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FAMILIES OF ULTRAFILTERS, AND HOMOMORPHISMS ON INFINITE DIRECT PRODUCT ALGEBRAS

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Abstract. Criteria are obtained for a filter $\mathcal{F}$ of subsets of a set I to be an intersection of finitely many ultrafilters, respectively, finitely many κ-complete ultrafilters for a given uncountable cardinal κ. From these, general results are deduced concerning homomorphisms on infinite direct product groups, which yield quick proofs of some results in the literature: the Łoś-Eda theorem (characterizing homomorphisms from a not-necessarily-countable direct product of modules to a slender module), and some results of N. Nahlus and the author on homomorphisms on infinite direct products of not-necessarily-associative k-algebras. The same tools allow other results of Nahlus and the author to be nontrivially strengthened, and yield an analog to one of their results, with nonabelian groups taking the place of k-algebras.

We briefly examine the question of how the common technique used in applying the general results of this note to k-algebras on the one hand, and to nonabelian groups on the other, might be extended to more general varieties of algebras in the sense of universal algebra.

In a final section, the Erdős-Kaplansky Theorem on dimensions of vector spaces D^I (D a division ring) is extended to reduced products $D^I/\mathcal{F}$, and an application is noted.

1. Results on filters and ultrafilters

The definition below recalls some standard concepts. Readers not familiar with some of these might skim those parts of the definition now, and return to them as one or another concept is called on. (For a thorough development of ultrafilters and related topics, see works such as [7] or [8].)

Definition 1. *If I is a set, then a* filter *on I means a set $\mathcal{F}$ of subsets of I, such that* (i) $I \in \mathcal{F}$, (ii) *if $J \in \mathcal{F}$ and $J \subseteq K \subseteq I$, then $K \in \mathcal{F}$, and* (iii) *if $J, K \in \mathcal{F}$, then $J \cap K \in \mathcal{F}$. A filter $\mathcal{F}$ on I is* proper *if it is not the set of all subsets of I, equivalently, if $\emptyset \notin \mathcal{F}$. A filter $\mathcal{F}$ on I is* κ-complete, *for κ an infinite cardinal, if $\mathcal{F}$ is closed under intersections of families of $< \kappa$ elements.* (*Thus, every filter is $\aleph_0$-complete.*) *A filter which is $\aleph_1$-complete, i.e., closed under countable intersections, is called* countably complete.

A maximal proper filter on I is called an ultrafilter. *An ultrafilter of the form $\{J \subseteq I \mid i_0 \in J\}$ for some $i_0 \in I$ is called* principal*; all other ultrafilters are called* nonprincipal.

An infinite cardinal κ is called measurable *if there exists a nonprincipal κ-complete ultrafilter on κ.*

The use to which filters will be put in this note arises from the following observation.

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Lemma 2. *Let I be a set, $(A_i)_{i\in I}$ an I-indexed family of nonempty sets, and $h: \prod_I A_i \to C$ a map from their direct product to another set. Then the set*

(1) $\mathcal{F} = \{J \subseteq I \mid h$ *factors as* $\prod_{i\in I} A_i \to \prod_{i\in J} A_i \to C,$ *where the first map is the natural projection*$\}$

is a filter on I.

Conversely, given any filter $\mathcal{F}$ on I, and any I-indexed family $(A_i)_{i\in I}$ of sets each having more than one element, there exists a map h from $\prod_I A_i$ to a set C, such that the filter $\mathcal{F}$ is given by (1).

Proof. It is clear that the set (1) satisfies conditions (i) and (ii) in the first sentence of Definition 1. To see (iii), note that if h factors through the subproducts indexed by J and by K, this means $h(a)$ is unaffected by any change in a that either modifies only coordinates outside of J, or only coordinates outside of K. Now a change affecting only coordinates outside $J\cap K$ can be achieved by first modifying coordinates outside J, then coordinates in $J-K$. If neither of these changes affects $h(a)$, then the combined change doesn't, so $J\cap K\in\mathcal{F}$, as required.

To get the converse, one takes for C the *reduced product* $(\prod_I A_i)/\mathcal{F}$, that is, the factor-set of $\prod A_i$ by the relation making $(a_i)_{i\in I}\sim(b_i)_{i\in I}$ if $\{i \mid a_i=b_i\}\in\mathcal{F}$. It is straightforward to check that this is an equivalence relation, and (remembering that each A_i has more than one element), that the filter induced by the factor map $\prod A_i \to (\prod A_i)/\mathcal{F}$ is precisely $\mathcal{F}$. □

It is easily shown that a proper filter $\mathcal{F}$ on I is an ultrafilter if and only if for every $J\subseteq I$, either $J\in\mathcal{F}$ or $I-J\in\mathcal{F}$; equivalently, if and only if for all $J, K\subseteq I$, if $J\cup K\in\mathcal{F}$ then either $J\in\mathcal{F}$ or $K\in\mathcal{F}$; and that the filters on I are precisely the *intersections* of sets of ultrafilters. (In this last statement, I am following the convention that, among sets of subsets of I, we regard the set of all subsets, i.e., the improper filter, as the intersection of the empty family of sets of subsets. If we did not allow the empty intersection, then the intersections of ultrafilters would be the *proper* filters on I.)

The next result characterizes those filters that are intersections of *finitely many* ultrafilters. In the statement, a *partition* of a set means an expression of it as the union of a family of pairwise disjoint subsets. We do not require these subsets to be nonempty, so under our definition, a partition may involve one or more occurrences of the empty set.

Lemma 3. *Let I be a set, and $\mathcal{F}$ a filter on I. Then the following conditions are equivalent.*

(2) *For every partition of I into infinitely many subsets J_s ($s\in S$, S an infinite set) there is at least one $s\in S$ such that $I-J_s\in\mathcal{F}$.*

(3) *For every partition of I into a* countably *infinite family of subsets J_m ($m\in\omega$), there is at least one $m\in\omega$ such that $I-J_m\in\mathcal{F}$.*

(4) *There exists $n\in\omega$ such that for every partition of I into $n+1$ subsets $J_0,\dots,J_n$, there is at least one $m\in n+1$ such that $I-J_m\in\mathcal{F}$.*

(5) *$\mathcal{F}$ is the intersection of finitely many ultrafilters on I.*

When these conditions hold, the finite set of ultrafilters having $\mathcal{F}$ as intersection is unique, and its cardinality is the least n as in (4).

Proof. We shall prove (5) $\implies$ (4) $\implies$ (2) $\implies$ (3) $\implies$ (5), then the final sentence.

To see (5) $\implies$ (4), let $\mathcal{F}$ be an intersection of n ultrafilters, and note that two disjoint sets cannot belong to a common ultrafilter. Hence in any partition of I into $n+1$ sets, at least one will belong to none of our n ultrafilters; hence its complement belongs to all of them, hence to $\mathcal{F}$. For (4) $\implies$ (2), take n as in (4), partition the infinite index-set S of (2) into $n+1$ nonempty subsets $S_0,\dots,S_n$, and for $m=0,\dots,n$, let $J_m=\bigcup_{s\in S_m} J_s$. By (4), for some m the complement of J_m lies in $\mathcal{F}$. Hence, for that m, if we take any $s\in S_m$, the complement of J_s, an overset of the complement of J_m, also lies in $\mathcal{F}$. (2) $\implies$ (3) is clear.

We shall prove (3) $\implies$ (5) in contrapositive form, $\neg$(5) $\implies$ $\neg$(3):

Since $\mathcal{F}$ is a filter, it is the intersection of a set U of ultrafilters. Suppose U were infinite. Take any two distinct members of U. Then there is a subset of I belonging to one but not to the other; hence the complement of that subset belongs to the other ultrafilter. Every ultrafilter on I must contain one of these two sets, so at least one of them belongs to infinitely many members of U. Let us write I_0 for such a one (making an arbitrary choice if both do), and let $J_0 = I - I_0,$ recalling that this still belongs to at least one member of $U.$

We now repeat the process on $I_0,$ decomposing it into a subset I_1 which belongs to infinitely many of the members of U that contain $I_0,$ and a complementary subset J_1 which belongs to at least one; then repeat the process on $I_1,$ and so forth. We thus get a countably infinite family $J_0,\ J_1, \dots$ of disjoint subsets of $I,$ each of which belongs to a member of $U.$ If $\bigcup J_m \neq I,$ we enlarge one of them, say $J_0,$ by attaching $I - \bigcup J_m$ to it. We then have a partition of I into sets J_i each belonging to a member of $U.$ Hence none of their complements belongs to all members of $U,$ i.e., belongs to $\mathcal{F},$ proving $\neg(3).$

To get the final sentence, use (5) to write $\mathcal{F} = \mathcal{U}_0 \cap \dots \cap \mathcal{U}_{n-1}$ with the $\mathcal{U}_m$ distinct. For any ultrafilter $\mathcal{U}$ distinct from each of the $\mathcal{U}_m,$ we can find sets $J_m \in \mathcal{U}_m - \mathcal{U}$ $(m = 0, \dots, n-1).$ Hence $J_0 \cup \dots \cup J_{n-1}$ belongs to all $\mathcal{U}_m$ but not to $\mathcal{U},$ showing that $\mathcal{F} \not\subseteq \mathcal{U}.$ Thus any other set of ultrafilters with intersection $\mathcal{F}$ must be a subset of $\{\mathcal{U}_0, \dots, \mathcal{U}_{n-1}\};$ and reversing the roles of the two sets of ultrafilters, we get equality. Our proof of (5) $\implies$ (4) showed that this n can be used as the n of (4). On the other hand, the conclusion of (4) does not hold for any smaller value than $n,$ since we can partition I into n sets, one in each $\mathcal{U}_m.$ $\square$

(One can get still more conditions equivalent to those of the above lemma by replacing the partition of I in each of (2)-(4) either by a family of disjoint subsets J_s of $I,$ or by a family of sets J_s having union $I.$ In the former case, one keeps the conclusions as in (2)-(4); in the latter, one replaces them by statements that the union of all but one of the sets J_s lies in $\mathcal{F}.$ These conditions are easily shown equivalent to (2)-(4), using the observation that the members of any family of disjoint subsets of I can be enlarged so that they give a partition, and the members of any family with union I can be shrunk down to give a partition. Two more equivalent conditions, of a different flavor, are proved toward the end of this note, in Lemma 12.)

The next lemma gives a condition for the finitely many ultrafilters of (5) to be κ-complete, for a specified uncountable cardinal $\kappa.$ We recall that a κ-complete ultrafilter on I can be nonprincipal only if I has cardinality at least some measurable cardinal $\geq \kappa$ [5, Proposition 4.2.7]. (Following [5], I have worded Definition 1 so that $\aleph_0$ counts as a measurable cardinal. I therefore write "uncountable measurable cardinal" for what many authors simply call a measurable cardinal.) It is known that if uncountable measurable cardinals exist, they are very large, and very rare; in particular, that if the standard set-theory, ZFC, is consistent, then it is consistent with the nonexistence of such cardinals [8, Chapter 6, Corollary 1.8]. Thus, under weak assumptions on the size of $I,$ or reasonable assumptions on our set theory, the ultrafilters of the next lemma must be principal. In the proof of that lemma, we will use the fact that an ultrafilter $\mathcal{U}$ on I is κ-complete if and only if for every partition of I into $< \kappa$ subsets, one of these subsets lies in $\mathcal{U}.$

Lemma 4. *Let I be a set, $\mathcal{F}$ a filter on $I,$ and κ an uncountable cardinal. Then the following statements are equivalent.*

(6) *For every partition of I into $< \kappa$ subsets J_s $(s \in S),$ there exist finitely many indices $s_0, \dots, s_{n-1} \in S$ such that $J_{s_0} \cup \dots \cup J_{s_{n-1}} \in \mathcal{F}.$*

(7) *$\mathcal{F}$ is the intersection of finitely many κ-complete ultrafilters.*

Proof. Assuming (7), let $\mathcal{F} = \mathcal{U}_0 \cap \dots \cap \mathcal{U}_{n-1}$ with all $\mathcal{U}_m$ κ-complete. Given a partition of I into sets J_s as in (6), κ-completeness implies that each $\mathcal{U}_m$ contains one $J_s;$ say $J_{s_m} \in \mathcal{U}_m.$ Thus $J_{s_0} \cup \dots \cup J_{s_{n-1}} \in \mathcal{F}.$

Conversely, assume (6). Since κ is uncountable, (6) applies in particular to countable decompositions, hence implies (3), which is equivalent to (5), i.e., to (7) without the specification of κ-completeness. Now suppose some ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ were not κ-complete. Then there would exist a partition of I into fewer than κ subsets $J_s \notin \mathcal{U}.$ The union of any finite subfamily of these is still $\notin \mathcal{U},$ hence $\notin \mathcal{F},$ so (6) fails. This contradiction completes the proof. $\square$

(Incidentally, the condition on a filter $\mathcal{F}$ that one might naively hope would imply that $\mathcal{F}$ is an intersection of κ-complete ultrafilters – namely, that $\mathcal{F}$ itself be κ-complete – definitely does not. E.g., if κ is a regular infinite non-measurable (hence uncountable) cardinal, then the filter $\mathcal{F}$ of complements in κ of subsets of cardinality $< \kappa$ is κ-complete, but there are no nonprincipal κ-complete ultrafilters on $\kappa.$ A cardinal κ such that every κ-complete filter extends to a κ-complete ultrafilter is called *strongly compact*; cf. [1].)

Digression: If U is a (not necessarily finite) set of ultrafilters on a set $I,$ then the four sets

(8) $\mathcal{F} = \bigcap_{\mathcal{U} \in U} \mathcal{U}, \qquad \mathcal{G} = \bigcup_{\mathcal{U} \in U} \mathcal{U}, \qquad \mathcal{H} = \bigcup_{\mathcal{U} \in U} {}^{\mathrm{c}}\mathcal{U}, \qquad \mathcal{I} = \bigcap_{\mathcal{U} \in U} {}^{\mathrm{c}}\mathcal{U},$

though they do not, in general, uniquely determine U, do all determine one another. Indeed, on the one hand, $\mathcal{F}$ and $\mathcal{H}$ are complements of one another, as are $\mathcal{G}$ and $\mathcal{I}$. On the other hand, from the fact that for any ultrafilter $\mathcal{U}$, the complement of $\mathcal{U}$ is the set of complements (in I) of members of $\mathcal{U}$, one sees that $\mathcal{I}$ is the set of complements of members of $\mathcal{F}$, and vice versa. (This makes each of $\mathcal{F}$ and $\mathcal{G}$ the sets of complements of members of the other's complement. It is not hard to show that each can also be described as the set of subsets of I having nonempty intersection with all members of the other. Likewise, $\mathcal{H}$ and $\mathcal{I}$, in addition to being the sets of complements of members of each other's complements, are each the set of subsets of I whose union with every member of the other is a proper subset of I.)

The description of filters in Definition 1 translates into equally elementary characterizations of the sorts of sets that can occur as $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{I}$ in (8); and since each set in (8) conveys the same information, each of these sorts of sets can, mutatis mutandis, serve the same mathematical function as filters. Sets of the form $\mathcal{I}$ are called *ideals* of subsets of I, since they are the ideals in the Boolean ring of all its subsets. Sets having the form $\mathcal{G}$ were named *grills* in [6] (cf. [15]), and are sometimes used under that name in topological contexts.

When I first obtained the results of this note, I formulated them in terms of finite unions $\mathcal{G}$ of ultrafilters. I finally realized that what I was doing could be restated in terms of filters, and rewrote the note accordingly, since filters are the most familiar of these four sorts of structures.

2. Ultrafilters, and maps on direct products

Suppose, as in Lemma 2, that $h : A = \prod_I A_i \to C$ is a map on a direct product of nonempty sets, and $\mathcal{F}$ the filter of subsets of I corresponding to those sub-products through which the map factors. Thus, h factors in a natural way through the canonical map $A \to A/\mathcal{F}$, where $A/\mathcal{F}$ denotes the reduced product of the A_i with respect to $\mathcal{F}$, defined in the last paragraph of the proof of that lemma. (The factoring map $A/\mathcal{F} \to C$ is not in general one-to-one. E.g., if $I = \{0, 1\}$, $A_0 = A_1 = C$ is a nontrivial abelian group G, and $h : G \times G \to G$ its group operation, then $\mathcal{F}$ is the trivial filter $\{I\}$, so $A \to A/\mathcal{F}$ is a bijection, though $A \to C$ is not one-to-one.)

Now suppose we write the filter $\mathcal{F}$ as $\bigcap_{\mathcal{U}\in U} \mathcal{U}$ for U some set of ultrafilters on I. Can we similarly factor h through the natural map $A \to \prod_{\mathcal{U}\in U} A/\mathcal{U}$?

Yes; but this time not, in general, in a natural way. Elements of A fall together in $\prod_{\mathcal{U}\in U} A/\mathcal{U}$ if and only if they fall together in $A/\mathcal{F}$, but $\prod_{\mathcal{U}\in U} A/\mathcal{U}$ is typically much larger than the embedded image of $A/\mathcal{F}$. One can extend the induced map into C from the image of $A/\mathcal{F}$ to all of $\prod A/\mathcal{U}$ by letting it act in arbitrary ways on elements not in that image; but there is no guarantee that such an extension can be made to respect further structure on our sets, e.g., structures of group.

Let us now show, however, that in the context of Lemmas 3 and 4, where we have only finitely many ultrafilters, the image of A, and hence of $A/\mathcal{F}$, is the full product $\prod_{\mathcal{U}\in U} A/\mathcal{U}$, so that the above problem does not arise.

Lemma 5. *Let I be a set, $(A_i)_{i\in I}$ an I-tuple of nonempty sets, and $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ $(n \in \omega)$ distinct ultrafilters on I. Then the natural map $A = \prod A_i \to A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1}$ is surjective; equivalently, the natural embedding $A/(\mathcal{U}_0 \cap \cdots \cap \mathcal{U}_{n-1}) \hookrightarrow A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1}$ is a bijection.*

Proof. Since the $\mathcal{U}_m$ are distinct, we can find a partition $I = J_0 \cup \cdots \cup J_{n-1}$ with each $J_m \in \mathcal{U}_m$. Now given any $(x_0, \dots, x_{n-1}) \in A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1}$, let us choose a representative $a^{(m)} \in A$ of each x_m, and let $a \in A$ be the element which agrees on each J_m with $a^{(m)}$. This will map to $(x_0, \dots, x_{n-1})$ in $A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1}$, as desired, proving the first assertion. The equivalence of this with the second assertion is clear. $\square$

In most of the remainder of this note, the A_i and C of Lemma 2 will have, inter alia, group structures (usually abelian and written additively). In this situation, let us define the *support* of an element $a = (a_i)_{i\in I} \in A$ as the set

(9) $\quad \mathrm{supp}(a) = \{i \in I \mid a_i \neq 0\}$ (or if our groups are written multiplicatively, $\{i \in I \mid a_i \neq e\}$).

Then we can make the natural identifications,

(10) $\quad$ For $J \subseteq I$, we identify the subalgebra $\{a \in \prod_{i\in I} A_i \mid \mathrm{supp}(a) \subseteq J\} \subseteq A$ with $\prod_{i\in J} A_i$.

Note that for $a,\, a' \in A,$ the set of indices at which these two elements differ can be described as the support of $a-a'$ (respectively aa'^{-1}). Combining this observation with the identification (10), we see that in the context of Lemma 2, if the A_i and C are groups and h a homomorphism, then (1) becomes

(11) $\quad \mathcal{F} \;=\; \{J \subseteq I \;\mid\; \prod_{i\in I-J} A_i \,\subseteq\, \ker(h)\}.$

Let us now apply Lemma 3 to the above situation. We could give a translation of each of conditions (2)-(5), but for brevity, we focus on (3) and (5).

Corollary 6 (to Lemma 3)**.** *Suppose I is a set, $(A_i)_{i\in I}$ a family of groups, C a group, and $h: A = \prod_I A_i \to C$ a group homomorphism. Then the following conditions are equivalent:*

(12) *For every partition of I into a countably infinite family of subsets J_m $(m \in \omega)$, at least one of the subgroups $\prod_{i\in J_m} A_i \subseteq A$ lies in* $\ker(h)$.

(13) *The homomorphism h factors $A \to A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1} \to C$ (where the first arrow is the product of the quotient homomorphisms) for some finite family of ultrafilters $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ on I.*

In this situation, the filter $\mathcal{F}$ of (11) *is the intersection of the unique least set of ultrafilters that can be used in* (13).

Proof. Defining $\mathcal{F}$ by (11), equivalently, by (1), Lemma 2 tells us that $\mathcal{F}$ is a filter on I. Condition (12) then translates to (3), which by Lemma 3 is equivalent to (5), i.e., the condition that $\mathcal{F}$ is an intersection of finitely many ultrafilters, $\mathcal{U}_0 \cap \dots \cap \mathcal{U}_{n-1}$. Let us show that such an expression for $\mathcal{F}$ is equivalent to a factorization of h as in (13).

On the one hand, if $\mathcal{F} = \mathcal{U}_0 \cap \dots \cap \mathcal{U}_{n-1}$, then by Lemma 5 and the discussion preceding it, h has the desired factorization (as a group homomorphism). Conversely, given (13), an element of A whose support lies in none of the $\mathcal{U}_m$ will belong to $\ker(h)$, hence by (11), $\mathcal{U}_0 \cap \dots \cap \mathcal{U}_{n-1} \subseteq \mathcal{F}$. Hence, as in the last paragraph of the proof of Lemma 3, $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ are the only ultrafilters that contain $\mathcal{F}$; so since we know it is an intersection of ultrafilters, it must be the intersection of some subset of this finite family. Hence it is, as required, a finite intersection of ultrafilters. By the final sentence of Lemma 3, the resulting set of ultrafilters is unique, and we get the final sentence of the present lemma. ☐

Combining the above with Lemma 4, we likewise get

Corollary 7 (to Lemma 4)**.** *Suppose I is a set, $(A_i)_{i\in I}$ a family of groups, C a group, $h: A = \prod_I A_i \to C$ a group homomorphism, and κ an uncountable cardinal. Then the following conditions are equivalent:*

(14) *For every partition of I into $<\kappa$ subsets J_s $(s \in S)$, there exist finitely many indices $s_0, \dots, s_{n-1} \in S$ such that* $\prod_{i\in I-J_{s_0}\cup\dots\cup J_{s_{n-1}}} A_i \subseteq \ker(h)$.

(15) *The homomorphism h factors $A \to A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1} \to C$ for some finite family of κ-complete ultrafilters $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ on I. Hence, if* $\mathrm{card}(I)$ *is less than every κ-complete measurable cardinal (in particular, if no such cardinals exist), then h factors through the projection of A to the product of finitely many of the A_i.*

Again, the $\mathcal{F}$ of (11) *is the intersection of the least family of κ-complete ultrafilters as in* (15). ☐

3. Strengthening some results about homomorphisms on product algebras

If k is a commutative ring (by which we will always mean a commutative associative unital ring), then a k-*algebra* (often shortened to "an algebra" when there is no danger of ambiguity) will here mean a k-module A given with a k-bilinear map $A \times A \to A$, written as multiplication, but *not* assumed associative, or commutative, or unital. As in [3] and [4], for A an algebra, we define its *total annihilator ideal* by

(16) $\quad Z(A) \;=\; \{x \in A \mid xA = Ax = \{0\}\}.$

Given algebras A_i $(i \in I)$ and B, and a surjective homomorphism $f: A = \prod_I A_i \to B$, N. Nahlus and the present author study in [3] and [4] conditions under which

(17) f can be written as the sum, $f_1 + f_0$, of a k-algebra homomorphism $f_1 : A \to B$ that factors through the projection of A onto the product of finitely many of the A_i, and a k-algebra homomorphism $f_0 : A \to Z(B)$.

In particular,

(18) (From [4, Theorem 9]) Given a surjective homomorphism $f : A = \prod_I A_i \to B$ of algebras over an infinite field k, there will exist a decomposition $f = f_0 + f_1$ as in (17) if either
(i) $\dim_k(B) < \mathrm{card}(k)$, and $\mathrm{card}(I) \leq \mathrm{card}(k)$, or
(ii) $\dim_k(B) < 2^{\aleph_0}$, and $\mathrm{card}(I) = \aleph_0$, or
(iii) $\dim_k(B)$ is finite, and $\mathrm{card}(I)$ is less than every measurable cardinal $> \mathrm{card}(k)$.

It occurred to me (while correcting the galley proofs to [4]!) that even if I does *not* satisfy one of the cardinality bounds of (i)–(iii) above, one can look at partitions

(19) $I \;=\; \bigcup_{s\in S} J_s$

where S does satisfy that bound, and apply (18) to the resulting product expressions

(20) $A \;=\; \prod_{s\in S} (\prod_{i\in J_s} A_i)$.

That approach yielded improvements on (18), culminating in the present note. The set-theoretic arguments underlying those improvements have been abstracted in §§1-2 above. Combining those with (18), we can now get

Theorem 8 (strengthening of (18))**.** *Suppose k is an infinite field, $(A_i)_{i\in I}$ a family of k-algebras, B a k-algebra, and $f : A = \prod_I A_i \to B$ a surjective k-algebra homomorphism. Suppose also either that*
(i) $\dim_k(B) < \mathrm{card}(k)$, *or that*
(ii) $\dim_k(B) < 2^{\aleph_0}$.

Then the composite homomorphism

(21) $A \;\to\; B \;\to\; B/Z(B)$

can be factored

(22) $A \;\to\; A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1} \;\to\; B/Z(B) \quad (n \in \omega)$,

where the $\mathcal{U}_m$ are ultrafilters on I, which in case (i) *are* $\mathrm{card}(k)^+$*-complete, and in case* (ii) *countably complete.*

Thus, if $\mathrm{card}(I)$ *is less than every measurable cardinal* $> \mathrm{card}(k)$ *in case* (i)*, or less than every uncountable measurable cardinal in case* (ii) (*in particular, in either case, if no such measurable cardinals exist*)*, then* (21) *factors through the projection of A to a finite subproduct $A_{i_0} \times \cdots \times A_{i_{n-1}}$* $(i_0, \ldots, i_{n-1} \in I)$. *In that situation, the original homomorphism $f : A \to B$ can be written $f = f_1 + f_0$ as in* (17).

Proof. In case (i), let $\kappa = \mathrm{card}(k)^+$, and in case (ii), $\kappa = \aleph_1$. Then in each case, given any partition of I into $< \kappa$ subsets J_s, if we write A as the product of products (20), then (18) tells us that f decomposes as the sum of a map that factors through a subproduct $\prod_{i\in J_{s_0}\cup\cdots\cup J_{s_{n-1}}} A_i$, and another with image in $Z(B)$. Hence, on composing with the factor map $B \to B/Z(B)$, we get a factorization of (21) through a subproduct $\prod_{i\in J_{s_0}\cup\cdots\cup J_{s_{n-1}}} A_i$. Corollary 7 now gives us everything but the last sentence of the theorem.

To get that sentence let us (as in [4, end of proof of Theorem 9]) define f_1, respectively, f_0, to be the maps $A \to B$ obtained by first projecting A to $\prod_{i=i_0,\ldots,i_{n-1}} A_i$, respectively, $\prod_{i\in I-\{i_0,\ldots,i_{n-1}\}} A_i$, regarded as subalgebras of A, and then (in each case) following that projection with our given $f : A \to B$. We see that these composite maps have the properties asserted in (17). $\square$

Remark: Case (iii) of (18) has disappeared from the above statement. That case was obtained in [4] by a different method from (i) and (ii), for which a trick that allows one to get κ-complete ultrafilters was easier to see, resulting in the measurable-cardinal bound in the statement of (iii). However, once (i) is strengthened as above, the resulting statement majorizes (iii).

(But I still find striking the property of vector spaces underlying case (iii) of (18), namely [4, Lemma 7], which implies that for any linear map f from k^I (I infinite) to a finite-dimensional k-vector-space V, there exist finitely many $\mathrm{card}(k)^+$-complete ultrafilters $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$ such that for every $I' \subseteq I$ belonging to none of the $\mathcal{U}_m$, there is a member of $\ker(f)$ with support containing I'. In contrast, I do not see any way to strengthen the results [4, Lemmas 3 and 5], which concern supports of elements in kernels of maps $k^I \to V$ for larger-dimensional V, and which underlie cases (i) and (ii) of (18), so as to raise the upper bounds on $\mathrm{card}(I)$ to a measurable cardinal. To do this in Theorem 8 above, we had to use results about *algebra* homomorphisms and their composites with $B \to B/Z(B)$, proved from those lemmas of [4].)

Case (i) of Theorem 8 above also subsumes [3, Theorem 19], a result which has the same measurable-cardinal bound on $\mathrm{card}(I)$ as Theorem 8, but stronger assumptions on B (countable dimensionality, plus a chain condition).

Returning to (18), the methods by which that result was obtained in [4, §§1-4] are extended in [4, §6] to get an almost parallel result for a direct product of algebras over a valuation ring, [4, Theorem 14]. Since a field may be regarded as a valuation ring with trivial value group, the latter result formally subsumed the former (except that it did not contain a case (iii)). However, since the proof was more difficult – but could be shortened by referring to aspects of the earlier proof – and the statement was somewhat more complicated, and algebras over fields are more familiar than algebras over valuation rings, the two results were stated separately. Here, likewise, let us state separately our strengthening of that result.

Theorem 9 (strengthening of [4, Theorem 14])**.** *Let R be a commutative valuation ring with infinite residue field k, and $f : A = \prod_I A_i \to B$ a surjective homomorphism from a direct product of R-algebras to an R-algebra B which is torsion-free as an R-module. Let us write* $\mathrm{rk}_R(B)$ *for the rank of B as an R-module; i.e., the common cardinality of all maximal R-linearly independent subsets of B. Suppose that either*

(i) $\mathrm{rk}_R(B) < \mathrm{card}(k)$, *or*

(ii) $\mathrm{rk}_R(B) < 2^{\aleph_0}$.

Then the composite homomorphism

(23) $\quad A \ \to\ B \ \to\ B/Z(B)$

can be factored

(24) $\quad A \ \to\ A/\mathcal{U}_0 \times \cdots \times A/\mathcal{U}_{n-1} \ \to\ B/Z(B),$

where the $\mathcal{U}_m$ are ultrafilters on I, which are $\mathrm{card}(k)^+$*-complete in case* (i), *and countably complete in case* (ii).

Thus, again, if $\mathrm{card}(I)$ *is less than every measurable cardinal* $> \mathrm{card}(k)$ *in case* (i), *or less than every uncountable measurable cardinal in case* (ii), *then* (23) *factors though the projection of A to the product of finitely many of the A_i, and $f : A \to B$ can be written as the sum of a homomorphism $f_1 : A \to B$ that factors through these projections, and a homomorphism $f_0 : A \to Z(B)$.*

Proof. Proved from [4, Theorem 14] exactly as Theorem 8 is proved from [4, Theorem 9]. □

Addendum: some new results of Maalouf [14] make it possible to weaken the bound $< \mathrm{card}(k)$ to $< \mathrm{card}(k)^{\aleph_0}$ in parts (i) of Theorems 8 and 9.

4. Three further applications

Subsections 4.1-4.2 below show how two results in the literature, which were proved there by arguments about particular sorts of structures, can be obtained via the general results of §§1-2 above. Subsection 4.3 notes an analog of the result of subsection 4.2 with k-algebras replaced by nonabelian groups.

4.1. **The Łoś-Eda Theorem.** Let R be any associative, unital, not necessarily commutative ring. By an R-module M we will mean either a left or a right R-module. (We will not be writing down the action, so we do not have to choose between right and left; but all modules are understood to be on the same side.)

Recall that an R-module M is called *slender* if every homomorphism from a countable direct product of R-modules $\prod_{i\in\omega} N_i$ into M annihilates all but finitely many of the N_i. (See [10], [12, §§94-95], [11, Chapter III]. The classical example is the $\mathbb{Z}$-module $\mathbb{Z}$.) In this situation, one can show that such a homomorphism must in fact factor through the projection to a finite sub-product, $N_{i_0} \times \cdots \times N_{i_{n-1}}$ [11, Theorem III.1.2]. Homomorphisms from a direct product of modules indexed by a *not necessarily countable* set to a slender module have a description analogous to the results of the preceding section:

(25) (Łoś-Eda Theorem, [9], [11, Theorem III.3.2]) Any homomorphism from a direct product $N = \prod_{i\in I} N_i$ of R-modules to a slender R-module M factors through the natural map of N to a finite direct product of ultraproducts, $N/\mathcal{U}_0 \times \cdots \times N/\mathcal{U}_{n-1}$, where $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1}$ are countably complete ultrafilters on I.

This now follows from Corollary 7 and the definition of slender module, via the same "product of products" trick used to deduce the theorems of the preceding section from results of [4].

(For further set-theoretic results about homomorphisms on direct products of abelian groups, see [1].)

4.2. **Almost direct factors.** We shall look next at a result proved by N. Nahlus and the present author in [3]. The hypothesis was of a weaker sort than for the results of [4] – no assumption of an infinite base-field k, and a weaker kind of condition on the codomain algebra than a bound on its k-dimension – so we also get a weaker conclusion, a case of Corollary 6 rather than Corollary 7.

We need some definitions to formulate the hypothesis. Given an algebra (defined as in the preceding section) B over a commutative ring R, we will say that a pair of ideals $B_0, B_1 \subseteq B$ are *almost direct factors* of B if they sum to B, and each is the 2-sided annihilator of the other. We call each such ideal *an almost direct factor of* B. The following three observations are easy. (The first is a special case of the fact that the 2-sided annihilator of every subset of B contains $Z(B)$; the second is seen by noting that if $B_0 + B_1 = B$, and B_0 annihilates both itself and B_1, then it annihilates B; the third by writing an element of the annihilator of $B_0 + Z(B)$ (resp. $B_1 + Z(B)$) as $x_0 + x_1$ with $x_i \in B_i$, and noting that x_0 (resp. x_1) must lie in $Z(B)$.)

(26) Every almost direct factor of B contains $Z(B)$.

(27) If an almost direct factor of B is strictly larger than $Z(B)$, it does not annihilate itself.

(28) Whenever B is the sum of two mutually annihilating ideals B_0 and B_1, the ideals $B_0 + Z(B)$ and $B_1 + Z(B)$ are almost direct factors.

We shall say that B has *chain condition on almost direct factors* if every ascending chain of almost direct factors of B terminates; equivalently, if every descending chain of such ideals terminates. (The equivalence follows from the order-reversing relation between pairs of almost direct factors.) A trivial but important class of algebras with chain condition on almost direct factors are the finite-dimensional algebras over fields.

The result from [3] that we will recover is

(29) [3, part of Proposition 16] If $f : A = \prod_I A_i \to B$ is a surjective homomorphism of algebras over a commutative ring R, and B has chain condition on almost direct factors, then there exist finitely many ultrafilters $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ on I such that the composite map $A \to B \to B/Z(B)$ factors through the natural map $A \to A/\mathcal{U}_0 \times \dots \times A/\mathcal{U}_{n-1}$.

To get this, we shall show that the composite map $A \to B \to B/Z(B)$ satisfies (12), and hence the desired conclusion (13). ((13) only refers to factorization as a map of abelian groups. However, the maps $A \to A/\mathcal{U}_i$ are algebra homomorphisms, hence we in fact get a factorization as an algebra homomorphism.)

Note that by (28), and the surjectivity assumption of (29), for any $J \subseteq I$ the ideals $f(\prod_{i\in J} A_i) + Z(B)$ and $f(\prod_{i\in I-J} A_i) + Z(B)$ of B are almost direct factors.

Suppose, now, in contradiction to (12), that we had a partition of I into subsets J_m $(m \in \omega)$ such that none of the ideals $\prod_{J_m} A_i$ belonged to the kernel of $A \to B \to B/Z(B)$. I claim that the chain of almost direct factors

(30) $$Z(B) \subseteq f(\textstyle\prod_{i\in J_0} A_i) + Z(B) \subseteq \dots \subseteq f(\prod_{i\in J_0\cup\dots\cup J_{n-1}} A_i) + Z(B) \subseteq \dots$$

would be strictly increasing. Indeed, the step where a given J_n first comes in cannot equal the preceding step, because $f(\prod_{i\in J_n} A_i)$ annihilates the latter, but not the former, by (27). This would contradict the chain condition assumed in (29). Thus (12) holds, as claimed.

4.3. **Nonabelian groups.** If K is a group, we can similarly call normal subgroups K_0 and K_1 "almost direct factors" of K if each is the centralizer of the other and their product is all of K, and so define chain condition on almost direct factors for groups. The same reasoning as above, with mutually centralizing normal subgroups in place of mutually annihilating ideals, and products of normal subgroups in place of sums of ideals, yields the analogous result. Namely, letting $Z(K)$ now denote the center of K, the reader can verify that we get

Proposition 10. *If $f : H = \prod_I H_i \to K$ is a surjective homomorphism of groups, and K has chain condition on almost direct factors, then there exist ultrafilters $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ on I such that the composite map $H \to K \to K/Z(K)$ factors through the natural map $H \to H/\mathcal{U}_0 \times \dots \times H/\mathcal{U}_{n-1}$.* $\square$

For further results along these lines, see [2].

5. Further thoughts on the above results

The statement and proof of Proposition 10 above are exactly modeled on those of (29), but the proof of (29) used properties specific to k-algebras, while Proposition 10 uses properties specific to nonabelian groups. Can we set up a general context which embraces these two cases, and leads to more examples?

Say we are working in a general variety $\mathbf{V}$ of algebras, in the sense of universal algebra. We have the minor complication that if $\mathbf{V}$ does not involve a group structure, we lose the simplification of interpreting the filter determined by a homomorphism h on a direct product algebra via $\ker(h)$, as in (11). But I don't think this should make a big difference; we still have (1); we must simply expect certain statements to involve twice as many variables as when we have a group structure, since we must deal with the condition that pairs elements fall together under a map, rather than the condition that individual elements lie in the kernel.

A less trivial problem is what should replace annihilators in algebras, and centralizers in groups. I think something like the following might work.

Let us understand a *formal relation* in variables $x_0, \dots, x_{n-1}$, written

(31) $\quad R(x_0, \dots, x_{n-1}),$

to mean a symbolic equation

(32) $\quad R_0(x_0, \dots, x_{n-1}) \ = \ R_1(x_0, \dots, x_{n-1}),$

where R_0 and R_1 are terms in n variables and the operations of $\mathbf{V}$.

Let us now consider formal relations $R(x, x'; y, y'; z_0, \dots, z_{n-1})$ in $n+4$ variables – which we will mostly abbreviate to $R(x, x'; y, y')$, suppressing the final n variables – with the property that

(33) $\quad$ Both $R(x, x, y, y')$ and $R(x, x', y, y)$ are identities of $\mathbf{V}$.

Thus, (33) says that $R(x, x'; y, y')$ becomes an identity of $\mathbf{V}$ if *either* the two variables x and x', *or* the two variables y and y', are set equal.

(For example, if $\mathbf{V}$ is the variety of k-algebras, then two examples of formal relations satisfying (33) are $(x - x')(y - y') = 0$ and $((x - x')z)(y - y') = 0$. In the variety of groups, examples are $[xx'^{-1},\, yy'^{-1}] = e$ and $[x^2 x'^{-2},\, (yy'^{-1})^3] = e$. For an example in a variety not involving a group structure, we may take for $\mathbf{V}$ the variety of lattices, and for R the formal relation

(34) $\quad (x \vee y) \wedge (x' \vee y') \ = \ (x \vee y') \wedge (x' \vee y).$

This last example generalizes to any variety with two derived operations, denoted $\wedge$, respectively, $\vee$, each in two "distinguished" variables x, y, and possibly additional variables $z_0, \dots, z_{n-1}$, respectively $w_0, \dots, w_{m-1}$, such that $\wedge$, but not necessarily $\vee$, is commutative in the distinguished variables.)

The following observation generalizes a property of products in k-algebras, and commutators in groups, that we used in the last two sections.

(35) $\quad$ Suppose a formal relation $R(x, x', y, y')$ in the operations of a variety $\mathbf{V}$ satisfies (33). Then on a direct product $A = A_0 \times A_1$ of algebras in $\mathbf{V}$, the relation $R(a, a', b, b')$ holds whenever a, a' agree in their A_0-components and b, b' agree in their A_1-components.

(Cf. the property of k-algebras that if one element of $A_0 \times A_1$ has zero first component, and another has zero second component, then their product is zero.) The idea is that such relations should help "detect" direct product decompositions.

Now let $\mathcal{R}_{\mathbf{V}}$ be the set of all formal relations R satisfying (33) in $\mathbf{V}$. Then for any binary relation C on the underlying set of an algebra $A \in \mathbf{V}$, we can define a binary relation $C^{\perp}$, by

(36) $\quad C^{\perp} = \{(a, a') \in A \times A \mid$ for all formal relations $R(x, x'; y, y'; z_0, \dots, z_{n-1}) \in \mathcal{R}_{\mathbf{V}}$, all pairs of elements $(b, b') \in C$, and all choices of $c_0, \dots, c_{n-1} \in A$, the relation $R(a, a'; b, b'; c_0, \dots, c_{n-1})$ holds in $A\}$.

The hope is that this construction, applied to *congruences* C, will play a role analogous to *annihilators of ideals* of a k-algebra, and *centralizers of subgroups* of a group, and so allow us to prove a general analog of (29) and Proposition 10. But exactly how this should be done is not clear. For instance, though one can show that the relation $C^{\perp}$ defined in (36) will be a subalgebra of $A \times A$, and as a binary relation it is easily seen to be reflexive and symmetric, I see no reason why it should be transitive, and hence a congruence on A, even if C was a congruence.

One can, of course, consider two congruences C_0 and C_1 to have a relation analogous to being "almost direct factors" of an algebra or a group if they simultaneously satisfy $C_0^{\perp} = C_1$ and $C_1^{\perp} = C_0$, are mutually commuting, and have for join the improper congruence. But to even state the analog of (29) and Proposition 10, one needs to be able to say that the analog of $Z(B)$, namely the relation $(B \times B)^{\perp}$ (where $B \times B$ is the improper congruence on B) is a congruence.

Perhaps one needs to find additional conditions on the variety $\mathbf{V}$ that make such conclusions hold; and/or replace (36) by a construction using, not all of $\mathcal{R}_{\mathbf{V}}$, but some subset $\mathcal{R}$ with appropriate properties. (Note, however, that for an arbitrary subset $\mathcal{R} \subseteq \mathcal{R}_{\mathbf{V}}$, some of the things I've noted hold for $\mathcal{R}_{\mathbf{V}}$ may fail: $C^{\perp}$ need not be symmetric or a subalgebra.) I leave these ideas for others to investigate.

Incidentally, not all situations to which we have applied the results of §§1-2 are based on ideas like those of "annihilator" and "centralizer", whose possible generalization we have just examined. The facts that $\mathbb{Z}$ and various other modules are slender are true for (so far as I can see) very different sorts of reasons.

6. Extending the Erdős-Kaplansky Theorem to reduced products

The Erdős-Kaplansky Theorem [13, Theorem IX.2, p. 247] says that if D is a division ring and I an infinite set, then $\dim_D D^I = \mathrm{card}(D^I)$. Combining the method of proof of that result with Lemma 4 above, we shall now prove a statement of which that theorem is essentially the case $\mathcal{F} = \{I\}$. ("Essentially" because, to avoid complications, we here assume D infinite. The case of finite D will covered by Corollary 13.)

Theorem 11. *Let D be an infinite division ring, I a set, and $\mathcal{F}$ a filter on I which is not the intersection of finitely many* $\mathrm{card}(D)^+$*-complete ultrafilters.* (*So in particular, $\mathcal{F}$ is not the principal filter generated by a finite set.*) *Then*

(37) $\quad \dim_D D^I/\mathcal{F} \ = \ \mathrm{card}(D^I/\mathcal{F}).$

(*where* $\dim_D$ *can be taken to mean the dimension either as a right or as a left vector space*).

Proof. Without loss of generality, let vector spaces (in particular, $D^I/\mathcal{F}$) be left vector spaces.

For any vector space V, since a basis for V is a subset of V, we have

(38) $\quad \dim\ V \ \leq \ \mathrm{card}(V).$

For D infinite, as in the present situation, equality is easily proved in (38) if $\mathrm{card}(V) > \mathrm{card}(D)$. (E.g., by [13, Lemma X.1, p. 245], and the fact that a product of two infinite cardinals equals the larger of the two.) So to prove (37) it suffices to show that under our assumptions on $\mathcal{F}$, if $\mathrm{card}(D^I/\mathcal{F}) = \mathrm{card}(D)$, then

(39) $\quad \dim_D(D^I/\mathcal{F}) \ \geq \ \mathrm{card}(D).$

To do this, let us slightly strengthen [13, Lemma IX.2, p. 246]. Namely, we shall show that there exists a $\mathrm{card}(D) \times \mathrm{card}(D)$-tuple $((x_{\alpha\beta}))_{\alpha,\beta\in \mathrm{card}(D)}$ of elements of D such that

(40) For every natural number n, and every pair of n-tuples $(\alpha_0, \ldots, \alpha_{n-1})$ and $(\beta_0, \ldots, \beta_{n-1})$ of elements of $\mathrm{card}(D)$ such that $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$, the n elements $(x_{\alpha_i\beta_j})_{j=0,\ldots,n-1} \in D^n$ $(i = 0, \ldots, n-1)$ are linearly independent; equivalently, the $n \times n$ matrix $((x_{\alpha_i\beta_j}))$ is nonsingular.

(When (40) holds, the family of vectors $(x_{\alpha\beta})_\beta$ $(\alpha \in \mathrm{card}(D))$ is called *strongly* linearly independent in [13, Lemma X.2, p. 246], though there, the index we call β is restricted to a countable range; i.e., only strongly linearly independent families of ω-tuples are considered.)

Mimicking the proof in [13], we can choose the elements $x_{\alpha\beta} \in D$ by a recursion over the index-set $\mathrm{card}(D) \times \mathrm{card}(D)$, lexicographically ordered. Given α, β, assume recursively that all $x_{\alpha'\beta'}$ with $(\alpha', \beta') < (\alpha, \beta)$ have been chosen so as to satisfy all cases of (40) involving only elements with subscripts $< (\alpha, \beta)$. In particular, for every natural number n and every pair of increasing n-tuples $(\alpha_0, \ldots, \alpha_{n-1})$ and $(\beta_0, \ldots, \beta_{n-1})$ with $\alpha_{n-1} = \alpha$, $\beta_{n-1} = \beta$, the values $x_{\alpha_i\beta_j}$ other than $x_{\alpha_{n-1}\beta_{n-1}}$ have been chosen; so we have an $n \times n$ matrix with one entry missing; and by our recursive assumption, its upper left $n{-}1 \times n{-}1$ minor is nonsingular. In this situation, one sees by linear algebra that one and only one value of the missing element will make the matrix singular. (Indeed, a unique left linear combination of the first $n-1$ rows will have first $n-1$ coordinates agreeing with those specified in the n-th row; and the last coordinate of that linear combination will be the value of $x_{\alpha\beta}$ in question.)

Now since $\alpha,\,\beta<\mathrm{card}(D),$ there are fewer than $\mathrm{card}(D)\,\mathrm{card}(D)=\mathrm{card}(D)$ choices for the integer n and the values $\alpha_0,\dots,\alpha_{n-2}$ and $\beta_0,\dots,\beta_{n-2}$ in the preceding paragraph. Since each such choice leads to only one value of $x_{\alpha\beta}\in D$ making the corresponding matrix singular, we may choose $x_{\alpha\beta}$ so as to make all these matrices nonsingular. Proceeding recursively, we get values of $x_{\alpha\beta}$ for all $\alpha,\beta\in\mathrm{card}(D)$ which together satisfy (40).

Now by assumption, our filter $\mathcal{F}$ is not a finite intersection of $\mathrm{card}(D)^+$-complete ultrafilters; so Lemma 4 tells us that there exists a partition of I into $<\mathrm{card}(D)^+,$ i.e., $\leq\mathrm{card}(D)$ subsets no finite union of which belongs to $\mathcal{F}.$ If that partition involves fewer than $\mathrm{card}(D)$ sets, let us throw in empty sets to reach that value. Thus, we can write our partition $(J_\alpha)_{\alpha\in\mathrm{card}(D)}.$ Let us now define elements $y_\beta\in D^I$ $(\beta\in\mathrm{card}(D))$ by the conditions that on each $J_\alpha,$ the element y_β has constant value $x_{\alpha\beta}.$

Then (40) tells us that for any positive integer $n,$ a nontrivial linear combination of n of these elements cannot be zero on n of the sets J_α; i.e., its zero-set must be a union of $<n$ of those sets. So as no union of finitely many J_α belongs to $\mathcal{F},$ no nontrivial linear combination of the y_β has zero image in $D^I/\mathcal{F}.$ Thus, we have a $\mathrm{card}(D)$-tuple of linearly independent elements of $D^I/\mathcal{F},$ proving (39), as required. $\square$

In contrast, if $\mathcal{F}$ *is* an intersection of $n\in\omega$ $\mathrm{card}(D)^+$-complete ultrafilters, then $D^I/\mathcal{F}\cong D^n,$ which has dimension less than its cardinality.

Remarks for the reader familiar with [4]: Let me note how, with the help of the above theorem, one can strengthen some of the results of [4] from the case of vector spaces over a field k to that of vector spaces over a division ring D. We begin with [4, Lemma 7]. (I will not to repeat here the statement of that technical result, but merely note how to extend its proof. I will, however, recall something of the content of the results that follow from that lemma.) One finds that all steps of the proof of that lemma *except* the paragraph following [4, (38)] go over unchanged to the division ring case. The result of that paragraph says (after putting D for k) that a certain ultrafilter $\mathcal{U}$ on a certain set J is $\mathrm{card}(D)^+$-complete [4, (39)]. This is vacuous if D is finite; to prove it when D is infinite, note that [4, (38)] says that $D^J/\mathcal{U}$ can be mapped injectively into the finite-dimensional D-vector-space $g(D^{J_0\cup J})/g(D^{J_0})$; so $D^J/\mathcal{U}$ is finite-dimensional. Thus, by Theorem 11 above, $\mathcal{U}$ is a finite intersection of $\mathrm{card}(D)^+$-complete ultrafilters, which, given that it is an ultrafilter, simply says it is $\mathrm{card}(D)^+$-complete, as desired.

From this we get the corresponding generalization of the corollary to that lemma, which in particular tells us that for D infinite and I a set having cardinality less than every measurable cardinal $>\mathrm{card}(D),$ every subspace of finite codimension in D^I contains an element of cofinite support in $I.$

This result, in turn, allows us to generalize [4, Theorem 9(iii)], i.e., roughly, case (iii) of (18) above, to the situation where k-algebras A are replaced by (D,D)-bimodules A given with balanced D-bilinear maps $A\times A\to A.$ In the generalized statement, one should assume both left and right finite-dimensionality of $B.$ One adapts the supporting result [4, Lemma 2] by associating to each $a=(a_i)\in\prod_I A_i$ both the left-vector-space map $g_a:D^I\to B$ defined by $g_a((u_i))=f((u_ia_i))$ and the right-vector-space map $g'_a:D^I\to B$ defined by $g'_a((v_i))=f((a_iv_i)).$ In display (5) of the proof of that lemma, the key step becomes $f((a_i\,x_i))=f((a_i\,v_i\,v_i^{-1}x_i)),$ and in the dual calculation, $f((x_i\,a_i))=f((x_i\,u_i^{-1}\,u_i\,a_i)).$

(I wonder whether in this version of [4, Theorem 9(iii)], the left and right finite-dimensionality assumption can somehow be weakened to one-sided finite-dimensionality, or even ascending chain condition on the (D,D)-bimodule structure of $B.$)

Returning to Theorem 11, of which, we saw, the only nontrivial case was when $\mathrm{card}(D^I/\mathcal{F})=\mathrm{card}(D),$ it is natural to ask how common this equality is – in other words, how common it is for an infinite set X to satisfy $\mathrm{card}(X^I/\mathcal{F})=\mathrm{card}(X)$ when the filter $\mathcal{F}$ is not a finite intersection of $\mathrm{card}(X)^+$-complete ultrafilters.

For some values of $\mathrm{card}(X),$ it is indeed common. For instance, if $\kappa=\lambda^\mu,$ where λ and μ are infinite cardinals, then $\kappa^\mu=\kappa,$ hence for any X of cardinality $\kappa,$ and nonempty I of cardinality $\leq\mu,$ we have $\mathrm{card}(X^I)=\mathrm{card}(X);$ hence for any proper filter $\mathcal{F}$ on $I,$ $\mathrm{card}(X)=\mathrm{card}(X^I)\geq\mathrm{card}(X^I/\mathcal{F})\geq\mathrm{card}(X)$ (the last inequality because X embeds diagonally in $X^I/\mathcal{F}),$ giving the desired equality.

On the other hand, if X is countably infinite, we always get $\mathrm{card}(X^I/\mathcal{F})\geq 2^{\aleph_0}.$ For, following the idea of the proof of [4, Lemma 6], consider the continuum many functions $f_r:\omega\to\omega$ given by $f_r(n)=\lfloor r\,n\rfloor$ for positive real numbers r. We see that any two of these agree only at finitely many n. Now given a filter $\mathcal{F}$ on I that is not countably complete, we can, by the same sort of application of Lemma 4 as in the next-to-last paragraph of the proof of Theorem 11 above, construct from the f_r continuum many functions $y_r : I \to \omega$ that have distinct images in $\omega^I/\mathcal{F}$; and using a bijection between X and ω, we get the asserted conclusion.

I do not know whether one of the above two examples is more "typical" than the other.

So far, this discussion has assumed X infinite. We may also ask when a reduced power $X^I/\mathcal{F}$ of a finite set X is finite. The answer (and a bit more) is given in

Lemma 12. *Let $\mathcal{F}$ be a filter on a set I, and X a finite set with more than one element. Then the equivalent conditions of Lemma 3 (in particular, condition (5), that $\mathcal{F}$ is the intersection of finitely many ultrafilters on I) are also equivalent to each of*

(41) *The reduced power $X^I/\mathcal{F}$ is finite,*

(42) *The reduced power $X^I/\mathcal{F}$ has cardinality $< 2^{\aleph_0}$.*

Proof. If (5) holds, with $\mathcal{F} = \mathcal{U}_0 \cap \dots \cap \mathcal{U}_{n-1}$, then $X^I/\mathcal{F}$ embeds in $X^I/\mathcal{U}_0 \times \dots \times X^I/\mathcal{U}_{n-1}$, and it is well known that an ultrapower of a finite set X is isomorphic to X; so $X^I/\mathcal{F}$ is finite, giving (41) and hence (42).

Conversely, if the equivalent conditions of Lemma 3 fail, then the failure of (3) says that we have a partition of I into subsets J_m $(m \in \omega)$ none of whose complements lies in $\mathcal{F}$. In this case, let us take two elements $x \neq y \in X$, and consider the $2^{\aleph_0}$ elements of $\{x, y\}^I$ which are constant on each subset J_m. If f and g are distinct members of this set, then they disagree throughout at least one J_m; so as the complement of J_m does not lie in $\mathcal{F}$, f and g yield distinct elements of $X/\mathcal{F}$. This contradicts (42), and hence (41). $\square$

Corollary 13. *In Theorem 11, the condition that D be infinite can be dropped.*

Proof. Assume D finite. Thus, the hypothesis that $\mathcal{F}$ is not the intersection of finitely many $\mathrm{card}(X)^+$-complete ultrafilters simply says it is not the intersection of finitely many ultrafilters, which by Lemma 12 tells us that the vector space $D^I/\mathcal{F}$ is infinite. But for an infinite vector space V over a finite field, one indeed has equality in (38). $\square$

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