text{ converges,}
\end{align*}
\]

and

\[
\begin{align*}
(34) \quad \text{every decreasing Cauchy sequence } x_0 \geq x_1 \geq \ldots \geq x_n \geq \ldots \text{ converges.}
\end{align*}
\]

Then every Cauchy sequence in \(L\) converges; i.e., \(L\) is complete as a metric space.

**Proof.** It will suffice to prove the case where \(L\) is an upper semilattice satisfying \((31)\), since this includes the case where \(L\) is a lattice satisfying \((31)\), while the cases where \(L\) is a lower semilattice or lattice satisfying \((32)\) follow by duality. So assume \(L\) such an upper semilattice.

In proving \(L\) complete, it suffices to show convergence of sequences \(x_0, x_1, \ldots\) such that

\[
\sum_{i \geq 0} d(x_i, x_{i+1}) < \infty,
\]

since every Cauchy sequence has such a subsequence (e.g., one chosen, as in [4], so that \(d(x_i, x_{i+1}) \leq 2^{-i}\)); and if a subsequence of a Cauchy sequence converges, so does the whole sequence.

So let such a sequence given, and let us define

\[
x_{h,j} = x_h \lor x_{h+1} \lor \cdots \lor x_j \quad \text{for } h \leq j.
\]
Note that if in (31) we put $x = x_{h,j}$, $y = x_j$, $z = x_{j+1}$, we get

\begin{equation}
(37) \quad d(x_{h,j}, x_{h,j+1}) \leq d(x_j, x_{j+1}).
\end{equation}

Now for $h \leq j \leq k$, the triangle inequality (applied $k-j$ times) gives

\begin{equation}
(38) \quad d(x_{h,j}, x_{h,k}) \leq \sum_{j \leq \ell < \infty} d(x_{\ell}, x_{\ell+1}).
\end{equation}

Applying (37) to each term of this summation, we get

\begin{equation}
(39) \quad x_{h,\infty} = \lim_{j \to \infty} x_{h,j}.
\end{equation}

Note that, by the way we obtained convergence, and the continuity of $d$ in the topology it defines, we have

\begin{equation}
(40) \quad d(x_{h,j}, x_{h,\infty}) \leq \sum_{j \leq \ell < \infty} d(x_{\ell}, x_{\ell+1}) \quad \text{for } h \leq j.
\end{equation}

Note next that

\begin{equation}
(41) \quad x_{h,j} \geq x_{i,j} \quad \text{for } h \leq i \leq j.
\end{equation}

I claim that this implies that

\begin{equation}
(42) \quad x_{h,\infty} \geq x_{i,\infty} \quad \text{for } h \leq i.
\end{equation}

Indeed, (41) and (42) are respectively equivalent to the conditions $d(x_{h,j}, x_{h,j} \lor x_{i,j}) = 0$ and $d(x_{h,\infty}, x_{i,\infty} \lor x_{i,\infty}) = 0$, and the latter can be obtained from the former using (39).

So the elements $x_{h,\infty}$ form a decreasing sequence. I claim that this sequence, too, is Cauchy; in fact, that

\begin{equation}
(43) \quad d(x_{h,\infty}, x_{i,\infty}) \leq \sum_{h \leq \ell < i} d(x_{\ell}, x_{\ell+1}) \quad \text{for } h \leq i.
\end{equation}

Namely, by essentially the same argument used to prove (38), one sees that for every $j \geq i$, $d(x_{h,j}, x_{i,j}) \leq \sum_{h \leq \ell < i} d(x_{\ell}, x_{\ell+1})$; and by continuity of $d$, this carries over to the limit as $j \to \infty$. Hence by (34), our sequence converges, and we can define

\begin{equation}
(44) \quad x_{\infty,\infty} = \lim_{h \to \infty} x_{h,\infty}.
\end{equation}

Finally, note that for every $h \geq 0$,

\begin{equation}
(45) \quad d(x_h, x_{\infty,\infty}) = d(x_{h,h}, x_{\infty,\infty}) \leq d(x_{h,h}, x_{h,\infty}) + d(x_{h,\infty}, x_{\infty,\infty}) \leq \sum_{h \leq \ell < \infty} d(x_{\ell}, x_{\ell+1}) + \sum_{h \leq \ell < \infty} d(x_{\ell}, x_{\ell+1}) = 2 \sum_{h \leq \ell < \infty} d(x_{\ell}, x_{\ell+1}),
\end{equation}

and that this approaches 0 as $h \to \infty$. Hence our original sequence converges,

\begin{equation}
(46) \quad \lim_{h \to \infty} x_h = x_{\infty,\infty},
\end{equation}

completing the proof of the theorem. \hfill \square

The first assertion of the following corollary clearly includes the result we called on in §§2-3. The remaining two assertions are further generalizations.

**Corollary 10.** Let $M$ be a measure space of finite total measure. Then the Boolean ring $B$ of measurable subsets of $M$ modulo sets of measure zero is complete with respect to the metric

\begin{equation}
(47) \quad d([S], [T]) = \mu(S + T),
\end{equation}

where $[S]$ and $[T]$ denote the equivalence classes of measurable sets $S$ and $T$ modulo sets of measure zero.

More generally, if $M$ is a measure space not necessarily of finite total measure, and $C$ a positive real constant, and if for $S$ and $T$ as above we define

\begin{equation}
(48) \quad d_C([S], [T]) = \min(\mu(S + T), C),
\end{equation}

then the Boolean ring $B$ is complete with respect to $d_C$.

Alternatively, if in the same situation we define $B_{\text{fin}}$ to be the nonunital Boolean ring of measurable sets of finite measure modulo sets of measure zero, then $B_{\text{fin}}$ is again complete with respect to the metric $d$ of (47).

In all these cases, the operations of our Boolean ring are continuous with respect to the metric defined.
Indeed, (52) gives
\[
(i > \leq d(54)
\]
The situation with respect to these is curious. From (52), for instance, one can still prove conditions arise, which would make it worth trying to prove such a generalization of Theorem 9.

inequality in the proof. I don’t know whether there are situations where metrics satisfying such weakened

property after taking into account the effect of the \(u_i\)th and
\[
(55)
\]

like \(d(55)\)

respectively,
\[
(51) \quad d(x \land y, x \land z) \leq u(d(y, z)) \quad (x, y, z \in L).
\]
The idea would be to choose from a general Cauchy sequence a subsequence such that the distances between \(i\)-th and \(i+1\)-st terms decrease rapidly enough not only to give a convergent sum, but to have the corresponding property after taking into account the effect of the \(u\) in (50) or (51) under the iterated application of that inequality in the proof. I don’t know whether there are situations where metrics satisfying such weakened conditions arise, which would make it worth trying to prove such a generalization of Theorem 9.

(A different pair of weakenings of conditions (31) and (32), which I also thought at first might work, are

\[
(52) \quad d(x, x \lor y) \leq d(x, y) \quad (x, y \in L),
\]

respectively,
\[
(53) \quad d(x, x \land y) \leq d(x, y) \quad (x, y \in L).
\]
The situation with respect to these is curious. From (52), for instance, one can still prove
\[
(54) \quad d(x_0, x_0 \lor \cdots \lor x_i) \leq \sum_{0 \leq \ell < i} d(x_\ell, x_{\ell+1}).
\]
Indeed, (52) gives (if \(i > 0\)) \(d(x_0, x_0 \lor \cdots \lor x_i) \leq d(x_0, x_1 \lor \cdots \lor x_i)\); by the triangle inequality this is
\[
\leq d(x_0, x_1) + d(x_1, x_1 \lor \cdots \lor x_i).
\]
The second of these terms can (if \(i > 1\)) be similarly bounded by \(d(x_1, x_2) + d(x_2, x_2 \lor \cdots \lor x_i)\), and this procedure, iterated, gives (54). But it does not appear that one can get a result like
\[
(55) \quad d(x_0 \lor \cdots \lor x_i, x_0 \lor \cdots \lor x_j) \leq \sum_{i \leq \ell < j} d(x_\ell, x_{\ell+1}) \quad (0 \leq i \leq j),
\]
as would be needed to carry out the argument used in proving Theorem 9.)

We remark, also, that in any partially ordered set with a metric, the conjunction of conditions (33) and (34) is easily shown to be equivalent to the single statement that every Cauchy sequence whose members form a chain converges. But the pair of statements seems easier to work with. In particular, it is easy to see that they hold for measurable sets in a measure space, modulo sets of measure zero.

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References


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