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On lattices and their ideal lattices, and posets and their ideal posets

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Abstract

For P a poset or lattice, let $\mathrm{Id}(P)$ denote the poset, respectively, lattice, of upward directed downsets in P, including the empty set, and let $\mathrm{id}(P) = \mathrm{Id}(P) - \{\varnothing\}$. This note obtains various results to the effect that $\mathrm{Id}(P)$ is always, and $\mathrm{id}(P)$ often, "essentially larger" than P. In the first vein, we find that a poset P admits no $<$-respecting map (and so in particular, no one-to-one isotone map) from $\mathrm{Id}(P)$ into P, and, going the other way, that an upper semilattice P admits no semilattice homomorphism from any subsemilattice of itself onto $\mathrm{Id}(P)$.

The slightly smaller object $\mathrm{id}(P)$ is known to be isomorphic to P if and only if P has ascending chain condition. This result is strengthened to say that the only posets P_0 such that for every natural number n there exists a poset P_n with $\mathrm{id}^n(P_n) \cong P_0$ are those having ascending chain condition. On the other hand, a wide class of cases is noted where $\mathrm{id}(P)$ is embeddable in P.

Counterexamples are given to many variants of the statements proved.

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1 Definitions.

Recall that a poset P is said to be *upward directed* if every pair of elements of P is majorized by some common element, and that a *downset*

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in P means a subset d such that $x \leq y \in d \implies x \in d$. The downset generated by a subset $X \subseteq P$ will be written $P \downarrow X = \{y \in P \mid (\exists x \in X)\ y \leq x\}$. A *principal* downset means a set of the form $P \downarrow \{x\}$ for some $x \in P$.

Definition 1.1. If P is a poset, an *ideal* of P will mean a (possibly empty) upward directed downset in P. The set of all ideals of P, partially ordered by inclusion, will be denoted $\mathrm{Id}(P)$, while we shall write $\mathrm{id}(P)$ for the subset of *nonempty* ideals.

The subposet of $\mathrm{Id}(P)$, respectively $\mathrm{id}(P)$, consisting of ideals generated by chains, respectively, nonempty chains, will be denoted $\text{ch-Id}(P)$, respectively $\text{ch-id}(P)$.

This use of the term "ideal" is common in lattice theory, where an ideal of a lattice L can be characterized as a subset (often required to be nonempty) that is closed under internal joins, and under meets with arbitrary elements of L. For general posets, "ideal" is used in some works, such as [6], with the same meaning as here; in others, such as [13], "(order) ideal" simply means downset, while in still others, such as [8] and [5], an "(order) ideal" means a *Frink ideal*, which can be characterized as a directed union of intersections of principal downsets. (We shall not consider Frink ideals here. In upper semilattices, they are the same as our ideals. For a general study of classes of downsets in posets, see [5].)

If S is an upper semilattice, its ideals are the closed sets with respect to a closure operator, so $\mathrm{Id}(S)$ is a complete lattice.

If L is a lattice (or a downward directed upper semilattice), $\mathrm{id}(L)$ is a sublattice of $\mathrm{Id}(L)$, though not a complete one unless L has a least element. For S any upper semilattice, $\mathrm{id}(S)$ at least forms an upper subsemilattice of the lattice $\mathrm{Id}(S)$.

In a poset P, the principal downsets (which we can now also call the principal ideals) form a poset isomorphic to P. If P has ascending chain condition, we see that every nonempty ideal is principal, so $\mathrm{id}(P) \cong P$. (This yields easy examples where $\mathrm{Id}(P)$ is neither an upper nor a lower semilattice.)

The operators ch-Id and ch-id are not as nicely behaved as Id and id. Even for L a lattice, $\text{ch-Id}(L)$ need be neither an upper nor a lower semilattice. For instance, regarding ω and ω_1 (the first infinite and the first uncountable ordinals), with their standard total orderings, as lattices, let L be the direct product lattice $(\omega+1)\times(\omega_1+1)$. (Recall that each ordinal is taken to be the set of all lower ordinals. Thus, $\omega+1=\omega\cup\{\omega\}$ and $\omega_1+1=\omega_1\cup\{\omega_1\}$.) Then the chains $x_0=\omega\times\{0\}$ and $x_1=\{0\}\times\omega_1$ belong to $\text{ch-Id}(L)$, but their join in $\mathrm{Id}(L)$, namely $\omega\times\omega_1$, has no cofinal subchain (because ω and ω_1 have different cofinalities), hence does not lie in ch-Id(L). Indeed, x_0 and x_1 have no least upper bound in ch-Id(L), since their two common upper bounds $y_0 = L\downarrow(\omega\times\{\omega_1\})$ and $y_1 = L\downarrow(\{\omega\}\times\omega_1)$ intersect in the non-chain-generated ideal $\omega\times\omega_1$. One also sees from this that the latter two elements, y_0 and y_1, have no greatest lower bound in ch-Id(L).

Why are we considering these badly behaved operators? Because the method of proof of our first result involves, not merely an ideal, but an ideal generated by a chain, and it seemed worthwhile to formulate the result so as to capture the consequences of this fact. In a final section, §5, I shall note some variants of these constructions that are better behaved.

We will also use

Definition 1.2. A map $f: P\to Q$ of posets will be called *strictly isotone* if $x<y$ in P implies $f(x)<f(y)$ in Q.

Thus, the strictly isotone maps include the embeddings of posets, and so in particular, the embeddings of lattices and of upper semilattices.

2 Nonembeddability results.

S. Todorcevic has pointed out to me that my first result, Theorem 2.1 below, is a weakened version of an old result of Đ. Kurepa [11], which says that the poset of well-ordered chains in any poset P, ordered by the relation of one chain being an initial segment of another, cannot be mapped into P by a strictly isotone map. (A still stronger version appears in Todorcevic [15].) One could say that the one contribution of Theorem 2.1, relative to Kurepa's result, is that by weakening this assertion about chains to one about the downsets they generate, it gives us a statement about ideals of P.

All the versions of this result have essentially the same proof; it is short and neat, so I include it. I give the result for posets; that statement implies, of course, the corresponding statements for semilattices and lattices.

Theorem 2.1 (cf. [11, Théorème 1], [15, Theorem 5])**.** Let P be any poset. Then there exists no strictly isotone map $f:\text{ch-Id}(P)\to P$. Hence (weakening this statement in two ways) there exists no embedding of posets $f:\mathrm{Id}(P)\to P$.

Proof. Suppose $f:\text{ch-Id}(P)\to P$ is strictly isotone. Let us construct a chain of elements $x_\alpha\in P$, where α ranges over all ordinals of cardinality $\leq \mathrm{card}(P)$ (i.e., over the successor cardinal to $\mathrm{card}(P)$), by the single recursive rule

(1.1) $\quad x_\alpha \;=\; f(P\downarrow\{x_\beta \mid \beta<\alpha\}).$

Given α, and assuming recursively that $\beta\mapsto x_\beta$ $(\beta<\alpha)$ is a strictly isotone map $\alpha\to P$, we see that $P\downarrow\{x_\beta\mid\beta<\alpha\}$ is a member of ch-Id(P), so (1.1) makes sense. We also see that for all $\beta < \alpha$, the chain in P occurring in the definition of x_β is a proper initial segment of the chain in the definition of x_α; so the strict isotonicity of f insures that $x_\beta < x_\alpha$, and our recursive assumption carries over to $\alpha + 1$. It is also clear that if that assumption holds for all β less than a limit ordinal α, it holds for α as well.

This construction thus yields a chain of cardinality $> \mathrm{card}(P)$ in P, a contradiction, completing the proof. Q.E.D.

In the above proof, we restricted (1.1) to ordinals α of cardinality $\leq \mathrm{card}(P)$ only so as to have a genuine set over which to do recursion. The reader comfortable with recursion on the proper class of all ordinals can drop that restriction, ending the proof with an all the more egregious contradiction.

Theorem 2.1 is reminiscent of Cantor's result that the power set of a set X always has larger cardinality than X. (Cf. the title of [9], where a similar result is proved with the poset of all downsets in place of the smaller poset of ideals.) In some cases, for instance when P is the chain of rational numbers, Id(P) in fact has larger cardinality than P; but in others, for instance when P is the chain of integers, or of reals, it does not. For the latter case, one can verify by induction that for every natural number n, the result of iterating this construction n times, $\mathrm{Id}^n(\mathbb{R})$, may be described as the chain gotten by taking $\mathbb{R} \times (n+1)$, lexicographically ordered, and attaching an extra copy of the chain n to each end. So the above theorem yields the curious fact that the chain so obtained using a larger value of n can never be embedded in the chain obtained using a smaller value. (The copies of n at the top and bottom are irrelevant to this fact, since by embedding $\mathbb{R}$ in, say, the interval $(0,1)$, one can get an identification of $\mathrm{Id}^n(\mathbb{R})$ with a "small" piece of itself, hence in particular, an embedding into itself minus those add-ons.)

Since the proof of Theorem 2.1 is based on constructing chains, one may ask whether ch-Id(P) always contains a chain that cannot be embedded in P. That is not so; to see this, let us form a disjoint union of chains of finite lengths $1, 2, 3, \ldots,$ with no order-relations between elements of different chains, and – to make our example not only a poset but a lattice – throw in a top element and a bottom element. The resulting lattice L has ascending chain condition, hence Id(L), and so also ch-Id(L), consists of the principal ideals and the empty ideal; in other words Id(L) = ch-Id(L) is, up to isomorphism, the lattice obtained by attaching one new element to the bottom of L. Hence, like L, it has chains of all natural number lengths and no more, though as Theorem 2.1 shows (and a little experimenting confirms), it cannot be mapped into L by any strictly isotone map.

In contrast to what Theorem 2.1 says about Id(P), we noted in §1 that

$\mathrm{id}(P)$ is canonically isomorphic to P whenever the latter has ascending chain condition. D. Higgs [10], answering a question of G. Grätzer, showed for lattices L that it is *only* in this case that $\mathrm{id}(L)$ can be isomorphic in *any* way to L, and M. Erné [6] (inter alia) generalized this statement to arbitrary posets. But our next result, extending the trick of the preceding paragraph, shows that the class of lattices L such that $\mathrm{id}(L)$ can be *embedded* as a lattice in L (and hence the class of posets P such that $\mathrm{id}(P)$ can be embedded as a poset in P) is much larger.

Proposition 2.2. Every lattice L is embeddable as a lattice in a lattice L' such that $\mathrm{id}(L')$ is embeddable as a lattice in L'.

Hence the same is true with "lattice" everywhere replaced by "upper semilattice" or "poset".

Proof. Without loss of generality, assume L nonempty. Let L' be the poset obtained by taking the disjoint union of the lattices L, $\mathrm{id}(L)$, $\dots$, $\mathrm{id}^n(L)$, $\dots$ $(n \in \omega)$, with no order-relations among elements of distinct pieces, and then throwing in a top element 1 and a bottom element 0. It is easy to see that L' is a lattice, and that every nonempty ideal of L', other than L' and $\{0\}$, contains elements of the sublattice $\mathrm{id}^n(L)$ for one and only one value of n. For each n, the ideals of this sort containing elements of $\mathrm{id}^n(L)$ form a sublattice of $\mathrm{id}(L')$ isomorphic to $\mathrm{id}(\mathrm{id}^n(L)) = \mathrm{id}^{n+1}(L)$. One sees from this that $\mathrm{id}(L')$ is isomorphic to the sublattice of L' obtained by deleting the original copy of L.

This proves the assertion about lattices. The corresponding statements for upper semilattices and for posets follow, since every semilattice or poset can be embedded as a subsemilattice or subposet in a lattice; e.g., in its lattice of ideals in the former case, in its lattice of downsets in the latter. (In fact, there exist embeddings preserving all least upper bounds and greatest lower bounds that exist in the given structures: [12], [4, Theorem V.21].)

Q.E.D.

On the other hand, there are many posets P for which we can deduce from Theorem 2.1 the nonembeddability of $\mathrm{id}(P)$ in P.

Corollary 2.3 (to Theorem 2.1)**.** Suppose P is a poset which admits a strictly isotone map g into a nonmaximal principal up-set within itself, i.e., into $P \uparrow x$ for some nonminimal $x \in P$. Then there exists no strictly isotone map $f : \mathrm{id}(P) \to P$. In particular for P the lattice of all subsets of an infinite set, or of all equivalence relations on an infinite set, there is no such f.

Proof. By assumption we have a strictly isotone map $g : P \to P \uparrow x$, where x is not minimal. Take $y < x$ in P. If there existed a strictly isotone map

$f: \mathrm{id}(P) \to P$, then gf would be another such map, with image consisting of elements $> y$. Hence we could extend it to $\mathrm{Id}(P)$ by sending $\varnothing$ to y, contradicting Theorem 2.1. This proves our general assertion.

If X is an infinite set, take distinct elements $x_0, x_1 \in X$. Then the lattice of all subsets of X is isomorphic to its sublattice consisting of subsets that contain x_0, and the lattice of equivalence relations is isomorphic to its sublattice of equivalence relations that identify x_0 with x_1. Thus, both lattices satisfy the hypothesis of our main assertion, giving the final statement. Q.E.D.

F. Wehrung [16] shows that the lattice L of equivalence relations on a set of infinite cardinality κ contains a coproduct of two copies of itself (and hence, by results of [3], a coproduct of 2^κ copies of itself). His proof uses the description of L, up to isomorphism, as $\mathrm{id}(L_{\mathrm{fin}})$, where $L_{\mathrm{fin}} \subseteq L$ is the sublattice of finitely generated equivalence relations. This led me to wonder whether L might also contain a copy of $\mathrm{id}(L)$, and so initiated the present investigation. The above corollary answers that question in the negative.

3 Nonexistence of surjections.

Another version of the idea that a lattice L is essentially smaller than its ideal lattice would be to say that there are no surjective homomorphisms $L \to \mathrm{Id}(L)$. The next theorem shows that this is true. We again get the result for a wider class of objects than lattices, in this case upper semilattices. We shall see that the result does not extend to general posets or isotone maps, nor can we replace ideals by chain-generated ideals; in these ways it is of a weaker sort than Theorem 2.1. On the other hand, it is stronger in a different way.

Theorem 3.1. Let S be an upper semilattice. Then there exists no upper semilattice homomorphism from any *subsemilattice* $S_0 \subseteq S$ onto $\mathrm{Id}(S)$.

Proof. Suppose $f : S_0 \to \mathrm{Id}(S)$ were such a surjective homomorphism. Then we could map $\mathrm{Id}(\mathrm{Id}(S))$ to $\mathrm{Id}(S)$ by taking each $I \in \mathrm{Id}(\mathrm{Id}(S))$ to $S \downarrow f^{-1}(I)$. Because f is onto, distinct ideals I of $\mathrm{Id}(\mathrm{Id}(S))$ yield distinct ideals $f^{-1}(I)$ of S_0, and these will generate distinct ideals of S. This leads to an embedding $\mathrm{Id}(\mathrm{Id}(S)) \to \mathrm{Id}(S)$ as posets, contradicting Theorem 2.1. Q.E.D.

We cannot replace the semilattice S and semilattice homomorphism f in Theorem 3.1 by a poset and an isotone map, because the inverse image of an ideal under an isotone map f need not be an ideal. Indeed, we can get a counterexample to the resulting statement in which the given poset is a lattice L, and f is a strictly isotone bijection $L \to \mathrm{Id}(L)$: Let L consist of a greatest element 1, a least element 0, and countably many mutually incomparable elements a_n $(n \in \omega)$ between them, and let f act by

$$(1.2)\quad f(1) = L, \quad f(a_{n+1}) = \{a_n, 0\}\ (n \in \omega), \quad f(a_0) = \{0\}, \quad f(0) = \varnothing.$$

(If we try to apply the construction in the proof of Theorem 2.1 to the map $S \downarrow f^{-1}(-)$ from $\mathrm{Id}(\mathrm{Id}(S))$ to subsets of S, the values we get for x_0, x_1, x_2, x_3 are respectively $\varnothing$, $\{0\}$, $\{a_0, 0\}$, and $\{a_1, a_0, 0\}$, of which the last is not an ideal, so the construction cannot be continued further.)

We could, of course, get a version of Theorem 3.1 for posets by restricting our morphisms to isotone maps under which inverse images of ideals are ideals.

Alternatively, we can escape these difficulties if we are willing to replace ideals by downsets, getting the first sentence of the next result. But in fact, we can deduce using Theorem 3.1 a stronger statement, the second sentence.

Corollary 3.2. No isotone map from a subset P_0 of a poset P to the lattice $\mathrm{Down}(P)$ of all downsets of P is surjective.

In fact, no isotone map f from a poset P_0 to any upper semilattice T containing $\mathrm{Down}(P_0)$ as a subsemilattice has the property that $f(P_0)$ generates T as an upper semilattice.

Sketch of proof. Clearly the first assertion is a case of the second. To prove the latter, let us, for any poset P, write $\mathrm{fdown}(P)$ for the upper semilattice of finite nonempty unions of principal downsets of P. Then one can verify that

$$(1.3)\quad \mathrm{fdown}(P) \;\cong\; \text{upper semilattice freely generated by the poset } P.$$

$$(1.4)\quad \mathrm{Down}(P) \;\cong\; \mathrm{Id}(\mathrm{fdown}(P)).$$

Now given a poset P_0 and an upper semilattice T containing $\mathrm{Down}(P_0)$, we see from (1.3) with P_0 for P that isotone maps $f : P_0 \to T$ such that $f(P_0)$ generates the semilattice T are equivalent to surjective semilattice homomorphisms $f' : \mathrm{fdown}(P_0) \to T$. Hence, given such a map f, if T contains $\mathrm{Down}(P_0)$ as a subsemilattice, then the inverse image under f' of that subsemilattice will be a subsemilattice of $\mathrm{fdown}(P_0)$ which f' maps surjectively to $\mathrm{Down}(P_0) \cong \mathrm{Id}(\mathrm{fdown}(P_0))$. But this is impossible, by Theorem 3.1 with $S = \mathrm{fdown}(P_0)$. Q.E.D.

We mentioned that one cannot replace $\mathrm{Id}(S)$ by $\text{ch-Id}(S)$ in Theorem 3.1. Indeed, even if we bypass the problem that $\text{ch-Id}(S)$ is not in general an upper semilattice, by restricting ourselves to cases where it is, the proof of that theorem fails because f^{-1} of an ideal generated by a chain need not be generated by a chain. Here is a counterexample to that generalization of the theorem.

Lemma 3.3. Let κ be an infinite cardinal, and S the lattice of all finite subsets of κ. Then ch-Id(S) forms a lattice, and if $\kappa = \lambda^{\aleph_0}$ for some cardinal λ, then ch-Id(S) is a homomorphic image of S as an upper semilattice.

Sketch of proof. Note that as an upper semilattice with 0, S is free on κ generators, and that it has no uncountable chains. From the latter fact one can verify that ch-id(S) is isomorphic to the poset of all countable subsets of κ, which is again a lattice, and has cardinality $\kappa^{\aleph_0}$. Hence ch-Id(S) is also a lattice of that cardinality. If $\kappa = \lambda^{\aleph_0}$, then $\kappa^{\aleph_0} = \kappa$, so as an upper semilattice, ch-Id(S) is a homomorphic image of the free upper-semilattice-with-0 on κ generators, namely S. Q.E.D.

But I do not know whether, if L is a lattice such that ch-Id(L) is again a lattice, the latter can ever be a *lattice-theoretic* homomorphic image of L, or of a sublattice thereof.

As another way of tweaking our results, we might go back to Theorem 2.1, and try replacing P on the right side of the map f by an isotone or (if P is a lattice or upper semilattice) a lattice- or semilattice-theoretic homomorphic image of P – the dual of our use of a subsemilattice S_0 on the left-hand side of the map in Theorem 3.1. It turns out that the sort of statements one can express in this way are weakened versions of statements of the sort exemplified by Theorem 3.1. For to embed an algebraic structure A in a homomorphic image of a structure B is equivalent to giving an isomorphism between A and a subalgebra of that homomorphic image of B; and the subalgebras of homomorphic images of B are a subclass of the homomorphic images of subalgebras of B, so we end up looking at homomorphisms from subalgebras of B onto A, as in Theorem 3.1.

So, for instance, it follows from Theorem 3.1 that if we restrict Theorem 2.1 to semilattices S and semilattice homomorphisms, and replace ch-Id with Id, then we *can* replace the codomain S of our map by an arbitrary semilattice homomorphic image of S. In the opposite direction, Lemma 3.3 shows that if we *keep* the operator ch-Id in Theorem 2.1, and again assume P and ch-Id(P) to be semilattices and restrict f to be a semilattice homomorphism, we cannot replace the codomain by such an image of itself. (In this case, the distinction between "subalgebra of a homomorphic image" and "homomorphic image of a subalgebra" makes no difference, for two reasons: semilattices have the Congruence Extension Property, and in that example, the subalgebra was the whole semilattice anyway. So our counterexample to the statement modeled on Theorem 3.1 is indeed a counterexample to what would otherwise be the weaker statement modeled on Theorem 2.1.)

For posets, one has many possible variants of our results, because of the many sorts of poset maps one can define. E.g., we found it natural to prove Theorem 2.1 for strictly isotone (but not necessarily one-to-one) maps; while the authors of [9] show that no poset P admits a *one-to-one* map $\mathrm{Down}(P) \to P$ that is *either* $\leq$-preserving (i.e., isotone), *or* $\not\leq$-preserving. By Lemma 3.3, one cannot, in Theorem 2.1, replace the codomain poset P by a general isotone image of itself; but such a result might be true for images of other sorts.

4 $P_0 \cong \ldots \cong \mathrm{id}^n(P_n) \cong \ldots$ can only happen "in the obvious way".

We have mentioned that by Erné's generalization [6] of a result of Higgs [10], the only posets P admitting any isomorphism with $\mathrm{id}(P)$ are those for which the canonical embedding $P \to \mathrm{id}(P)$ is an isomorphism, namely the posets with ascending chain condition. We prove below a further generalization of this statement. Rather than assuming an isomorphism between P and its *own* ideal poset, we shall see that it suffices to assume P simultaneously isomorphic to an ideal-poset $\mathrm{id}(P_1)$, a double ideal-poset $\mathrm{id}^2(P_2)$, and generally to an n-fold ideal-poset $\mathrm{id}^n(P_n)$ for each n. I will give two proofs: one based on the ideas of Higgs' and Erné's proofs, and one that obtains the result from Erné's (via a version of the trick of Proposition 2.2 above).

First, some terminology and notation. Generalizing slightly the language of [7], let us call an element x of a poset P *compact* if for every directed subset $S \subseteq P$ which has a least upper bound $\bigvee S$ in P, and such that $\bigvee S \geq x$, there is some $y \in S$ which already majorizes x. For P any poset, the compact elements of $\mathrm{id}(P)$ are the principal ideals. (These are clearly compact, while a nonprincipal ideal is the join of the directed system S of its principal, hence proper, subideals.) Thus, defining $d_P : P \to \mathrm{id}(P)$ by

$$(1.5) \quad d_P(x) \;=\; P \downarrow \{x\},$$

the map d_P is an isomorphism between P and the poset of compact elements of $\mathrm{id}(P)$. Since the set of compact elements of a poset is determined by the order structure of the poset, this shows that P and the map $d_P : P \to \mathrm{id}(P)$, are recoverable, up to isomorphism, from the order structure of $\mathrm{id}(P)$.

Lemma 4.1. Let us call the compact elements of a poset P the 1-*compact* elements, and inductively define the n-*compact* elements of P to be the elements of the subposet of $n{-}1$-compact elements that are compact in that subposet. Then in a poset of the form $\mathrm{id}^n(P)$ where $n > 1$, every non-compact element a_0 yields a chain

$$(1.6) \quad a_0 \;<\; a_1 \;<\; \ldots \;<\; a_{n-1},$$

where for $i = 1, \ldots, n-1$, a_i is the least i-compact element of $\mathrm{id}^n(P)$ majorizing a_{i-1}.

Proof. From our preceding observations, we see that the 1-compact elements of $\mathrm{id}^n(P)$ are, in the notation of (1.5), the members of $d_{\mathrm{id}^{n-1}(P)}(\mathrm{id}^{n-1}(P))$, the 2-compact elements are the members of $d_{\mathrm{id}^{n-1}(P)} d_{\mathrm{id}^{n-2}(P)}(\mathrm{id}^{n-2}(P))$, and so on, through the n-compact elements, which are the members of $d_{\mathrm{id}^{n-1}(P)} \ldots d_{\mathrm{id}(P)} d_P(P)$.

Note also that for any poset Q, if I is a nonprincipal ideal of $\mathrm{id}(Q)$, then $d_Q^{-1}(I)$ i.e., $\{x \in Q \mid d_Q(x) \in I\}$, must be a nonprincipal ideal of Q (though the converse is not true). Moreover, that ideal, regarded as a member of $\mathrm{id}(Q)$, will majorize all members of I, and will be the least element that does so; hence in $\mathrm{id}^2(Q)$, the element $d_{\mathrm{id}(Q)} d_Q^{-1}(I)$ will be the least compact element majorizing the noncompact element I. Thus, in $\mathrm{id}^2(Q)$, every *noncompact* element has a least compact element majorizing it, and that *compact* element is again *noncompact* within the subposet of compact elements.

Hence in the situation of the lemma, where a_0 is a *noncompact* element of an n-fold ideal poset $\mathrm{id}^n(P)$, we have a least compact element a_1 majorizing it, which is the image under $d_{\mathrm{id}^{n-1}(P)}$ of a *noncompact* element of $\mathrm{id}^{n-1}(P)$, for which we can repeat the argument if $n-1 > 1$, giving the desired chain (1.6). Q.E.D.

Theorem 4.2 (cf. [10], [6]). Suppose P_0 is a poset such that for each natural number n there exists a poset P_n with $P_0 \cong \mathrm{id}^n(P_n)$. Then P_0 has ascending chain condition.

Proof 1. For notational simplicity, let us assume without loss of generality that $P_0 = \mathrm{id}(P_1)$. If P_0 does not have ascending chain condition, then the poset P_1 clearly cannot have ascending chain condition either; hence it has a nonprincipal ideal, hence by Zorn's Lemma we can find a maximal nonprincipal ideal, so P_0 will have a maximal noncompact element a_0. Applying the preceding lemma for all positive integers n, we get an infinite chain

(1.7) $\quad a_0 < a_1 < \ldots < a_n < \ldots .$

These form an infinite chain of ideals of P_1 above a_0, and the union of this chain will be a nonprincipal ideal strictly larger than a_0, contradicting the assumed maximality. Q.E.D.

Proof 2. By the observations at the beginning of the proof of Lemma 4.1, for each $n > 0$ the posets of $n-1$-compact elements and of n-compact elements of $P_0 \cong \mathrm{id}^n(P_n)$ are isomorphic respectively to $\mathrm{id}(P_n)$ and to

P_n; comparing these statements for two successive values of n, we conclude that $\mathrm{id}(P_n) \cong P_{n-1}$. This suggests that we extend the sequence of posets P_n to allow negative subscripts by writing $\mathrm{id}^n(P_0) = P_{-n}$. Now let Q be the disjoint union $\coprod_{n\in\mathbb{Z}} P_n$, where elements from distinct posets P_n are taken to be incomparable. No ideal of Q can contain elements of more than one of the P_n, hence

$$\mathrm{id}(Q) \ = \ \textstyle\coprod_{n\in\mathbb{Z}} \mathrm{id}(P_n) \ \cong \ \coprod_{n\in\mathbb{Z}} P_{n-1} \ \cong \ \coprod_{m\in\mathbb{Z}} P_m \ = \ Q.$$

Hence by Erné's result, Q has ascending chain condition; hence so does $P_0 \subseteq Q$. Q.E.D.

It is interesting to compare the situation of the preceding theorem with what we get if we start with any poset P with a nonprincipal ideal I, and consider the posets

(1.8) $\quad P \ \to \ \mathrm{id}(P) \ \to \ \mathrm{id}^2(P) \ \to \ \dots \ ,$

with connecting maps $d_{\mathrm{id}^{n-1}(P)} : \mathrm{id}^{n-1}(P) \to \mathrm{id}^n(P)$. Here I can be regarded as an element $b_1 \in \mathrm{id}(P)$, which is the least upper bound therein of the set $d_P(I)$. On the other hand, the ideal of $\mathrm{id}(P)$ generated by that set, since it consists of elements $< b_1$, can be regarded as an element $b_2 \in \mathrm{id}^2(P)$ which is $< d_{\mathrm{id}(P)}(b_1)$; this element in turn will strictly majorize all elements of $d_{\mathrm{id}(P)} d_P(I)$, and so the ideal generated by that set in $\mathrm{id}^2(P)$ will be an element of $\mathrm{id}^3(P)$ which is $< d_{\mathrm{id}^2(P)}(b_2)$; and so on. Letting P_∞ denote the direct limit of (1.8), and writing $d_{\infty,n} : \mathrm{id}^n(P) \to P_\infty$ for the induced maps to that object, we get a *descending* chain $d_{\infty,1}(b_1) > d_{\infty,2}(b_2) > \dots$ above the set $d_{\infty,0}(I)$ in P_∞. On the other hand, if we stop after n steps, and consider the chain $d_{\mathrm{id}^n(P)} \dots d_{\mathrm{id}(P)}(b_1) > d_{\mathrm{id}^n(P)} \dots d_{\mathrm{id}^2(P)}(b_2) > \dots > b_n$, this is essentially the finite chain described in Lemma 4.1, used there in building up the *ascending* chain (1.7).

I don't know whether the analog of Theorem 4.2 with ch-id in place of id is true. (This seems related to the problem stated at the end of [6].) The natural approach to adapting the above argument to that case would start by defining an element x of a poset to be *chain-compact* if every chain S having a least upper bound $\bigvee S$ which majorizes x contains an element s that already does so. However, it turns out that elements of $d_P(P) \subseteq \text{ch-id}(P)$ are not necessarily chain-compact: If, slightly modifying the example by which we showed in §1 that ch-Id of a lattice need not be a lattice, we let $P = (\omega \times \omega_1) \cup \{(\omega, \omega_1)\}$, i.e., that original example, with the chains $\omega \times \{\omega_1\}$ and $\{\omega\} \times \omega_1$ deleted, but the top element (ω, ω_1) retained, we find that $\text{ch-id}(P)$ can be identified with $(\omega+1) \times (\omega_1+1)$, in which the image of that top element is the least upper bound of each of the chains $\omega \times \{\omega_1\}$ and $\{\omega\} \times \omega_1$, hence not chain-compact (though, in fact, it was chain-compact in P). This is related to the fact that the inverse image under d_P of the ideal generated by either of these chains is a non-chain-generated ideal of P. One encounters similar phenomena on taking for P the poset of finite subsets of a set of cardinality $\aleph_1$, together with the improper subset.

These examples used uncountable chains; might the analog of Theorem 4.2 hold with id replaced by the operator taking P to its poset of nonempty ideals generated by countable chains; equivalently, nonempty ideals with countable cofinal subsets? Example 3 of [6] shows that this, too, fails: the poset P of that example, the totally ordered set ω_1, is easily seen to be isomorphic its own poset of countably generated ideals, equivalently, bounded ideals (whether or not we include the empty ideal). The reason Proof 1 of Theorem 4.2 fails to give a contradiction in this case lies not in the phenomena sketched above (indeed, the inverse image in ω_1 of a bounded ideal of $\text{ch-id}(\omega_1)$ will again be a bounded ideal), but in the fact that Zorn's lemma cannot produce a *maximal* bounded ideal.

A question suggested by juxtaposing the present considerations with those of [2, §7] is: What can be said about lattices L such that $\text{id}(L)$ is (not necessarily equal to, but at least) *finitely generated over* its sublattice $d_L(L)$; and similarly for upper semilattices? (In these questions it makes no difference whether we refer to $\text{id}(L)$ or $\text{Id}(L)$.)

Since dropping the bottom element $\varnothing$ from $\text{Id}(P)$ makes such a difference in the properties we have studied, it might be interesting to investigate the effect on these questions of dropping the top element, P, of $\text{id}(P)$ or $\text{Id}(P)$ if P is a directed poset (e.g., a lattice or semilattice); or of adding an extra top element; though these constructions are admittedly less natural than that of dropping $\varnothing$. One might also investigate the variants of some of the questions we have considered that one gets by using the opposite structures, $\text{Id}(P)^{\text{op}}$ etc., in place of $\text{Id}(P)$ etc..

I will mention one other interesting result on the relation between L and $\text{id}(L)$ for any lattice L: It is shown in [1] that $\text{id}(L)$ is a homomorphic image of a sublattice of an ultrapower of L.

5 Tangential note on chains and products of chains.

We observed in §1 that for L a lattice, the poset $\text{ch-Id}(L)$ of ideals of L generated by chains need neither be an upper nor a lower semilattice; our counterexample was based on the fact that a direct product of two chains of distinct infinite cofinalities has no cofinal subchain. Let us put this phenomenon in a more general light.

Lemma 5.1. Let X be a class of posets, and for any poset P, denote by $X\text{-Down}(P) \subseteq \text{Down}(P)$ the set of all downsets $d \subseteq P$ of the form

$d = P \downarrow f(Q)$, for $Q \in X$ and $f : Q \to P$ an isotone map. Then

(i) The following conditions on X are equivalent.

(i.a) X-Down$(P) \subseteq \mathrm{Id}(P)$ for all posets P.

(i.b) Every member of X is upward directed.

(ii) Among the following conditions on X, we have the implication (ii.a) $\Longrightarrow$ (ii.b), and, if the equivalent conditions of (i) above hold, also (ii.a) $\Longrightarrow$ (ii.c) $\wedge$ (ii.d).

(ii.a) For all $Q, Q' \in X$, there exists $R \in X$ which admits an isotone map to the product poset $Q \times Q'$, with cofinal image.

(ii.b) X-Down(S) is a lower subsemilattice of Down(S) for all lower semilattices S.

(ii.c) X-Down(S) is an upper subsemilattice of $\mathrm{Id}(S)$ for all upper semilattices S.

(ii.d) X-Down(L) is a sublattice of $\mathrm{Id}(L)$ for all lattices L.

Proof. Since an isotone image of an upward directed set is upward directed, and the downset generated by an upward directed set is an ideal, we clearly have (i.b) $\Longrightarrow$ (i.a). Conversely, if (i.b) fails, let $Q \in X$ not be upward directed. Then $Q = (Q \downarrow Q) \in X$-Down$(Q)$ is not an ideal, so (i.a) fails.

To get (ii), consider any $Q, Q' \in X$, any poset P, and any isotone maps $f : Q \to P$, $f' : Q' \to P$. If our P is a lower semilattice, then the intersection of downsets $(P \downarrow f(Q)) \cap (P \downarrow f'(Q'))$ can be described as $P \downarrow \{f(q) \wedge f'(q') \mid q \in Q,\ q' \in Q'\}$, while if P is an upper semilattice and $P \downarrow f(Q)$ and $P \downarrow f'(Q')$ are ideals, then their join in $\mathrm{Id}(P)$ can be described as $P \downarrow \{f(q) \vee f'(q') \mid q \in Q,\ q' \in Q'\}$. In these statements, note that the sets $\{f(q) \wedge f'(q') \mid q \in Q,\ q' \in Q'\}$, respectively $\{f(q) \vee f'(q') \mid q \in Q,\ q' \in Q'\}$, are isotone images of the poset $Q \times Q'$. Hence if X contains a poset R which admits an isotone map $g : R \to Q \times Q'$ with cofinal image, the composite of g with the above maps $Q \times Q' \to P$ are maps $R \to P$ whose images generate the indicated meet-downset and join-ideal respectively. This gives (ii.a) $\Longrightarrow$ (ii.b), and, assuming (i.a), also (ii.a) $\Longrightarrow$ (ii.c); together these give (ii.a) $\Longrightarrow$ (ii.d). Q.E.D.

To avoid awkward statements, I have not attempted to formulate if-and-only-if versions of the implications of (ii). That the converses to the present statements do not hold arises from the fact that on members of X, we are only assuming a poset structure, but we are mapping them into sets with lattice or semilattice structure. For instance, since in a lower semilattice S every downset is a connected poset, we see that the class C of all connected posets satisfies C-Down$(S) = $ Down(S); hence taking $X = C \cup \{Q\}$, where Q is the disconnected poset consisting of two incomparable elements, we find that X-Down(S) is still Down(S), so (ii.b) holds. But (ii.a) does not, since no member of X can be mapped into $Q \times Q$ so as to have cofinal image.

However, the above lemma shows why the choice for X of the class of all chains (or even the set consisting of the two chains ω and ω_1) can fail to have properties (ii.c) and (ii.d), and points to some variants that will have those properties. Any class of upward directed posets closed under taking pairwise products will satisfy (i.b) and (ii.a), and hence (ii.b)-(ii.d); in particular, this will be true of the class of all finite products of chains (cf. [14]). A singleton whose one member is a chain, Q, will also satisfy these properties, since the diagonal image of Q in $Q \times Q$ is cofinal. Both of these classes yield variants of the construction ch-Id that are, in this respect, better behaved than that construction.

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