Student 3-Manifold Seminar: Chern-Simons theory and Wess-Zumino Witten model

(Largely based on · Witten - "QFT and the Jones polynomial"
· Atiyah - "The Geometry and Physics of Knots"
· Kohno - "Conformal Field Theory and Topology")

Recall: On 3-mfld can study action for connections on trivial G ($SU(2)$) - bundle

$$CS(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A)$$

Define partition function

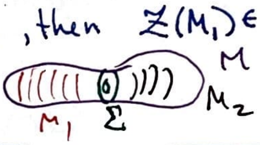
$$\mathcal{Z}(M) = \int \mathcal{D}A\, e^{iCS(A)}$$

For 3-mfld w/ oriented link L w/ components $K_1, \dots, K_n$. Let $W_{R_i}(K_i)$ be trace of holonomy around K_i in a rep^n R_i.

Then $\langle W_{R_1}(K_1)\cdots W_{R_n}(K_n)\rangle = \int \mathcal{D}A\, W_{R_1}(K_1)\cdots W_{R_n}(K_n)\, e^{iCS(A)}$ defines link inv.

For $M = S^3$ and std rep^{n}s, this is Jones polynomial.

To understand better, study on $\Sigma\times\mathbb{R}$, for Σ oriented surface. Associated to Σ is a Hilbert space of the theory $\mathcal{H}$. If we decompose $M = M_1 \cup_\Sigma M_2$, then $\mathcal{Z}(M_1)\in\mathcal{H}$, $\mathcal{Z}(M_2)\in\overline{\mathcal{H}}$ and $\mathcal{Z}(M) = \langle \mathcal{Z}(M_1), \mathcal{Z}(M_2)\rangle$.



Geometric Quantization: Consider surface Σ and space of G-connections $\mathcal{A}$ on bundle over Σ. This space has a symplectic form:

$$\omega(\alpha,\beta) = -\int_\Sigma \underbrace{\mathrm{Tr}}_{\to \text{really Killing form}}(\alpha\wedge\beta) \quad \text{for } \alpha,\beta\in\Omega^1(\Sigma,\mathfrak{g})$$

We may act on connections by the gauge group $\mathcal{G} = C^\infty(\Sigma, G)$. This will preserve ω. In this circumstance where we have symmetry of symplectic mfld, can often define momentum map $\mu : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*$ which generalizes giving linear or angular momentum of point in phase space.

In our case $\mu(A) = F_A$ the curvature under identification $F_A(\underset{\mapsto C^\infty(\Sigma,\mathfrak{g})}{X}) = \int_\Sigma \mathrm{Tr}(XF_A)$.

Given symmetry w/ conserved momenta, can quotient by symmetry to simplify problem. Generalization of this is Marsden-Weinstein quotient (also called symplectic reduction)

$$\mathcal{A}//\mathcal{G} \overset{\text{def.}}{=} \mu^{-1}(0)/\mathcal{G} \;\dots$$ this is moduli space of flat G-connections

This space has a natural symplectic structure. Moreover, if we pick J a conformal / complex structure on Σ, then $\mathcal{A}//\mathcal{G}$ will have a Kähler structure.

Fact: The symplectic form has $[\frac{\omega}{2\pi}]\in H^2(\Sigma,\mathbb{Z})$. A integral second cohomology class uniquely determines a hol. line bundle $\underset{\underset{\mathcal{A}//\mathcal{G}}{\downarrow}}{\mathcal{L}}$ w/ $c_1(\mathcal{L}) = [\frac{\omega}{2\pi}]$. Equivalently any connection on $\mathcal{L}$ has $[F] = [-i\omega]$.

At level k, should modify symplectic form to $k\omega \Rightarrow$ line bundle obtained is $\mathcal{L}^{\otimes k}$.

The quantum Hilbert space at level k is $\Gamma_{hol}(\mathcal{L}^{\otimes k}, \mathcal{A}//\mathcal{G})$.

we can make this move explicit (and connect to Chern-Simons)

If $g \in \mathcal{G}$, then $CS(g^*A) - CS(A) = -2\pi k \int_M g^*\sigma$ where $\sigma = \frac{1}{24\pi^2}\mathrm{Tr}(\mu^3)$ and $\mu(g) = g^{-1}dg$ Maurer-Cartan form

$\underset{H^3(SU(2),\mathbb{Z})}{\cap}$

$\Rightarrow \exp(ikCS): \mathcal{A}/\mathcal{G} \to \mathbb{R}/\tfrac{1}{k}\mathbb{Z}$ well-defined.

On $\Sigma' \times \mathbb{R}$: given $a \in \mathcal{A}_{\Sigma'}$ extend to $A \in \mathcal{A}_M$. Extend $g \in \mathcal{G}_{\Sigma'}$ to $\tilde{g} \in \mathcal{G}_M$.

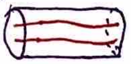
let $c(a,g) = \exp(i(CS(\tilde{g}^*A) - CS(A))) = \exp(ik(\frac{1}{4\pi}\int_{\Sigma'} a \wedge \mu - 2\pi \int \tilde{g}^*\sigma))$ indep. of extension

Given trivial bundle $\mathcal{A}_{\Sigma'} \times \mathbb{C} \to \mathcal{A}_{\Sigma'}$, can quotient by $(a,1) \sim (g^*a, c(a,g))$.

This defines a line bundle as fibre product $\mathcal{L} = \mathcal{A}_{\Sigma'} \times_{\mathcal{G}} \mathbb{C}$ over $\mathcal{A}_{\Sigma'}/\mathcal{G}$

Restricting to moduli space of flat connections gives line bundle above.

Notes (i) In presence of marked points on Σ', should reduce structure group of

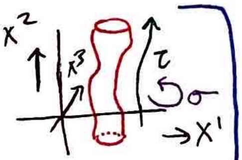


flat connection at these points.

(ii) Is Hilbert space fin diml? what is its size? A priori, our Hilbert space $\mathcal{H}$ depends on complex structure (J). Is it independent of J?

Conformal Field Theory and Wess-Zumino-Witten:

Given two manifolds (possibly w/ extra structure) X, M, suppose we have a functional $S: \mathrm{Map}(X,M) \to \mathbb{R}$. Such a theory is called a σ-model. If $M \neq \mathbb{R}^n$, this is a non-linear σ-model.



E.g. let $X = S^1_\sigma \times \mathbb{R}_\tau$ w/ metric $h_{\mu\nu}$. Let $M = \mathbb{R}^{1,D-1}$ Minkowski space.

Define action $\boxed{S(X, h_{\mu\nu}) = -\frac{1}{4\pi\alpha'}\int_{\Sigma'} d\sigma d\tau \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}$ → Polyakov Action

Roughly measures surface area of string world sheet. This is starting point for string theory.

Note: This action is invariant under a conformal change of metric $h \mapsto \Omega^2 h$. Hence S defines a conformal field theory.

Let's now define a non-linear σ-model for $f \in \mathrm{Map}(\Sigma, G)$ w/ Σ compact Riemann surface and $G = SU(2)$.

Let $E_{\Sigma'}(f) = -i\int_{\Sigma'} \mathrm{Tr}(f^{-1}\partial f \wedge f^{-1}\bar{\partial} f) = \int_{\Sigma'} |df|^2$ = "Dirichlet Energy"

↳ Killing norm

Clearly this term is conformally inv. (since depends on complex str. of Σ' only).

Let $S_{WZW}(f) = \frac{k}{4\pi}E_{\Sigma'}(f) - 2\pi i k\int_B \tilde{f}^*\sigma$. Here, $k \in \mathbb{Z}$ is the level,

B is 3-mfld w/ $\partial B = \Sigma'$, $\tilde{f}$ is extension to B of f and $\sigma = \frac{1}{24\pi^2}\mathrm{Tr}(\mu^3)$ as before.

Lemma: $\exp(-S_{WZW}(f))$ is independent of B and $\tilde{f}$.

Proof: If $\tilde{f}_1: B_1 \to G$, $\tilde{f}_2: B_2 \to G$ two choices, let $M = B_1 \cup_{\Sigma} (-B_2)$ and $F: M \to G$ extending $\tilde{f}_{1,2}$.

Then $S_{WZW}(\tilde{f}_1) - S_{WZW}(\hat{f}_2) = 2\pi i k\int_M F^*\sigma \in 2\pi i\mathbb{Z}$. □

Corollary: S_{WZW} defines a 2D CFT.

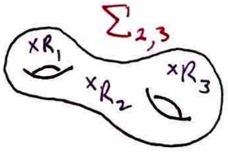
This is the Wess-Zumino-Witten model, describes string living in Lie group w/ topological gauge force.

Quantizing WZW Model:

2D CFT's highly structured, so can understand lots just from algebra of operators derived from field $f:\Sigma\to G$. Actually enough to consider primary operators which obey $\phi(z')=\left(\frac{dz'}{dz}\right)^h\left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{h}}\phi(z)$ under conformal COV $z\mapsto z'(z)$, $\bar{z}\mapsto\bar{z}'(\bar{z})$ for $(h,\bar{h})\in\mathbb{Q}^2$.

For WZW, these primaries are "integrable highest weight" irreducible rep^n s of affine Lie algebra $\hat{\mathfrak{g}}$ given by central extension prop. to k of loop algebra of $\mathfrak{g}=\mathfrak{su}(2)$.

Suppose we are studying theory on surface $\Sigma_{g,n}$ of genus g w/ n punctures associated to marked rep^n s. Central goal is to compute correlators:



$$\langle R_1(z_1,\bar{z}_1)\cdots R_n(z_n,\bar{z}_n)\rangle=\int\mathcal{D}f\,R_1(z_1,\bar{z}_1)\cdots R_n(z_n,\bar{z}_n)\,e^{-S_{WZW}[f]}/Z(\Sigma)$$

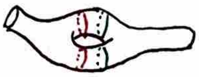
WZW is a rational CFT, meaning there is finite basis of primaries $\phi_1,\ldots,\phi_N$. A key tool is the operator product expansion which gives:

$$\langle\phi_i(z,\bar{z})\,\phi_j(w,\bar{w})\,\mathcal{A}_1(z_1,\bar{z}_1)\cdots\mathcal{A}_n(z_n,\bar{z}_n)\rangle=\sum C_{ij}^k\langle\phi_k(w,\bar{w})\mathcal{A}_1\cdots\mathcal{A}_n\rangle$$

→ structure consts.

→ arbitrary insertions at radii $|z_i|>|z|,|w|,|z-w|$.

Finding correlators: It will be enough to work on S^2. Higher genera can be obtained from gluing spheres along propogators. Let's study for different no. of punctures.



p=0: $\langle\mathbb{I}\rangle=1$. p=1: $\langle\phi(z,\bar{z})\rangle=0$ unless ϕ is the triv rep^n.

p=2: Use OPE: $\langle R_i(z,\bar{z})R_j(w,\bar{w})\rangle\propto\frac{1}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}}\delta_{R_i,\bar{R}_j}$

p=3: Conformal inv. $\Rightarrow$ $\langle\phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\phi_k(z_3,\bar{z}_3)\rangle=\mathcal{A}_{ijk}/(z_1-z_2)^{h_{123}}(z_1-z_3)^{h_{132}}(z_2-z_3)^{h_{231}}\ldots$ a.h.

p=4: No longer fixed by conformal invariance. Have:

$$\langle\phi_i\phi_j\phi_k\phi_l\rangle=\sum_m C_{ij}^m C_{kl}^m\,\mathcal{F}_{ij}^{kl}(m,z)\,\bar{\mathcal{F}}_{ij}^{kl}(m,\bar{z})$$

, z is cross-ratio i.e. PGL(2,ℂ) inv.

For p>4 can again expand as ~~prod~~ sum of hol times antihol f^n in p-3 variables.

These functions $\mathcal{F}_{m;ij}$ are called conformal blocks. On genus g w/ n punctures, conformal blocks generate finite dim^l vector space $V_{\Sigma_{g,p}}$. Varying conformal structure, $V_{g,p}$ defines vector bundle over moduli space of complex structures on $\Sigma_{g,p}$ w/ conformal blocks as hol. sections.

Fact: Knizhink-Zamoldchikov eq^n shows this bundle $V_{g,p}$ has flat connection w/ conformal blocks as covariantly constant sections.

Thm: (Segal, Witten) The Hilbert space $\mathcal{H}^{(J)}_{\Sigma_{g,n}}$ of Chern-Simons theory at level k is canonically iso. to space of conformal blocks $V^{(J)}_{\Sigma_{g,n}}$ for WZW model at level k. In particular:

(a) Hilbert space is fin dim^l w/ dim. computed by Verlinde.

(b) Hilbert spaces for different complex structures are canonically identified.

Holographic Principle: This theorem represents simple example of important "holographic principle" in physics which says that ~~many~~ quantum field theories can be recovered from a theory on their boundary. The most well known example of this principle is the many conjectural cases of the AdS-CFT correspondence which says that quantum gravity theories in Anti-DeSitter space should be recovered by a conformal field theory on the boundary.