THE YANG-MILLS EQUATIONS: A PDE PERSPECTIVE

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1. Introduction

In this survey we discuss the Yang-Mills equations, one of the central non-linear PDEs in theoretical physics and differential geometry. We will define the equations and explain their importance in quantum field theory and the study of four-manifolds. Then we will discuss some relations of Yang-Mills theory to many aspects of non-linear PDE theory, like variational problems and solitons. Finally, we also summarize an array of results in understanding properties of these equations and the existence and explicit construction of solutions in special geometries. This includes an in depth look at known instanton solutions in $\mathbb{R}^4$ with some helpful graphics.

2. Background on Bundles and Connections

We review some standard geometric material needed to define the equations. This can all be found in Kobayashi and Nomizu's standard text [15].

Recall a *principal G-bundle* is a fibre bundle $\pi : P \to M$ with a right G-action of a Lie group G that preserves the fibres and acts freely and transitively on them. A *principal connection* on a principal bundle P is an element $A \in \Omega(P, \mathfrak{g})$; i.e. a one-form on P with values in the Lie algebra of G. One can think of this as a projection from TP to the vertical subbundle $VP = \ker \pi_* \cong P \times \mathfrak{g}$.

Given a principal G-bundle P and a vector space V with a G-representation, we may define an *associated bundle*:

$$E = P \times_G V := P \times V/(p \cdot g, v) \sim (p, g \cdot v),$$

which is a vector bundle with fibre isomorphic to V and a structure group G.

Example 2.1. The group G acts naturally on its Lie algebra via the adjoint action Ad. So for a pricipal G bundle P, we obtain the *adjoint bundle* $P \times_G \mathfrak{g}$, which we denote $\mathrm{ad}(P)$.

A *connection* or covariant derivative on a vector bundle E is a linear map $\nabla : \Gamma(E) \to \Omega^1(E)$ satisfying a Leibniz rule,

$$\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s, \quad \forall\, f \in C^\infty(M),\ s \in \Gamma(E).$$

A principal connection A on P defines a connection ∇_A on any associated bundle E. In a local trivialization τ, ∇_A may be written as the usual exterior derivative on vector-valued functions plus some $\mathfrak{g}$-valued 1-form A^τ. Because of this, the space of connections on E is an infinite dimensional affine space modelled on $\Omega^1(\mathrm{ad}(P))$.

A *gauge transformation* ψ is a bundle automorphism respecting the G-structure. Such a gauge transformation acts on connections via,

$$\nabla_{\psi(A)} s = \psi \nabla_A(\psi^{-1} s).$$

The collection of gauge transformations form the *gauge group*, denoted $\mathscr{G}$.

For a fixed connection A, the connection extends to family of maps $\mathrm{d}_A : \Omega^k(E) \to \Omega^{k+1}(E)$, collectively called the *exterior covariant derivative*, determined by $\mathrm{d}_A = \nabla_A$ for $k = 0$ and the Leibniz rule,

$$\mathrm{d}_A(\omega \wedge \mu) = (\mathrm{d}_A\omega) \wedge \mu + (-1)^k \omega \wedge \mathrm{d}_A\mu,$$

for $\omega \in \Omega^k(E)$.

We have a map $\mathrm{d}_A^2 : \Omega^0(E) \to \Omega^2(E)$ which we may regard as an $\mathrm{ad}(P)$-valued (or more generally $\mathrm{End}(E)$-valued) 2-form F_A called the *curvature*. The curvature always satisfies the Bianchi identity: $\mathrm{d}_A F_A = 0$. We see that F_A in a trivialization τ has the form,

$$F_A^\tau = \mathrm{d}A^\tau + A^\tau \wedge A^\tau. \tag{2.1}$$

Also under a gauge transformation ψ, F_A is mapped to $\psi F_A \psi^{-1}$.

3. The Yang-Mills Equations

Suppose X is a Riemannian n-manifold. Recall the Hodge-star operator $\star : \Omega^k(X) \to \Omega^{n-k}(X)$ defines a natural inner product on k-forms,

$$(\omega, \mu) = \int_X \omega \wedge \star\mu = \int_X \langle \omega, \mu \rangle \,\mathrm{dvol}\,,$$

where $\langle , \rangle$ is the inner product on tangent spaces defined by the metric and dvol is the Riemannian volume form. If $\mathfrak{g}$ is endowed with an invariant symmetric bilinear form (for example the Killing form if G compact) then $\mathrm{ad}(P)$ has an induced inner product on its fibres. Combining this with the inner product on forms induced by X's metric, we have a metric on spaces of forms valued in $\mathfrak{g}$ and $\mathrm{ad}(P)$.

For a vector bundle $E \to M$ with connection A, we may define a formal adjoint to the exterior covariant derivative $\mathrm{d}_A^* : \Omega^k(E) \to \Omega^{k-1}(E)$ given by,

$$\int_X \langle \mathrm{d}_A\omega, \mu \rangle \,\mathrm{dvol} = \int_X \langle \omega, \mathrm{d}_A^*\mu \rangle \,\mathrm{dvol}$$

for $\omega \in \Omega^k(E), \mu \in \Omega^{n-k-1}(E)$.

The *Yang-Mills equations* are the non-linear PDEs for a connection A on a vector bundle E over a Riemannian manifold X in terms of a condition on A's curvature:

$$\boxed{\mathrm{d}_A^* F_A = 0.} \tag{3.1}$$

This can be equivalently expressed as $\mathrm{d}_A \star F_A = 0$. The non-linearity of (3.1) arises from both non-linear terms in F_A (see (2.1)) and the dependence of d_A on A.

Using the inner product on $\mathfrak{g}$, we may define the L^2 norm of the curvature:

$$\|F_A\|_{L^2}^2 = \int_A \langle F_A, F_A \rangle_{\mathfrak{g}} \, \mathrm{dvol}\,.$$

That is, we wedge the 2-form part of F_A, multiply the norm squared of the $\mathrm{ad}(P)$ part, and then integrate. We call the integrated term,

$$\mathscr{L}_{YM} = \langle F_A, F_A \rangle_{\mathfrak{g}},$$

the *Yang-Mills Lagrangian*. We will show in section 5 that the corresponding Euler-Lagrange equations reproduce the Yang-Mills equations.

Example 3.1. In the case where X is four dimensional, the Hodge star operator $\star$ defines an automorphism of the bundle of two forms and the Bianchi identity $\mathrm{d}_A F_A = 0$ then gives that *(anti-)self dual connections* or *instantons* A, defined by $\star F_A = \pm F_A$, solve the Yang-Mills equations. These are an example of the phenomena of *solitons*, which appear frequently in the study of certain classes of non-linear PDE; we will explore this fact later. We frequently refer to $\star F_A = \pm F_A$ as the ASD equations and solutions as ASD connections for short.

Example 3.2. Consider a trivial $\mathrm{U}(1)$ complex line bundle over $\mathbb{R}^{1,3}$, meaning $\mathbb{R}^4$ with the Minkwoski metric. Suppose A is a connection, i.e. an ordinary one-form and its curvature is $F = \mathrm{d}A$, since the non-linear term vanishes as $\mathrm{U}(1)$ is abelian. We may expand F in the following way,

$$F = (E_1\,\mathrm{d}x + E_2\,\mathrm{d}y + E_3\,\mathrm{d}z) \wedge \mathrm{d}t + B_1\,\mathrm{d}y \wedge \mathrm{d}z + B_2\,\mathrm{d}z \wedge \mathrm{d}z + B_3\,\mathrm{d}x \wedge \mathrm{d}y\,,$$

for 3-vector valued functions $\vec{E}, \vec{B}$. We can also let,

$$A = W_1\,\mathrm{d}x + W_2\,\mathrm{d}y + W_3\,\mathrm{d}z - \varphi\,\mathrm{d}t\,.$$

We find that $\vec{E} = \nabla\varphi$ and $\vec{B} = \nabla \times \vec{W}$.

Enforcing the Yang-Mills equations and Bianchi identity, $\mathrm{d}F = \mathrm{d} \star F = 0$ gives the following four equations for $\vec{E}$ and $\vec{B}$:

$$\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \partial_t \vec{E}, \quad \text{and} \quad \nabla \times \vec{E} = -\partial_t \vec{B}.$$

These are the vacuum Maxwell's equations for $\vec{E}$ and $\vec{B}$: the electric and magnetic fields. In this interpretation, the covariant version of A, $(\varphi, \vec{W})$, is what is usually called the electromagnetic four-potential and F is the electromagnetic field tensor.

Example 3.3. Consider a trivial $\mathrm{SU}(2)$-bundle over $\mathbb{R}^4$. We may write the connection A as a $\mathfrak{su}(2)$-valued field in four-vector notation with matrix-valued components A_μ. The curvature tensor then has components,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The Yang-Mills Lagrangian is written as $\langle F_{\mu\nu}, F^{\mu\nu}\rangle_{\mathfrak{g}}$ and the associated Yang-Mills equations can be expressed as,

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0.$$

The ASD equations in this Euclidean case are,

$$F_{\mu\nu} = \pm * F_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}.$$

4. Some Motivation

4.1. **Physics.** The Yang-Mills equations are important in physics because they are in some sense the simplest example of a gauge theory. As we saw above, in the U(1) case the Yang-Mills equations reduces to the vacuum Maxwell's equations [9, pp. 38]. It was the insight of physicists Chen-Ning Yang and Robert Mills in the 1950s that this could be extended fruitfully to non-abelian groups [22]. Indeed, the equations of motion of the fundamental forces (or gauge bosons) of particle physics are described in quantum field theory by the quantized versions of the Yang-Mills equations for gauge groups $\mathrm{SU}(2) \times \mathrm{U}(1)$ in electroweak theory and SU(3) in quantum chromodynamics (see part three of [17] for an in-depth treatment of the physics and [12] for a mathematical description). The complete Lagrangian of the standard model of particle physics can be written as:

$$\boxed{\mathscr{L}_{SM} = -\frac{1}{4}\left\langle F^{W+B}_{\mu\nu}, F^{\mu\nu}_{W+B}\right\rangle_{\mathfrak{su}(2)\oplus\mathfrak{u}(1)} - \frac{1}{4}\left\langle F^{G}_{\mu\nu}, F^{\mu\nu}_{G}\right\rangle_{\mathfrak{su}(3)} + \mathscr{L}_D + \mathscr{L}_H + \mathscr{L}_Y.}$$

The first two kinetic terms are the Yang-Mills Lagrangians for gauge fields: for the W and Z bosons and photon (electroweak theory), and for the gluons (QCD). The remaining potential terms are the Dirac Lagrangian governing fermions like electrons, neutrinos, and quarks and their interactions with gauge fields, the Higgs Lagrangian governing the interactions of gauge fields and the Higgs field that endows them with mass, and the Yukawa Lagrangian governing the interaction of fermions with the Higgs field [12, Chapter 8]. We describe the general Yang-Mills Lagrangian in the next section; the form of this Lagrangian essentially implies that Yang-Mills theory provides the kinetic input of fundamental physics.

While these physical theories concern the quantized version of Yang-Mills theory, the classical theory is still of interest to better understand the quantum case. One is often interested in properties of the equations which are preserved or broken under quantization, requiring an understanding of the classical equations. Additionally, as we discuss in the next section, classical solutions like instantons on $\mathbb{R}^4$ can correspond directly to phenomena in the quantum theory of $\mathbb{R}^{1,3}$; hence an understanding of the classical theory gives predictions for the quantum one. Lastly, the formalization of Yang-Mills theory has also been of profound interest to mathematical physicists. For instance, one of the seven Millenium Prize Problems deals with finding a rigorous construction of a quantum Yang-Mills theory in $\mathbb{R}^4$ for any compact simple structure group G so that there is a "mass gap," meaning a positive minimal mass for a particle in the theory [14].

4.2. **Geometry.** The Yang-Mills equations became of interest to differential geometers in part due to the work of Simon Donaldson. As we prove in 5.1, the Yang-Mills equations and their four-dimensional refinement the ASD equations are invariant under the action of $\mathscr{G}$. Letting $\mathscr{B}$ denote the space of solutions to the ASD equations on a given vector bundle (with topology induced from metric on space of connections), we may define a quotient space,

$$\mathcal{M}(X, g, E) = \mathscr{B}/\mathscr{G},$$

called the *moduli space of ASD Yang-Mills connections.* It turns out that under conditions on the vector bundle E, for a generic metric g, $\mathcal{M}$ is a smooth manifold apart from finitely many classifiable singularities. The space $\mathcal{M}$ can also be shown to be orientable with a natural compactification (these facts rely on many analytic results about the equations developed by Uhlenbeck, Taubes, and others) [9, Chapters 4,5]. Donaldson used properties of these moduli spaces to prove novel theorems about and construct invariants of smooth four-manifolds. This theory is thoroughly exposited in Donaldson and Kronheimer's well known monograph [9], and detailed more accessibly in the book of Freed and Uhlenbeck [11]. The central result is as follows. Recall for X four dimensional, taking the cup product of two elements of $H^2(X;\mathbb{Z})$ and pairing with $[X] \in H_4(X;\mathbb{Z})$ defines a symmetric bilinear pairing on $H^2(X;\mathbb{Z})$. We may interpret this as an invertible integer matrix Q called the *intersection form.*

Theorem 4.1 (Donaldson's Theorem [7])**.** *If a simply connected smooth four manifold X has a definite intersection form Q_X, i.e. the eigenvalues of Q are all positive or all negative, then Q is diagonalizable over $\mathbb{Z}$.*

This result rules out many possible forms from being the intersection form of a smooth simply connected manifold. Since manifolds realizing these other forms exist in the topological category, we are led to many examples of non-smoothable topological four-manifolds. This result led to the proof that $\mathbb{R}^4$ admits (uncountably many) non-standard smooth structures, in contrast to Euclidean space in all other dimensions [18]. Donaldson won a Fields medal for his work and spawned increasing interest among geometers in studying PDEs coming from particle physics, developing the field of mathematical gauge theory. Other gauge theories have later been developed, also of interest to geometers, including Seiberg-Witten and Chern-Simons theories. Gauge theory has been one of the most fruitful approaches to resolving problems in low-dimensional topology and geometry in the last thirty years (see [8] for a short summary).

Remark 4.2. In the following sections we will mostly restrict to the theory of when X is four dimensional. This is obviously natural from a physics viewpoint, and as we have seen it turns out to be where many of the geometric applications of gauge theory have been.

5. PDE Phenomena

5.1. **A Variational Approach.** Recalling the Yang-Mills Lagrangian, we can define a natural functional on the space of connections, analogous to the Dirchlet energy, namely:

$$S(A) = \|F_A\|_{L^2}^2 = \int_X \mathscr{L}_{YM} \,\mathrm{dvol}\,. \tag{5.1}$$

Proposition 5.1. *The Euler-Lagrange equations for the functional $S(A)$ in* (5.1) *are the Yang-Mills equations.*

Proof. Suppose A is a critical point of S and suppose we vary A in the direction of $a \in \Omega^1(\mathrm{ad}(P))$. We calculate,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S(A+ta) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_X \langle F_{A+ta}, F_{A+ta}\rangle_{\mathfrak{g}} \,\mathrm{dvol}\,.$$

By (2.1),

$$\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_X \langle \mathrm{d}_A(A+ta) + (A+ta)\wedge(A+ta), \\
&\qquad\qquad\qquad \mathrm{d}_A(A+ta) + (A+ta)\wedge(A+ta)\rangle_{\mathfrak{g}} \,\mathrm{dvol} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_X \langle F_A + t\mathrm{d}_A a + t^2 a\wedge a, F_A + t\mathrm{d}_A a + t^2 a\wedge a\rangle_{\mathfrak{g}} \,\mathrm{dvol} \\
&= \int_X 2\langle \mathrm{d}_A a, F_A\rangle_{\mathfrak{g}} \,\mathrm{dvol} \\
&= 2\int_X \langle a, \mathrm{d}_A^* F_A\rangle_{\mathfrak{g}} \,\mathrm{dvol}\,.
\end{aligned}$$

We conclude A is a critical point if and only if this vanishes for all $a \in \Omega^1(\mathrm{ad}(P))$. From definiteness of our inner product, we obtain the desired result. □

From information on characteristic classes, one can deduce the following result which makes the simplification of Yang-Mills to the first order ASD equations quite natural.

Proposition 5.2. *If X is four dimensional, the absolute minima of the action functional S occur precisely at the (anti-)self dual solutions of the Yang-Mills equations.*

Corollary 5.3. *The Yang-Mills equations are a* gauge theory*, by which we mean* (3.1) *is invariant under the action of the gauge group $\mathscr{G}$. The ASD equations also are gauge invariant.*

Proof. Since we choose an invariant inner product on $\mathrm{ad}(\mathfrak{g})$, for $\psi \in \mathscr{G}$,

$$S(\psi\cdot A) = \|\psi F_A \psi^{-1}\|_{L^2}^2 = \|F_A\|_{L^2}^2 = S(A).$$

The result for Yang-Mills cleary follows.

The result for the ASD equation follows either from Proposition 5.2 above or noticing that the star operator and gauge transformations commute. □

5.2. **Sobolev Estimates and Elliptic Operators.** To study analytic details of these equations, one is often led to generalize from the smooth setting and work with connections and curvature in larger Sobolev spaces. In these spaces, one has many standard results from functional analysis and PDE theory at one's disposal to apply in proving theorems. We state a few standard results in Sobolev theory that are critical to gauge theory. Our presentation is based on that in [9, Appendix]; a standard reference for these results is [6].

Proposition 5.4 (Sobolev Embedding)**.** *If X is an n-dimensional compact manifold, then there is a bounded inclusion map from the Sobolev space of sections L^2_k to the space of r-continuously differentiable section C^r whenever $k - \frac{n}{2} > r$.*

Corollary 5.5. *A section on a compact manifold which lies in L^2_k for each k is smooth.*

Proposition 5.6 (Elliptic Regularity)**.** *Suppose D is an order ℓ elliptic operator on a vector bundle $E \to X$ over a compact manifold. For each $k \geq 0$, there is some constant C depending on k, so that for all sections $s \in \Gamma(E)$,*

$$\|s\|_{L^2_{k+1}} \leq C(\|Ds\|_{L^2_k} + \|s\|_{L^2}).$$

Proposition 5.7 (Sobolev Inequality)**.** *Let $E \to X$ be a vector bundle over a compact n-manifold. Suppose $k \geq \ell$ and*

$$k - \frac{n}{p} \geq \ell - \frac{n}{q}.$$

Then there is a bounded inclusion map of spaces of sections $L^p_k \hookrightarrow L^q_\ell$.

From this, one can deduce various results about Soblev multiplication. I.e. that the product of a section in some Sobolev space and a function in another Soblev space must also lie in some Sobolev space.

Proposition 5.8 (Fredholm Alternative)**.** *Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic operator between sections of vector bundles with given metrics over a compact manifold X. Then D has an elliptic adjoint D^* and D is Fredholm (its kernel and cokernel are finite dimensional). Moreover, a section $s \in \Gamma(F)$ is in the image of D if and only if $\langle s, t\rangle = 0$ for all $t \in \ker D^*$.*

Let d^+ denote the usual exterior derivative on forms composed with the projection to the space of self-dual forms with respect to a metric on X. One can show that both $\mathrm{d} + \mathrm{d}^*$ and $\mathrm{d}^+ + \mathrm{d}^*$ are elliptic operators. For a given connection A, the operator $\mathrm{d}^*_A \mathrm{d}_A$ is a second order elliptic operator. Hence we may apply the elliptic regularity and Fredholm alternative results to study these operators.

A common application of these ideas in gauge theory (and more broadly in studying weak solutions of many elliptic PDEs) is the technique of "elliptic bootstrapping." Here, one possesses some section $\psi \in L^2_k$ so that for some (say first order) elliptic operator D, $D\psi$ can be written in terms of ψ and perhaps some other known quantities which are bounded pointwise or belong to certain Sobolev spaces. Using Sobolev multiplication and Sobolev inequalities, we may be able to bound the expression for $D\psi$ in L^2_k. Then, elliptic regularity implies that ψ is bounded in its L^2_{k+1} norm. In certain circumstances, we may be able to extend this bootstrapping inductively so that $\psi \in L^2_i$ implies $\psi \in L^2_{i+1}$ for each i, and hence ψ is in L^2_k for every k. The Sobolev embedding theorem then implies that ψ is actually smooth. This technique is often useful to show, modulo changes of gauge, we may take our solutions to a certain gauge theory PDE like the ASD or Yang-Mills equations to be smooth objects and to obtain some sort of compactness for the moduli space of solutions.

5.3. **Solitons of the Theory.** We have described instanons as (anti-)self dual solutions of the Yang-Mills equations. These solutions are so named because they are spatially and temporally localized in $\mathbb{R}^4$ with the Euclidean metric. The local nature of instantons, combined with the fact that two instantons may scatter off each other [10, §8.2.1], gives these solutions the same character as localized wave packet solutions that occur for many PDEs like the KdV equations and the nonlinear Schrödinger equation. The important class of solutions to nonlinear problems with these properties are commonly called *solitons*. One can often produce solitons from the inverse scattering transform and a Lax pair. There is a more complicated analogue for instantons called *twistor theory*, first developed by Roger Penrose. A detailed account of this theory is given in Chapter 7 of [10].

These insantons have several other important properties from a physics perspective. As we described above, they are absolute minima of the action. This has the interpretation of being zero energy solutions, and so instantons may be thought of as vacuum fluctuations. One can also associate to these solutions an integer called its topological charge (see (6.1)).[1] This integer can be thought of as the number of instantons in our solution, which corresponds physically to how many local wave packets it contains [1, pp. 463]. We construct some of these N-instanons in the next section. It is a general fact that classical Euclidean solutions influence the quantum theory of tunneling for Minkowski solutions. The consequence of this is that in quantum Yang-Mills theory there are a family of topological vacua, parameterized by the integers, and the N-instanon classical solutions corresponds to tunneling between these vacua, with the change in state dependent on N [1, pp. 463].

A related family of solutions are *merons*, which are localized and carry half a unit of topological charge. These again have physical significance in the quantum theory of tunneling. There are predictions that the charge-one instanton solutions should be decomposable into two merons [1, pp. 464]. At isolated moments of time, the meron solutions will appear like magnetic monopoles. So, merons resemble short-lived monopoles that appear out of the vacuum [1, pp. 496]. There are also more general one-parameter families of *elliptic solutions* which interpolate between a charge one instanton and a pair of charge one half merons [1, pp. 497].

A physically interesting extension of the Yang-Mills equations are the Yang-Mills-Higgs equations. In addition to the usual structures of Yang-Mills theory, we consider Φ to be a section of E. The Yang-Mills-Higgs Lagrangian is,

$$\mathscr{L}_{YMH} = -\frac{1}{2}\langle F_A, F_A\rangle_{\mathfrak{g}} + \langle d_A\Phi, d_A\Phi\rangle_E.$$

The corresponding *Yang-Mills-Higgs equations* are:

$$\mathrm{d}_A \star F_A + [\Phi, \mathrm{d}_A\Phi] = 0 \quad \text{and} \quad \mathrm{d}_A \star \mathrm{d}_A\Phi = 0$$

subject to the boundary condition $\lim_{x\to\infty} |\Phi|(x) = 1$. Physically, these equations describe a gauge field which is coupled to the Higgs field [12, 7.6]. Thus it gives a massive version of the Yang-Mills equations.

The Yang-Mills-Higgs equations have several interesting classes of solutions. The theory gives rise to *monopoles*, which generalize the hypothetical magnetic monopoles of the U(1) theory. These solutions are localized and can be made with finite energy. A related class of solutions are *dyons*, which are essentially monopoles allowed to have electric charge [1, pp. 464]. A third family consisting of the solitons in the abelian (U(1)) two dimensional Yang-Mills-Higgs theory are *vortices*. This theory has applications to the theory of superconductivity, where these solutions appear naturally. The vortices are so named because the field Φ, which lies in a two dimensional bundle, acts like a rotational vector field around the soliton of the theory [13, 1.6]. A thorough introduction to the Yang-Mills-Higgs equations and these special solutions is provided in Jaffe and Taube's monograph [13].

6. Explicit Constructions and Existence Results

6.1. **Gauge Fixing and Removable Singularities.** Since the Yang-Mills equations are gauge invariant, it can be useful to find representatives in a certain gauge equivalence class of solutions with special properties. One may think of this as a "gauge fixing," where we select a certain slice of solutions from the total space and remove the extra degrees of freedom. A

[1]This is called a *topological* charge because the charge relates to the homotopy type of maps $S^3 \to G$ (this comes from mapping the asymptotic data of our solution at infinity, thought of as a copy of S^3, to G). It is a non-trivial fact that $\pi_3(G) = \mathbb{Z}$ for simple compact Lie groups [3] and so the charge is typically integer valued.

particular common choice generalizes the notion of Coloumb gauge from electromagnetism where the magnetic vector potential is chosen to be divergenceless. We say a connection B is in the *Coloumb gauge* relative to A if $\mathrm{d}_A^*(B-A)=0$. The central result here is that on the n-ball we can place connections with small enough curvature into Coulomb gauge.

Theorem 6.1 (Uhlenbeck's Lemma, 1982 [20])**.** *Let $X = B^n$ and G be a compact subgroup of $\mathrm{SO}(\ell)$. Let $2p \geq n$ and consider a connection one-form $\widetilde{A} \in L_1^p(B^n, \mathbb{R}^\ell \times \mathrm{ad}(P))$. There are constants $\varepsilon(n) > 0, C(n) < \infty$ so that if,*

$$\|F\|_{L^{n/2}}^{n/2} \leq \varepsilon,$$

then $\nabla_{\widetilde{A}} = \mathrm{d} + \widetilde{A}$ is gauge equivalent via $\psi \in L_2^p(B^n, G)$ to a connection ∇_A so that,

$$\mathrm{d}^* A = 0 \quad \textit{and} \quad \|A\|_{L_1^p} \leq C\|F\|_{L^p}.$$

One motivation for the utility of this result in Yang-Mills theory is that the ASD equation, which we can write locally as $\mathrm{d}^+ A + (A \wedge A)^+$, is not elliptic at the highest order since d^+ is not elliptic. But if we impose the Coulomb condition $\mathrm{d}^* A = 0$, then we can rewrite the first order linearization of the ASD equations as $(\mathrm{d}^+ + \mathrm{d}^*)A = 0$. Since $\mathrm{d}^+ + \mathrm{d}^*$ is elliptic, the theory of elliptic regularity described above can now be wielded in studying the ASD equations.

In the same year, Uhlenbeck also published work detailing how, up to a change of gauge, the domain of solutions can be extended in four-dimensions.

Theorem 6.2 (Uhlenbeck's Removable Singularities Theorem [21])**.** *Let A be a Yang-Mills connection on a G-bundle E, with $G \subset SU(\ell)$, over the punctured 4-ball $B^4 \setminus \{0\}$. Suppose $\|F_A\|_{L^2}$ is finite. Then there is a gauge transformation ψ under which $\psi(A)$ extends to a smooth Yang-Mills connection on a bundle $\overline{E}$ over B^4.*

Corollary 6.3 ([21])**.** *Suppose A is a Yang-Mills connection on a bundle E over $\mathbb{R}^4$ so that $\|F_A\|_{L^2}$ is finite. Let $f : S^4 \setminus \{0\} \to \mathbb{R}^4$ be the stereographic projection map. Then f^*A is a Yang-Mills connection on f^*E over $S^4 \setminus \{0\}$ extending under some gauge transformation to a Yang-Mills connection on a bundle $\overline{f^*E}$ over S^4.*

6.2. **The BPST Construction and its Generalizations.** The most well known explicit Yang-Mills solution is a charge one instanton solution due to Belavin, Polyakov, Schwartz, and Tyupkin from 1975, commonly called the BPST instanton [5]. This is a self dual solution on $\mathbb{R}^4$ with structure group SU(2). We define 2×2 matrices,

$$\omega_a = i\sigma_a, \quad a = 1,2,3, \quad \text{and} \quad \omega_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where σ_a are the Pauli matrices. Then we define the *'t Hooft tensors*,

$$\eta_{\mu\nu} = -\frac{1}{4}(\omega_\mu^\dagger \omega_\nu - \omega_\nu^\dagger \omega_\mu).$$

The BPST solution is then given by,

$$A_\mu(x) = \frac{2x_\nu}{x^2 + \lambda^2}\eta_{\mu\nu}$$

for a parameter $\lambda > 0$. The corresponding curvature is given by,

$$F_{\mu\nu}(x) = \frac{4\lambda^2}{(x^2+\lambda^2)^2}\eta_{\mu\nu}.$$

The topological charge we discussed earlier is given by the second Chern class,

$$c_2(F) = -\frac{1}{16\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}(F_{\mu\nu} \star F_{\mu\nu})\,\mathrm{d}x\,. \tag{6.1}$$

Different topological charges correspond to different asymptotic behaviours of solutions. On S^4, different topological charges mean the connection A is associated with a different isomorphism class of vector bundle over S^4. For the BPST instanton we calculate,

$$c_2(F) = \frac{6}{\pi^2}\int_{\mathbb{R}^4} \frac{\lambda^4}{(x^2+\lambda^2)^4}\,\mathrm{d}x = 1.$$

In Figure 1, we graph the tensor coefficients of the BPST solution restricted to two input variables. We can also graph the curvature as in Figure 2.

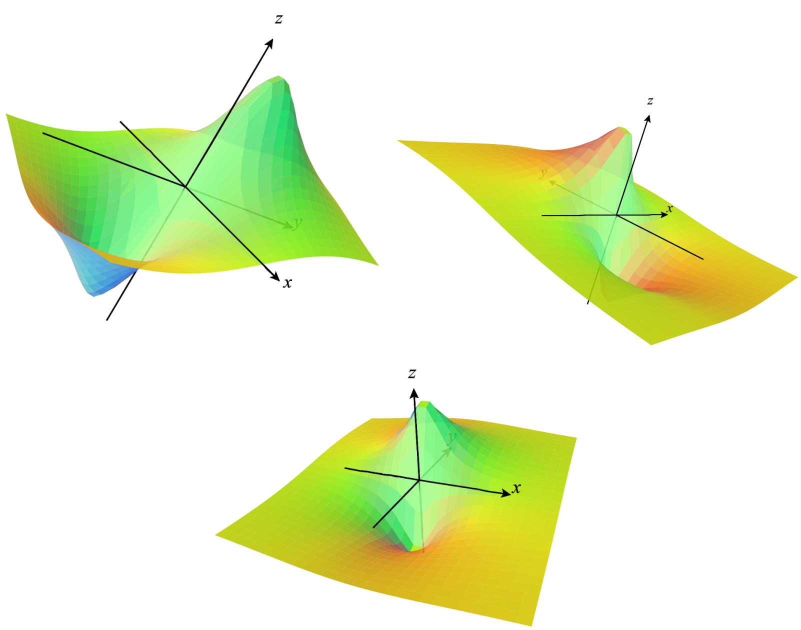


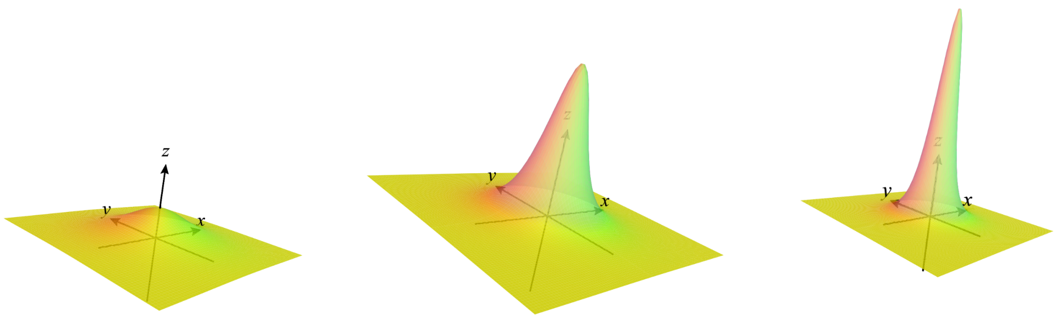
FIGURE 1. Three plots of the xy slice of the η_{i1} coefficient of the BPST instanton for $\lambda = 1/4$. The solution is localized in space and time.

These one instanton solutions are parameterized by five variables, their *centre* $x_0 \in \mathbb{R}^4$ and their *scale* $\lambda > 0$. After a gauge transformation, we may write our BPST solution as,

$$\widetilde{A}_\mu(x) = (\partial_\nu \log f)\eta_{\mu\nu} \quad \text{where} \quad f(x) = 1 + \frac{\lambda^2}{x^2}. \tag{6.2}$$

The details of this are presented in the book of Yang and the paper of Actor [1, 23]. One can generalize to a wider class of functions,

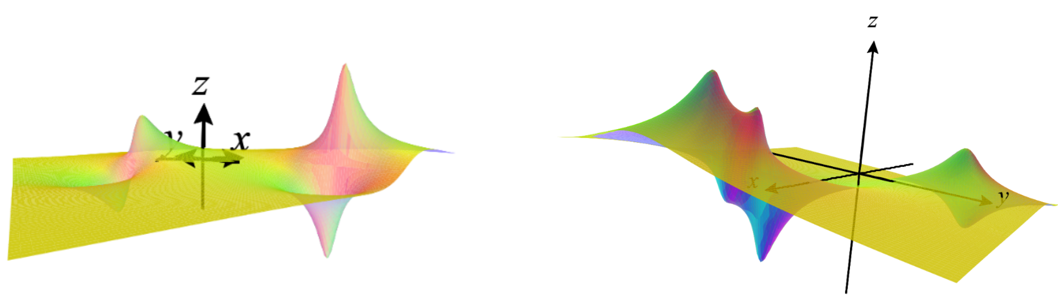
$$f(x) = 1 + \sum_{j=1}^{N} \frac{\lambda_j^2}{(x-p_j)^2}, \quad \lambda_j > 0, \quad p_j \in \mathbb{R}^4. \tag{6.3}$$



(A) $\lambda = 2$ (B) $\lambda = 1$ (C) $\lambda = 0.75$

FIGURE 2. Three plots of the xy slice of the η_{ij} coefficient of the BPST instanton's curvature for different values of λ. We see how the solutions have a localized focus, appearing like a point-particle or soliton. Their size depends quite strongly on the magnitude of λ.

In 1976, 't Hooft realized that these functions give N-instanton solutions when f is plugged into the formula for $\widetilde{A}$ in (6.2) [23, §3.1.3]. This gives solutions of topological charge N consisting of N pointlike solitons with centres p_j and scales λ_j. A couple examples are given in Figure 3. The curvatures of these N-instantons add linearly, as seen in Figure 4.



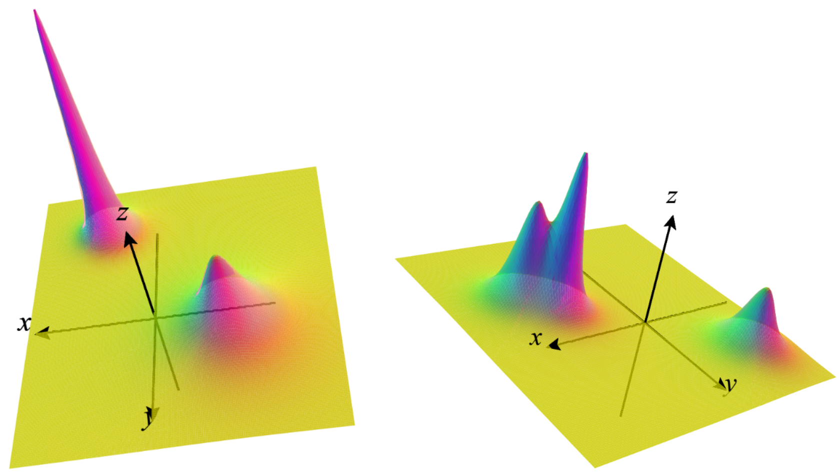
(A) $(p_i, \lambda_i) = ((2, -5), 0.3), ((-3, 1), 0.5)$ (B) $(p_i, \lambda_i) = ((3, -2), 0.4), ((-3, 3), 0.6), ((4, -3), 0.5)$

FIGURE 3. Plots of coefficients for $N = 2$ and $N = 3$ instanton solutions. Far apart instantons do not interact, while nearby ones add in a form of non-linear superposition.

There is also an explicit two-meron solution of the Yang-Mills SU(2) equation on $\mathbb{R}^4$ due to deAlfaro, Fubini, and Furlan from 1976 [1, pp. 505]. Given $a, b \in \mathbb{R}^4$ we construct a singular two-meron solution with merons centered at a and b as,

$$A_0(x) = \frac{1}{2}\left(\frac{(x-a)_\mu}{(x-a)^2}\sigma_0 + \frac{(x-b)_\mu}{(x-b)^2}\right)\sigma_\mu,$$

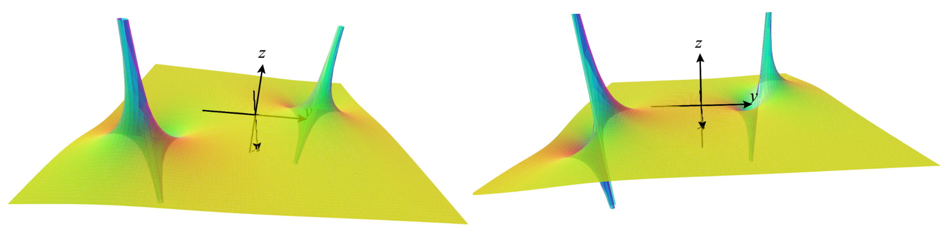
$$A_i(x) = -\left[\epsilon_{i\mu\nu}\left(\frac{(x-a)_\nu}{(x-a)^2} + \frac{(x-b)_\nu}{(x-b)^2}\right) + \delta_{i\mu}\left(\frac{(x-a)_0}{(x-a)^2} + \frac{(x-b)_0}{(x-b)^2}\right)\right]\sigma_\mu, \quad i = 1, 2, 3.$$



(A) (B)

FIGURE 4. Plots of coefficients for the curvatures corresponding to the connections from Figure 3; the scales have all been doubled to aid in visibility. Here we can see that the curvatures add linearly.

The corresponding formula for the curvature is too complicated to state here, but one can deduce it from the equations given in [1]. The topological charge density of this connection (i.e. the integrand of (6.1)) will be one half times the sum of two δ functions centered at a and b. A plot of a two-meron connection is given in Figure 5.



(A) σ_0 coefficient of A_0 (B) σ_1 coefficient of A_0

FIGURE 5. Plots of coefficients for the two meron connection with merons centered at $(t, x) = (-1, 3)$ and $(2, -4)$. The corresponding curvatures should be highly localized at these points. Our solution is singular at the meron's centres, similar in form to a monopole. We may also think of our two merons as a one-instanton connection which has been split into two charge-1/2 pieces.

A one-meron solution is constructed by moving one of our merons to infinity. We can similarly constuct anti-meron solutions and meron-anti-meron pairs. While they are known to exist, no one has found general explicit formulae for higher numbers of meron-anti-meron solutions [1, pp. 506]. To see the connection to instantons, one can compute that for a single instanton centred at 0, as the scale λ approaches zero, the solution becomes two merons both at the origin [1, pp. 496]. To better understand merons, their connection to instantons, and whether they possess a physical interpretation in the quantum theory, may require an explicit construction of general meron solutions.

6.3. **The ADHM Construction.** One can observe from (6.3) that 't Hooft's solutions give a $5N$ parameter of N-instanton solutions. This has been enlarged to an explicit family of $5N+4$ solutions, and index theory implies there should in fact be an $8N-3$ parameter family of solutions [23, pp. 87].

Since our instantons have finite curvature density, Uhlenbeck's Removable Singularities theorem implies that our instantons on $\mathbb{R}^4$ can be compactified to instantons on S^4. We now briefly discuss this problem of classifying instantons on S^4.

The classification of instantons on S^4 was discovered independently by Atiyah and Hitchin, and Drinfield and Manin in 1976. They classify these instantons for SU(2) and indeed any compact Lie group in a well known two page paper [2]. Their work is now referred to as the *ADHM construction.*

Despite being so terse, the result is rather technical and so we do not repeat their methods of construction here (a pedagogical account is given in chapter three of [9] for the interested reader). The result is proved by a mixture of algebraic geometry and linear algebra. In an earlier work, Atiyah, Hitchin, and Singer show that instanons correspond to real algebraic bundles on $\mathbb{C}P^3$.[2] The ADHM construction essentially re-expresses this algebro-geometric data in terms of a pair of complex Hermitian vector spaces, and certain chosen linear maps [2]. The 't Hooft N-instanton solutions above are one output of this more general classification.

A certain interesting related result is due to Taubes, who shows that instantons exist on a wide class of four-manifolds.

Theorem 6.4 (Taubes, 1982, [19]). *Let X be a compact oriented manifold and G be a compact semi-simple Lie group. Suppose that X has no non-trivial self dual closed (harmonic) two form. Then there is a principal G-bundle $P \to M$ for which M admits an anti-self dual connection.*

The proof of this result is an interesting mix of the techniques and ideas we have encountered so far. Taubes uses the ADHM construction of instantons we have just seen and essentially pastes the localized region of high-curvature into a neighbourhood of the manifold. To show this can be smoothed into a proper instanton requires various results involving elliptic operators and Sobolev theory of the kind we considered in section 5.2.

6.4. **Existence and Uniqueness Theory in $\mathbb{R}^{1,3}$.** The perpetual quest of PDE theory is proving existence and uniqueness for equations of interest. We would be remiss without mentioning at least one result in this direction. The Yang-Mills equations consist of a wide range of equations and so the general question of existence and uniqueness is not settled. However, in Minkwoski space $\mathbb{R}^{1,3}$ with a general compact structure group G, well-posedness has been demonstrated by Klainerman and Machedon [16].

We need a few preliminaries; everything here is done for a trivial G-bundle over $\mathbb{R}^{1,3}$. Associated to the curvature of a Yang-Mills connection is the *energy-momentum tensor*,

$$T_{\mu\nu} = \frac{1}{2}(\langle F_{\mu\alpha}, F_{\nu}{}^{\alpha}\rangle + \langle \star F_{\mu\alpha}, \star F_{\nu}{}^{\alpha}\rangle).$$

[2] This work is in part a consequence of Penrose's twistor theory; see [4].

We consider the following definitions with respect to some point $x_0 \in \mathbb{R}^3$ which we keep implicit in our notation. We define a *past causal domain* as a set,

$$\mathcal{T}^{T,L} = \{(x,t) \in \mathbb{R}^{1,3} : 0 \leq t \leq T,\ |x - x_0| + t \leq L\}.$$

We also denote the balls $\{x : |x - x_0| \leq L - t\}$ at a given t by $B_{t,L}$. We can define the *local energy* at x_0 by,

$$\mathcal{E}(t, B_{t,L}) = \int_{B_{t,L}} T^{00}(t,x)\,\mathrm{d}x\,.$$

An initial condition is determined by a pair of $\mathfrak{g}$-valued 1-forms in $\mathbb{R}^3$: $(\widehat{A}, \widehat{E})$ so that $\mathrm{div}\widehat{E} + [\widehat{A}, \widehat{E}] = 0$. The corresponding Yang-Mills connection $\mathbf{A} = (A_0, A)$ should satisfy $A = \widehat{A}$ and $\partial_t A_i - \partial_i A - 0 + [A_0, A_i] = \widehat{E}_i$ when restricted to $t = 0$. The existence and uniqueness result is the following lengthy theorem.

Theorem 6.5 (Klainerman and Machedon, [16]). *Let $(\widehat{A}, \widehat{E})$ be an initial data set in $\mathbb{R}^3$ which has finite energy $\mathcal{E}(0, \mathbb{R}^3) = \mathcal{E}_0 < \infty$ and which is locally in L^2_2. Then there exists a unique global solution $A(t,x)$ in $[0,\infty) \times \mathbb{R}^3$ to the Yang-Mills equations satisfying the following properties.*

(1) Given a past causal domain $\mathcal{T}^{T,L}$ with $L \geq 1$, there is a constant C_1 dependent on $\mathcal{E}_0$, $\|A(0,\cdot)\|_{L^3(B_{0,L+1})}$, and L so that,

$$\|A(t,\cdot)\|_{L^2_1(B_{t,L})} + \|\partial_t A(t,\cdot)\|_{L^2(B_{t,L})} \leq C_1\Big(\|A(0,\cdot)\|_{L^2_1(B_{0,L+1})} + \|\partial_t A(0,\cdot)\|_{L^2(B_{0,L+1})}\Big)$$

and $\mathcal{E}(t, B_{t,L}) \leq \mathcal{E}(0, B_{0,L})$.

(2) Given initial conditions $(\widehat{A}, \widehat{E})$ and $(\widehat{A}', \widehat{E}')$ meeting the conditions above, there is a constant C_2 depending on T, L, and the $L^2_1(B_{0,L+1})$-norms of the initial conditions so that,

$$\begin{aligned}\|(A - A')(t,\cdot)\|_{L^2_1(B_{t,L})} &+ \|\partial_t(A - A')(t,\cdot)\|_{L^2(B_{t,L})} \\ &\leq C_2\Big(\|(A - A')(0,\cdot)\|_{L^2_1(B_{0,L+1})} + \|\partial_t(A - A')(0,\cdot)\|_{L^2(B_{0,L+1})}\Big).\end{aligned}$$

(3) The solution obtained from the initial condition $(\widehat{A}, \widehat{E})$ satisfies for some constant C_3,

$$\|A(t,\cdot)\|_{L^2_2(B_{t,L})} + \|\partial_t A(t,\cdot)\|_{L^2_1(B_{t,L})} + \|\partial_t^2 A(t,\cdot)\|_{L^2(B_{t,L})} \leq C_3\|(\widehat{A}, \widehat{E})\|_{L^2_2(B_{0,L+1})}.$$

The result requires 70 pages of hard analysis, including elliptic estimates and Sobolev theory like that considered in section 5.2. Central to the proof is an application of various versions of Uhlenbeck's lemma. The authors then prove sophisticated estimates in the Coulomb gauge before transferring the results to the temporal gauge ($A_0 = 0$).

7. Conclusion

In this review, we have met and studied the Yang-Mills equations. Despite their quite technical definition, these equations are critical to the foundations of modern physics and have been wielded to great effect in differential geometry. We have investigated the PDE-theoretic aspects of the theory and stated some well known results in the subject. Additionally, we have discussed the role of solitons in the theory and constructed some explicitly known solutions in four dimensions. Of course there is much more to discuss in Yang-Mills theory and many remaining open questions in the field. We hope that this summary has provided a compelling introduction to this important equation for the reader.

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