## Chapter XIII. Matrices and Linear Maps.

[These notes on Chapters XIII, XIV, and XVI were written after teaching from this material in Spring 1997. Until I get to teach from it again and polish them further, they are liable to be somewhat rough. As with everything else in this Companion, I welcome corrections to these notes, suggestions on points that could use clarifying, etc..]

*Re §XIII.1. Matrices.*

P.504, first display [>].

*Some background and some notation.*

The definitions Lang has just given, of matrices and their additive and multiplicative structures, are of course standard. But where do they come from?

Consider a linear map $f$ from the standard free module of rank $n$, $R^n$, to the standard free module of rank $m$, $R^m$. If we write the natural bases of these modules $\{e_1,, \dots, e_n\}$ and $\{e_1,, \dots, e_m\}$, then

$f: R^n \to R^m$ is determined by the $n$ elements $f(e_1), \ldots, f(e_n)$, each of which is in turn determined by the coefficients with which the $m$ elements $e_1, \ldots, e_m$ occur in it. Thus, writing $f(e_j) = \Sigma\, a_{ij}\, e_i$, we see that $f$ is determined by the $mn$ elements $a_{ij} \in R$ $(1 \leq i \leq m,\ 1 \leq i \leq n)$. It is natural to arrange these in a rectangular array, so that those coefficients expressing the image of a given $e_j$ form a column, while those occurring as coefficients of a given $e_i$ in these images form a row. The laws of addition and multiplication of matrices express addition and composition of linear maps in terms of these coefficients.

There are other standard uses of matrices: to describe changes of bases of free modules, to represent bilinear forms, etc.. When these come up below, however, we shall see how to relate them all to the above interpretation.

Note that the hom-set $\mathrm{Hom}(R^1, R^m)$ can be identified with $R^m$ by associating to each homomorphism the element to which it sends the basis element $e_1 = 1 \in R$. This has the pleasant consequence that rather than setting up distinct notations for matrices and for elements of the free modules on which they act, we can treat these elements as special cases of matrices, namely as matrices with only one column.

There is a convenient notation, though not used by Lang, in which one writes the set of $m \times n$ matrices over a ring $R$ as ${}^mR^n$. One also generally drops $m$ and/or $n$ if its value is 1. I will use this notation for the remainder of the notes on this chapter. In particular, the modules denoted by $R^n$ and $R^m$ above will henceforth be written as ${}^nR$ and ${}^mR$.

Note that ${}^nR^n$ is the ring Lang writes $\mathrm{Mat}_n(R)$. Matrix multiplication makes the set ${}^mR^n$ a left module over ${}^mR^m$, and a right module over ${}^nR^n$. This multiplication corresponds to composing linear maps ${}^nR \to {}^mR$ on the left with endomorphisms of ${}^mR$ and on the right with endomorphisms of ${}^nR$. Associativity of composition of functions implies that the *left* module structure acts by endomorphisms of ${}^mR^n$ as a *right* module, and vice versa. An abelian group with such a pair of module structures is called an $({}^mR^m, {}^nR^n)$-*bimodule*. I will occasionally mention bimodules parenthetically below, but will not introduce the concept formally until a later chapter (discussion entitled *Tensor products in the noncommutative context*, at end of comments to §XVI.1).

So far, I have tacitly followed Lang and assumed $R$ commutative. If we drop this assumption (though always assuming our rings have unity), what we have said above works, as long as we regard ${}^mR$ as a free *right* $R$-module. (So where I wrote $f(e_j) = \Sigma\, a_{ij}\, e_i$, this should read $\Sigma\, e_i\, a_{ij}$.) Indeed, its right $R$-module structure is the $n=1$ case of the right ${}^nR^n$-module structure on ${}^mR^n$.

If $M$ and $N$ are right modules over a noncommutative ring $R$, then $\mathrm{Hom}(M, N)$, though it is an abelian group, does not form an $R$-module, since when one multiplies a homomorphism $f$ by an element $a \in R$, the resulting map $x \mapsto f(x)a$ does not in general respect multiplication by elements of $R$: $f(xr)a = f(x)ra \neq f(x)ar$. However, when $N = R$, it makes sense to *left*-multiply elements $f(x)$ by members of $R$, and it is easy to verify that this makes $\mathrm{Hom}(M, R)$ a left $R$-module. (More generally, if $M$ is a right $S$-module and $N$ an $(R, S)$-bimodule, then the abelian group of right $S$-module homomorphisms $\mathrm{Hom}(M, N)$ acquires in this way a natural structure of left $R$-module.) It is easy to verify that if $M$ is a free right module of finite rank with basis $B$, then $\mathrm{Hom}(M, R)$ is free as a left module on the "dual basis". If we take for $M$ the standard free right module ${}^nR$, we find that $\mathrm{Hom}(M, R) = \mathrm{Hom}({}^nR, {}^1R) = {}^1R^n = R^n$. That is, the set of *row* vectors of length $n$ has a natural structure of free *left* module of rank $n$, and we may take it as our "standard" free left module of that rank. Its left module structure is given by the action of $1 \times 1$ matrices on the left of row vectors.

Since the rules for matrix multiplication are symmetric, ${}^mR^n$ can be regarded equally as representing the right module homomorphisms ${}^nR \to {}^mR$ and the left module homomorphisms $R^m \to R^n$. Note, however, that in using these conventions, we must understand homomorphisms of right modules to be written to the *left* of their arguments, while homomorphisms of left modules are written to the *right* of their arguments, with each sort being composed in the way natural to the side on which it is written: $A(Bx) = (AB)x$ while $(yA)B = y(AB)$. (This is a convention that is often followed in noncommutative ring theory generally: Homomorphisms among right modules are written to the left of their arguments, and vice versa.

Note that it makes the equations saying that such homomorphisms respect scalar multiplication into formal associative laws: $f(xr) = (fx)r$, respectively $(rx)f = r(xf)$ for $f$ a homomorphism, $x$ a module element, and $r$ a ring element.)

Of course, if $R$ is commutative, it does not mean much to say ''the set ${}^nR$ of column vectors represents the free right module on $n$ generators while the set $R^n$ of row vectors represents the free left module''. In that case, we simply regard ${}^nR$ and $R^n$ as two copies of the free $R$-module of rank $n$, subject to different notational conventions regarding linear maps.

Now back to Lang.

P.505, paragraph near top, defining the **transpose** of a matrix [~]: Although, as I have noted, the definitions of matrix addition and multiplication are as useful in studying modules over noncommutative rings as in the commutative case, the concept of the transpose of a matrix is only natural when $R$ is commutative. In that case, it can be motivated by taking the abovementioned isomorphism of (right = left) modules ${}^nR \cong R^n$, and noting that this leads to an identification between $\mathrm{Hom}({}^nR, {}^mR) \cong {}^mR^n$ and $\mathrm{Hom}(R^n, R^m) \cong {}^nR^m$, which we denote ${}^t$. The difference between the notations for composition of these two sorts of morphisms leads to the identity ${}^t(AB) = {}^tB\,{}^tA$. For $R$ noncommutative, both the motivation and the identity fail; indeed, the $1\times 1$ case of the identity is the condition of commutativity.

Nevertheless, the symbol ${}^t$ is sometimes used in noncommutative ring theory for typographical convenience: If one wants to refer to the column with entries $x_1, \dots, x_m$, without leaving a lot of space between lines, one can write it ${}^t(x_1, \dots, x_m)$.

Two other definitions that are natural only for matrices over commutative rings are those of the *trace*, defined later on this page of Lang, and the *determinant*, introduced in §XIII.4.

In general, when definitions or results in Lang apply only for $R$ commutative, I will say so; likewise, I will point when there are nontrivial adjustments to be made in extending results to the noncommutative context. When I refer to a result in Lang as translating to, say, a certain statement about right modules, it should be understood that the obvious corresponding statement about left modules is also true, and may indeed be called on in subsequent arguments. Where I say nothing to the contrary, assume the results and proofs he gives are valid in the noncommutative case, perhaps with some straightforward notational adjustment. (But if you find some difficulty, let me know – I may have missed a point.)

P.505, middle [=]: For ''A similar remark'' read ''This definition of $\varphi A$''.

P.505, bottom [>]. *Further remarks.* There is a kind of matrix notation more general than those we have discussed here. (We mentioned it briefly in our discussion of §III.5.) Rather than considering the free right $R$-modules ${}^nR$ and ${}^mR$, suppose we look at any two right $R$-modules given with finite direct-sum decompositions: $N = N_1 \oplus \dots \oplus N_n$, $M = M_1 \oplus \dots \oplus M_m$. A homomorphism $f: N \to M$ is clearly determined by its restrictions $f_j: N_j \to M$ to the $n$ summands of $N$, and each of these is determined by its composites with the projections of $M$ onto its $m$ components $f_{ij}: N_j \to M_i$. If we write these $mn$ homomorphisms in a rectangular array, it is easy to see that the formal laws of matrix addition and multiplication correctly describe addition and composition of maps among such direct sums of modules. In particular, the ring of endomorphisms of a module of the form $N = N_1 \oplus \dots \oplus N_n$ may be expressed as a square ''matrix ring'', with homomorphisms among these modules as the matrix entries.

There are still other sorts of generalized matrices that are occasionally useful, which I won't go into here. For instance, if $B$ is an $(R, S)$-bimodule, one finds that formal matrix operations make the set $\begin{pmatrix} R & B \\ 0 & S \end{pmatrix}$ a ring.

*Re §XIII.2. The rank of a matrix.*

P.506, first paragraph [>]: The result to be proved here, that the row and column rank of a matrix over a field are equal, is true, more generally, for matrices over a division ring, where, of course, one defines the row rank of such a matrix as the dimension of the *left* vector space spanned by the rows, and the column rank as the dimension of the *right* vector space spanned by the columns. The same proof works, if we assume the duality theory of §III.6 verified for modules over not necessarily commutative rings, the dual of a right module being regarded as a left module and vice versa.

However, let me give a proof that does not call on duality theory. We will need

**Definition XIII.2.c1.** *Let $R$ be a ring, $m$ and $n$ nonnegative integers, and $A \in {}^mR^n$. Then the* inner rank *of $A$ is defined to be the least nonnegative integer $r$ such that $A$ can be factored $A = BC$, with $B \in {}^mR^r$ and $C \in {}^rR^n$.*

(*Here we allow the possibility $r = 0$, understanding an $m \times 0$ or a $0 \times n$ matrix to have no entries, and to represent the unique homomorphism $\{0\} \to {}^mR$, respectively ${}^nR \to \{0\}$. Clearly, $A$ has rank $0$ if and only if it is the zero $m \times n$ matrix.*)

Now a matrix $A$ has column rank $r$ if and only if all of its columns can be expressed as right linear combinations of some $r$ column vectors, $b_1, \ldots, b_r$, but cannot be so expressed using fewer. But such an expression is equivalent to a factorization $A = BC$ where $B$ is the matrix with those $r$ vectors as its columns. Thus, the column rank and the inner rank of $A$ are equal. Looking at rows instead of columns, we similarly deduce that the row rank and the inner rank are equal. Thus we have

**Lemma XIII.2.c2.** *The row rank, the inner rank, and the column rank of a matrix over a division ring are all equal.* □

Incidentally, the definitions of row and column rank are based implicitly on the fact that a vector space has a well-defined dimension, i.e., that all its bases have the same number of elements. This was proved for vector spaces over fields in §III.5. As I noted there, the proof is equally valid for vector spaces over division rings.

*Re §XIII.3. Matrices and linear maps.*

P.507, first paragraph of §XIII.3 [>]: This shows that a basis $\xi_1, \ldots, \xi_n$ of a right module $E$ determines an isomorphism ${}^nR \to E$. This isomorphism can be characterized as the unique module homomorphism sending the $i$th standard basis vector $e_i$ of ${}^nR$ to $\xi_i$. Inversely, any isomorphism ${}^nR \to E$ determines a basis of $E$ indexed by $\{1, \ldots, n\}$, namely the image under the homomorphism of the standard basis of ${}^nR$. Thus, a choice of an $n$-element ordered basis for $E$ is *equivalent* to an isomorphism ${}^nR \to E$. This observation will be a key tool below in translating results about arbitrary free modules into matrix language.

P.507, second paragraph of §XIII.3 [>]: This proof that a free module over a commutative ring has unique dimension uses the fact that vector spaces (i.e., modules over *fields*) have unique dimension, and that every commutative ring admits a homomorphism into a field. Cf. this Companion, discussion at end of §III.5, which also shows how to construct noncommutative rings with free modules having *nonunique* ranks. (Apologies for having written homomorphisms of left modules on the left of their arguments there; but I didn't want to introduce unfamiliar notation for that brief digression.)

Despite Lang's use of the word *dimension*, it is more usual to speak of the *rank* of a free module, if the base ring is not a division ring, and I will do so below.

P.508, Proposition XIII.3.1 [~]: Note that the column vectors considered here are $n$-tuples of elements of a free module, which is itself of rank $n$. Moreover, since a matrix $A \in {}^nR^n$ acts on the left on elements of ${}^nE$, we should, in formulating this result for $R$ not necessarily commutative, take $E$ to be a left $R$-module.

In particular, if we take for $E$ the standard free left $R$-module of rank $n$, consisting of row vectors, we find that our column vectors of elements of $E$ are height-$n$ columns of length-$n$ rows; in other words, they look like elements of ${}^nR^n$ itself. If we take for $x_1, \ldots, x_n$ the standard basis of $R^n$, then the column that they form is the identity matrix, and the left-hand side of the displayed equation is just $A$. So this case of the proposition says that $A$ is invertible if and only if its rows form a basis of $R^n$.

Some quick exercises in the use of the proposition: Suppose $V$ is a 3-dimensional $k$-vector-space with basis $\{x, y, z\}$. Under what conditions, if any, on $k$ will $\{x+y, y+z, z+x\}$ be a basis of $V$? What about $\{x-y, y-z, z-x\}$? If $\mathcal{B}$ is a $\mathbf{Z}/p\mathbf{Z}$-vector-space basis of ${}^n(\mathbf{Z}/p\mathbf{Z})$, and the matrix having the elements of $\mathcal{B}$ for rows has determinant other than $\pm 1 \in \mathbf{Z}/p\mathbf{Z}$, show (assuming standard properties of the determinant, though we will not formally see these till the next section) that $\mathcal{B}$ is not the image of any basis of the free abelian group ${}^n\mathbf{Z}$ under the natural map ${}^n\mathbf{Z} \to {}^n(\mathbf{Z}/p\mathbf{Z})$.

P.509, bottom [>]: As noted earlier, a choice of an $n$-element basis for a right $R$-module $E$ induces an isomorphism between ${}^nR$ and $E$. Hence given bases $\mathcal{B}$ and $\mathcal{B}'$ of right modules $E$ and $F$, say with $n$ and $m$ elements respectively, a module-homomorphism $f\colon E \to F$ is equivalent to a homomorphism of standard free modules:

$$\begin{array}{ccc} {}^nR & & {}^mR \\ \downarrow \mathcal{B} & & \downarrow \mathcal{B}' \\ E & \xrightarrow{f} & F. \end{array}$$

Here the arrows marked $\mathcal{B}$, $\mathcal{B}'$ represent the module *isomorphisms* taking the standard bases of ${}^nR$ and ${}^mR$ to these bases of $E$ and $F$. The resulting homomorphism ${}^nR \to {}^mR$ corresponds to a matrix $A \in {}^mR^n$, which Lang denotes $M^{\mathcal{B}}_{\mathcal{B}'}(f)$. If you draw the appropriate diagrams, you will now see that the equation in the proposition at the beginning of the next page is immediate! Likewise, given two bases $\mathcal{B}$ and $\mathcal{B}'$ of a right module $E$ (say of $n$ and $m$ elements respectively, where $m=n$ if $R$ is commutative), we get a diagram

$$\begin{array}{ccc} {}^nR & & {}^mR \\ \downarrow \mathcal{B} & & \downarrow \mathcal{B}' \\ E & = & E. \end{array}$$

from which we see that $M^{\mathcal{B}}_{\mathcal{B}'}(\mathrm{id})$ represents the linear map ${}^nR \to {}^mR$ that carries the expression for an element of $E$ with respect to the basis $\mathcal{B}$ to its expression with respect to the basis $\mathcal{B}'$.

P.510, to end of section [~]: If it is hard to see what the first two corollaries on p.510 are asserting, this is because Lang has included the proofs as part of the statements. The real assertion of Corollary XIII.3.4 is that a change-of-basis matrix $M^{\mathcal{B}}_{\mathcal{B}'}(\mathrm{id})$ is invertible; the displayed equation is the proof. Likewise, the point of Corollary XIII.3.5 is that the left side of the display equals the right side. The middle term gives the proof.

Everything in this section is valid for noncommutative as well as for commutative rings, except the last sentence of Corollary XIII.3.5 (since for noncommutative $R$ one cannot speak of $R$-algebras), and the remark about the trace function at the very end of the section.

*Re §XIII.4. Determinants.*

P.511, start of the section [<]:

*I will assume in my comments on the remaining sections of this chapter that* $R$ *is commutative.*

(There is a concept of multilinear map appropriate to noncommutative ring theory, which I will introduce in the notes on §XVI.1, but there is no natural concept of *alternating* multilinear maps even in that context.)

P.514, line 3 [=]: ''Using a previous lemma ... '' – namely, Lemma XIII.4.3. On the same page, two lines above the corollary, ''by the lemma'' again refers to Lemma XIII.4.3.

P.516, first sentence beginning after middle display [~]: Another proof of this result is noted in the comments to §VI.4 in this Companion.

P.517, line 1 [>]: In general, I favor more mnemonic notation; so where Lang writes $L^n_a(E, F)$, I would prefer something like $\mathrm{Lin}^n_{\mathrm{alt}}(E, F)$. Of course, when in the creative throes of discovery, use in your scratch-work whatever notation you are comfortable with. But in a good write-up, try for notation whose meaning is fairly transparent, if you can do so without making it excessively bulky. In this Companion, I will follow Lang's notation $L^n_a(E, F)$.

P.518, Proposition XIII.4.16 [=]: Note that this is essentially Cramer's rule (Theorem XIII.4.4), restated.

P.520, Proposition XIII.4.19 [=]: Note that this proposition characterizes $D(f)$ in a way that does not depend on a matrix representation of $f$, and hence a choice of basis of $E$.

There is a slightly more abstract, but very elegant variant to this characterization of the determinant, in which one associates to every module $E$ and nonnegative integer $d$ a module $\wedge^d E$, the ''$d$th exterior power of $E$'', having a universal $d$-linear alternating map of $E\times\ldots\times E$ into it. One finds that if $E$ is free of rank $n$, then $\wedge^d E$ is free of rank $\binom{n}{d}$. In particular, $\wedge^n E$ is free of rank 1. The construction $\wedge^d$ is a functor, hence any endomorphism $f$ of $E$ induces an endomorphism $\wedge^n f$ of this rank-1 free module, which must be given by multiplication by an element $D(f)\in R$. Some of the theory of the functors $\wedge^d$ (and of $\wedge = \bigoplus_d \wedge^d$, which is not only module- but algebra-valued) is developed by Lang in Chapter XIX (not presently covered in this Companion); cf. also a handout of mine, *Tensor algebras, exterior algebras, and symmetric algebras*, accessible through my web page.

P.521, second line of second paragraph [=]: For ''view it multiplicatively, as'' read ''restrict it to automorphisms, getting''.

P.522, end of §XIII.4 [>]. Here is a curious question: Suppose $k$ is a field and $n$ a positive integer, and we form the polynomial ring in $n^2$ indeterminates, $k[x_{11},\ldots,x_{nn}]$, and let $X = (x_{ij})$ be the $n\times n$ matrix formed from these indeterminates. Then as shown in Proposition XIII.4.16, we get a factorization (for which I will use different notation from Lang): $\det(X)\,I = X\,\mathrm{adj}(X) = \mathrm{adj}(X)\,X$. Is any further factorization possible, and if so of what sort?

This question is answered completely for $k$ of characteristic 0 in *Transformation Groups*, **11** (2006) pp.7-15; but for prime characteristic it is still open.

*Re §XIII.5. Duality.*

P.522, last six lines [>]: Where Lang writes **perpendicular**, the more common term is ''orthogonal''. Note that the notation $S^\perp$ is unambiguous if $E$ and $F$ are disjoint; but if, for example, they are the same space, then for a subset $S\subseteq E$, the symbol $S^\perp$ could mean either $\{x\in E \mid (\forall\, s\in S)\ s\perp x\}$ or $\{x\in E \mid (\forall\, s\in S)\ x\perp s\}$; and if $f$ is neither symmetric or antisymmetric, these may be distinct. In setting up the abstract theory, as is done here, one can treat these spaces as though they were disjoint without actually making this a hypothesis, since in any given context a set $S$ will be described as a subset of $E$ or as a subset of $F$, so that the sense of $S^\perp$ will be clear. But if one applies this theory to a particular case in which $E$ and $F$ are not disjoint, one may need to introduce distinct notations for the above sets. This sort of situation is not uncommon in mathematics.

P.523, first boxed display [=]: If we adopt ''exponential notation'', writing $B^A$ for $\mathrm{Hom}(A,B)$, and using the *ad hoc* notation $C^{A\cdot B}$ for $L^2(A,\ B;\ C)$, then (replacing the $R$ in this display by a general module $D$, and interchanging the roles of $E$ and $F$ on the right, which is OK because we are still assuming our ring commutative, so that the construction $L^2(E,\ F)$ is symmetric in $E$ and $F$), the display takes the suggestive form $D^{E\cdot F}\cong (D^E)^F$. (In noncommutative ring theory, one can formulate a generalization valid for bimodules, but we won't look at it here.)

P.523, last paragraph [>]: Actually, this only needs $f$ to be nonsingular on the left. For in that case, we can apply the isomorphism $\mathrm{Hom}_R(F,R)\cong E$ to the previous boxed display to turn the right-hand side into $\mathrm{Hom}_R(E,E) = \mathrm{End}_R(E)$. The top three paragraphs on the next page just make this construction explicit.

P.524, 7 lines from bottom [<]: Suppose we are given two modules $E$ and $F$, together with right-nonsingular bilinear forms relating them to modules $E'$ and $F'$:

$$E\times E'\to R,\qquad F\times F'\to R.$$

These forms allow us to identify $E'$ and $F'$ with $E^\vee = \mathrm{Hom}_R(E,R)$ and $F^\vee = \mathrm{Hom}_R(F,R)$ respectively. Now any homomorphism $A\colon E\to F$ induces a homomorphism $A^\vee\colon F^\vee\to E^\vee$, hence, via the above identifications, a homomorphism $F'\to E'$, which we can call ${}^tA$. The preceding discussion has concerned the case where $E = F'$ and $F = E'$. The paragraph beginning here concerns the case where $E = E'$ and $F = F'$. (So it is not, as Lang claims, ''more general'' than the preceding case.)

P.525, **Note** following Proposition XIII.5.1 [>]: This is true because ${}^tAA = \mathrm{id} \Rightarrow A$ has invertible determinant $\Rightarrow A$ is invertible.

One may ask whether the same implication holds without the assumption that $E$ is free of finite rank. Here is an almost-counterexample, where the given bilinear form is non-degenerate, though not, as in the proposition, nonsingular. Let $R$ be a field, $E$ a countable-dimensional vector space with basis $\{x_0, x_1, \ldots\}$, and let our bilinear form be given by $\langle x_i, x_j \rangle = \delta_{ij}$. Let $A x_i = x_{i+1}$. Knowing only that $\langle\ ,\ \rangle$ is non-degenerate, we do not know a priori that there exists an operator ${}^tA$ related to $A$ by the equation $\langle Ax, y\rangle = \langle x, {}^tAy\rangle$; but in this case there is such an operator, given by ${}^tA x_{i+1} = x_i$ $(i \geq 0)$, ${}^tA x_0 = 0$. We find that ${}^tAA = \mathrm{id}$ but that $A\,{}^tA \neq \mathrm{id}$.

To get an example where $\langle\ ,\ \rangle$ is in fact nonsingular, one must go out of the realm of vector spaces, since for finite-dimensional vector spaces the implication is true, while the dual of an infinite-dimensional vector space $E$ is always of larger dimension than $E$, hence cannot be isomorphic to $E$. Here are some hints for the student interested in constructing a module-theoretic counterexample. Note that for any module $E$, there is a natural homomorphism $E \to E^{\vee\vee}$. If this is an isomorphism, then $E \oplus E^{\vee}$ will be isomorphic to its dual, hence will admit a non-singular bilinear form. Now try your hand at Exercise I.7:1(b); a solution to this, together with the above observations, will provide the tool needed to find the desired example with $R = \mathbf{Z}$.

*Re §XIII.6. Matrices and bilinear forms.*

In this and the remaining sections of this chapter, we shall adopt Lang's assumption that $R$ is commutative. (After going through the first few sections of Chapter XVI and my comments on these, you might want to revisit this section and think about how to give generalizations of its ideas to free right and left modules over general rings.)

P.527, display at bottom [>]: This shows that a bilinear form $f$ can be described by a matrix $g_{ij}$. This is the matrix (with respect to the given basis of $E$, and the basis of $F^{\vee}$ dual to the given basis of $F$) of the linear map $E \to F^{\vee}$ which takes $x \in E$ to $f(x, -) \in F^{\vee}$. Thus, the use of matrices to describe bilinear forms reduces to that of describing linear maps.

*Re §XIII.7. Sesquilinear duality.*

P.533, definition of **sesquilinear form** [>]: Clearly, this is just a form that is *bilinear* with respect to the given module structure on $E$ and a modified module structure on $F$ (which Lang will call, a few paragraphs from now, the ''anti-module'' structure). So in this generality, there is nothing new. It is only on adding the condition $E = F$ (in results XIII.7.{1,5,6}) that one gets a theory that doesn't reduce to the corresponding theory for bilinear forms.

P.534, line preceding Proposition XIII.7.2 [<]: Ignore this line; or let me know if you can figure out what it means!

Also, the last line on this page should be indented like the three that precede it, being one of the equivalent conditions.

*Re §XIII.8-9.* No notes on these at present.

## Chapter XIV. Representation of One Endomorphism.

*Re §XIV.1. Representations.*

P.553, bottom [>]: A ''representation of $R$ in $E$'', as here defined, is the same as a left $R$-module structure on $E$ extending the given $k$-module structure. This alternative language probably comes from the theory of group representations, where actions of a group $G$ by linear automorphisms on a vector space $V$, which are equivalent to $kG$-module structures on $V$, were studied before the concept of module was defined, so that the group-theorists developed their own language. Hopefully, this duplication of terminology will eventually be eliminated.

P.554, second paragraph [=]: In the last two sections of this chapter, the ring $R$ will in fact be $k[x]$. But the definitions in this section should be understood for general $R$, not necessarily commutative.

*Re §XIV.2. Decomposition over one endomorphism.*

P.556, near bottom, definition of **principal** [>]: What Lang calls a ''principal module'' is more often called a ''cyclic module'' (cf. the definition of a cyclic group), and I will use the latter term.

*Remarks:* If $E$ is a cyclic left module over any ring $R$, with generator $v$, then the *annihilator* of $v$ in $R$, $\mathrm{Ann}_R\, v = \{r \in R \mid rv = 0\}$, is a left ideal of $R$, and it is easy to see that $E \cong R/\mathrm{Ann}_R\, v$ as a left $R$-module. (This is analogous to the description of the general transitive $G$-set in terms of the isotropy subgroup of an element.) If $R$ is *commutative*, then any element of $\mathrm{Ann}_R\, v$ also annihilates every element $av \in E$, hence $\mathrm{Ann}_R\, v$ can also be characterized as $\{r \in R \mid \forall\, x \in E,\ rx = 0\}$, i.e., as the annihilator of the whole module $E$, equivalently, as the kernel of the given map $R \to \mathrm{End}(E)$, and so is independent of a choice of cyclic generator. This is not true in the noncommutative case, where the annihilator of the whole module will be a 2-sided ideal, while the annihilator of a single element can be any left ideal. (Given $a \in R$, you should find it easy to write down the formula for $\mathrm{Ann}_R\, av$ in terms of $\mathrm{Ann}_R\, v$.) But for $R$ commutative, we thus see that the isomorphism classes of cyclic $R$-modules $E$ are classified by the ideals of $R$. If $R$ is a principal ideal domain, then these ideals are in turn classified by elements of $R$, up to unit factors. In the case $R = k[t]$, such ideals, when nonzero, correspond to monic polynomials $q(t)$.

Given a monic polynomial $q(t)$, it is easy to write down a matrix for the endomorphism of the $k[t]$-module $k[t]/q(t)$ given by multiplication by $t$; this is done on the next page; the result is generally called the *companion matrix* of $q(t)$ (but is not named by Lang). As Lang then notes, the structure theorem for finitely generated modules over principal ideal domains allows one to reduce the study of arbitrary finitely generated modules over such rings to that of cyclic modules, and hence to classify $k[t]$-modules finite-dimensional over $k$, i.e., finite-dimensional $k$-vector-spaces given with endomorphisms $A$, in terms of families of monic polynomials.

P.557, Corollary XIV.2.2 [<]: We now want to translate our classification of vector spaces given with an endomorphism into a classification of matrices. Two pairs $(k^{(n)}, A)$ and $(k^{(n)}, B)$ (using Lang's notation here instead of my ${}^n k$) are isomorphic if and only if some change-of-basis transformation converts $A$ into $B$; equivalently, if and only if $B = CAC^{-1}$ for some invertible matrix $C$; so Theorem XIV.2.1, which shows that every pair $(E, A)$ is isomorphic to a unique member of a certain family of such pairs, shows that every square matrix over $k$ is conjugate to a unique member of a certain family of matrices; namely, those given by diagonal blocks of companion matrices of a sequence of monic polynomials in which each polynomial is divisible by the next.

Now if two matrices are conjugate over $k$ to *distinct* members of this family, they are still conjugate to those distinct matrices over any larger field $k'$; hence by the unicity part of the above result, they cannot be conjugate to one another over this extension field. This is the result of Corollary XIV.2.2. The last sentence of the proof is vaguely worded because Lang has not introduced the term ''companion matrix'', which would allow him to state it more precisely.

P.558, Theorem 2.4 [<]: As Lang has just noted, we have another structure theorem for finitely generated modules over a principal ideal domain, from which one can get a different canonical form, expressing a matrix as a diagonal sum of companion matrices of powers $p^e$ of monic irreducible polynomials.

But for such polynomials there is in fact a matrix representation nicer than the companion matrix. Observe that the submodules of the $k[t]$-module $k[t]/(p^e)$ form a chain

$$(1)/(p^e) \supseteq (p)/(p^e) \supseteq \ldots \supseteq (p^e)/(p^e) = \{0\}.$$

A $k$-basis for the first term of this chain modulo the second is given by $1, \ldots, t^{d-1}$, where $d$ is the degree of $p$; a basis for the second modulo the third is given by $p, tp, \ldots, t^{d-1}p$, and so on. Taking the union of these bases, we get a basis for the whole module $k[t]/(p^e)$, in terms of which the matrix for multiplication by $t$ consists of $e$ diagonally arranged copies of the companion matrix for $p$, together with single entries ''1'' just below and to the left of the corners where these blocks touch, representing the fact that when one multiplies $t^{d-1}p^i$ by $t$, and subtracts off an appropriate linear combination of lower terms $t^j p^i$, one gets, not 0, but $p^{i+1}$. (I suggest you work out the details.)

The form of these matrices is especially simple when $p$ is a linear polynomial $t-\alpha$, and is given in Theorem XIV.2.4. If $k$ is algebraically closed, then every irreducible polynomial is linear, and the resulting canonical form for matrices is the well-known Jordan Canonical Form (Corollary XIV.2.5, p.559).

P.560, Corollary XIV.2.7 [>]: Actually, this can be proved more easily, without the assumption that $K$ is separable (or even commutative), and without the use of the preceding theorem. Given a finite-dimensional division algebra $K$ over $k$, and two representations of $K$ on $V$, let us denote by $V_1$ and $V_2$ the corresponding $K$-vector-spaces. The $K$-dimension of each is clearly $(\dim_k V)/[K:k]$, hence, having the same dimension, they are isomorphic as $K$-vector-spaces, which is equivalent to the stated conclusion.

*Re §XIV.3. The characteristic polynomial.*

P.561, second line after second display [=]: $R$ should be $k$. (And for ''in'' I would say ''over'', both here and three lines before that display.)

P.561, five lines above Theorem XIV.3.1 [=]: Why do we define the characteristic polynomial of the unique endomorphism of the zero-module to be 1? This is a particular case of a more general convention Lang should have made earlier, that the *determinant* of the unique endomorphism of the zero module is 1. A pragmatic justification for this is that it makes various formulas come out right; e.g., given endomorphisms $e$ and $f$ of free modules $E$ and $F$, one gets an endomorphism $e \oplus f$ of the module $E \oplus F$. When these modules have positive rank, one sees that the determinant of $e \oplus f$ is the product of the determinants of $e$ and of $f$. To make this true if $E$ or $F$ is the trivial module, one should define the determinant of its unique endomorphism to be 1.

One can also justify this formally, by examining closely what ''$n$-linear alternating map'' means when $n = 0$. One finds that it is equivalent to a choice of element of the codomain module, and that the unique endomorphism of the zero module has no effect on the module of 0-linear alternating forms, $L_a^0(\{0\}, k) \cong k$, i.e., it acts on it by multiplication by 1. I leave the details to the interested reader.

P.561, third from bottom line [=]: Change ''Let $\widetilde{B}(t)$ be'' to ''Let $B(t) = tI_n - M$ and let $\widetilde{B}(t)$ be''.

Pp.561-562, proof of Theorem XIV.3.1 [=]:

*Proof of the Cayley-Hamilton Theorem.*

This is a somewhat mysterious proof in any case, made more difficult to follow here by notational problems; but it has an intriguing quality once one has penetrated the obscurity. Let me try to show what is going on.

Let us assume without loss of generality that $E = k^{(n)}$, so that $A$ can be regarded as a matrix. As in the preceding sections, we make $E$ a $k[t]$-module by letting $t$ act by $A$. We then form the matrix $B = tI_n - A \in \mathrm{Mat}_n(k[t])$, and let it act on $E^{(n)}$, i.e., on columns of elements of that module.

Now let us step back and note that, if we are stacking elements of $E$ up in columns, then we should regard these elements as *row* vectors; thus elements of $E^{(n)}$ take the form of $n \times n$ matrices. Since $t$ acts on $E^{(n)}$ via an action on $E$, it acts on each row separately; so it should be regarded as *right* multiplication of these $n\times n$ matrices by $A$. Thus, if we left multiply one of these stacks of elements of $E$ by $B = tI_n - A \in \mathrm{Mat}_n(k[t])$, the first term acts as right multiplication by $A$, while the second term, which does not involve $t$, but is a matrix of elements of $k$ acting on our column, has the effect of *left* multiplication by $A$. Thus, $B$ represents the difference between right and left multiplication by $A$. In general, this operator does not annihilate all of $E^{(n)}$; but it clearly annihilates the identity matrix; i.e., the stack whose $i$th term is the $i$th standard basis vector. Now if we can find a left multiple of $B$ in $\mathrm{Mat}_n(k[t])$ that is a *scalar* matrix $c(t)I_n$ $(c\in k[t])$, then as a left multiple of $B$, it will still annihilate the identity matrix, hence the element $c(t)$ will annihilate all basis elements of $E$, hence will act trivially on $E$. Now in fact, the first line of Proposition XIII.4.16 tells us that for any matrix $A$ over a commutative ring, the scalar matrix $\det(A)I_n$ is a left multiple of $A$; so the above argument shows that the operator $\det(B) = \det(tI_n - A) = P_A(t) \in \mathrm{Mat}_n(k[t])$ acts trivially on $E$, i.e., that $P_A(A) = 0$, as required.

There is another standard proof of this theorem, which uses ''brute force'' instead of ''magic'', but also involves a trick that is worth having under one's belt. Let me sketch it; you should not find it hard to fill in the details. (a) Verify that the theorem is true for a matrix $A$ whose only nonzero entries lie in square diagonal blocks (as on p.555) *if* it is true for each of those blocks. (Use the fact that the characteristic polynomial of $A$ is the product of the characteristic polynomials of those blocks.) (b) Verify that the theorem is true for a matrix of the form shown at the bottom of p.558, whose characteristic polynomial is $(t-\alpha)^e$. Deduce that it is true for any matrix in Jordan canonical form. (c) Verify that if it is true for a matrix $A$, it is true for any conjugate matrix $CAC^{-1}$, and deduce that it is true for any matrix over an algebraically closed field. (d) Since any integral domain embeds in a field, and any field embeds in an algebraically closed field, deduce that it is true for any matrix over an integral domain. (e) Since any commutative ring is a homomorphic image of an integral domain (for instance, a polynomial ring over $\mathbf{Z}$ in a sufficiently large set of indeterminates), deduce that it is true for any matrix over any commutative ring.

Note that steps (d) and (e) work because the Cayley-Hamilton Theorem for $n\times n$ matrices is equivalent to a set of identities for commutative rings ($n^2$ identities in $n^2$ variables. This is because the coefficients of the characteristic polynomial $P_A$ of an $n\times n$ matrix $A$ are polynomials in the entries of $S$, and the computation of $P_A(A)$ is in turn done by polynomial operations.) Clearly, if an identity holds in a ring, it holds in any subring and in any homomorphic image, and this is what we have used.

P.562, Theorem XIV.3.2 [~]: Since the minimal polynomial of a linear transformation $A$ is the generator of the ideal of polynomials that $A$ satisfies, and the Cayley-Hamilton Theorem shows that the characteristic polynomial belongs to that ideal, the characteristic polynomial is divisible by the minimal polynomial. It is not hard to verify that the roots of the minimal polynomial are precisely the eigenvalues of $A$, hence these are also among the roots of the characteristic polynomial. Strikingly, Theorem XIV.3.2 shows that the eigenvalues of $A$ are *all* the roots of its characteristic polynomial, and does so without using the Cayley-Hamilton Theorem.

In fact, none of the remaining results in this section use the Cayley-Hamilton Theorem in their proofs. Rather, the Cayley-Hamilton Theorem shows the importance of the characteristic polynomial of a matrix, and the remaining results give additional information about that important polynomial.

Incidentally, note that where we previously considered $tI_n - A$, we are now looking at $A - \lambda I_n$. The former order is more convenient in defining the characteristic polynomial, since it is nice to have that polynomial monic; the latter is slightly more natural when thinking about eigenvalues, since to say a vector $x$ is in the kernel of this matrix is to say that ''$A$ acts on $x$ like multiplication by $\lambda$''. $A - \lambda I_n$ is also more convenient to write down in explicit cases, since one doesn't have to change the signs of all $n^2$ entries of $A$ (and be plagued by computational errors if one forgets to change one of them).

P.562, Proof of Theorem XIV.3.2 [~]: Note the similarity of this proof to that of Theorem VI.4.1

(Artin's Theorem on Independence of Characters, p.283). In fact, not only the proofs but the results are related. For consider a $k$-vector-space $E$ (not necessarily finite-dimensional) and a family $(A_i)_{i\in I}$ of linear maps $E \to E$. Suppose $w_1, \dots, w_m \in E$ are *simultaneous eigenvectors* of all the $A_i$ (in general, with different eigenvalues for different $A_i$), and that for each $j \neq j'$, there is some $A_i$ with respect to which the eigenvectors $w_j$ and $w_{j'}$ have different eigenvalues. Then one can deduce from the present theorem (or more easily, adapt the proof of the theorem to show) that the vectors $w_1, \dots, w_m$ are linearly independent. I claim that this result, the linear independence of simultaneous eigenvectors, is equivalent to Artin's Theorem.

For let $G$ be a monoid, and $E$ the space of all $k$-valued functions on $G$. For each $g \in G$, let $A_g: E \to E$ denote the *shift operator* defined by $(A_g w)(h) = w(hg)$. It is not hard to show that an element $w \in E$ is a character if and only if $w$ is an eigenvector of each of the operators $A_g$, and $w(1) = 1$. When this is so, the eigenvalue of the action of each $A_g$ on $w$ is $w(g)$, so distinct characters are associated with distinct families of eigenvalues. Hence by the result on linear independence of simultaneous eigenvectors, distinct characters are linearly independent.

Conversely, given a family of operators $A_i$ on a vector space $E$, note that the monoid $M$ of operators generated by the $A_i$ acts on $E$, and each simultaneous eigenvector $w$ of the $A_i$ generates a 1-dimensional invariant subspace. The action of $M$ on that subspace determines a character $M \to k$, and if we had a linear relation among eigenvectors inducing distinct characters, we could, by applying a linear functional, get a linear relation among these characters, a contradiction. So such eigenvectors are linearly independent.

P.564, Theorem XIV.3.7 [~]: This looks simpler if you only draw the exact sequence once, and express the condition that the squares in Lang's diagram commute as saying that the arrows are homomorphisms of $k[t]$-modules, where $t$ acts on $E'$, $E$ and $E''$ by $A'$, $A$ and $A''$ respectively. Note that the theorem does not require $k$ to be a field; but it is easiest to apply in that case, since the freeness hypothesis then holds automatically.

P.567, first two sentences of Proof of Theorem XIV.3.10 [>]: By ''the canonical decomposition in terms of matrices given in Theorem 2.4'' Lang means the Jordan canonical form. This gives an easy verification of the result when $k$ is a field. Now, as in the alternative proof I sketched for the Cayley-Hamilton Theorem, the result for fields immediately implies the same result for integral domains, since every integral domain embeds in a field, and if the characteristic polynomial of $M$ splits into linear factors over the integral domain, it certainly splits into the same factors over its field of fractions.

Now in the alternative proof of the Cayley-Hamilton Theorem, we passed from the case where $k$ was an integral domain to the case where it was a general commutative ring using the observation that given any $n \times n$ matrix $A$ over a commutative ring $k$, there existed an $n \times n$ matrix $U$ over an integral domain $R$ and a homomorphism $R \to k$ which carried the entries of $U$ to the corresponding entries of $A$. To argue analogously here, we would need to know that given an $n \times n$ matrix $A$ over a commutative ring $k$ such that the characteristic polynomial of $A$ is a product of linear factors, there exists an $n \times n$ matrix $U$ over an integral domain $R$ whose characteristic polynomial is also a product of linear factors, and a homomorphism $R \to k$ which carries $U$ to $A$, and carries the factorization of the characteristic polynomial of $U$ to the given factorization of the characteristic polynomial of $A$. (One would also need some elements to be mapped to the coefficients of the polynomial $f$ of the statement of the theorem, but these cause no problem, so I will leave them out of the discussion here.)

Will there in general exist an integral domain $R$ that can be so mapped? A natural way to attempt to get one would be to start with the polynomial ring in $n^2$ indeterminates, which is a commutative ring with a universal $n \times n$ matrix $U$, and then ''force'' the characteristic polynomial of $U$ to factor, by adjoining to this ring $n$ more indeterminates $y_1, \dots, y_n$ and imposing relations that equate the coefficients of the characteristic polynomial of $U$ with the corresponding coefficients of $(t-y_1) \dots (t-y_n)$. Lang comments that the statement that this ring will be an integral domain ''is obvious to the reader who read the chapter on integral ring extensions'', and that ''the reader who has not can forget about this part of the theorem'' (i.e., the case of the theorem where $k$ is not an integral domain).

Fair enough. But thinking about this, I see a way of getting around the difficulty, which I will present below. The method is instructive, but since it represents a digression from the main subject, I may or may not, in a given year, wish to assign it as reading.

*Alternative completion of the proof of Theorem XIV.3.10.* (Optional.)

The complication in the above attempt to complete the proof arose in trying to weld together two universal constructions: the commutative ring with a universal $n\times n$ matrix, which is $\mathbf{Z}[x_{11}, x_{12}, \ldots, x_{nn}]$, and the commutative ring with a universal degree-$n$ polynomial that splits into linear factors $t-y_i$, which is $\mathbf{Z}[y_1, \ldots, y_n]$. Each of these separately is a polynomial ring, hence an integral domain; but it is nontrivial to prove that the ring one gets by combining these constructions, to give the universal ring with an $n\times n$ matrix whose characteristic polynomial splits into linear factors, is also an integral domain. The trick below will be to use these constructions separately rather than together. We begin with the following result, which makes no mention of matrices.

**Lemma XIV.3.c1.** *Let $n$ and $d$ be positive integers. Then there exist polynomials $p_0, \ldots, p_{n-1}$ in $n+d+1$ indeterminates $W_0, \ldots, W_{n-1}$, $Z_0, \ldots, Z_d$ over $\mathbf{Z}$, with the property that if $k$ is any commutative ring, $g(t)\in k[t]$ is any monic polynomial of degree $n$ which factors into linear factors over $k$,*

$$g(t) \;=\; t^n + w_{n-1}t^{n-1} + \ldots + w_0 \;=\; \prod_i (t-s_i) \qquad (w_i,\, s_i \in k),$$

*and $f(t)\in k[t]$ is any (not necessarily monic) polynomial of degree $\leq d$ over $k$,*

$$f(t) \;=\; z_d t^d + z_{d-1}t^{d-1} + \ldots + z_0,$$

*then the monic polynomial $h(t)$ of degree $n$ gotten by ''applying $f$ to the roots of $g$'', i.e.,*

$$h(t) \;=\; \prod_i (t - f(s_i))$$

*has the form*

$$t^n \;+\; p_{n-1}(w_0, \ldots, w_{n-1}, z_0, \ldots, z_d)\, t^{n-1} \;+\; \ldots \;+\; p_0(w_0, \ldots, w_{n-1}, z_0, \ldots, z_d)\,.$$

**Sketch of Proof.** Form the polynomial ring in $n+d+2$ indeterminates $S_1, \ldots, S_n$, $Z_0, \ldots, Z_d$, $t$ over $\mathbf{Z}$, and in it define the polynomial

$$F(t) \;=\; Z_d t^d + Z_{d-1}t^{d-1} + \ldots + Z_0,$$

and, using this, the polynomial

$$H(t) \;=\; \prod_i (t - F(S_i)).$$

Clearly, $H(t)$ is invariant under all permutations of the variables $S_i$, hence each of its coefficients is symmetric as a polynomial in those variables; hence, by Theorem IV.6.1 (p.191), can be expressed in terms of the $Z_i$ and the elementary symmetric functions in the $S_i$, the latter being precisely the coefficients of the polynomial

$$G(t) \;=\; \prod_i (t - S_i).$$

Let $p_0(W_0, \ldots, W_{n-1}, Z_0, \ldots, Z_d)$, ..., $p_{n-1}(W_0, \ldots, W_{n-1}, Z_0, \ldots, Z_d)$ be the polynomials over $\mathbf{Z}$ such that on substituting the elementary symmetric functions in the $S_i$ for the $W$'s, one gets the coefficients of $H(t)$. It is easy to verify that these polynomials have the asserted property.

(You might wonder whether something stronger was needed where we called on Theorem IV.6.1, since we were talking about polynomials in $S_1, \ldots, S_n$, $Z_0, \ldots, Z_d$ that were only assumed symmetric in the first $n$ indeterminates. But what we needed follows by applying that theorem to the coefficient in $\mathbf{Z}[S_1, \ldots, S_n]$ of each monomial in $Z_0, \ldots, Z_d$.) $\square$

To prove Lang's Theorem XIV.3.10, it will clearly suffice to show the stronger statement that given *any* $n \times n$ matrix $M$ over a commutative ring $k$ (not necessarily having a characteristic polynomial that splits into linear factors), and any polynomial $f$ over $k$ of degree $\leq d$, the coefficients of the characteristic polynomial of $f(M)$ are the values given by the above polynomials $p_i$ on substituting for the $W_i$ the coefficients of the characteristic polynomial of $M$, and for the $Z_i$, the coefficients of $f$. Since this assertion of equality is equivalent to a family of identities, it suffices to verify it in a universal situation, namely where $M$ is a matrix of $n^2$ independent indeterminates, and the coefficients of $f$ are $d+1$ independent indeterminates. Thus, we are reduced to the case where $k$ is a polynomial ring over $\mathbf{Z}$. This is an integral domain; embedding it in its field of fractions, and that in its algebraic closure, we may factor the characteristic polynomial of $M$ into linear factors, and apply the preceding lemma and the fact that Theorem XIV.3.10 holds over fields, to get the desired result.

## Chapter XVI. The Tensor Product.

*Re §XVI.1. Tensor product.*

P.601, first paragraph [<]: Don't worry if you don't follow the latter half of this paragraph; Lang is talking about things that will be defined later on.

P.602, second paragraph [=]: Given a family $E_1, \ldots, E_n$, Lang talks here about forming a category whose objects are the $n$-linear maps on this family. In doing so, he is relying on the convention that a map determines its codomain (CAT 1, p.53). More intuitively, the objects of this category should be regarded as modules $F$ *given with* multilinear maps of $E_1 \times \ldots \times E_n$ into them. (I would prefer actually to define the category that way. I made a similar remark concerning p.58, where Lang set up the category in terms of which he characterized the free abelian group on a set $S$.)

P.602, four lines below definition of **tensor product** [=]: After ''we know, of course, that a tensor product'', add ''if it exists''.

P.602, first two lines of next paragraph [=]: I recommend replacing the words ''generated by'' by the single word ''on'' in both lines. That is, we are not taking these $n$-tuples and using them to generate a submodule of an existing module, but taking the free module on this generating set.

P.604, two paragraphs beginning **Warning.** [>]. Here is a still more important warning: Not every member of a tensor product $E_1 \otimes \ldots \otimes E_n$ is of the form $x_1 \otimes \ldots \otimes x_n$. Elements of this form are called *decomposable tensors*, and they *generate* the tensor product module; consequently, one often proves that two homomorphisms on the tensor product are equal by verifying that they agree on all elements of that form. Also, the universal property of the tensor product tells us that one can construct a morphism out of the tensor product module by specifying its values on elements $x_1 \otimes \ldots \otimes x_n$, as long as the value specified is multilinear in $x_1, \ldots, x_n$. Seeing such arguments leads some students every year to imagine that the general element of $E_1 \otimes \ldots \otimes E_n$ *is* of the form $x_1 \otimes \ldots \otimes x_n$, and to give incorrect arguments based on this assumption.

As an example where such an argument leads to an incorrect conclusion, let $k \subseteq K$ be a field extension with $1 < [K:k] < \infty$, let us regard $K$ as a $k$-module, and let us form the tensor product module $K \otimes_k K$. There is a natural $k$-module map from this module to $K$, given by $x \otimes y \mapsto xy$ (since the field multiplication of $K$ is $k$-bilinear). Now it is easy to see that this map does not annihilate any nonzero element of the form $x \otimes y \in K \otimes K$; but if we concluded from this that it has trivial kernel, we would be quite wrong, since by Proposition XVI.2.3 in the next section, $K \otimes K$ has $k$-dimension $[K:k]^2 > [K:k]$.

P.604, Proposition XVI.1.1 [>]: In the proof, Lang will construct the map going in the opposite direction. But by the symmetry of the statement, this doesn't matter.

P.605, end of proof of Proposition XVI.1.1 [=]: Why is the map Lang has constructed an isomorphism? Because one can construct, in the same way, a map going the opposite way, and these two maps take the generators $(x\otimes y)\otimes z$ and $x\otimes(y\otimes z)$ to one another. Hence their composites, in either order, are the identity.

*Remark.* When an interesting binary operation $*$ turns out to be associative, one generally finds that the function taking three elements $x, y, z$ to the common value of $(x*y)*z$ and $x*(y*z)$ represents some elegant ternary analog of the given binary operation. (Example: look at the formula for the common value of the two sides in the proof of associativity of multiplication in a group ring.) So we should ask here what the common isomorphism class of $(E_1\otimes E_2)\otimes E_3$ and $E_1\otimes(E_2\otimes E_3)$ represents. The answer is, of course, $E_1\otimes E_2\otimes E_3$. Thus, it would have been better to prove the proposition in the stronger form

$$(E_1\otimes E_2)\otimes E_3 \cong E_1\otimes E_2\otimes E_3 \cong E_1\otimes(E_2\otimes E_3),$$

via isomorphisms characterized by the behavior

$$(x\otimes y)\otimes z \leftrightarrow x\otimes y\otimes z \leftrightarrow x\otimes(\otimes y\otimes z).$$

By symmetry, it suffices to show the existence of such an isomorphism between the center object and the left-hand object. The map going outward from the center object is very easy to construct: since $(x,y,z)\mapsto(x\otimes y)\otimes z$ is 3-linear, it induces a homomorphism $x\otimes y\otimes z\mapsto(x\otimes y)\otimes z$ by the universal property of $E_1\otimes E_2\otimes E_3$. One gets the map going the other way by the same method as in Lang's proof of the proposition: fixing one variable, proving bilinearity in the other two, using this to get a map on the 2-fold tensor product, etc..

Strictly speaking, even the result described above is just the simplest case of the result one should really prove, which is that given a finite family of finite families of modules $E_{1,1},\dots,E_{1,n(1)}$, $\cdots$ $E_{m,1},\dots,E_{m,n(m)}$, one has

$$\otimes_i(\otimes_j E_{ij}) \cong \otimes_{i,j} E_{ij}.$$

This is tedious to verify, but not really harder than the case sketched above.

P.607, end of §XVI.1 [>]. I will end the notes on this section with a brief digression, followed by a longer discussion of a topic to which I will return repeatedly in this chapter.

*Digression: the duality game.*

Suppose $k$ is a field, and $E$ and $F$ are $k$-vector spaces, and we want to construct $E\otimes_k F$, but consider the system of generators and relations used on p.602 too ugly to dirty our hands with. Now though we can't easily capture $E\otimes_k F$ directly, it is easy to say what the space of linear functionals on that space should look like: By the universal property of $E\otimes_k F$ they will correspond to *bilinear forms* $E\times F\to k$. So the dual of the space we want can be identified with $L^2(E,F;k)$. Moreover, every vector space has a natural embedding in its double dual; so the space we want should embed in a natural way in $L^2(E,F;k)^\vee$. We find, in fact, that the image of $x\otimes y\in E\otimes_k F$ will be the functional $L^2(E,F;k)\to k$ that takes $\varphi\in L^2(E,F;k)$ to $\varphi(x,y)\in k$. Hence we may identify $E\otimes_k F$ with the subspace of $L^2(E,F;k)^\vee$ spanned by functionals of the form $\varphi\mapsto\varphi(x,y)$ $(x\in E,\ y\in F)$. (For a finite-dimensional vector space, the natural map into its double dual is an isomorphism, so if $E$ and $F$ are finite-dimensional, we can identify $E\otimes_k F$ with the whole space $L^2(E,F;k)^\vee$.)

You may have seen this given as the *definition* of a tensor product of vector spaces in some other course. But though the construction is ingenious, and worth seeing, I think it really misses the point of what tensor products are about. It is also inapplicable to general modules over a ring that is not a field.

A still more primitive way of constructing the tensor product of two vector spaces $E$ and $F$ is to take bases $(\xi_i)_{i\in I}$ and $(\eta_j)_{j\in J}$ of these spaces, and define $E\otimes_k F$ to be a vector space having for basis a set of symbols $(\xi_i\otimes\eta_j)_{i\in I,\,j\in J}$. This is very concrete, but does not make it clear how the spaces one gets using different choices of bases are ''naturally isomorphic''. (They are isomorphic because they have the same dimension, but one wants to specify the isomorphism that for every $x\in E,\ y\in F$ sends the element named $x \otimes y$ in one space to the element named $x \otimes y$ in the other.) It is also, like the previous construction, inapplicable to general modules over rings that are not fields.

I believe that this was the earliest definition of tensor product; the definition in terms of the dual of the space of bilinear forms was seen as an advance over this, because it gave a ''basis-free'' definition. But I think the generators-and-relations construction is preferable to both of these.

*Tensor products in the noncommutative context.*

If we want to define a concept of tensor product of modules over a not necessarily commutative ring $S$, we need an appropriate version of the concept of bilinear map. This is given in the following definition (though as I will mention later, there is not unanimity as to the language used).

**Definition XVI.1.c1.** *Let $S$ be a (not necessarily commutative) ring, $E$ a right $S$-module, $F$ a left $S$-module, and $A$ an abelian group. Then a map $\beta: E \times F \to A$ is called* bilinear *if it satisfies*

(c43) $$\beta(x+x', y) = \beta(x,y) + \beta(x',y) \qquad (x, x' \in E,\ y \in F),$$

(c44) $$\beta(x, y+y') = \beta(x,y) + \beta(x,y') \qquad (x \in E,\ y, y' \in F),$$

(c45) $$\beta(xs, y) = \beta(x, sy) \qquad (x \in E,\ s \in S,\ y \in F).$$

(Examples: If $S$ is any ring, then we may regard $S$ as a right $S$-module, a left $S$-module, and an abelian group, and the multiplication operation $S \times S \to S$ clearly satisfies the above definition. If $k$ is any ring and $m, n, p$ are positive integers, then in the matrix notation I introduced in the comments to §XIII.1, we may regard ${}^m k^n$ as a right ${}^n k^n$-module, ${}^n k^p$ as a left ${}^n k^n$-module, and ${}^m k^p$ as an abelian group. Then the matrix multiplication operation ${}^m k^n \times {}^n k^p \to {}^m k^p$ is bilinear with respect to these structures.)

We can now construct from any $E$ and $F$ as in the above definition an abelian group $A$ with a universal map having the above properties. The construction is exactly like that on p.602, mutatis mutandis, so I will leave the details to you, and simply make

**Definition XVI.1.c2.** *Let $S$, $E$ and $F$ be as in the preceding definition. Then $E \otimes_S F$ will denote the abelian group having a universal bilinear map $\otimes: E \times F \to E \otimes_S F$ (constructed using generators $x \otimes y$ for all $x \in E$, $y \in F$, and relations corresponding to* (c43)-(c45) *above).*

Note that the tensor product construction is a *functor* in each variable. I.e., given a right $S$-module homomorphism $f: E \to E'$, we get a homomorphism of abelian groups $E \otimes F \to E' \otimes F$ which acts by $x \otimes y \mapsto f(x) \otimes y$, and this construction respects composition and identity homomorphisms; and the same is true for maps of the second argument. These constructions can also be seen to respect *addition* of module homomorphisms. Finally, given both a homomorphism $f: E \to E'$ and a homomorphism $g: F \to F'$, the diagram of induced maps

$$\begin{array}{ccc} E \otimes F & \to & E' \otimes F \\ \downarrow & & \downarrow \\ E \otimes F' & \to & E' \otimes F' \end{array}$$

commutes, since both composite maps take $x \otimes y$ to $f(x) \otimes g(y)$. These facts will soon be important to us.

It may seem disappointing that the definition of bilinear map only involves an abelian group structure on $A$, and hence that our tensor product object is likewise only an abelian group. We might hope that if in the context of Definition XVI.1.c1 we also had, say, a right $S$-module structure on $A$, there would be a natural condition relating this to the right $S$-module structure on $E$, which we could expect to hold under favorable circumstances; for instance, the condition $\beta(xs, y) = \beta(x,y)s$. But in fact, this condition turns out to be unnatural; for if it holds, we see that for any $s_1, s_2 \in S$ we have

$$\beta(xs_1s_2, y) \;=\; \beta(xs_1, s_2y) \;=\; \beta(x, s_2y)s_1 \;=\; \beta(xs_2, y)s_1 \;=\; \beta(xs_2s_1, y).$$

Thus, $\beta(x(s_1s_2 - s_2s_1), y) = 0$, so that $\beta$ involves $E$ only via the factor-module $E/E[S,S]$, which is a module over the commutative ring $S/[S,S]$. The same would hold by similar arguments for the left module structure of $F$, and for the right $S$-module structure of the submodule of $A$ generated by the image of $\beta$. Thus we would really be looking at a bilinear map of modules over the commutative ring $S/[S,S]$, and this would exclude all essentially noncommutative examples, such as those indicated above.

What was wrong here, however, was only that we were trying to imitate too slavishly the commutative situation! To be able to state the correct extension of the concept of bilinear map, we first need

**Definition XVI.1.c3.** *Let $R$ and $S$ be rings. Then an* $(R,S)$-bimodule *will mean an abelian group $A$ given with both an action of $R$ by endomorphisms on the left* (*a left $R$-module structure*) *and an action of $S$ by endomorphisms on the right* (*a right $S$-module structure*), *related by the identity*

(c46) $$(rx)s \;=\; r(xs) \qquad (r \in R,\ x \in A,\ s \in S),$$

*i.e., such that $R$ acts by endomorphisms of the $S$-module structure, equivalently, such that $S$ acts by endomorphisms of the $R$-module structure.*

Examples: Any ring $S$ is an $(S,S)$-bimodule, via its internal multiplication. We can get further interesting examples by starting with examples of the above sort, taking two subrings $R$ and $T$ of $S$, and noting that $S$ will then also be an $(R,T)$-bimodule, and may have interesting $(R,T)$-subbimodules.

For example, consider the division ring $\mathbf{H}$ of quaternions (Exercise XIII:L7, p.545) as a $(\mathbf{C},\mathbf{C})$-bimodule. One sees that it is the direct sum of two sub-bimodules, namely $\mathbf{C}$ itself, and $\mathbf{C}j$. Each of these is 1-dimensional as a right and as a left $\mathbf{C}$-module, but they have nonisomorphic bimodule structures, since right and left multiplication by $i \in \mathbf{C}$ agree on one, but not on the other.

If we take for $S$ the *Weyl algebra*, generated over $\mathbf{R}$ by elements $X$ and $Y$ satisfying the relation $XY - YX = 1$ (corresponding to the differential operators of multiplication by $x$ and differentiation with respect to $x$ on polynomial functions on the real line; see this Companion, notes to p.85), then we can regard this as an $(\mathbf{R}[X], \mathbf{R}[X])$-bimodule. You may find it interesting to examine the structure of the $(\mathbf{R}[X], \mathbf{R}[X])$-subbimodule generated by $Y$ (whose general element can be written $Yf(X) + g(X)$). These two examples both show that even a bimodule over a commutative ring can have more structure than just that of a ''module'' in the sense of commutative ring theory.

Finally, as I mentioned in the notes to §XIII.1 (though I had not yet formally defined the concept of bimodule), a matrix ring ${}^m k^n$ is an $({}^m k^m, {}^n k^n)$-bimodule.

Note that in the examples of *bilinear maps* given after Definition XVI.1.c1, the objects $E$, $F$ and $A$ actually had natural bimodule structures, and the bilinear maps ''respected'' these. The sense in which this held is formalized in

**Definition XVI.1.c4.** *Suppose that in the situation of Definition XVI.I.c1, $E$ is in fact an* $(R,S)$-*bimodule, and $A$ a left $R$-module, for some ring $R$. Then we will call $\beta$ bilinear as a map from the* $(R,S)$-*bimodule $E$ and the left $S$-module $F$ to the left $R$-module $A$ if in addition to* (c43)-(c45), *it satisfies*

(c47) $$\beta(rx,y) \;=\; r\beta(x,y) \qquad (r \in R,\ x \in E,\ y \in F).$$

*If, instead, $F$ is an* $(S,T)$-*bimodule and $A$ a right $T$-module for some ring $T$, then we will call $\beta$ bilinear as a map from the right $S$-module $E$ and the* $(S,T)$-*bimodule $F$ to the right $T$-module $A$ if, in addition to* (c43)-(c45), *it satisfies* (c48) $$\beta(x, yt) \;=\; \beta(x,y)t \qquad (x\in E,\ y\in F,\ t\in T).$$

*Finally, if $E$ is an $(R,S)$-bimodule, $F$ an $(S,T)$-bimodule, and $A$ an $(R,T)$-bimodule, then we shall say that $\beta$ is bilinear with respect to these bimodule structures if* (c47) *and* (c48) *both hold.*

This would seem to call for further universal constructions, e.g., ''the tensor product of an $(R,S)$-bimodule $E$ with a left $S$-module $F$'', which would be a left $R$-module, and so forth. But marvelously, no new constructions are needed. For let $E$ be an $(R,S)$-bimodule and $F$ a left $S$-module, and let us form their tensor product $E\otimes_S F$ as a right and a left $S$-module, as before. Then since $R$ acts on $E$ by endomorphisms of its right $S$-module structure, and the tensor product construction is functorial, each element of $R$ induces an endomorphism of the abelian group $E\otimes_S F$. It is easy to verify (from the observations we made earlier concerning the functoriality of $\otimes$) that these endomorphisms constitute a left $R$-module structure on $E\otimes_S F$, with respect to which the map $(x,y)\mapsto x\otimes y$ satisfies (c47), and the universal property of that map among maps bilinear for the $S$-module structures on $E$ and $F$ immediately gives the universal property for maps that are bilinear in the stronger sense. Likewise, if, instead, $F$ has an $(S,T)$-bimodule structure, then $E\otimes_S F$ acquires a right $T$-module structure making $\otimes$ universal in the appropriate strengthened sense, and if both $E$ and $F$ have bimodule structures as above, then $E\otimes_S F$ becomes an $(R,T)$-bimodule, and $\otimes$ becomes a universal bilinear map of bimodules.

It may still seem awkward to have four different cases to consider. Fortunately, even this is only apparent. Just as an abelian group is equivalent to a $\mathbf{Z}$-module, so a right $S$-module is equivalent to a $(\mathbf{Z},S)$-bimodule, a left $S$-module is equivalent to an $(S,\mathbf{Z})$-bimodule, and an abelian group is equivalent to a $(\mathbf{Z},\mathbf{Z})$-bimodule, in the sense that any structure of the indicated ''weaker'' sort extends *uniquely* to a structure of the ''stronger'' sort, and any homomorphism of the ''weaker'' structure is in fact a homomorphism of the ''stronger'' structure. So all of the concepts of bilinearity introduced in this section can be subsumed in the concept of a bilinear map from an $(R,S)$-bimodule cross an $(S,T)$-bimodule to an $(R,T)$-bimodule; and all our versions of the tensor product can be subsumed in the statement that the tensor product with respect to $S$-module structures of an $(R,S)$-bimodule and an $(S,T)$-bimodule has a natural structure of $(R,T)$-bimodule, with respect to which the map $(x,y)\mapsto x\otimes y$ is bilinear in our strongest sense, and is universal for that property. Of course, we will continue to refer to left and right modules and to abelian groups; we will simply know that any statement we make about bimodules applies to these as special cases.

A module $E$ over a *commutative* ring $k$ can be regarded either as a left $k$-module, as a right $k$-module, or as a $(k,k)$-bimodule in which $xc = cx$ for all $x\in E$ and $c\in k$. We find that the concept of bilinear map $E\times F\to A$ of $k$-modules as defined in Lang for modules over commutative rings agrees with our above definition of bilinear map when the ''bimodule'' interpretations are used, and that the concept of tensor product, which we have seen in our context to depend essentially only on a right module structure on $E$ and a left module structure on $F$, also agrees with the definition for modules over commutative rings whenever $k$-modules $E$ and $F$ are interpreted in terms of at least this much structure.

I should mention here a point of language that I alluded to earlier. Some writers feel that since ''bilinear'' means ''twice linear'', the condition that a map from an $(R,S)$-bimodule cross an $(S,T)$-bimodule to an $(R,T)$-bimodule be ''bilinear'' should only mean that it is $R$-linear in the first argument and $T$-linear in the second argument, i.e., that it satisfies (c43), (c44), (c47) and (c48). Though these authors agree that (c45) is equally important, they feel compelled to give it a different name, calling a map that satisfies it ''balanced'' or ''middle-linear''. So they would refer to as a ''bilinear and balanced'' or ''bilinear and middle-linear'' map what I call a ''bilinear map''. I will continue to use the term ''bilinear'' to include condition (c45).

An easy exercise, which you should think through, is to verify that the object having the universal property of a *free* $(R,S)$*-bimodule on one generator* is $R\otimes_{\mathbf{Z}} S$, the generator being $1\otimes 1$.

Though it would not have been apparent from Definition XVI.I.c1 that there was any way to generalize the concept of bilinear map to get concepts of $n$-linear maps $E_1\times\ldots\times E_n\to F$ for higher $n$, we now see that these will make sense if for rings $R_0, \ldots, R_n$, each $E_i$ is an $(R_{i-1}, R_i)$-bimodule and $F$ is an $(R_0, R_n)$-bimodule. And (just as for modules over commutative rings) one can define the $n$-fold tensor product of such a family, and verify that it can be obtained by iterating the 2-fold tensor product construction. In fact, everything Lang does in the above section goes over to this context, except Proposition XVI.1.2 (commutativity of the tensor product construction), which is meaningless in the general context where $E$ is an $(R, S)$-bimodule and $F$ an $(S, T)$-bimodule, and still false (Exercise XVI.I:5) even if we happen to have $R = S = T$ so that the two sides could be compared.

In my comments on the remaining sections of this chapter of Lang, I will often note that definitions and results that he gives for tensor products $E \otimes_R F$ of modules over a commutative ring $R$ go over to the context of tensor products $E \otimes_R F$ of a right $R$-module $E$ and a left $R$-module $F$, for a not necessarily commutative ring $R$. I will seldom refer to bimodules, but I hope that the above discussion of these has shown that tensor products of the sort just described are the key to a wider theory. I will generally call the base-ring $R$, as Lang does; it was only the desire to put it in the middle of a 3-letter sequence that led me to call it $S$ in most of the above development.

Incidentally, if we assume a commutative ring $k$ to be given, then throughout the above discussion we can replace ''ring'' by ''$k$-algebra'', ''abelian group'' by ''$k$-module'', and ''bimodule'' by ''$k$-centralizing bimodule'', i.e., bimodule such that the actions of elements of $k$ on the left and on the right agree. It is easy to show that if $R$, $S$, $T$ are $k$-algebras, and $E$ and $F$ are respectively a $k$-centralizing $(R, S)$-bimodule and a $k$-centralizing $(S, T)$-bimodule, then the tensor product $E \otimes_S F$ constructed as above will be a $k$-centralizing $(R, T)$-bimodule, and will be universal among $k$-centralizing $(R, T)$-bimodules given with bilinear maps of $E \times F$ into them. But for simplicity, I will not speak of this again.

*Re §XVI.2. Basic properties.*

P.607, boxed display [=]: The isomorphism between the left-hand and right-hand sides can be expressed as saying that for a fixed $R$-module $F$, the functor $E \mapsto E \otimes F$ is *left adjoint* to the functor $G \mapsto L(F, G)$, or in more compact notation, $- \otimes F$ is left adjoint to $L(F, -)$. By interchanging the roles of $E$ and $F$, we see that $E \otimes -$ is also left adjoint to $L(E, -)$.

If you have had a bit more category theory than Lang gives, you may know that every left adjoint functor respects colimits, in particular, coproducts and direct limits. The fact that $E \otimes -$ respects direct limits was noted at the top of p.604; the fact that it respects coproducts (which in categories of modules are direct sums) will be noted on p.608. The explicit verifications given are unnecessary if one knows the general category-theoretic fact.

What form does this boxed display take in the noncommutative context? There are several versions of the statement; one good enough for our present purposes is to take $E$ to be a right $R$-module, $F$ to be a left $R$-module, and $G$ to be an abelian group. It turns out that $F$ being a left $R$-module makes the set $L(F, G)$ of abelian group homomorphisms $F \to G$ into a *right* $R$-module, by the definition $(fr)x = f(rx)$ $(f \in L(F, G),\ r \in R,\ x \in F)$. (Have I violated here my rule of writing homomorphisms of left modules on the right? No, because though $F$ is a left $R$-module, $f$ is not a left $R$-module homomorphism. In fact, if we regard $F$ as an $(R, \mathbf{Z})$-bimodule, we can describe $f$ as a homomorphism of right $\mathbf{Z}$-modules, and $fr$ as the result of composing it on the right with the $\mathbf{Z}$-module endomorphism $r$ of $F$.) If we then let $L(E,\ L(F, G))$ denote the abelian group of module homomorphisms between these *right* $R$-modules, the boxed result is easily established as a functorial isomorphism of abelian groups. In particular, $- \otimes F$ is still a left adjoint functor.

P.609, Proposition XVI.2.3 [~]: This says that if $E$ is free on a basis indexed by a set $I$, then $E \otimes F$ is isomorphic to the direct sum of an $I$-tuple of copies of $F$.

In the noncommutative context, one can say that if $E$ is free as a right $R$-module on a basis indexed by $I$, then as an *abelian group*, $E \otimes F$ is isomorphic to the direct sum of an $I$-tuple of copies of $F$. If $F$ is in fact an $(R, S)$-bimodule, then this is an isomorphism of right $S$-modules.

The corollary to this result can likewise be deduced in the noncommutative context, in the form ''If a right $R$-module $E$ is free on a basis indexed by $I$, and an $(R,S)$-bimodule $F$ is free as a right $S$-module on a basis indexed by $J$, then $E\otimes_R F$ is free as a right $S$-module, on the obvious set indexed by $I\times J$''. Note the curious fact that nothing is assumed about the left $R$-module structure on $F$, yet it has to be there for the result to make sense.

As usual, each of these statement also holds with the roles of right and left reversed; e.g., if we tensor a right $R$-module $E$ with a free left $R$-module $F$, the result has the abelian group structure of a direct sum of copies of $E$.

P.610, Proposition XVI.2.5 [>]: This is the one result of this section for which there is no evident noncommutative version. (If $E$ is a right $R$-module, that does not create any sort of module structure on $\mathrm{End}_R(E)$, so it is not clear over what one could take such a tensor product.)

Note that the homomorphism displayed in this proposition will exist without the assumption that $E$ and $F$ are free of finite rank, but in such cases it need not be one-to-one or onto. Some examples where it is not, and an additional class of cases where it *is* an isomorphism, are given in Exercise XVI.2:1. Incidentally, in the context of the material we have had so far, this map can only be considered a homomorphism of modules, but §XVI.6 will show how to make a tensor product of $R$-algebras an $R$-algebra, and in that context it will be clear that the map is an algebra homomorphism.

P.612, lines 3-5 [>]: Why must the above sequence be exact if the original sequence splits? Because a split short exact sequence $0\to E'\to E'\oplus E''\to E''\to 0$ is a termwise direct sum of two short exact sequences $0\to E'\xrightarrow{\mathrm{id}} E'\to 0\to 0$ and $0\to 0\to E''\xrightarrow{\mathrm{id}} E''\to 0$. Each of these clearly remains of the same form under tensoring with $F$, hence remains exact; and since tensoring with $F$ is a left adjoint functor, it preserves direct sums. Hence it takes the given sequence to the direct sum of two exact sequences, which is an exact sequence.

P.612, Proposition XVI.2.7 [>]: This result does not need a separate proof – we can get it by tensoring the short exact sequence $0\to\mathbf{a}\to R\to R/\mathbf{a}\to 0$ with $E$ and applying Proposition XVI.2.6 (with the roles of first and second tensor factor reversed). Indeed, this gives an exact sequence $\mathbf{a}\otimes E\to R\otimes E\to (R/\mathbf{a})\otimes E\to 0$; the middle term can be identified with $E$, and the image of the left-most term in that term is $\mathbf{a}E$, so the rightmost term is isomorphic to $E/\mathbf{a}E$, as claimed.

In the noncommutative context, the result makes sense for $E$ a left $R$-module and $\mathbf{a}$ a right ideal; in that case, $E/\mathbf{a}E$ is an abelian group. If $\mathbf{a}$ is in fact a 2-sided ideal, i.e., a sub-$(R,R)$-bimodule of $R$, then the isomorphism described will be an isomorphism of left $R$-modules (or left $(R/\mathbf{a})$-modules).

P.612, end of §XVI.2 [>]. We have seen that the functor $F\otimes -$ takes short exact sequences to sequences that are *almost* short exact, with possible failure of exactness at the left end. Can we find an example where this failure actually occurs? As Lang observed earlier on this page, in such an example the original sequence must be non-split; so we first need an example of a non-split short exact sequence of modules. We know that the rightmost module of such a sequence must be non-projective; in particular, non-free. The simplest examples of non-free modules are the $\mathbf{Z}$-modules $\mathbf{Z}/p\mathbf{Z}$. There are two easy examples of nonsplit short exact sequences with this module as their rightmost term:

$$0\ \to\ \mathbf{Z}\ \xrightarrow{p}\ \mathbf{Z}\ \to\ \mathbf{Z}/p\mathbf{Z}\to\ 0, \tag{c49}$$

$$0\ \to\ \mathbf{Z}/p\mathbf{Z}\ \xrightarrow{p}\ \mathbf{Z}/p^2\mathbf{Z}\ \to\ \mathbf{Z}/p\mathbf{Z}\to\ 0. \tag{c50}$$

What shall we tensor these with? Not a free module; tensoring with a free module sends any short exact sequence to a direct sum of copies of itself. The non-free $\mathbf{Z}$-module that suggests itself in relation to the above two sequences is obviously $\mathbf{Z}/p\mathbf{Z}$. And we in fact find that on tensoring either of them with this module (and dropping the initial 0, so as not to write down a sequence that looks like a short exact sequence but isn't), we get

$$\mathbf{Z}/p\mathbf{Z}\ \xrightarrow{0}\ \mathbf{Z}/p\mathbf{Z}\ \xrightarrow{\mathrm{id}}\ \mathbf{Z}/p\mathbf{Z}\to\ 0,$$

in which the left-hand arrow is indeed not one-to-one.

The examples (c49) and (c50) are useful test-cases whenever a non-split exact sequence is needed.

*Re §XVI.3. Flat modules.*

Pp.612-613, conditions **F1** to **F3** [>]: These are all versions of the condition ''$F\otimes -$ preserves exactness''. It is useful to know the equivalence of strong and weak forms of a condition, so that, for instance, when one has to prove a module flat, one only needs to establish the weak statement **F3**, while when flatness is known, one has available the strong statement **F1**, which, applied simultaneously to each step in an exact sequence, says that $F\otimes -$ takes arbitrary exact sequences to exact sequences. (The significance of **F2** is indicated in Exercise XVI:3.1.) You should think through the proof of equivalence that Lang sketches, noting that it is valid in the context where $F$ is a right module and the $E$'s are left modules. Most of the other results proved in this section will also be true in the not-necessarily-commutative context, so let us make the convention that as far as possible, the modules which Lang tensors with exact sequences will be taken to be right modules in our discussion, and those comprising the exact sequences will be taken to be left modules. (The corresponding results with right and left modules interchanged will of course also be true, but we will not refer to them. Unfortunately, Lang vacillates between writing the modules for which he is considering the flatness property on the right and on the left of the tensor product sign, so that about half the time you will have to implicitly reverse his order to follow our convention of discussing flat *right* modules.)

P.613, Proposition XVI.3.1 [>]: Statement (i) is immediate, since $R\otimes -$ is isomorphic to the identity functor. Statement (ii) is a consequence of the facts that $(\oplus F_i)\otimes E \cong \oplus(F_i\otimes E)$, and that a direct sum of sequences is exact if and only if all summands are exact. From (i) and (ii) we see that a direct sum of copies of $R$, i.e., any free module, is flat, hence a direct summand in a free module, i.e., a projective module, is flat, which is (iii).

Tensor products also respect direct limits (p.604, top, also p.639, Exercise XVI:L12), and direct limits of exact sequences are exact (immediate), hence we also have

**Lemma XVI.3.c1.** *A direct limit of flat modules is flat.* □

The facts that tensor products respect direct sums and direct limits, used in proving the results we have just been discussing, are instances of the general fact mentioned earlier that tensor products respect colimits. However, most sorts of colimits do not respect exactness, and so do not carry flat modules to flat modules. For instance, the *cokernel* of a module-map $f: F\to F'$ can be characterized as the colimit of a diagram based on a parallel pair of maps, $f$ and the zero homomorphism; but $\mathbf{Z}/p\mathbf{Z}$, which is the cokernel of the map $\mathbf{Z}\xrightarrow{p}\mathbf{Z}$ of flat $\mathbf{Z}$-modules, is not flat, since we saw that (c49) and (c50) do not remain exact under tensoring with it. In this connection you may find it instructive to write down the case of the snake lemma (pp.158-159) in which the middle horizontal sequences are two copies either of (c49) or of (c50), and the connecting vertical arrows are multiplication by $p$.

Let us now turn to some important

*Further criteria for flatness*

There are a couple of characterizations of flat modules rather different from **F1-F3**, which Lang relegates to an exercise (XVI:L13). Since it will be useful to refer to these in discussing the results to come, but the exercise is a hard one and the hint Lang gives for a key step is inadequate, let me develop the result here.

From Lemma XVI.3.c1 above, we see in particular that *a direct limit of free modules of finite rank is flat*. Our main result below will be the converse. In getting this, we shall use a third equivalent condition, which I will now motivate.

Given a *relation* holding among elements $x_1,\dots,x_n$ in any sort of algebraic structure, let us say (in this discussion) that the relation ''holds for good reason'' if there exist elements $y_1,\dots,y_m$ such that the $x$'s can be expressed by certain formulas in the $y$'s, in such a way that any $n$-tuple of elements of an algebraic structure of the given type that can be expressed by those formulas in terms of an $m$-tuple of elements must satisfy the given relation. As an example of what I mean, suppose two elements $x_1$, $x_2$ of a group $G$ commute with each other. In any group, any two powers of an element commute, so if we can find an element $y \in G$ and integers $s$ and $t$ such that $x_1 = y^s$ and $x_2 = y^t$, then $x_1$ and $x_2$ commute ''for good reason''. (In contrast, you will not be able to find any such ''good reason'' for the elements (1,0) and (0,1) of $\mathbf{Z}\times\mathbf{Z}$ to commute – if we regard $\mathbf{Z}\times\mathbf{Z}$ simply as a group. If we regard it as an *abelian* group, then of course any pair of its elements commutes ''for good reason''.) For another example, consider a pair of elements $x_1$, $x_2$ of an abelian group $A$ which satisfy $4x_1 = 6x_2$. If we can write $x_1 = 3y$, $x_2 = 2y$, then they satisfy the above relation for good reason; and in fact, if $A$ is torsion-free, you will find that taking $y = x_1 - x_2$, we will indeed have $x_1 = 3y$ and $x_2 = 2y$; on the other hand, if $A = \mathbf{Z}/2\mathbf{Z}$ and $x_1 = x_2 = 1$, there will be no $y$ with that property.

Rather than going further into these examples (and providing examples where the number of $y$'s used must be greater than 1), let us note how to make the above loosely stated condition precise. I spoke of a relation holding among a family of elements $x_1, \dots, x_n$ that follows from their being expressible by certain formulas in terms of another family $y_1, \dots, y_m$. A precise way to say this is that there exist elements $X_1, \dots, X_n$ in the *free* object (free group, free abelian group, free $R$-module, or whatever) $F$ on $m$ generators $Y_1, \dots, Y_m$, such that $X_1, \dots, X_n \in F$ satisfy the given relation, and such that there exists a homomorphism from $F$ into our given object which carries $X_1, \dots, X_n$ to $x_1, \dots, x_n$.

Let us at this point restrict attention to the specific sort of object we are interested in here, right modules over a fixed ring $R$. Suppose elements $x_1, \dots, x_n$ of a right $R$-module $E$ satisfy a relation $\Sigma_j\ x_j a_j = 0$. This will hold for good reason if and only if there exist an integer $m$, elements $y_1, \dots, y_m \in E$, and equations $x_j = \Sigma\ y_i b_{ij}$ holding in $E$, such that if, in the corresponding free right module ${}^mR$, we form the corresponding linear combination of the free generators $e_i$, namely the columns $b_j$ of the matrix $B = (b_{ij})$, then these will also satisfy the relation $\Sigma\ b_j a_j = 0$. If we write $A$ for the column vector $(a_j)$, then the latter relation becomes the matrix equation $BA = 0$.

Lemma XVI.3.10 (which you should read now – although it is on p.621, it does not call on any of the results proved in the intermediate pages) thus says precisely that *any* relation holding among elements of a *flat* module holds ''for good reason'', in this sense! (Actually, it says this for a finitely generated flat module, but finite generation is never used, and should be dropped from the statement.) Note that, unfortunately, to maintain our practice of making the module we are testing for flatness a right module, we must reverse the orders of all multiplications in the proof, both of elements and of tensor factors. In the discussion below I will also, for convenience, take the subscripts $i$ and $j$ to be interchanged relative to the way Lang uses them in that proof, and, for compatibility with Exercise XVI:L13, rename $M$ as $E$. Note that since we are taking $E$ to be a right module, the first displayed sequence in the proof should be regarded as an exact sequence of *left* $R$-modules,

$$0 \to K \to R^n \xrightarrow{A} R.$$

Assuming you have gone through the proof of the lemma, I now claim a little more than the lemma asserts; namely, that given elements $x_1, \dots, x_n \in E$ as above, and any *finite family* of relations that they satisfy,

$$\text{(c51)} \qquad \Sigma\, x_j a_{jk} = 0 \qquad (k = 1, \dots, r),$$

these relations must *simultaneously* hold for good reason; that is, it is possible to write the $x_j$ as linear combinations of other elements $y_i$ in such a way that the corresponding elements $X_j$ of the free module on generators $Y_i$ satisfy all $r$ of these relations.

On the one hand, this result could be proved inductively from the lemma as stated. For given a family of $r$ relations (c51) satisfied by the $x_j$, the lemma allows us to express the $x_j$ in terms of other elements $y_i$ so that *one* of these relations becomes a consequence of this way of expressing them. The remaining $r-1$ relations, on substituting for the $x$'s these expression in the $y$'s, translate to $r-1$ relations satisfied by the $y$'s. We now use the lemma again, to express the $y$'s in terms of elements $z_h$ so that one more relation is eliminated, and so on.

However we can, more briefly, get this result by going through the proof of Lemma XIV.3.10 with the $n\times 1$ matrix $(a_j)$ replaced by the $n\times r$ matrix $(a_{jk})$; in particular, with the first display replaced by

$$0 \to K \to R^n \xrightarrow{A} R^r.$$

Let us now make one further restatement of the above result. To choose a family of elements $x_1, \dots, x_n$ of a right module $E$ satisfying a family of $r$ relations $\Sigma_j\, x_j a_{jk} = 0$ is equivalent to choosing a homomorphism into $E$ from the right module $P$ presented by $n$ generators $X_1, \dots, X_n$ and $r$ relations $\Sigma_j\, X_j a_{jk} = 0$, i.e., the right module with presentation

$$^rR \xrightarrow{A} {}^nR \to P \to 0.$$

Our result that every family of elements satisfying such a system of relations is the image of a family of elements in a free module which satisfies the same system of relations thus translates to the statement that every homomorphism from a *finitely presented* right $R$-module $P$ into $E$ factors through a free module $F$ (without loss of generality, of finite rank).

Turning to Exercise XVI:L13 on p.639, we see that we have proved (i) ⇒ (ii) ⇒ (*) ⇔ (iv), where (*) is the result of Lemma XIV.3.10, generalized to finite families of relations (c51).

(We have sidestepped condition (iii) of the exercise. If you want to work that condition in, note that in our context, the displayed homomorphism thereof should be written

$$E \otimes_R \mathrm{Hom}(P, R) \to \mathrm{Hom}(P, E),$$

and that the condition that it be surjective is equivalent to saying that every homomorphism $P \to E$ is a sum of finitely many homomorphisms that each factor through $R$. But such a finite sum is equivalent to a single homomorphism that factors through a direct sum of copies of $R$, and the condition, so translated, is easily seen to be equivalent to (iv).)

Before approaching the final implication, (iv)⇒(i), let us note that any module $E$ is a direct limit of finitely presented modules. Indeed, take any presentation of $E$ by a generating set $S$ (e.g., all elements of $E$) and a set $T$ of relations satisfied by the elements of $S$ in $E$ (e.g., all such relations). Now consider the set $I$ of all pairs $(S_0, T_0)$ such that $S_0$ is a finite subset of $S$, and $T_0$ is a finite subset of $T$ consisting of relations which involve only elements of $S_0$. We may partially order $I$ by componentwise inclusion, and we see that as such it is *directed* (every pair of elements has a common upper bound). Now for every $(S_0, T_0)\in I$, let us define $P_{(S_0,\,T_0)}$ to be the module presented by generators $S_0$ and relations $T_0$. Whenever $(S_0, T_0) \leq (S_1, T_1)$ in $I$, we clearly get a unique module homomorphism $P_{(S_0,\,T_0)} \to P_{(S_1,\,T_1)}$ taking the image in $P_{(S_0,\,T_0)}$ of each element of $S_0$ to the image in $P_{(S_1,\,T_1)}$ of the same element, and these maps make the $P_{(S_0,\,T_0)}$ a directed system of modules. These modules all have natural maps into $E$ which make commuting triangles with the above maps, and the universal property of $E$ as the module presented by the given generators and relations immediately implies its universal property as a direct limit of these finitely presented modules.

We now seem very close to having the implication (iv)⇒(i). Given a module $E$ that satisfies (iv), let us express it as $\varinjlim_I P_i$ for some directed system $(P_i)_{i\in I}$ of finitely presented modules, denoting the connecting homomorphisms by

$$p_{j,i}: P_i \to P_j \qquad (i < j \text{ in } I),$$

and the limit homomorphisms by

$$p_i: P_i \to E.$$

We may use (iv) to factor each of the $p_i$ through a free module of finite rank:

(c52) $$P_i \xrightarrow{h_i} F_i \xrightarrow{f_i} E.$$

But here we run into a problem. Condition (iv) is not a functorial statement – it says that for *each* map from a finitely presented module $P_i$ into $E$ we can get a factorization through an $F_i$, but not that we can do this so that for each map between such modules $P_i \to P_j$ ''over'' $E$ (i.e., making a commuting triangle with the maps to $E$) we can get a map $F_i \to F_j$ ''over'' $E$, and certainly not that we can do this in a way that respects composition on our infinite family of modules and maps.

Indeed, once the factorizations (c52) have been chosen, it may not be possible to get any corresponding maps among the $F$'s. For given $i<j$, note that $p_i(P_i) = p_j\, p_{j,i}(P_i) \subseteq p_j(P_j)$; but $F_i$ and $F_j$ may have been chosen so that $f_i(F_i)$ is not similarly contained in $f_j(F_j)$; in which case we cannot get a map $F_i \to F_j$ over $E$.

I claim, however that (assuming the factorizations (c52) chosen in an arbitrary but fixed way), if we take $j$ ''far enough above $i$'', this problem vanishes. Precisely, for each $i\in I$, the facts that $E$ is the direct limit of the $P_j$ and that $f_i(F_i)$ is finitely generated tell us that there will exist some $i'>i$ such that $f_i(F_i) \subseteq p_{i'}(P_{i'})$. Then for any $j\geq i'$, we will have $f_j(F_j) \supseteq p_j(P_j) \supseteq p_{i'}(P_{i'}) \supseteq f_i(F_i)$. Since $F_i$ is free, this allows us to map $F_i$ to any such $F_j$ over $E$; in particular, to $F_{i'}$. In fact, we will find it useful to do something stronger, which the condition $f_i(F_i) \subseteq p_{i'}(P_{i'})$ also shows is possible, namely, to choose a homomorphism $\theta_i: F_i \to P_{i'}$ over $E$.

Unfortunately, the map $\theta_i$ may still not be very ''good'', in the sense that the composite $P_i \xrightarrow{h_i} F_i \xrightarrow{\theta_i} P_{i'}$ may not agree with the map $p_{i',i}: P_i \to P_{i'}$ of the original directed system. However, note that for every $x\in P_i$, the two elements $\theta_i h_i(x)$ and $p_{i',i}(x)$ will be elements of $E_{i'}$ mapping to the same element of $E$, hence (again by the fact that $E$ is the direct limit of our given system), if we take an $i''$ sufficiently high above $i'$, we will have

$$p_{i'',i'}\theta_i h_i(x) \;=\; p_{i'',i'}p_{i',i}(x) \;=\; p_{i'',i}(x).$$

Moreover, if we choose $i''$ so that the above relation holds for all members of a finite generating set for $P_i$, it will hold for all $x\in P_i$. Renaming things, we conclude that for each $i\in I$ we can choose an $\alpha(i) \geq i$ (what we called $i''$ above), and a homomorphism

$$\varphi_i: F_i \;\to\; P_{\alpha(i)}$$

(called $p_{i'',i'}\theta_i$ above) such that

$$\varphi_i\, h_i \;=\; p_{\alpha(i),\,i}.$$

Let us now define a relation $\leq'$ on $I$ by

$$j \geq' i \text{ if and only if } j=i \text{ or } j \geq \alpha(i).$$

It is easy to verify that this is a partial ordering, and again makes $I$ a directed partially ordered set. Further, for $j\geq' i$ let us define $f_{j,i}: F_i \to F_j$ by

$$f_{j,i} \;=\; \begin{cases} \mathrm{id}_{F_i} & \text{if } j=i, \\ h_j\, p_{j,\alpha(i)}\, \varphi_i & \text{if } j \geq \alpha(i). \end{cases}$$

You should now be able to verify that these maps make the free modules $F_i$ a directed system, with $E$ as its direct limit. This completes the proof of

**Theorem XVI.3.c2** (D. Lazard; cf. Lang, Exercise XVI:L13)**.** *Let $E$ be a right module over a ring $R$. Then the following conditions are equivalent.*

(a) *$E$ is flat.*

(b) *Every homomorphism from a finitely presented right $R$-module $P$ into $E$ factors through a free module.* (*Equivalently, in the ad hoc language introduced earlier in this discussion, every finite family of* *relations holding among a finite family of elements of* $E$ *holds ''for good reason''*).

(c) $E$ *is the direct limit of a directed system of free right modules of finite rank.* □

The above proof of (b)⇒(c) follows the Bourbaki reference given in Lang. The proof in Lazard's original paper (*Sur les modules plats,* Comptes Rendus Acad. Sci. Paris, **258** (1964) pp.6313-6316) is sketched as Exercise XVI.3:3.

Let us note that from one of the relatively easy parts of the above lemma, the implication (a)⇒(b) (equivalent to Lemma XVI.3.10), we get the

**Corollary XVI.3.c3** (D. Lazard)**.** *If a flat module* $E$ *is finitely presented, it is projective.*

**Proof.** The identity map of $E$ is in this situation a homomorphism from a finitely presented module to a flat module, hence it factors through a free module $F$:

$$E \xrightarrow{b} F \xrightarrow{a} E \quad \text{with} \quad ab = \mathrm{id}_E.$$

It follows that the other composite, $ba$, is a retraction of $F$ onto a direct summand isomorphic to $E$. □

Incidentally, suppose that $E$ is a finitely generated projective module, say the image of an idempotent endomorphism $e$ of a free right module ${}^nR$. By Lang's Proposition XIV.3.1(iii), we know that $E$ is flat, and we may ask how to get an explicit directed system of free modules of finite rank having $E$ as direct limit. Amusingly, such a system is given by

$${}^nR \xrightarrow{e} {}^nR \xrightarrow{e} {}^nR \xrightarrow{e} \ \ldots \ .$$

Now back to the initial pages of this section of Lang.

P.613, Proposition XVI.3.2 [=]: In this proposition, we must assume $R$ commutative. (As noted in my comments on the end of §II.4, there are various noncommutative concepts of ''localization''. Some of these do and others don't have the property that a localization of a ring $R$ is flat as a right $R$-module; Exercise XVI.3:11 gives examples.) For the definitions of the symbols used in (ii) and (iii), see Exercises III:L9-10 (pp.167-168).

The proofs of these three statements don't seem to me as obvious (in terms of the material given so far) as Lang claims. I would guess that the proof of statement (i) that Lang had in mind was to combine Exercise III:L9(b) with the fact that $S^{-1}M \cong M \otimes_R S^{-1}R$. However, that isomorphism hasn't been noted yet; in fact, the right context for it is the next section.

A proof appropriate here would be the following. Let $u\colon X \to S$ be a homomorphism from a free abelian monoid $X$ (written multiplicatively) onto $S$. Now a free abelian monoid can be ordered by *divisibility*, and is directed under this ordering. For every pair of elements related under this ordering, $x \leq xy \in X$, let us define $f_{xy,x}\colon R \to R$ to be the $R$-module homomorphism given by multiplication by $u(y)$. This defines a directed system of $R$-modules (all equal to $R$) indexed by $X$, and it is easy to verify that the direct limit of this system can be identified with $S^{-1}R$ by sending the copy of $r \in R$ in the member of this system indexed by $x \in X$ to $u(x)^{-1}r$. Hence $S^{-1}R$, being a direct limit of free modules, is flat.

In statement (ii), the implication that if $M$ is flat then so is $M_{\mathbf{p}}$ is true in the more general form ''if $M$ is flat, then so is $S^{-1}M$ for any multiplicative subset $S$''. (Recall that $M_{\mathbf{p}}$ means $(R-\mathbf{p})^{-1}M$.) This can be proved exactly like (i), by writing $S^{-1}M$ as a direct limit of copies of $M$. For the other direction, first note that localization commutes with tensor products, i.e.,

$$(S^{-1}M) \otimes_R E \ \cong \ S^{-1}(M \otimes_R E) \ \cong \ M \otimes_R (S^{-1}E). \tag{c53}$$

This can easily be seen from the above direct limit description of localization. Now if $M$ is not flat, find a short exact sequence whose exactness is not preserved under $M \otimes_R -$. Then note that by

Exercise III:L10(b), there must be some prime $\mathbf{p}$ such that this nonexactness remains true on localizing at $\mathbf{p}$; and finally, use (c53) to turn this nonexactness into the statement that $M_{\mathbf{p}}$ is not flat.

Turning to (iii), to get the ''if'' direction, assume $F$ is torsion free, regard it as the directed union of its finitely generated submodules, and note that this expresses it as a direct limit of finitely generated torsion free modules, which by the theory of finitely generated modules over a PID are free. For the ''only if'' direction, observe that if $F$ has torsion, then the calculation by which we showed that the $\mathbf{Z}$-module $\mathbf{Z}/p\mathbf{Z}$ was non-flat can be adapted to show $M$ non-flat. Namely, if $F$ has a nonzero element annihilated by a nonzero $r \in R$, then tensoring with $M$ does not preserve the one-one-ness of the map $R \xrightarrow{r} R$. This argument, indeed, gives the more general statement on the last two lines of the page. (Alternatively, we can get this using Lazard's Theorem, i.e., Theorem XVI.3.c2 above, together with the observation that a free module over an integral domain is torsion-free, hence so is a direct limit of such modules.)

P.614, first two paragraphs [<].

*Examples of non-flat torsion-free modules.*

Lang describes several classes of examples, but these assume a nontrivial amount of number theory and/or commutative algebra; so let me give some cases where non-flatness can be verified ''by hand''.

First, let $R$ be a polynomial ring $k[x,y]$ in two indeterminates over a field. (This is not the $k[x,y]$ of Lang's first paragraph, since his $x$ and $y$ satisfy a polynomial relation.) Now it follows from unique factorization in $R$ that in a free $R$-module, if we have two elements $\xi$ and $\eta$ such that $\xi y - \eta x = 0$, there must be an element $\alpha$ such that $\xi = \alpha x$ and $\eta = \alpha y$; hence the same will be true in any direct limit of free modules, i.e., any flat module. Within $R$ itself, if we take $\xi = x$, $\eta = y$, then of course such an element is given by $1$. But in the ideal $(x,y)$ of $R$, there is no such element, hence that ideal is not flat as an $R$-module. (In other words, $x$ and $y$ satisfy the indicated module-relation ''for good reason'' in $R$, but not in $(x,y)$.)

This argument can be formulated so as not to use the hard part of Lazard's Theorem, but only the implication (a)⇒(b). For this shows that if $(x,y)$ were flat, we could factor its identity map through a free module. Comparing the coordinates of the images of $x$ and $y$ in that free module, we can deduce by unique factorization that those images have the forms $\alpha x$ and $\alpha y$ respectively; then, considering the image of $\alpha$ under the map from this free module back into $(x,y)$, we get a contradiction.

Can we give an explicit example where $(x,y) \otimes_R -$ does not preserve exactness? Yes; tensor $(x,y)$ with the exact sequence

$$0 \to (x,y) \to R \to R/(x,y) \to 0.$$

The element $x \otimes y - y \otimes x$ of $(x,y) \otimes_R (x,y)$ belongs to the kernel of the map into $(x,y) \otimes_R R \cong (x,y)$, but is nonzero. The former fact is immediate; to show the latter, make $k$ an $R$-module by letting $x$ and $y$ act trivially, and define an $R$-bilinear map from $(x,y) \times (x,y)$ to the direct sum of four copies of this module, so as to send each member of $\{x,y\} \times \{x,y\}$ to a different basis element. One finds that the induced $R$-linear map from $(x,y) \otimes_R (x,y)$ into that $k$-vector space takes $x \otimes y - y \otimes x$ to a difference of basis elements. (One copy of $k$ would have sufficed instead of four, but this would have required using a contrived-looking bilinear map, while the map used in this case looked, I hope, natural.) The same method works for the power series ring $k[[x,y]]$. (Lang gives this example in the second paragraph of this page, with a different argument.)

From the point of view of algebraic geometry, the above examples used rings whose spectra have dimension > 1. (The student who is not familiar with algebraic geometry should not worry; this is just an aside to those who are. The ring-theoretic arguments below will be self-contained. My later reference to the concept of integral closure can likewise be ignored by the reader not familiar with that concept.) There are also rings whose spectra are 1-dimensional, but have a singularity at some point that make them ''fat'' enough there to get a similar example. For example, consider the subring $k[x^2, x^3] \subseteq k[x]$. One can show that its ideal $(x^2,x^3)$ is nonflat as a module by using $x^2$ and $x^3$ in place of $x$ and $y$ in the arguments for the preceding example. (Instead of unique factorization, which does not hold, one must consider explicitly the factorization properties of powers of $x$ in this ring; but this is not difficult; the equation $x^2\cdot x^3 = x^3\cdot x^2$, when looked at as a module relation, still holds without ''good reason'' in the ideal.) Note further that if we regard $k[x]$ as a module over its subring $k[x^2,x^3]$, then the linear map $f(x) \mapsto x^2 f(x)$ is an isomorphism between this module and the above ideal of $k[x^2,x^3]$; so having proved that ideal non-flat, we can say that the ring $k[x]$, which is the integral closure of $k[x^2,x^3]$, is non-flat as a module over that subring. Thus, this is an example in the spirit of the one with which Lang begins this paragraph.

The above subring of $k[x]$ can be described as consisting of those functions whose derivative at $x = 0$ is zero; from the point of view of algebraic geometry, its spectrum is the line ''pinched'' at that point. Another example of non-integral-closure is gotten by taking the line and ''tying together'' the two points $x = 1$ and $x = 0$, getting the ring $\{f\in k[x] \mid f(0) = f(1)\}$. A non-flat ideal of this ring is given by the set of polynomials whose common value at those two points is 0; in showing non-flatness, one may use the elements $x^2(x-1)$ and $x(x-1)^2$ in the role of ''$x$ and $y$''. Again, $k[x]$ is $R$-module-isomorphic to this ideal, this time by $f(x) \mapsto x(x-1)f(x)$.

Rather than going through separate but parallel arguments for nonflatness in each of the above cases as sketched above, one can handle them all with one lemma; this is done in Exercise XVI.3:4. Exercise XVI.3:9 gives another condition which quickly shows the non-flatness of the last two examples.

P.614, Lemma XVI.3.3 [>]: Note that this result is easy when $F$ is projective, since the given exact sequence splits; this illustrates the fact that flat modules can be considered ''close to'' projective. An alternative proof of the lemma, which uses Lazard's Theorem to reduce to the case where $F$ is in fact free, and hence projective, is sketched in Exercise XVI.3:5 (which also does the same for Proposition XVI.3.4 on the next page).

In Lang's proof, note that he does not show the full diagram of the snake lemma, but only those parts that are relevant to the argument at hand. He will follow this practice in subsequent proofs as well.

P.615, Proposition XVI.3.4 [>]: Strictly speaking, for the phrase ''*More generally*'' (preceding the last display) to be correct, the last line should say that if $F^2,\dots,F^n$ are flat, then $F^0$ is flat if and only if $F^1$ is. This is true, and may be proved by induction from the first part of the proposition: Let $n>2$, let us assume inductively that the above statement is true for smaller $n$, and let $K$ denote the kernel of the map $F^2 \to F^3$, equivalently, the cokernel of the map $F^0 \to F^1$, in the given exact sequence. Thus, that sequence decomposes into two exact sequences

$$0 \to F^0 \to F^1 \to K \to 0, \qquad 0 \to K \to F^2 \to \dots \to F^{n-1} \to F^n.$$

Our inductive hypothesis, applied to the second sequence, shows that $K$ is flat, and the first part of the proposition now shows the asserted conclusion about $F^0$ and $F^1$.

The above is what Lang means by the second paragraph on the next page. The longer paragraph which precedes it proves the first assertion of the result. In following that proof, note that where, just below the diagram, Lang says ''The 0 on the right'' he means the 0 at the top; and that for the remainder of the proof, he is using what the snake lemma says about the kernels of the three vertical maps. The importance of the 0 on the left in the first row is that it shows that the map of kernels above is also one-to-one.

P.617, five lines above bottom [=]: The phrase ''it becomes 0 already in a finite subsum'' is vague. A more precise formulation of the argument is that any element of the indicated kernel is a finite sum $z = \Sigma\, x_j \otimes y_j$ $(x_j\in F,\ y_j\in E')$, and each $y_j\in E' \subseteq E$ is a sum of elements from finitely many of the $E_i$. If we write $E_{I_0} \subseteq E$ for the sum of the finitely many $E_i$ that are thus involved in expressing $z$, and let $E'_{I_0} = E'\cap E_{I_0}$, then we can show that the element $\Sigma\, x_j \otimes y_j \in F\otimes E'_{I_0}$ is in the kernel of the map to $F\otimes E_{I_0}$, and is nonzero if $z$ is, because one can map it to $z$. Thus, if the result is true for finite direct sums $E_{I_0}$, it is true for arbitrary direct sums $E$.

But an easier approach is to forget about dealing with individual elements, and note that the exact sequence $0 \to E' \to E$ is the direct limit, over the directed partially ordered set of all finite subsets $I_0 \subseteq I$, of the exact sequences $0 \to (E' \cap E_{I_0}) \to E_{I_0}$, and that direct limits respect tensor products and exactness. In either case, we are reduced to finite direct sums, for which Lang's proof is fine, except for a pair of misprints, noted below:

P.618, end of proof of Lemma XVI.3.6 [>]: Now observe that if an $R$-module $F$ is ''$R$-flat'', then by the above lemma, it is flat with respect to any *free* $R$-module, hence by the preceding lemma, it is flat with respect to any homomorphic image of a free $R$-module, i.e., with respect to any $R$-module; i.e., it is flat. As the converse is trivially true, we see that flat is equivalent to $R$-flat. This is the culmination of the preceding study of ''$E$-flatness'', and is the content of the second assertion of the next result, Proposition XVI.3.7. (For the noncommutative version, we should assume $F$ a right $R$-module, and let $\mathbf{a}$ range over all left ideals of $R$.)

Now the exactness condition associated with the short exact sequence of that assertion is equivalent to saying that the map $F \otimes \mathbf{a} \to F$ is one-to-one. The image of this map is $F\mathbf{a}$, so this says that the natural map $F \otimes \mathbf{a} \to F\mathbf{a}$ is an isomorphism, which is the first assertion of the proposition (adjusted to make modules we are testing for flatness right modules).

The above is essentially Lang's proof of the proposition, but I think more clearly stated.

P.619, Lang's **Remark on abstract nonsense** [<]: Skip this (except the last one-sentence paragraph!) unless you know about, and like, abelian categories.

If you do read it, note that there is an unnamed third abelian category, which we could call $\mathbb{C}$, in which $T$ is assumed to take values; and that ''right exactness'' is the name of the property proved for $F \otimes -$ in Proposition XVI.2.6 (p.610).

P.619, Theorem XVI.3.8 [=]: Lang will first prove this in the case where $R$ is Noetherian, then show how to modify the proof to work under the weaker assumption that $M$ is finitely presented, and finally, he will give a different proof for the general case.

Concerning his proofs in the first two cases, I will merely mention that Lemma XVI.3.9 can be considered an immediate consequence of an easily proved but powerful result given in a later chapter, Schanuel's Lemma (Lemma XXI.2.4).

The big tool that he brings to bear to get the general case of Theorem XVI.3.8 is Lemma XVI.3.10, which we have already discussed. His explanation of how to get the theorem from that lemma (next-to-last paragraph of p.621) is sketchy, so I will give a proof here.

Suppose $M$, $R$ and $x_1, \dots, x_n$ are as in the theorem, where we will as usual understand $M$ to be a right module. By Nakayama's Lemma (say the version given as Lemma X.4.3), $x_1, \dots, x_n$ generate $M$. Let $f: F \to M$ be a homomorphism from a free module of rank $n$ onto $M$ taking a basis of $F$ to $x_1, \dots, x_n$. The fact that $x_1, \dots, x_n$ are linearly independent modulo $M\mathbf{m}$ tells us that $\mathrm{Ker}(f) \subseteq F\mathbf{m}$. We wish to show that this kernel is $0$.

So say $y \in \mathrm{Ker}(f)$. Then $f$ factors

$$F \to F/yR \to M$$

Here the right-hand map is from a finitely presented module to a flat module, so by our translation of Lemma XVI.3.10, it factors through a free module $G$. This means that our map $f$ factors through $G$

$$F \xrightarrow{g} G \xrightarrow{h} M$$

in such a way that $y \in \mathrm{Ker}(g)$. On the other hand, since $f$ is surjective, we can factor $h$ through it (since the domain of $h$ is free):

$$F \xrightarrow{g} G \xrightarrow{e} F \xrightarrow{f} M.$$

Thus $feg = f$, so the endomorphism $eg - \mathrm{id}_F$ of $F$ has range in $\mathrm{Ker}(f) \subseteq F\mathbf{m}$, so the image under

$eg$ of a basis $\mathcal{B}$ of $F$ is congruent elementwise to $\mathcal{B}$ modulo $F\mathbf{m}$, so by Theorem X.4.4, $eg(\mathcal{B})$ is a basis. In particular, its elements are linearly independent, so $eg$ is one-to-one. But $y \in \mathrm{Ker}(g)$, so $y = 0$, as required.

P.622, [<]: Skip this unless you have read Chapter XX or have the equivalent background.

*Re §XVI.4. Extension of the base.*

P.623, beginning of §XVI.4 [<]: What Lang does here is valid in the noncommutative context (say for left $R$-modules, since he is tensoring on the left with extension rings), up to a point on the next page where I will note the contrary.

Before we go further, let us play some

*Games with adjoints*

Given a ring homomorphism $f: R \to R'$, note that there is a very simple construction taking left $R'$-modules to left $R$-modules: Regard the underlying abelian group of the given $R'$-module $F$ as an $R$-module by defining $rx$ (for $r \in R$, $x \in F$) to be $f(r)x$. This is called *restriction of scalars* from $R'$ to $R$ (a term adapted from the special case where $f$ is the inclusion of a subring $R$ in a larger ring $R'$). You should not find it hard to verify that the construction of *extension of scalars*, described by Lang, is left adjoint to restriction of scalars. You should write out the universal property that this implies.

The next two results that Lang will prove can both be looked at as immediate consequences of the fact that composites of adjoint functors are adjoints of the composite functors (Lemma II.3.c3). Indeed, given a ring homomorphism $f: R \to R'$, consider the functors

$$R'\textbf{-modules} \xrightarrow{\text{restriction of scalars}} R\textbf{-modules} \xrightarrow{\text{forget module structure}} \textbf{Sets},$$

whose composite is the forgetful functor from $R'$-modules to sets. The left adjoints of the above two functors are

$$R'\textbf{-modules} \xleftarrow{\text{extension of scalars}} R\textbf{-modules} \xleftarrow{\text{free } R\text{-module}} \textbf{Sets},$$

so these compose to the left adjoint to the above composite, which is the free $R'$-module functor. Thus, extension of scalars takes free modules to free modules, which is Proposition XVI.4.1.

Likewise, if we have two ring homomorphisms $R \to R' \to R''$, and we consider successive restriction of scalars from $R''$-modules to $R'$-modules to $R$-modules, then the above observation on composites of adjoint functors gives the result that Lang on the next page calls transitivity of extension of base.

Going back to the definition of extension of scalars, let us note that where Lang has described ''by hand'' how to make $R' \otimes_R E$ into an $R'$-module, there is a neat formal approach. Given a homomorphism $R \to R'$, let us regard $R'$ as an $(R', R)$-bimodule, using its natural left module structure over itself, and getting the right $R$-module structure by starting with its right module structure over itself, and then restricting scalars. Then extension of scalars can be regarded as the operation of tensoring left $R$-modules with this $(R', R)$-bimodule, which we know gives left $R'$-modules.

Suppose instead that we look at $R'$ as an $(R, R')$-bimodule. Then we can tensor a left $R'$-module $E$ with it to get a left $R$-module. What is this module? Tensoring an $R'$-module with $R'$ is essentially the identity functor; so this construction associates to any left $R'$-module $E$ the ''same'' object, but considered as a left $R$-module. It is not hard to check that this gives the restriction of scalars functor!

We saw above that restriction of scalars was right adjoint to extension of scalars; but here we find that it can be expressed as a tensor product, hence as a left adjoint functor. What is its right adjoint? You can verify that it is the functor associating to every left $R$-module $E$ the abelian group $\mathrm{Hom}_R(R', E)$, where by this we understand the set of homomorphisms as left $R$-modules, and where we make this set a left $R'$-module with the help of the *right* $R'$-module structure of $R'$. (I.e., if $f \in \mathrm{Hom}_R(R', E)$, and we write $f$, according to our convention, to the right of its argument, then for any $s \in R'$, we define $sf$ by $x(sf) = (xs)f$.)

Thus, the restriction-of-scalars functor belongs to the rather special class of functors that have both left and right adjoints. (Another such functor is the forgetful functor from monoids to groups.)

P.624, a third of the way down, where Lang writes ''If $E$ has a multiplicative structure'' [<]: From this point to the bottom of the page, $R$ should be assumed commutative.

P.625, Proposition XVI.4.2 [=]: The two results that Lang labels ''Base change'' and ''Transitivity'' might better be labeled ''Extension of scalars'' and ''Restriction of scalars''.

Both of these results are easy to get using associativity of the tensor product. Unfortunately, Lang has only noted this associativity for modules over a common commutative ring, and this is not quite enough. The useful result here is associativity of tensor products of bimodules: If $E$ is a right $R$-module, $F$ an $(R,S)$-bimodule, and $G$ a left $S$-module, then

$$(E \otimes_R F) \otimes_S G \;\cong\; E \otimes_R F \otimes_S G \;\cong\; E \otimes_R (F \otimes_S G),$$

and, of course, if either or both of the modules $E$, $G$ is in fact a bimodule, these isomorphisms respect the resulting module or bimodule structures on these tensor products. If we now use the characterizations of extension and restriction of scalars as tensoring operations, then we immediately get noncommutative versions of the above two results, namely that if $R \to S$ is any ring homomorphism, then extension of scalars takes flat left $R$-modules to flat left $S$-modules, while if $S$ is flat as a left $R$-module, restriction of scalars takes flat left $S$-modules to flat left $R$-modules. In each case, to verify that the tensor product modules in question are flat, ''test'' for flatness by tensoring them with a one-one-map of the appropriate sort of module, and use associativity of the tensor product to deduce a positive answer to this test. (Why is a hypothesis on $S$ needed in the case of restriction, but no corresponding hypothesis needed in the case of extension?)

*Re §XVI.5. Some functorial isomorphisms.*

P.625, lines immediately below the diagram [=]: What Lang calls an ''additive category'' is more commonly called (and I will call it) an **Ab**-category. Here **Ab** is the standard symbol for the category of abelian groups; an **Ab**-category means a category whose hom-sets are objects of **Ab**, and such that for every morphism $f$, the composition operations $f \circ -$ and $- \circ f$ act by morphisms of **Ab**. The term ''additive category'' is commonly used for **Ab**-categories in which every pair of objects has a coproduct, i.e., a direct sum. Most of the **Ab**-categories that come up naturally satisfy a still larger set of important properties, which are abstracted in the definition of *abelian category*, Lang, p.133. However, the simpler concept of **Ab**-category is all that he needs for the definitions he makes here.

I don't know whether the term ''additive functor'' is standard; I will follow Lang in this usage.

To see that not every naturally occurring functor among **Ab**-categories is additive, let $R$ be a commutative ring, and consider the functor from $R$-modules to $R$-modules taking $E$ to $E \otimes_R E$.

P.626, Proposition XVI.5.1 [~]: This result and its proof are valid for right or left modules over not necessarily commutative rings. More generally, the argument shows that if a morphism of additive functors between module categories (not necessarily modules over the same ring) is an isomorphism on some family of modules, it is an isomorphism on their direct sum. Still more generally, one could prove such a result for additive functors among **Ab**-categories, but one would first need to develop tools for studying products and coproducts in such categories, rather than working with elements as Lang does here.

P.627, Corollary XVI.5.3 [~]: Here let's keep Lang's assumption that $R$ is commutative. (If we want a noncommutative version of this result, we need a context where it first makes sense for the free module of rank 1; but I don't see any very natural contexts where the sets $L(R,R)$ have module structures that allow us to take tensor products, and such that the two sides come out isomorphic.)

P.628, Corollary XVI.5.4 [~]: Calling this a ''functorial isomorphism'' is a little sticky, because $L(E',E)$ is covariant in the first variable and contravariant in the second variable, hence one cannot make $L(E,E)$ into either a covariant or a contravariant functor in its one variable.

A way to get around this is to restrict attention to the category whose objects are our modules, but whose morphisms are the *isomorphisms* among these modules. In that case, by taking isomorphisms to their inverses, one can interchange covariant and contravariant functors, and indeed, given an isomorphism $f: E \cong E'$, one gets an isomorphism $\mathrm{End}(E) \cong \mathrm{End}(E')$, namely $g \mapsto fgf^{-1}$. As a construction on this restricted category, the isomorphism of this corollary is in fact functorial.

P.628, Corollary XVI.5.5 [~]: This is valid for right modules over an arbitrary ring $R$. Recalling that $E^{\vee}$ should then be considered a left $R$-module, we see that both sides of the isomorphism can be regarded as abelian groups.

P.629, both results [<]: Assume $R$ commutative here.

*Re §XVI.6. Tensor product of algebras.*

Assume $R$ commutative throughout this section, but note that the $R$-algebras $A$ that Lang is considering need not be commutative.

P.630, Proposition XVI.6.1 [>]: This result, that in the category of commutative $R$-algebras the tensor product gives the coproduct, is actually a case of a characterization of the tensor product of two not-necessarily-commutative $R$-algebras $A$ and $B$: namely, that $A \otimes_R B$ is universal among $R$-algebras given with homomorphisms of $A$ and $B$ into them such that every element of the image of $A$ commutes with every element of the image of $B$.

P.631, line beginning **Graded Algebras.** [=]: There is no need to assume $G$ commutative. Probably Lang's reason was that the most frequent grading monoids are the nonnegative integers, the integers, and $\mathbf{Z}/2\mathbf{Z}$, all of which are written additively, so he wants to use additive notation. But gradings by noncommutative monoids, especially noncommutative groups, are also important, and people who work with graded rings are comfortable with both additive and multiplicative notation.

P.631, last paragraph [~]: Actually, if $A$ is $\mathbf{Z}$-graded, then the graded algebra Lang here denotes $A_{su}$ is isomorphic to $A$; but when $A$ is $\mathbf{Z}/2\mathbf{Z}$-graded, they can be nonisomorphic (Exercise XVI.6:1). On the other hand, the ''super tensor product'' discussed on the top half of the next page is in general not isomorphic to the ordinary tensor product for either grading group.

P.632, end of §XVI.6 [>]. You might like to try your hand at

**Open Question XVI.6.c1.** *If $k$ is an algebraically closed field, and $D$, $E$ are division algebras over $k$, must $D \otimes_k E$ be a $k$-algebra without zero-divisors?*

The answer is ''yes'' if $D$ and $E$ are commutative (or even if one of them is), ''no'' if we drop the assumption that $k$ is algebraically closed, also ''no'' if we keep that assumption but weaken the hypothesis on $D$ and $E$ from ''division algebras'' to ''algebras without zero-divisors''. (Cf. my paper *Zero-divisors in tensor products*, pp.32-82 of Springer Lecture Notes in Mathematics, v.545, R. Gordon, ed., *Noncommutative ring theory, papers presented at the international conference at Kent State University, 4-5 April 1975*, Springer Verlag, 1976.)

*Re §XVI.7. The tensor algebra of a module.*

I generally teach the subject of tensor algebras (this section), symmetric algebras (next section) and exterior algebras (§XIX.1) from notes of mine, which can be found through my web page. I will give here brief notes on Lang's presentation of this material.

P.633, line 10 [=]: I'm not sure what Lang means by ''the map of §1''. Perhaps ''the $r=1$ case of $T(f, \ldots, f)$''.

P.633, italicized sentence in middle of page [=]: When he calls $T(f)$ a ''homomorphism of degree zero'', Lang is using terminology which he has not introduced: If $M = \oplus_G M_r$ and $N = \oplus_G N_r$ are two modules, written as direct sums indexed by an abelian group $G$, then for $d \in G$ a ''homomorphism $M \to N$ of degree $d$'' is defined to mean a module homomorphism $M \to N$ which for each $r$ carries $M_r$ into $M_{r+d}$.

(One can set up similar language for $G$ noncommutative, but one must then distinguish between homomorphisms carrying $M_r$ into $M_{rd}$ and homomorphisms carrying $M_r$ into $M_{dr}$.)

P.633, next paragraph [=]: There's no reason for assuming $E$ to have finite rank; the objects which are sometimes called, as Lang does, noncommutative polynomial rings, and more often called free associative algebras, are generally studied without such an assumption.

The free associative algebra on a set $X$ of indeterminates is most often denoted $R\langle X\rangle$, and the tensor algebra on a module $E$ is often denoted $R\langle E\rangle$. The former is universal among $R$-algebras given with set-maps of $X$ into them; the latter among $R$-algebras given with $R$-module homomorphisms of $E$ into them. In other words, the two functors denoted $R\langle -\rangle$ are left adjoint to the forgetful functor from $R$-algebras to sets, and to the forgetful functor from $R$-algebras to $R$-modules, respectively.

P.634, third line, through end of the section [~]: This is devoted to proving Proposition XVI.7.2. I'm not sure why Lang considers this result important; perhaps he will make use of it in some section I haven't read. I will generally not assign this page.

P.634, end of §XVI.7 [>]: A large amount of interesting ring theory has been done involving free associative algebras. See P. M. Cohn, *Free Rings and Their Relations*, Academic Press, 1985. My doctoral thesis (though it proved most results in more general contexts) was inspired by questions about these objects. They are also the starting-point in the study of structures of algebras presented by generators and relations, on which I usually spend a few lectures, assigning reading from the first few sections of my paper, *The diamond lemma for ring theory,* Advances in Mathematics **29** (1978) 178-218.

There is a generalization of the concept of tensor algebra. If $R$ is a not-necessarily-commutative ring, one defines an *R-ring* to mean a ring given with a homomorphism of $R$ into it; thus, an $R$-ring has a natural structure of $(R,R)$-bimodule. Now if $E$ is any $(R,R)$-bimodule, one can define $R\langle E\rangle$ to be the direct sum over $r$ of the $r$-fold tensor products $E\otimes \ldots \otimes E$ of bimodules, where again the 0-fold product is defined to be $R$. This is universal among $R$-rings with $(R,R)$-bimodule homomorphisms of $E$ into them.

*Re §XVI.8. Symmetric products.*

P.635, bottom [>]: $S(E)$ is universal among *commutative* $R$-algebras given with $R$-module homomorphisms of $E$ into them.

P.636, bottom, Proposition XVI.8.2 [~]: This is true without the assumption that $E'$ and $E''$ be free of finite rank. (Though freeness is needed for the second proof that Lang suggests, based on identifying $S(E)$ with a polynomial algebra.)

P.647, statement of Lemma XVII.3.1 [=]: ''*Let* $x\in R$'' should be ''*Let* $x\in E$''.

P.689, third line after boxed display [>]: Lang hasn't introduced the concept of adjoint functor, so this remark can be ignored if you haven't seen it defined. However, I discussed some cases of the concept above in the notes on Lang's p.607 and p.623. For a full motivated development, with many varied examples, see §§7.1-7.3 of my Math 245 notes.

P.735, sentence before final Example [=]: It is not clear to me what Lang means by this, since the values of this map, 1-dimensional vector spaces, don't form an abelian group. Perhaps he means this to be an intuitive idea rather than a literal assertion.