String Topology and its Role in Floer Theory

Symplectic and Contact Geometry Seminar
17th September 2025

§1. String Topology

Let Q be a closed oriented n-manifold. Let ΛQ denote its free loop space $C^0(S^1,Q)$ or $C^\infty(S^1,Q)$ or $H^1(S^1,Q)$.

Defn.: (Chas-Sullivan '99) The <u>string topology</u> of Q is the homology $H_*(\Lambda Q)$.

There are two important operations on string topology:

(i) The <u>Chas-Sullivan product</u> $\star: H_*(\Lambda Q)\otimes H_*(\Lambda Q)\to H_{*-n}(\Lambda Q)$.

This is defined by concatenating loops when possible. Consider inclusion $i: \mathrm{Maps}(S^1\vee S^1, Q)\to \Lambda Q\times\Lambda Q$ and concatenation map $c: \mathrm{Maps}(S^1\vee S^1,Q)\to \Lambda Q$. Then,

$$\star: H_*(\Lambda Q)\otimes H_*(\Lambda Q)\to H_*(\Lambda Q\times \Lambda Q)\xrightarrow{i^!} H_{*-n}(\mathrm{Maps}(S^1\vee S^1,Q))\xrightarrow{c_*} H_{*-n}(\Lambda Q).$$

(ii) The <u>BV operator</u> $\Delta: H_*(\Lambda Q)\to H_{*+1}(\Lambda Q)$.

There is rotation action $S^1\times\Lambda Q\to\Lambda Q$ inducing $\rho: H_*(S^1)\otimes H_*(\Lambda Q)\to H_*(\Lambda Q)$.

Then $\Delta=\rho([S^1],-)$. One has $\Delta^2=0$. (This is 1st differential in spectral sequence computing $H_*^{S^1}(\Lambda Q)$.)

Together, these give string topology the structure of a <u>BV algebra</u>.

We'll also want to consider the equivariant loop space $\Sigma^1 Q := \Lambda Q /\!/ S^1$ and its relative homology $H_*(\Sigma Q, Q)$.

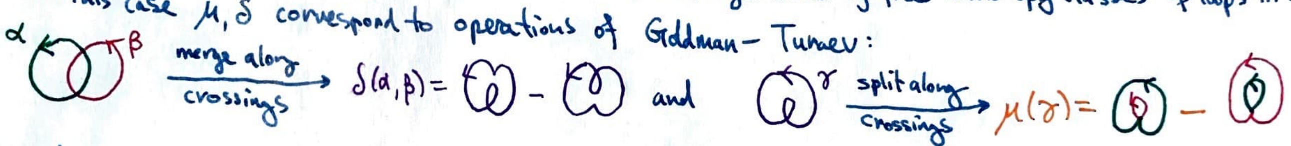
This has the structure of a <u>Lie bialgebra</u> with bracket/cobracket

$\mu: H_*(\Sigma^1 Q)\otimes H_*(\Sigma^1 Q)\to H_*(\Sigma^1 Q)$, $\quad\delta: H_*(\Sigma^1 Q)\to H_*(\Sigma Q)\otimes H_*(\Sigma Q)$

Roughly these correspond to concatenating/splitting loops at intersection points.

E.g. Let Q be surface of genus ≥ 2. Then $H_*(\Sigma^1 Q)$ generated by free homotopy classes of loops in Q.

In this case μ,δ correspond to operations of Goldman-Turaev:



signs determined by comparing orientations w/ Q.

A century ago Morse realized that $H_*(\Lambda Q)$ could be understood via 'Morse theory'.

Given metric g on Q, the functional on loop space $\mathcal{E}:\Lambda Q\to\mathbb{R}$

$\mathcal{E}(\gamma)=\int_\gamma \|\dot\gamma\|_g^2\,dt$ has critical points the closed geodesics of (Q,g).

This functional is bounded below, satisfies the Palais-Smale condition, and critical points have a well defined Morse index $\mu_{Morse}(\gamma)=\#$ conjugate points along γ (w/ multiplicity).

So we can use Morse theory to understand the topology of ΛQ. The algebraic operations on string topology also have Morse theoretic interpretations (c.f. Cohen-Schwarz '08)

This has applications to topology (e.g. Bott periodicity) and Riemannian geometry (e.g. Gromoll-Meyer).

§2. Symplectic Homology

We now consider another setting where we are led to Morse theory for geodesics.

Consider T^*Q with its natural symplectic structure. Given metric g, there is an associated cotangent disk bundle D^*Q. This is a Liouville domain w/ contact type boundary S^*Q.

The Reeb flow on S^*Q coincides with the co-geodesic flow of (Q,g).

In this setting, we may define the symplectic homology of D^*Q. We do Hamiltonian Floer homology on the symplectic completion $\widehat{D^*Q}$ for a Hamiltonian H^a which is C^2-small on D^*Q and linear of slope a on $\mathbb{R}_{\geq \varepsilon} \times S^*Q$. The resulting Floer complex has one generator per Morse critical point and two per Reeb orbit of period at most a. The differential counts solutions of Floer's equation. There are continuation maps $HF(H^a) \to HF(H^b)$ when $a \leq b$ and so we may take the colimit

$$SH(T^*Q) := \varinjlim_a HF(H^a).$$

This depends only on the symplectic completion.

We have defined two versions of Morse theory for closed geodesics. In fact they are the same.

Thm: (Viterbo Isomorphism) Suppose Q is spin, then $SH_*(T^*Q) \cong H_*(\Lambda Q)$.

Proof: (Viterbo '03) Generating Functions
(Salamon-Weber '06) Heat Flow
(Abbondandolo-Schwarz '06) Legendre Transform
(Abouzaid '15) Finite Dimensional Approximation $\square$

The spin assumption was originally overlooked. Observed by Kragh. One can also use a local coefficient system or work over $\mathbb{Z}/2$ without spin assumption.

What about the BV algebra structure?

SH has a pair of pants product which counts pairs of pants $\# \{$ 

$\}$ and a BV operator which counts cylinders where the Hamiltonian is rotated $\bigcup_{\theta \in S^1} \# \{$ 

$\}$

(Δ is 1st differential in spec. seq computing $SH^{S^1}(M)$)

Thm: (Abbondandolo-Schwarz '10, Abouzaid '15) For Q spin, $SH(T^*Q) \cong H(\Lambda Q)$ as filtered BV algebras.

A few applications:

(i) **Thm:** (Abouzaid '11) A cotangent fibre generates the wrapped Fukaya category of T^*Q. ↗ Can use this to compute $\mathcal{W}(T^*Q)$

(ii) **Thm:** (Viterbo '99) If $SH(M) = 0$, e.g. $M = \mathbb{C}^n$ or a subcritical Stein manifold, then M has no exact embedded Lagrangian Q.

Proof: An exact Lag. emb. $i: Q \hookrightarrow M$ equivalent to Liouville embedding $D^*Q \hookrightarrow M$. The low action subcomplex of $SH_*(M)$ recovers $H_*(M, \partial M)$ and we have cobordism map from Liouville embedding. So we get commutative square,

$$\begin{array}{ccc} H_*(M, \partial M) & \xrightarrow{i_!} & H_*(D^*Q, S^*Q) = H_*(Q) \\ \downarrow & & \downarrow j_* \\ 0 = SH_*(M) & \longrightarrow & SH_*(D^*Q) = H_*(\Lambda Q) \end{array}$$

where $j: Q \xrightarrow{\text{const.}} \Lambda Q$.

But $j_* \circ i_! \neq 0$ since j_* injective and $i_![M] = [Q]$. $\square$

(iii) **Conjecture:** (Weak Nearby Lagrangian) If $T^*Q_1 \cong_{\text{symp.}} T^*Q_2$ then $Q_1 \cong_{\text{diff.}} Q_2$.

We know at least that $H_*(\Lambda Q_1) \cong H_*(\Lambda Q_2)$ as BV algebras. This can detect homotopy type and sometimes even simple homotopy type using other string topology operations.

Thm: (Abouzaid-Kragh '18) $Q_1 \cong_{SHE} Q_2$.

Going beyond this may require gauge theory or Floer homotopy theory.

§3. SFT

Given a holomorphic curve in T^*Q or $\mathbb{R} \times S^*Q$ pos./neg. asymptotic to orbitsets $\Gamma_\pm$,

$$\mathrm{ind}(u) = (n-3)(2-2g-s_+-s_-) + \mu_{CZ}^\tau(\Gamma_+) - \mu_{CZ}^\tau(\Gamma_-) + 2c_1^\tau([u]) \overset{\text{In standard triv.}}{=} (n-3)(2-2g-s_+-s_-) + \mu_{Morse}(\Gamma_+) - \mu_{Morse}(\Gamma_-)$$

(s_+, s_-: no. of pos/neg punctures)

Thm: (Viterbo '94) Suppose X is uniruled symplectic manifold of $\dim \geq 6$. It admits no embedded Lagrangian of negative sectional curvature.

Proof: Uniruled means X has a rational curve through every point. Take this point near the Lag. Q and neck stretch along the boundary of a tubular neighbourhood D^*Q. We obtain a rational curve in T^*Q with a point constraint. Since Q has neg. curvature, all geodesics have Morse index zero and aren't contractible. So, WLOG u is simple and lives in a moduli space of dim $\mathrm{ind}(u) = (n-3)(2-s_+) \leq 0$. This curve can't exist for a generic pt. constraint. $\square$

Thm: (Bourgeois–Oancea '17, Cieliebak–Latschev '07) $H_*(\Sigma Q, Q) \cong SH^{+,S^1}(T^*Q) \cong LCH(T^*Q)$.

Note: one expects to identify the full SFT of S^*Q with a more complicated algebraic gadget associated to ΣQ, namely the IBL_∞ algebra of string topology.

We specialize to a case where things are better understood. Let Q be a hyperbolic surface. Geodesics all have Morse index zero and the index of a curve is $\mathrm{ind}(u) = 2g + s_+ + s_- - 2$.

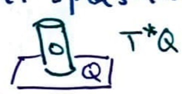
Claim: The SFT potential / cobordism map $\Phi_{D^*Q} : SFT(S^*Q) \to \mathbb{Q}$ sends pairs of a geodesic and the reverse geodesic to ± 1 and other generators to zero (extend multiplicatively).

Proof: The only possible curves are cylindrical. Since geodesics occupy unique free homotopy classes, the ends are γ and γ^{-1}. For Jstd, there's a unique such curve given by the line bundle over γ spanned by $\dot{\gamma}$.

$\square$

Thm: (CL) The SFT Hamiltonian is a linear function of the Goldman–Turaev operations.

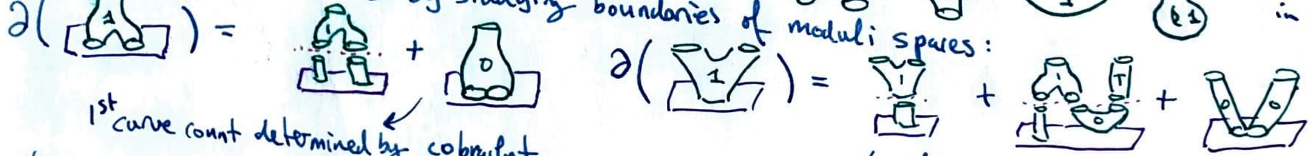
Proof: We can identify a geodesic with its underlying free homotopy class. In terms of moduli spaces this map is given by counting cylindrical curves with one boundary in Q



The SFT Hamiltonian counts four types of curves

in $\mathbb{R}\times S^*Q$

We can determine these by studying boundaries of moduli spaces:



1^{st} curve count determined by cobracket

↳ 2^{nd} curve count determined by 1^{st} and bracket.

Other computations are similar. $\square$

Thm: (Hutchings–Irie) The ECH differential is also determined by string topology. I.e. there's a combinatorial formula for the ECH index of a curve counted by the SFT Hamiltonian.

§4. Embeddings and Dynamics

Understanding SFT + ECH for cotangent bundles helps to describe Riemannian and Finsler geodesics.

Recall the ECH Weyl law implies:

Thm: (Irie '15) For Q a closed surface, a C^∞-generic metric on Q has a dense set of closed geodesics.

Irie's spectral gap criterion + string topology and SFT may allow this to generalize to higher dimensions.

It is also interesting to compute the capacities of disk cotangent bundles for applications to embeddings and dynamics.

Prop: On a surface Q of genus ≥ 2 the Gutt-Hutchings capacities $c_k^{GH}(D^*Q)$ are infinite.

Proof: c_1^{GH} measures the action of a Reeb orbit bounding a holomorphic disk.
But Q has no contractible geodesics for a hyperbolic metric. $\square$

Thm: (B.) With Q as above, the ECH capacities $c_k^{ECH}(D^*Q)$ are also infinite.

Proof: Uses the existence of Anosov-Liouville domains.

Corollary: The ECH capacities of T^4 and $K3$ are infinite.

Corollary: No four-dimensional Liouville domain with finite ECH capacity $c_1^{ECH}(M) < \infty$ admits a closed Lagrangian embedding of genus ≥ 2.

Q: Are the completed $\bar{c}_k^{ECH}$ and elementary alternative c_k^{Alt} ECH capacities finite for D^*Q?

I have been trying to prove this. Possibly this could be done by obtaining a combinatorial model for the U-map on $ECH(S^*Q)$ as with ∂^{ECH} above.