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A NUMBER SYSTEM WITH AN IRRATIONAL BASE

George Bergman

The reader is probably familiar with the binary system and the decimal system and probably understands the basis for any others of that type, such as the trinary or duodecimal. However, I have developed a system that is based, not on an integer, or even a rational number, but on the irrational number τ (tau), otherwise known as the "golden section", approximately 1.618033989 in value, and equal to $(1 + \sqrt{5})/2$.

In order to understand this system, one must comprehend two peculiarities of the number τ . They are based on tau's distincive property 1 that

$$\tau^n = \tau^{n-1} + \tau^{n-2}$$

(a.) Take any approximation (A_1) of τ . Taking the reciprocal, we get a number (a_1) that is proportionately the same distance from $1/\tau$ as A_1 was from τ , but arithmetically nearer. Adding $1,^2$ we get a number (A_2) that is proportionately nearer τ than a_1 was to $1/\tau$ but arithmetically just as near. Since a_1 is arithmetically nearer than A_1, A_2 is nearer in both respects to τ than A_1 . Repeating the process of taking the reciprocal and adding 1, we approach τ . Now, taking 1 as A_1 , and expressing our approximations of τ (i.e. A_1, A_2, A_3 , etc.) as fractions, we get

$$\frac{1}{1} \frac{2}{1} \frac{3}{2} \frac{5}{3} \frac{8}{5} \frac{13}{8} \dots$$

Taking either the numerators or the denominators, we get what is know as the Fibonacci Series, each term of which is formed by adding the two previous terms³; for

$$\frac{f_{n+2}}{f_{n+1}} = 1 + \frac{1}{\left(\frac{f_{n+1}}{f_n}\right)} = 1 + \frac{f_n}{f_{n+1}} = \frac{f_{n+1} + f_n}{f_{n+1}} \quad \text{so } f_{n+2} = f_{n+1} + f_n$$

(We designate the n^{th} term of the Fibonacci Series by f_n , setting $f_1 = 1$, $f_2 = 1$. This practice shall be used throughout the article.)

(b.) Any integral power of τ can be expressed in the form $\tau^n = AT + B$, where A and B are integers and, in fact, numbers in the Fibonacci Series. The explanation of this startling fact is really rather simple:

Since

$$\tau^{1} = 1\tau + 0$$
 and $\tau^{2} = 1\tau + 1$,

and since $\tau^3 = \tau^2 + \tau^1$; $\tau^3 = (1\tau + 0) + (1\tau + 1) = 2\tau + 1$.

In the same way

$$\tau^{4} = \tau^{2} + \tau^{3} = (1\tau + 1) + (2\tau + 1) = 3\tau + 2.$$

In general

$$\tau^{n} = \tau^{n-1} + \tau^{n-2} = (f_{n-1}\tau + f_{n-2}) + (f_{n-2}\tau + f_{n-3})$$
$$= (f_{n-1} + f_{n-2})\tau + (f_{n-2} + f_{n-3})$$
$$= f_{n}\tau + f_{n-1} \qquad (\text{see Note } 2.)$$

Can this be applied to negative powers of τ ? We don't know any Fibonacci numbers before 1, but it is easy to see how we can find them: Taking 1 and 1 as our first two, we can see that the term before must be 0, since that is the only number which, when added to 1 gives 1. In the same way, the number before that must be 1, since 1 is the only number that, when added to 0 gives 1; and the next term must be -1, since no other number gives 0 when added to 1. Continuing this process, we get 0, 1, -1, 2, -3, 5, -8, 13, -21... Obviously, this is alternately +1 and -1 times the corresponding Fibonacci numbers. But can this be proved to be true in all cases? It can by induction. The rule we want to prove, expressed as an equation, is:

$$f_{-y} = (-1)^{y+1} f_{y}$$

Let us assume it true for y = 1, 2, ... n. Now by the basic property of the Fibonacci Series:

$$f_{-n} + f_{-n-1} = f_{-n+1}$$

$$f_{-n-1} = f_{-n+1} \quad f_{-n}$$

$$f_{-(n+1)} = (-1)^n f_{n-1} - (-1)^{n-1} f_n$$

$$f_{-(n+1)} = (-1)^{n+2} (f_{n-1} + f_n)$$

$$f_{-(n+1)} = (-1)^{n+2} f_{n+1}$$

The inductive proof is completed by the examples already cited.

Applying this to powers of r, we make a list of them from r^{-5} to r^{5} :

$$r^{-5} = 5r - 8 \qquad r^{0} = 0r + 1 r^{-4} = -3r + 5 \qquad r^{1} = 1r + 0 r^{-3} = 2r - 3 \qquad r^{2} = 1r + 1 r^{-2} = -1r + 2 \qquad r^{3} = 2r + 1 r^{-1} = 1r - 1 \qquad r^{4} = 3r + 2 r^{5} = 5r + 3$$

Now, at last, we shall get back to our concept of a system based on τ . Like the binary system, it can have only two symbols: 1 and 0. But, unlike the binary system, it has the rule (2): $100 = 011^5$ (place the decimal point anywhere - it's a general rule). But how do we find the numbers? We know that 1 is τ^0 or 1.0. Next, looking at the table of powers of τ , one notices that

$$r^1 = 1r + 0$$
 and $r^{-2} = -1r + 2$.

Adding them together, one gets

$$\tau^{1} + \tau^{-2} = (1\tau + 0) + (-1\tau + 2) = 2$$

Therefore, 2 = 10.01 (in this system). Of course, because of rule (2), this can also be expressed as 1.11, ⁶ 10.0011, 10.001011, 1.101011, etc., but 10.01 is what I call the simplest form (that form in which there are no two 1's in succession, and which, therefore, cannot be acted upon by the reverse of rule (2), called simplification (11=100). To convert a number to its simplest form, repeatedly simplify the leftmost pair of consecutive 1's.

To continue with our "translation" of numbers into this system, we notice (after a careful examination of the table) that $r^2 = 1r + 1$ and $r^{-2} = -1r + 2$, and adding them together $r^2 + r^{-2} = 3$, and so 3 is 100.01 in this system. What about 4? Well, since $r^2 + r^{-2} = 3$, $r^2 + r^{-2} + r^0$ (101.01) must equal 4, since $r^0 = 1$. Can this method of adding 1 be used for other numbers? The answer is "yes"; just convert the number into the form in which there is a zero in the units column and place a 1 in it. If the method of conversion is not obvious, use this method:

- a. Change to the simplest form.
- b. If there is no 1 in the units column, you are finished. If there is, look in the r^{-2} column (there can't be any in the r^{-1} column because it is in its simplest form and there is a 1 in the column next to it); if there is a 0 there, expand 7 the 1 into the r^{-1} and r^{-2} columns (that's all); if there is 1, look in the r^{-4} column; if there is a zero there, expand the 1 in the r^{-2} column into the r^{-3} and r^{-4} column and the 1 in the units column into the r^{-1} and r^{-2} columns. If there is a 1, look in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} column; if there is a zero, expand the 1 in the r^{-6} columns, the 1 in the r^{-2} column into the r^{-2} columns.

column into the r^{-3} and r^{-4} column, and the l in the units column into the r^{-1} and r^{-2} columns; if on the other hand there is a l, look in the r^{-8} column, etc. If it is the endless fraction 1.0 10 10 10 1..., change it to 10.000000

We can now construct a table of integers in this system. Here are those from 0 to 14 (in their simplest forms)

> 0 - 05 - 1000.1001 10 - 10100.0101 1 - 16 - 1010.0001 11 - 10101.01012 - 10.017 - 10000.0001 12 - 100000, 101001 3 - 100.01 8 - 10001.0001 13 - 100010,0010014 - 101.019-10010.0101 14 - 100100.110110Examples of the basic processes

1. Change 100101.111001 (equals 16) to the simplest form. The first pair (farthest left) is in the units and r^{-1} column, so we simplify it into the r^{1} column, giving 100110.011001. This time the pair farthest to the left is the one we have just created with our new 1 in the r^{1} column (the result of our simplification), added to the 1 already in the r^{2} column. This we simplify into a 1 in the r^{3} column, which gives us 101000.011001. Finally, we change the last remaining pair (in the r^{-2} and r^{-3} columns) into a 1 in the r^{-1} column, arriving at our final answer: 101000.100001

2. Change 101.01 (4) to a form with a zero in the units column. (i.e. a form to which 1 can be added). One can see that it is already in its simplest form. However, there is a 1 in the units column, and we must remove it. The first thing we do is look in the r^{-2} column; since there is a 1 there, we look in the r^{-4} column. This is empty, and so we expand the 1 in the r^{-2} column into the r^{-3} and r^{-4} columns, getting 101.0011. Now that the r^{-1} columns, getting 100.1111, which has a zero in the units column. We can now add 1 to it:

100.1111+1=101.1111=110.0111=1000.0111=1000.1001 (five)

The Arithmetic Operations

The arithmetical operations, although they are basically the same as in any other system, are, in practice, quite different because of the peculiarities of this system. As our first step in all of them, we eliminate zeros, which would only hinder us, and show the place values of l's by actural placement in columns. For instance, four (101.01) would be |1| |1| |1|, the heavy line representing the "decimal point". The necessity for this step results from the fact that, though in the systems to which we are accustomed, the steps of addition are simple enough to be performed mentally, this is not so in the tau system; nor is each column non-dependent on the one to the left of it. It is thus necessary to have lined columns in which to carry out the work.

Now for the actual processes, we shall start with addition. The example

$$\begin{array}{c} 100\ 10.\ 010\ 1 \\ +\ 1010.\ 0001 \end{array} \begin{pmatrix} 9 \\ +\ 6 \end{pmatrix} \text{ would be represented by} \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$$

In this set-up it can be seen that we have a pair, consisting of a 1 in the τ^3 column and one in the τ^4 column. This we simplify into a 1 in the τ^5 column.

Now, however, we have no obvious way to continue. We are left with two l's in the same column. We can neither add them together to give 2 (as we would in the decimal system), nor is there any simple "carrying" operation. We must, therefore, change this to a form not having two l's in the same column. We will start by expanding one of the l's in the r^1 column:

Now we can simplify the pair we have just created in the r^1 and units columns:

1	A		1	1			1	1	
		ĭ		1	1	1		1	

and the one in the r^{-1} and r^{-2} columns:

We shall now use the same type procedure for the l's in the r^{-4} column. We expand one of the l's there: $\begin{vmatrix} 1 & \chi \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 & \chi \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} 1 \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \begin{vmatrix} \chi \\ & \chi \end{vmatrix} \end{vmatrix} \end{vmatrix}$

and simplify in the τ^{-4} and τ^{-5} columns:

and express our answer in ordinary form, writing l's in columns with an un-crossed-out l and 0's in the columns where all have been crossed out: 100101.001001 (15) For general rules as to procedure, I believe that these will do in most cases. (These general rules and the ones for the other processes are not the types of rules that, if disobeyed, would give the wrong answer, but merely guides to the quickest way to get the right one):

a) Expand only when that is the only way to remove a 1 from the same column as another, regardless of whether this will result in the same situation in another column, but only if no more simplification can be done.

Subtraction

Subtraction is the next process I shall describe. As in addition, we set up the numbers in columns, but here we shall assign negative values to the 1's from the subtrahend. For instance, to find (11-6) we set up

I	1	1		1	1	1
	-1		-1			- 1

We now "cancel" the 1 and the -1 in the τ^{-4} column, giving

:	1	1		1	1	1
	-1		-1			-/1

Next, we expand the 1 in the r^2 column, getting

 $\begin{vmatrix} 1 \\ -1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ -1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ -1 \end{vmatrix} \end{vmatrix}$

Again, we cancel, this time in the τ^1 column,

$$\begin{vmatrix} 1 \\ -1 \end{vmatrix} \begin{vmatrix} \cancel{1} \\ -\cancel{1} \end{vmatrix} \begin{vmatrix} \cancel{1} \\ -\cancel{1} \end{vmatrix} \begin{vmatrix} 1 \\ -\cancel{1} \end{vmatrix} \begin{vmatrix} 1 \\ -\cancel{1} \end{vmatrix} \begin{vmatrix} \cancel{1} \\ -\cancel{1} \end{vmatrix}$$

And again we expand, this time the 1 in the r^4 column, getting

1	1	Ă	1	1	1	Ă
	-1	1	-1	1		-1

For the third time we cancel, (in the τ^3 column) giving

 A
 A
 A
 1
 1
 I

 -A
 1
 -A
 1
 -A

This we treat just as we would if we were adding and arrived at this stage; by expanding one of the l's in the units column we get

Next we simplify the pair in the units and r^{-1} columns getting

$$\begin{vmatrix} A \\ A \\ -A \end{vmatrix} \begin{vmatrix} A \\ A \\ -A \end{vmatrix} \begin{vmatrix} A \\ -A \\ 1 \\ -A \end{vmatrix} \begin{vmatrix} A \\ -A \\ A \\ -A \end{vmatrix} \begin{vmatrix} A \\ A \\ -A \\ 1 \end{vmatrix} \begin{vmatrix} A \\ -A \\ -A \end{vmatrix}$$

and the pair we thereby form in the r^{1} and r^{2} column getting

Finally, we expand one of the l's in the $\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

and simplify the resulting pair in the τ^{-2} and τ^{-2} and τ^{-2} columns) getting as our final answer

For subtraction it is harder to formulate a general rule, but I think it would suffice to say: Cancel whenever possible and simplify or expand whenever that would permit cancellation (also remember not to confuse a 1 with a -1 ($| \cdot | 1 | -1 | \neq | 1 | \mathcal{I} | -\mathcal{I} |$) and not to make the mistake of "expanding" a -1 into two +1's.) After all -1's have been removed by cancellation, proceed as you would with an addition example.

Multiplication

Multiplication involves nothing new. We simply place the partial products as we do in the decimal system, and add. For instance:

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 $\begin{array}{c} 101.01 \\ \times 100.01 \quad (3 \times 4) \end{array}$

			Å	A		1	1		
1 1	a	1		A	Å	1			A REAL PROPERTY AND ADDRESS OF ADDRE

Next we expand one of the 1's in the r^{-2} column, and one of the 1's in the r^{-4} column:

 1
 A
 A
 A
 1
 1

Finally, we simplify the pair in the τ^{-2} and τ^{-3} columns, and then the one in the τ^{-4} and τ^{-5} columns, giving our final answer:

Division is quite different in this system, and is, in fact, rather odd. The only things it has in common with ordinary division are the basic principles behind it, the way the example looks, and the movement of the "decimal point" to eliminate any figures to the right of it in the divisor. It is best explained by an example: 12 divided by 2, or



which, after moving the "decimal point", is

Now, since we are dividing by $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$, if there are anywhere two 1's with two spaces between them (the spaces can be empty or full), they can be crossed out and a 1 placed in the quotient above the rightmost of the two. This crossing out in no way signifies that the 1's should not be there, but is merely equivalent to, in long division (decimal) the subtraction of the product of the number placed in the quotient and the divisor from the dividend. Since the number placed in the quotient can only be 1 (placed in any column, of course), we merely subtract the dividend (placed in that same column), i.e. cross it out. It is obvious that once the whole dividend has been crossed out, the group of 1's in the quotient, after being changed to the simplest form, will be the complete quotient. Getting back to our original problem, we see that we do have just such a set of 1's in the τ^{-1} and τ^{-4} column, and so we cross it off and place a 1 in the quotient, getting

But now, you may say, there are no more pairs of 1's spaced in that way; what shall we do? The answer is our old pair of friends, expansion and simplification. Since they do not change the value of a number, if either of those processes yields a set of 1's spaced correctly, that set can be crossed off and a 1 placed in the quotient just as though that set were part of the original number. Since in our problem it is so far impossible to simplify, we shall expand. Expanding the 1 in the τ^7 column, we get



No such set yet. However, when we expand the 1 we've just placed in the τ^5 column, giving



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we have not one but two such sets (remember that all we need is two 1's with that certain separation, regardless of intervening and crossed out 1's), one made of the 1's in the r^1 and r^4 columns, and the other of the 1's in the r^3 and r^6 columns. Crossing them both out and placing the 1's in the correct places in the quotient, we get

and since there are no more 1's in the dividend, our number in the quotient is the complete quotient, and so 1010.0001, or 6, is our answer. This time the general rule is: Always take that course of action that will place your next 1 (i.e., a 1 in the quotient—set in the dividend) farthest to the left. By a course of action, I mean a series of expansions and simplifications and the exchange of a set for a 1 in the quotient that follows; or simply that exchange, if the set is already there. (I did not obey this rule in my demonstration so that I could show the process in a simpler way.) This is so that the answer be in its simplest form.

By the way, the processes of addition, subtraction, and often multiplication, can be performed together by writing the addends, the subtrahends, and the partial products in one set of columns; for instance: $2 \times 3 + 4 + 3 - 5$:

$$10.01 \\ \times 100.01 + 101.01 + 100.01 - 1000.1001$$

and working it out:



Now that we know these four processes, we have a much better way of finding a number in this system than merely repeatedly adding 1's until we reach it. For instance, to find thirty-seven, we can multiply 6×6 and add 1; to check the arithmetic, we multiply 7×5 and add 2:

$$\begin{array}{cccc} 1010.0001 & (to check) & 1000.1001 \\ \times 1010.001 + 1 & \times 10000.0001 + 10.01 \end{array}$$

What's more, we are not only able to find integers, but, since we can divide, we ought to be able to find fractions also. Let us try. First, we shall attempt to find $\frac{1}{2}$. We begin by setting up our division:



Our first set, as can be seen, will have its leftmost 1 in the r^1 column. We therefore expand the 1 in the r^2 column:



The other one we need to complete the set is a 1 in the r^{-2} column; this we get by expanding the 1 in the units column:

After we exchange our set for a 1 in the quotient $(r^{-2} \text{ column})$, we notice that our remainder is 1 (in the r^{-1} column). Since 1 is the number we started with, the next figure in the quotient and the next remainder should be the same as these. However, the question is, where in the quotient shall we place it? Since our first 1 was in the r^2 column (because we moved the decimal point) and our remainder is three places to the right of it, in the r^{-1} column, our next 1 in the quotient should be three places to the right of the first 1 there. Since the next remainder will bear the same relationship to the first remainder as the first did to our original 1, the following 1 in the quotient will be three places to the right of our second 1. Since this can be carried on indefinitely, it appears that $\frac{1}{2}$ expressed in the Tau System is .01001001001...... (any "doubting Thomases" may carry it out a few places to see).

Before we go on to other fractions, it would be wise to mention something about 1 in the Tau System. As you can easily see, 1=.11=.1011=.1011=.101011=.10101011 etc. It is, therefore, equal to the endless "fraction" .10101010.... (just as in the decimal system 1=.99999999....). If we can now take this fraction and expand the leftmost 1, and then expand the 1 in the r^{-3} column, so as to prevent the occurrence of two 1's in the same column, and then expand the 1 in the r^{-5} column so that there are not two 1's in that column, etc., we will get .01111111.... If, on the other hand, we start by expanding 1's in other columns, we get: .100111111...., .10100111111....., 101010011111....., etc. Therefore, if you multiply .01001001001.... $(\frac{1}{2})$ by 10.01 (2) and get .1001111111...., this does not mean that $2 \times \frac{1}{2} = a$ fraction, but merely shows a different way of representing 1.

To get back to fractions, we can make a list of them just as we did of integers before:

- 1/2 = .010010010010.....
- 1/3 = .00101000001010000101000.....
- 1/4 = .001000001000001000001000.....

Of course, finding these fractions is immeasurably harder than finding $\frac{1}{2}$, and with 1/10 I had to work it out 5 or 10 times before I got the correct answer, as there is much room for error.

By the way, no fraction can be terminating in this system, since that would mean that it could be expressed as the sum of a group of integral powers of Tau. Since all the powers of Tau can be expressed as the sum of an integer and an integral multiple of Tau; if the integral multiples "cancel" (e.g. $r^{-1}+r^{-3}+r^{-4}=1r-1+2r-3-3r+5=1$) the result will be an integer, and if they don't (e.g. 2r-3-3r+5=r+2), it will naturally be irrational. However, when we have an endless series, this paradox is detoured by admitting the fact that Lim AT+B with A and B always integral can be a rational fraction if A and $B \rightarrow \infty$.

The Tau System has a good many other interesting and unusual characteristics, and investigation by the readers of some, such as the frequency, occurrence, and nature of numbers with a 1 in the units column (when in simplest form) might prove interesting. I do not know of any useful application for systems such as this, except as a mental exercise and pastime, though it may be of some service in algebraic number theory. For instance, the numbers expressible in the Tau System in terminating form consist of all the algebraic integers in $R(\sqrt{5})$, and some of the properties of numbers in this and other systems might correspond to facts about associated fields.

Definitions Invented for Work in the Tau System

- *Expand:* alter three successive figures of a number by changing ...100... to ...011... The result is the same in value as the original, because of rule (1). This does *not* mean change zeros to ones and ones to zeros; just this specific change.
- Simplify: the reverse of expand; alter the figures thus: change ...011... to ...100... One speaks of simplifying the 1's in the r^{n-1} and r^{n-2} columns into the r^n column. Also, one speaks of expanding the 1 in

the τ^n column into the τ^{n-1} and τ^{n-2} columns.

- Simplest form: that form of a number which has been simplified until no more simplification is possible. It therefore has no two 1's in succession. It also has the fewest 1's and is the easiest form to work with.
- Columns: just as in our decimal system we speak of a units column, a ten's column, a hundred's column, a tenth's column, etc., in the Tau system we speak of a units column, a r^1 column, a r^{-1} column, etc.

Pair: two 1's in succession.

Cancellation: a change of the form

1		1
-1	=	-1

Set: in division, two or more 1's arranged with the same spacing as the 1's in the divisor (regardless of intervening 1's). A set can be "exchanged" for a 1 in the quotient.

¹also true of $-1/\tau$; there are other numbers which have similar properties, e.g. there is a number S between 1 and 2 for which $S^3 = S^2 + S + 1$. Ed. ²because $\tau^{-1} + \tau^0 = \tau$.

³this is the basic property defining the Fibonacci Series.

⁴There is also a more complex proof which involves multiplying the expressions like $2\tau + 1$ by τ , giving $2\tau^2 + \tau$, and expanding, τ^2 into $\tau + 1$. ⁵This is a restatement of $\tau^n = \tau^{n-1} + \tau^{n-2}$.

⁶By changing the 1 in the τ column to 1.1.

⁷If you come across words (like "expand") used in an unfamiliar way, look for them in the list of definitions at the end of this article. I have put there all words which I have had to invent or alter for use in this system, so as not to break up the text by explaining them.

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