THE WHEELER-DEWITT EQUATION: CONSTRUCTING A CANONICAL THEORY OF QUANTUM GRAVITY

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Contents

In this expository essay we will provide a path to understanding the basics of canonical quantum gravity and the Wheeler-DeWitt equation. This theory was among the first attempts to combine general relativity with quantum field theory, and while it was largely unsuccessful, it is the basis for some of the major modern candidates for a theory of quantum gravity.

In order to cover substanial ground, we only quickly summarize relevant details of basic Hamiltonian mechanics, canonical quantization, and Riemannian submanifolds that we need; good sources to read about these things are [7], [10], and [8, Chapter 8] respectively. We will begin with some advanced details of Hamiltonian dynamics with constraints, followed by a study of the ADM formalism of general relativity. We will then attempt to quantize the equations of this theory, leading to the Wheeler-DeWitt equation, and also explain some its issues.

1. Constraints in Hamiltonian Mechanics

Let us review the standard procedure of Hamiltonian mechanics. We begin with a system described by spacial coordinates $q^1, \ldots, q^n$. We are given some Lagrangian $\mathcal{L}(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$. The Euler-Lagrange equation gives n-equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i} \quad i = 1, \ldots, n.$$

To obtain the Hamiltonian formulism, we Legendre transform; we define conjugate momenta,

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}^i}.$$

Then, we describe our system in terms of a $2n$ dimensional phase space with coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ and we define the Hamiltonian as,

$$\mathcal{H} := p_i \dot{q}^i - \mathcal{L}.$$

The equations of motion are then given by Hamilton's equations,

$$\dot{q}^i = \{\mathcal{H}, p_i\} \quad \text{and} \quad \dot{p}_i = \{\mathcal{H}, q_i\}$$

where $\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$ is the Poisson bracket.

To discuss constraints we follow the approach of [5, 1.1]. The Lagrangian and Hamiltonian formalisms should be equivalent; however, there may be issues stopping the inversion of our Legendre transform. If the Hessian matrix $(\partial^2\mathcal{L}/\partial\dot{q}^i\partial\dot{q}^j)$ is singular, then the momenta will not be invertible functions of the coordinate derivatives. In particular, we will have *constraints*,

$$\varphi_k(q^i, p_i) = 0,$$

i.e. functions φ_k that must vanish for our system. This means that only a particular submanifold of the phase space will be physically allowable. Such functions φ_k which immediately are seen to vanish from the choice of momenta are called *primary constraints* and we use the notation of *weak equality*, $\varphi_k \approx 0$, to denote that this equality holds only for physically allowable events in a constrained phase-space rather than in the whole $2n$ dimensional space defined by the coordinates.

As such, our Hamiltonian formalism is incomplete without additional constraint information. To rectify this, we can modify our Hamiltonian by using Lagrange multipliers: consider new variables λ^k associated to our constraints φ_k and let,

$$\widetilde{\mathcal{H}} = \mathcal{H} + \sum_k \lambda^k \varphi_k.$$

Note that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are weakly equal. From this Hamiltonian, the Euler-Lagrange equation in the variable λ^k will give the equation of motion $\varphi_k = 0$ and so the constraint is now a property of the dynamics. The modified Hamiltonian also gives new Hamilton's equations for arbitrary $f(q^i, p_i)$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{\mathcal{H}, f\} + \lambda^k\{\varphi_k, f\}$$

and we may now invert the transform to yield,

$$\dot{q}^i = \frac{\partial\mathcal{H}}{\partial p_i} + \lambda^j \frac{\partial\varphi_j}{\partial p_k}.$$

In return for fixing invertibility, we may need to pay for it with additional constraints. Since φ_k vanishes, we must have,

$$0 \approx \dot{\varphi}_k = \{\mathcal{H}, \varphi_k\} + \lambda^\ell\{\varphi_\ell, \varphi_k\}. \tag{1.1}$$

In most cases this imposes a restriction on the multipliers λ^ℓ, but if the right hand side happens to be independent of the λ^ℓ's and non-trivial, we obtain a new constraint on the original coordinates: $\psi(p, q) \approx 0$. So we may obtain a family of constraints ψ_k from some of the φ_k. We can thus add new Lagrange multipliers to the Hamiltonian and and then impose (1.1) for constraints ψ to obtain even more constraints, and so on for each new constraint derived. This yields potentially a whole collection of *secondary constraints* $\psi_k \approx 0$. These are secondary constraints because they are constraints derived from Hamilton's equations for other constraints rather than purely from the algebra of our coordinates. For us, primary versus secondary constraints will be a distinction without a difference as whether constraints

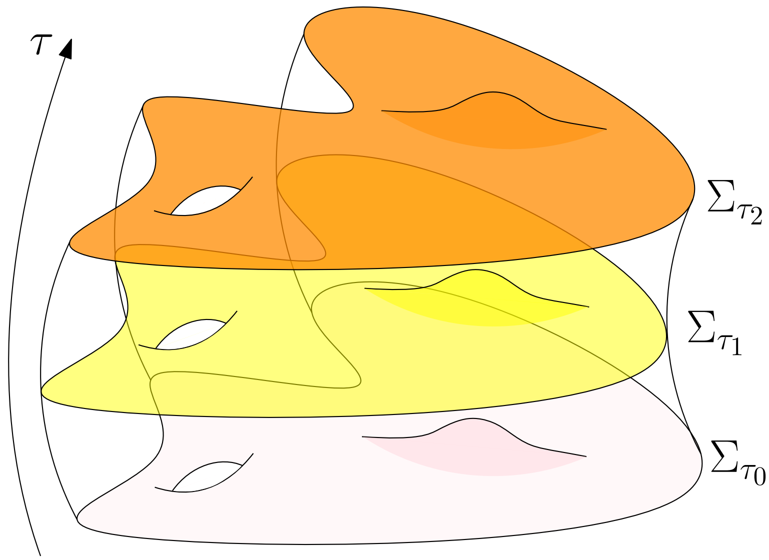

FIGURE 1
A certain spacetime which is foliated by spacelike slices Σ_τ, each with a Riemannian metric.

are primary or secondary is determined by our description of the system rather than the system itself.[1]

2. The ADM Formalism

The ADM formalism for general relativity, as developed by Arnowitt, Deser, and Misner in [1], begins from a familiar and relatively classical perspective on spacetime.

Foliating Spacetime. Assume that we can foliate our Lorentzian spacetime M by spacelike 3-manifolds.[2] Thus we have a diffeomorphism $\varphi : M \to \mathbb{R} \times \Sigma$. The preimage of each τ, $\varphi^{-1}(\{\tau\} \times \Sigma)$, will be a hypersurface Σ_τ which we think of as a 3-dimensonal spacial universe at a fixed moment in time, see Figure 1. Since each Σ_τ is spacelike, it is naturally a Riemannian manifold with metric $h(\tau)$. Along each slice Σ, we will have a unit normal vector field $n \in \Gamma(TM|_\Sigma)$ with $g(n, n) = 1$ and $g(n, X) = 0$ for $X \in \Gamma(T\Sigma)$ which is oriented in the positive τ direction. We write $N\Sigma$ for the normal bundle to Σ so that n is a unit norm section of $N\Sigma$.

We will work to translate Einstein's equations into equivalent statements in this formalism. We will find that four of the ten equations correspond to constraints on the intrinsic and extrinsic curvature of each slice while the remaining six describe the evolution of these slices in time.

We briefly review the construction of Riemannian submanifolds in our relevant context. Given vector fields X, Y on Σ, extend them smoothly to vector fields $\widetilde{X}, \widetilde{Y}$ on M. Now we may

[1] A more canonical and mathematically interesting division exists between *first class* and *second class constraints* as derived by Dirac. This will not be neccessary for us and so we invite the reader to read further in [5] on their own.

[2] If we accept our spacetime is globally hyperbolic, so that the metric everywhere can be inferred from its value and the extrinsic curvature on a particular hypersurface, then this assumption is automatic [3, pp 104].

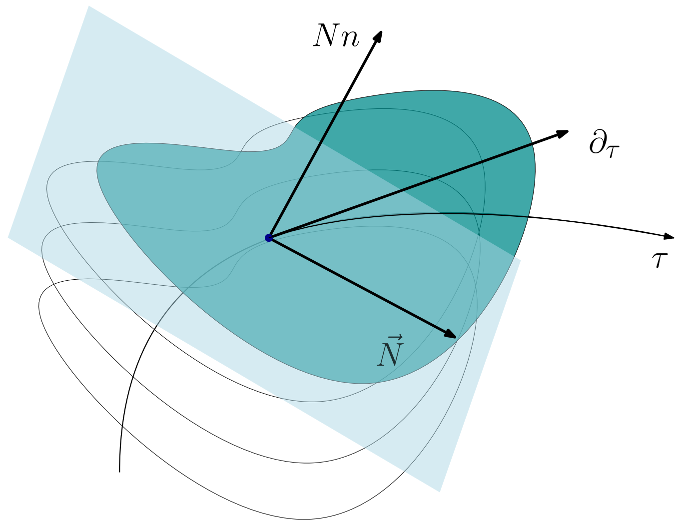

FIGURE 2
The lapse and shift vectors as components of the time directed unit normal.

take a covariant derivative in M and consider the component normal to Σ: $\nabla_{\widetilde{X}}\widetilde{Y}^{\perp}$. This is independent of our extension of X, Y [8, Prop. 8.1] and we refer to the resulting vector field as $K(X,Y)$. This defines a bilinear map,

$$K : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \Gamma(N\Sigma), \quad \text{s.t.} \quad K(X,Y) = {}^{(M)}\nabla_{\widetilde{X}}\widetilde{Y} - {}^{(3)}\nabla_X Y$$

which is called the *second fundamental form*. While h measures the intrinsic curvature measured by someone restricted to move on Σ, K measures the *extrinsic curvature*. For example, while a cylinder is locally isometric to flat space, when we embed it in $\mathbb{R}^3$, we can see how it curves within the larger space; this is what K measures.

Note we have a canonical choice of time-directed vector field $\partial_\tau = \varphi_*^{-1}(\partial_\tau)$, where ∂_τ is the standard coordinate vector field on $\mathbb{R}$. We can decompose this vector field into its direction parallel to Σ and perpendicular to it,

$$\partial_\tau = -g(\partial_\tau, n)n + (\partial_\tau + g(\partial_\tau, n)n) =: Nn + \vec{N}.$$

We define the *shift vector* to be $\vec{N}$ and the *lapse vector* to be Nn as given above, see Figure 2. An intuitive fact important for us later is that may write the Jacobian factor in terms of our 3-metric and lapse vector: $\sqrt{-g} = N\sqrt{h}$; see [3, 4.2] for the derivation.

A first result one proves in studying extrinsic curvature is the Gauss-Codazzi Equations. Henceforth, R will refer to the curvature tensor on M, while ${}^3R$ will be the curvature tensor on Σ_τ for arbitrary τ.

Theorem 2.1 (Gauss-Codazzi [8, Theorems 8.5, 8.9])**.** *The Gauss Equation says that,*

$$R^m{}_{ijk} = {}^3R^m{}_{ijk} + K_{jk}K_i{}^m - K_{ik}K_j{}^m. \tag{2.1}$$

Define a connection $\widetilde{\nabla}$ *acting on the bundle with sections given by bilinear maps* $\Gamma(T\Sigma)^2 \to \Gamma(N\Sigma)$ *as follows,*

$$(\widetilde{\nabla}_X F)(Y,Z) = \nabla_X F(Y,Z)^{\perp} - F(\nabla_X Y, Z) - F(Y, \nabla_X Z).$$

Then the Codazzi Equation says that,

$$(R(W,X)Y)^{\perp} = (\widetilde{\nabla}_W K)(X,Y) - (\widetilde{\nabla}_X K)(W,Y).$$

Here $R(\cdot,\cdot)$ is the curvature thought of as an $\mathrm{End}(TM)$ *valued 2-form which has been contracted as a two form with two vector fields to give a map $\Gamma(TM) \to \Gamma(TM)$.*

It is a straightforward computation given in [2, pp. 422-424] to see how these equations allow us to write components of the Einstein tensor in terms the properties of Σ. One computes, combining with Einstein's equations,

$$\begin{cases} 8\pi T_0^0 = G_0^0 = -\frac{1}{2}({}^3R + (K_i{}^i)^2 - K_{ij}K^{ij}) & \\ 8\pi T_i^0 = G_i^0 = {}^3\widetilde{\nabla}_j K_i{}^j - {}^3\widetilde{\nabla}_i K_j{}^j & i = 1,2,3. \end{cases}$$

The first equation imposes constraints on the scalar curvature of Σ in terms of its extrinsic curvature, while the other three restrict the extrinsic curvature. The other six Einstein equations will become second order differential equations that describe the time evolution of our 3-metric [2, pp 424].

A Hamiltonian Theory. From now on we assume that we work in the vacuum.[3] In classical field theory, one studies a Lagrangian density whose coordinate functions are (scalar, vector, tensor etc.) fields permeating space-time (the usual Lagrangian is the integral of the density over all of space). When we solve the equations of motion, they will describe the physical field configuration our theory predicts. The most familiar example is electromagnetism in which the Lagrangian density,

$$\mathcal{L}_{EM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad \text{where} \quad F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

reproduces the vacuum Maxwell's equations and predicts the electric and magnetic field propogate as waves [3, 3.4].

The natural choice of field in our theory of gravity is just the tensor field g_{ab} and the derivation is given in [11, Appendix 1] that the solution of the Einstein vacuum equations extremizes the *Einstein-Hilbert Action*,

$$S = \int \mathcal{L}_G \mathrm{d}^4x \quad \text{where} \quad \mathcal{L}_G = \frac{1}{16\pi}\sqrt{-g}R$$

is the Lagrangian density of our system (the Lagrangian is its integral over Σ). We will want the time variable to be explicit when we consider the Hamiltonian picture, so we use the foliation of our space and the corresponding curvature variables. Recalling $\sqrt{-g} = N\sqrt{h}$ and using the Gauss equation (2.1) to solve for the curvature, we may rewrite the Lagrangian density as,

$$\mathcal{L}_{ADM} = \frac{1}{16\pi}N\sqrt{h}({}^3R + K_{ij}K^{ij} - K^2) = \frac{1}{16\pi}N\sqrt{h}({}^3R + (h^{ir}h^{js} - h^{ij}h^{rs})K_{ij}K_{rs}).$$

We choose the fields of our field theory to be the 3-metric components h_{ij} and our lapse and shift vectors N, and N^i. For every fixed time τ, this defines ten independent scalar field components just as for our original 4-metric g. Note that K is independent of derivatives of

[3] Once we include matter and energy in the quantized theory, the gravitational field will have to interact with regular quantum field theories of matter. For us, the quantized gravitation is enough of a problem on its own!

lapse and shifts, so we have that the conjugate momenta associated to N and each N^i are constrained to vanish:

$$\Pi := \frac{\partial \mathcal{L}_{ADM}}{\partial(\partial_\tau N)} \approx 0 \quad \text{and} \quad \Pi_k := \frac{\partial \mathcal{L}_{ADM}}{\partial(\partial_\tau N^k)} \approx 0.$$

These are primary constraints in our system. The other conjugate momenta associated to components of h are,

$$\Pi^{ij} := \frac{\partial \mathcal{L}_{ADM}}{\partial(\partial_\tau h_{ij})}.$$

In [3, 4.2.1], the useful relation,

$$K_{ij} = \frac{1}{2N}(\partial_\tau h_{ij} - {}^3\nabla_i N_j - {}^3\nabla_j N_i)$$

is derived. From this it easily follows that,

$$\Pi^{ij} = \frac{1}{16\pi}\sqrt{h}(K^{ij} - K h^{ij}) = \frac{1}{16\pi}\sqrt{h}(h^{ir}h^{js} - h^{ij}h^{rs})K_{rs}.$$

If we define the tensor $G^*_{ijrs} := h_{ir}h_{js} - \frac{1}{2}h_{ij}h_{rs}$, then we may rewrite the Lagrangian density as,

$$\mathcal{L}_{ADM} = \frac{1}{16\pi} N\sqrt{h}\,{}^3R + \frac{16\pi N}{\sqrt{h}} G^*_{ijrs}\Pi^{rs}\Pi^{ij}.$$

Let the *Einstein Wheeler metric* G_{ijrs} be the symmetrization of G^*_{ijrs} divided by $\sqrt{h}$, i.e.,

$$G_{ijrs} = \frac{1}{2\sqrt{h}}(h_{ir}h_{js} + h_{is}h_{jr} - h_{ij}h_{rs}).$$

By the standard procedure (remembering our constraints and the Lagrange multipliers), we obtain the Hamiltonian density,

$$H_{ADM} = \lambda\Pi + \lambda^i\Pi_i + N^k\mathcal{H}_k + N\mathcal{H} + 2\,{}^3\nabla_i(\Pi^{ij}N^k k_{kj}). \tag{2.2}$$

where,

$$\mathcal{H} := 16\pi G_{ijrs}\Pi^{rs}\Pi^{ij} - \frac{1}{16\pi}\sqrt{h}\,{}^3R \quad \text{and} \quad \mathcal{H}_k := -2h_{kj}\,{}^3\nabla_i\Pi^{ij}$$

are called the *super-Hamiltonian* and *super-momenta* respectively [3, pp 112]. We can ignore the last term of (2.2) since it is a total derivative and so when we integrate it in the action it vanishes on any compact interval by Stoke's theorem (this is standard practice in constructing physical theories) [3, pp 147]. On the other hand, we require from imposing secondary constraints that,

$$0 \approx \{\Pi, H_{ADM}\} = -\mathcal{H} \quad \text{and} \quad 0 \approx \{\Pi_k, H_{ADM}\} = -\mathcal{H}_k. \tag{2.3}$$

We conclude that the Hamiltonian density of (2.2) is a sum of quantities which weakly vanish and so we deduce that it weakly vanishes as well,

$$H_{ADM} \approx 0.$$

By definition of the Poisson bracket, our coordinates and momenta satisfy the following:

$$\{N(x^0,x), \Pi(x^0,y)\} = \delta^{(3)}(x-y)$$
$$\{N^i(x^0,x), \Pi_k(x^0,y)\} = \delta^i{}_k\delta^{(3)}(x-y)$$
$$\{h_{ij}(x^0,x), \Pi^{rs}(y,x^0)\} = \delta^{(r}{}_i\delta^{s)}{}_j\delta^{(3)}(x-y),$$

with the brackets of any other pair of fields or momenta vanishing.

3. Wheeler-DeWitt Equation

We are ready to quantize, following the standard procedure of quantum field theory, as described in [9, Chapter 2]. The result will be the Wheeler-DeWitt Equation first given by Bryce DeWitt in 1967 [4, Eqn. 5.5] and our description will largely follow the exposition given in [3, Chapter 6].

To quantize, we promote the coordinates and momenta to operators on a space of states. What space should this be? In quantum mechanics, the state space is the Hilbert space of L^2 functions on the configuration space. The same is true in quantum field theory, except the configuration space is infinite dimensional, corresponding to the value of the field at every point in spacetime. When we quantize, the Hilbert space is *Fock space*, complex valued functionals on the space of all field configurations.

For us, the configuration space is all possible spacetime metrics h and values for the lapse and shift vectors. The quantized space of states should be the space of L^2 functionals on this space [2, pp 434]. One can informally think of this like traditional quantum mechanics which describe particles by wavefunctions that represent a probability distribution on possible positions of the particle; instead of the universe being composed of a single space time metric, it is a delocalized combination of many different metrics. This is an informal description of the idea of "quantum foam," in which the local structure of space is not fixed but a bubbling quantum superposition Ψ. This is schematically shown in Figure 3.

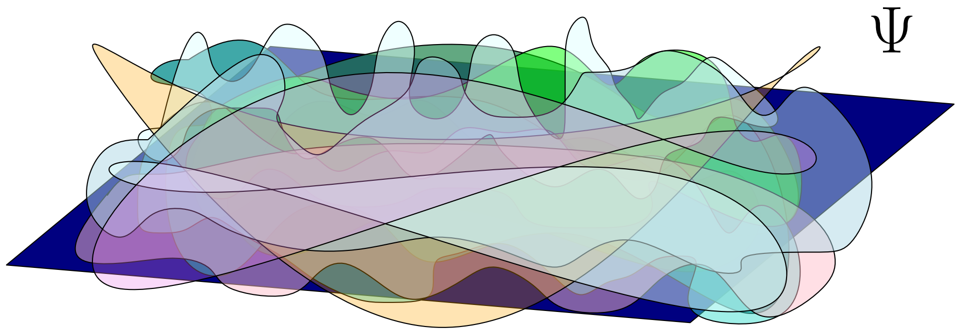

Figure 3

An artistic depiction of quantum foam—this is the conjectural local model for spacetime.

The procedure for giving our space of metrics and Lebesgue measure and hence a proper L^2 Hilbert space structure on states is *not currently known* [3, pp 155]. Nevertheless, we persist, treating the states and operators as purely formal expressions.

As in quantum field theory, we will make our fields of interest and their conjugate momenta into operators on our Hilbert space, denoted with a hat over the relevant classical quantity, and ask that the operators obey *equal time commutation relations* derived from the Poisson bracket relations above:

$$[\widehat{h}_{ij}(x^0,x),\widehat{\Pi}^{k\ell}(x^0,y)] = \frac{i}{2}({\delta_i}^k{\delta_j}^\ell + {\delta_i}^\ell{\delta_j}^k)\delta^{(3)}(x-y)$$
$$[\widehat{N}(x^0,x),\widehat{\Pi}(x^0,y)] = i\delta^{(3)}(x-y)$$
$$[\widehat{N}^i(x^0,x),\widehat{\Pi}_j(x^0,y)] = i{\delta^i}_j\delta^{(3)}(x-y),$$

with other commutators vanishing. In analogy with quantum mechanics, in a coordinate basis, we may consider our coordinate operators as multiplication operators and our momenta operators as functional derivatives acting on functionals $\Psi[h_{ij}, N^i, N]$:

$$\begin{aligned}
\widehat{h}_{ij}(x)(\Psi) = h_{ij}(x)\Psi \quad \widehat{\Pi}^{ij}(x)\Psi &= -i\frac{\partial \Psi}{\partial h_{ij}(x)} \\
\widehat{N}(x)(\Psi) = N(x)\Psi \quad \widehat{\Pi}(x)\Psi &= -i\frac{\partial \Psi}{\partial N(x)} \\
\widehat{N}^i(x)(\Psi) = N^i(x)\Psi \quad \widehat{\Pi}_i(x)\Psi &= -i\frac{\partial \Psi}{\partial N^i(x)}.
\end{aligned}$$

Given our total space of states, the constraints we found in the ADM theory will tell us that only certain states are physically allowable. This will restrict us to some subspace on which constraint equations are satisfied.

Recall we had primary constraints $\Pi \approx \Pi_k \approx 0$ and so for arbitrary physical state ψ we must have that,

$$-i\frac{\partial \Psi}{\partial N} = -i\frac{\partial \Psi}{\partial N^k} = 0.$$

In particular, $\Psi = \Psi[h_{ij}]$ must be a functional of the 3-metrics alone, independent of lapse and shift vectors. We also found that the whole Hamiltonian was constrained to vanish via our supermomentum and super-Hamiltonian. Imposing the vanishing of the components of the supermomentum from (2.3) gives,

$$\widehat{\mathcal{H}}_i\Psi = -2h_{ik}\,{}^3\nabla_j\left(\frac{\partial \Psi}{\partial h_{jk}(x)}\right) = 0.$$

One can show this equation corresponds roughly to the fact our system should be dependent only on the geometry of the slices Σ and not the coordinates we used [4, pp 1122]. According to DeWitt, one way to interpret this is to make Ψ a function whose infinite variables are the family of scalar invariants,

$$\int \sqrt{h}\,\mathrm{d}x\,, \int \sqrt{h}\,{}^3R\,\mathrm{d}x\,, \int \sqrt{h}\,{}^3R^2\,\mathrm{d}x\,, \text{ etc.},$$

which depend only on the metric and its derivatives.

Just as with the super-momenta, the super-Hamiltonian constraint of (2.3) gives the intimidating equation,

$$\boxed{\widehat{\mathcal{H}}\Psi = -\left[: 16\pi\widehat{G}_{ijk\ell}(x)\frac{\partial^2}{\partial h_{ij}(x)\partial h_{k\ell}(x)} : +\frac{1}{16\pi}\sqrt{h}\,{}^3\widehat{R}(x)\right]\Psi[h_{ij}] = 0.} \tag{3.1}$$

This, at last, is the **Wheeler-DeWitt equation**. The colons here refer to a proper reordering of the operators (a standard procedure in quantum field theory to avoid nonphysical results).[4] The equation gives a final dynamical constraint on what states Ψ we may have in our quantum theory of gravity.

[4]Actually executing this proper ordering poses major issues itself; this is the *operator ordering problem* [2, pp 434].

4. Theoretical Problems and Extensions

While we may naively hope that our canonical quantization has worked just like in the usual field theory case, unfortunately issues abound. A lengthy discussion of the many problems with our theory of quantum gravity is presented in [6], with a good summary of the major points in Chapter 2.4. We discuss in simple terms a few of the complications.

The Hilbert Space. As a first issue, we note as above that the space of states is not a Hilbert space in a well understood sense. I.e. it is unclear what scalar product we should expect. Understanding the measure-theoretic underpinnings of usual quantum field theory is difficult enough where Fock space is perfectly well understood, so this will also pose further difficulties.

The Problem of Time. There are a plethora of issues with considering time in our theory of quantum gravity. These centre around the fact that time should play the role of an internal parameter. For example, the Schrödinger equation essentially asserts (in the Heisenberg picture) that for a system with Hamiltonian operator $\widehat{\mathcal{H}}$ and an observable A, in analogy with Hamilton's equations,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -i[A, \mathcal{H}].$$

Equivalently, (in the Schrödinger Picture) the time evolution of states in the system is determined by acting on them with the unitary operator $\exp\left(-i\tau\widehat{\mathcal{H}}\right)$. But our Hamiltonian is always constrained to vanish and so we should expect no dynamics in the universe whatsoever, either in the observables or states! What this really represents is that our system should be relativistically invariant, with no external time dependence in its states. But this "frozen formalism" raises a number of issues, for example, finding what time invariant observables can be used to characterize the system. In fact, its not even agreed upon precisely what we mean by an observable in this theory [6, Chapter 5.1]. Given the special role time plays in our Hamiltonian construction, it is also unclear the correspondence between the quantized systems for two different choices of spacelike foliations and two different time variables.

There are many other similar variations to quantizing gravity, the main distinction being between whether we impose constraints before or after we quantize. While the different approaches have various pros and cons, technical issues surrounding interpreting time persist in all of them. It is also unknown if all of these quantization procedures are equivalent. Again, see [6] for a comprehensive summary for the various theories and their mutual issues.

Finding Solutions. Even if we discount the interpretation issues above, no one has been able to find anything more than a purely formal solution to the Wheeler-DeWitt equation as written in (3.1) [2, pp 435]. Of course a proper theory of gravity with testable predictions would also include studying the interaction between gravity and matter fields, which the theory as we have developed it is inadequate to handle.

Where Do We Go From Here? In the 1980's, Abhay Ashketar discovered a new collection of variables to describe the theory. In these "Ashketar variables," the constraints become much simpler and one can use Chern-Simons theory, a technique of gauge theory, to formally produce a solution (at least for non-zero cosmological constant) [2, Chapter 5]. Later, Lee Smolin and Carlo Rovelli built on these connections to gauge theory, and in particular the loop representation of Yang-Mills theory, to develop a framework that associates states in our quantum theory to systems of links (entangled knots) called "spin networks." This is the basis for the modern theory of *loop quantum gravity* which is a highly active area of research in trying to quantize gravity. One can find a rigorous introduction to the subject in the last chapter of [3].

Loop quantum gravity is likely only the second most popular modern approach to a "theory of everything," the most (in)famous being string theory. The basic premise of string theory is to replace point particles with one dimensional vibrating strings. The theory is very technical, an active area of math research, and of a very different flavour to what we have considered.

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