Appendix 2. Some Set Theory.

Re §A2.1. Denumerable sets.

P.876, last line of Proof of Proposition A2.1.1 [=]: To prove that the map $n \mapsto k_n$ enumerates D, one shows by induction that for every n, $\{k_1, \ldots, k_n\}$ includes all members of D that are $\leq n$.

P.877, proof of Proposition A2.1.3 [>]: This proof uses the Axiom of Choice, which Lang considers so obvious it need not be mentioned, but which I discuss in the handout ''The Axiom of Choice, Zorn's Lemma, and all that'' to be read in connection with §A2.2. One could prove this proposition in a way that hides the fact that it requires this Axiom, using a sequence of single choices (each of which could be done without the Axiom), and an argument ending with ''and so forth'' (which implicitly means one is making infinitely many choices, which requires the Axiom). Lang, however, makes all the choices in one step, which is just how a well-stated proof using the Axiom of Choice should go. The reader not familiar with the Axiom of Choice need not worry about this now, since this is not a course in set theory, but should simply be alerted that a set-theoretic assumption we will discuss later has been used implicitly here.

At the point of this proof where Lang says ''By induction'', it would be more proper to say ''by recursion'': an inductive proof proves the n+1st of a sequence of *statements* from the nth; a recursive construction constructs the n+1st of a sequence of *objects* from the nth.

Re §A2.2. Zorn's Lemma.

P.879, lines 3 and 4 [=]: ''If the ordering satisfies this additional condition ... ''. Delete this sentence. (What is said unclearly here is said again clearly near the bottom of the same page.)

P.879, 2/3 down, definition of **induced** ordering [>]: Note that Examples 1 and 2 can be obtained from Example 3 by taking such induced orderings.

P.880, middle, to end of section on p.884 [<]: I recommend that you read my handout *The Axiom of Choice, Zorn's Lemma, and all that* for this material, then simply look over this section of Lang to familiarize yourselves with his viewpoint. The module-theoretic result stated on p.880, and proved there using Zorn's Lemma, is discussed in the notes on §I.7 in this Companion, so if you are reading this material before that section, don't worry about it yet.

Re §A2.3. Cardinal numbers.

P.885, first half of page [<]: This is a bit disorganized and repetitive, but wade through it and absorb the elementary facts noted.

P.886, line 9, ''Note first ... '' [=]: This is by Proposition A2.1.3 (p.877).

P.887, Corollary A2.3.4 [=]: The hypothesis ''*Let A be an infinite set*'' from the preceding theorem is still assumed.

P.891, end of §A2.3 [>]: Lang has introduced the symbol ''card(X)'', and given meaning to equalities and inequalities of these symbols, but has not defined any mathematical entities that the symbols represent. Set-theorists define certain sets called *cardinals*, which are canonically chosen ''sample sets'' of every cardinality, and given a set X, they define card(X) to be the unique cardinal with which X can be put into bijective correspondence. I often sketch the construction of the cardinals in class; it depends on another construction, that of the *ordinals*, which are representatives of the isomorphism classes of well-ordered sets. Though there is exactly one *finite* ordinal of each finite cardinality, in the infinite case, the correspondence becomes many-to-one, e.g., the *denumerable* ordinals include ω, $\omega+1$, ... , $\omega+\omega$, ... The cardinals are defined as those ordinals that are *least* among the ordinals of their cardinality. For a more detailed sketch, see Chapter 4 of my Math 245 notes in particular, §4.5. For a full development, see a text on set theory.

P.891, Theorem A2.3.10 and Corollary A2.3.11, [>].

Some inequalities for cardinals (optional for my 250A).

Consider the following inequalities which are true for nonnegative integers:

(c54) $a_i < b_i \quad \Rightarrow \quad \Sigma\, a_i < \Sigma\, b_i \qquad (a_i \geq 0;\ \ 0 \leq i < n;\ \ n \geq 1)$

(c55) $a_i < b_i \quad \Rightarrow \quad \Pi\, a_i < \Pi\, b_i \qquad (a_i \geq 1;\ \ 0 \leq i < n;\ \ n \geq 1)$,

(c56) $a_i \leq b_i \quad \Rightarrow \quad \Sigma\, a_i < \Pi\, b_i \qquad (a_i \geq 3;\ \ 0 \leq i < n;\ \ n \geq 2)$.

As noted in A2:L8, there are also natural arithmetic operations on cardinals – but one finds that none of (c54)-(c56) hold for possibly infinite sums and products of possibly infinite cardinals! A counterexample to (c54) can be seen from the fact that any infinite sum of 1's is equal to the corresponding infinite sum of 2's (or of $\aleph_0$'s), although $1 < 2$. A counterexample to (c55) is similarly gotten by comparing countable products of 2's and 4's. To see that (c56) fails, take all a_i's and b_i's to be a fixed infinite cardinal, and take the sum and product over a finite set. Having resigned outselves to these features of cardinal arithmetic, it comes as a surprise that the law

(c57) $\alpha_i < \beta_i \quad \Rightarrow \quad \Sigma\, \alpha_i < \Pi\, \beta_i$

holds for arbitrary (zero, positive-integer, or infinite) cardinals α_i and β_i, indexed by an arbitrary (empty, finite nonempty, or infinite) set I. The theorem and corollary that you have just read in Lang, saying that for any set I one has $\mathrm{card}(I) < \mathrm{card}(2^I)$, is the case of this where all $\alpha_i = 1$ and all $\beta_i = 2$. The inequality (c57), known as the König-Zermelo Theorem, is in fact proved in essentially the same way as this special case, if the proof of that case is appropriately formulated. Namely, in proving the special case, consider a function f from the disjoint union U of an I-tuple of 1-element sets to the I-fold product P of the set $\{0, 1\}$. For each $i \in I$, there is just *one* element u_i in the Ith member of the union U, but there are *two* elements of $\{0, 1\}$ which are the possible values of the ith term of a member of P, hence, since $1 < 2$, for each i we can choose a member of $\{0, 1\}$ which is not the ith term of $f(u_i)$. From this one deduces that f is not surjective. You should easily be able to adapt that argument to prove (c57).

As another application of the König-Zermelo Theorem, consider any increasing sequence of cardinals $\alpha_0 < \alpha_1 < \dots$ indexed by the positive integers, and let $\alpha = \Sigma\, \alpha_i$. Then

$$\alpha \ = \ \Sigma\, \alpha_i \ < \ \Pi\, \alpha_{i+1} \ \leq \ \Pi\, \alpha \ = \ \alpha^{\aleph_0}.$$

In the language of Exercise A2:L13 (p.893), this says $\alpha < \alpha'$. Since α can be taken greater than any cardinal β (by letting $\alpha_0 = \beta$ and $\alpha_{i+1} = 2^{\alpha_i}$ for $i \geq 0$), this answers the question raised in brackets in that exercise. (Solovay showed me the König-Zermelo Theorem, and I was the ''grapevine'' that communicated it to Lang.)

Incidentally, the proof of the result $\mathrm{card}(I) < \mathrm{card}(2^I)$ is itself a generalization of the observation that a set theory which allows one to speak of ''the set of all sets'' leads to a contradiction. Namely, the contradiction when we have a ''set of all sets'' arises by noting paradoxical properties of its subset, ''the set of all sets that are not members of themselves''; in the proof that $\mathrm{card}(I) < \mathrm{card}(2^I)$ one considers a map $f: I \to 2^I$, and effectively notes that the set of all $i \in I$ that are not members of $f(i)$ cannot have the form $f(j)$ for $j \in I$; hence f is not surjective.

The paradox about the set of all sets that are not members of themselves can in turn be considered a descendent of paradoxes about the truthfulness of people who assert that they always lie, etc., which go back thousands of years!

Re §A2.4. Well-ordering. The material in this brief section is also included in the ''Axiom of Choice, etc.'' handout.

P.892, line 2 [=]: Both occurrences of A in this line should be S. (The A on the preceding line is correct.)

P.892, Theorem A2.4.1 [>]: The word ''nonempty'' in the result is unnecessary – the empty set can be well-ordered, too. In the fifth line of the proof, an ''initial segment'' of a totally ordered set X means a subset Y such that whenever $y \in Y$ and $z < y$ we also have $z \in Y$.