Structures on Manifolds and the Triangulation Conjecture

Student Low-Dimensional Topology Seminar
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§1. Structures on Topological Manifolds

Throughout, let M be a closed topological manifold (of varying dimension).

Def^n: A triangulation of M is a homeomorphism from M to a simplicial complex.

A PL structure on M is an atlas whose chart transitions are piecewise linear. This is equivalent to a combinatorial triangulation, i.e. a triangulation where the subcomplex adjacent to any vertex (its link) is PL homeomorphic to a sphere.

Whitehead proved every smooth manifold has a (unique) PL structure. Thus we have inclusions:

$$\text{Top Mflds} \supseteq \text{Triangulated Mflds} \supseteq \text{PL Mflds} \supseteq \text{Smooth Mflds}$$

In this talk, we address the following:

Q. In each dimension n, when are these inclusions strict? If they are strict, what obstructs the structures?

A. The answer is summarized in a chart below.

- For $n<8$, PL $=$ Diff. For $n\geq 8$, there are obstructions due to Munkres–Hirsch–Mazur. (e.g. Kervaire manifold)
- For $n<5$, Triang $=$ PL. For $n\geq 5$, there is obstruction from Kirby–Siebenmann invariant. (e.g. $E_8 \times T^{n-4}$)
- For $n<4$, Top $=$ Triang. For $n=4$, E_8 manifold is not triangulable using Casson invariant.
  For $n\geq 5$, [Manolescu '16] proved there are non-triangulable n-manifolds. This is his disproof of the triangulation conjecture.

We will describe the obstruction theory giving PL $\subsetneq$ Diff and Top $\subsetneq$ PL before surveying Manolescu's disproof, which uses an equivariant stable homotopy model of Seiberg–Witten Floer theory.

§2. Obstruction Theory

Consider the spaces $\mathrm{Top}(n)$, $PL(n)$, and $\mathrm{Diff}(n) \simeq O(n)$ of origin preserving invertible maps of $\mathbb{R}^n$ in the appropriate category. These are topological groups with classifying spaces $B\mathrm{Top}(n)$, $BPL(n)$, $B\mathrm{Diff}(n)$.

Given M, it has a topological tangent bundle TM, the normal bundle of the diagonal embedding $\Delta_M \hookrightarrow M\times M$. This is classified by some map $M \to B\mathrm{Top}(n)$. Similarly in the PL, Diff case.

As for any top. gps., the inclusions Diff $\leq$ PL $\leq$ Top give fibrations,

$$\mathrm{Top}(n)/PL(n) \hookrightarrow BPL(n) \to B\mathrm{Top}(n) \quad\text{and}\quad PL(n)/\mathrm{Diff}(n) \hookrightarrow B\mathrm{Diff}(n) \to BPL(n).$$

We might expect, as w/ vector bundles, a Top mfld having a PL structure and a PL mfld having a smooth structure comes from lifts,

$$\begin{array}{ccc} & & BPL(n) \\ & \nearrow & \downarrow \\ M & \longrightarrow & B\mathrm{Top}(n) \end{array} \quad\text{and}\quad \begin{array}{ccc} & & B\mathrm{Diff}(n) \\ & \nearrow & \downarrow \\ M & \longrightarrow & BPL(n) \end{array}$$

It turns out, we can reduce to the stable problem. I.e. let $\mathrm{Top} = \operatorname{colim}_n \mathrm{Top}(n)$, same for PL, Diff. We can study lifts of stabilized maps $M \to B\mathrm{Top}$, $M \to BPL$.

Thm: (Hirsch–Mazur, Kirby–Siebenmann) Upgrading structures corresponds to lifts of the stabilized maps in the PL/Diff case, and in the Top/PL case when $n \geq 5$.

Dimension	Top ⊆ Triangulated	Triangulated ⊆ PL	PL ⊆ Diff	When is a topological n-manifold smoothable?
$n \leq 3$	All notions coincide [Radó $n=2$, Moise $n=3$].			Always uniquely smoothable.
$n = 4$	E_8 not triangulated. [Casson]	Triangulated = PL [Perelman]		Only smoothable if $ks(M)$ vanishes. Further obstructions from gauge theory. Unclear how many smooth structures, sometimes infinitely many e.g. $S^2 \times S^2, K3$. (always???)
$n = 5$	For $n \geq 5$, there are always non-triangulable manifolds.	There is a single obstruction to topological manifolds having a PL structure, $ks(M) \in H^4(M;\mathbb{Z}/2)$	PL = Diff [Hirsch-Mazur]	Smoothable if and only if $ks(M)$ vanishes. Smooth structures parametrized by $H^3(M;\mathbb{Z}/2)$.
$n = 6$				
$n = 7$	[Manolescu]	[Kirby-Siebenmann] e.g. $E_8 \times T^{n-4}$ triangulable but not PL.		
$n \geq 8$			Obstructions in $H^n(M;\Theta_{n-1})$ e.g. Kervaire manifold. [HM]	Smoothable iff $ks(M)$ and Hirsch-Mazur obstructions vanish. Smooth structures parametrized by (finite) cohomology groups.

How can we understand lifts? Obstruction Theory!

Recall: Given fibration $F \hookrightarrow E \to B$ and a map $f: X \to B$, there are a sequence of obstructions for extending a lift of f:
$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r, "f"'] \arrow[ur, dotted] & B \end{tikzcd}$$
from the k- to the $(k+1)$-skeleton of X given by cohomology classes $c_f^{k+1} \in H^{k+1}(X; \pi_k(F))$. If this class vanishes, the lift extends and the possible extensions up to homotopy are parametrized by $H^{k+1}(X; \pi_{k+1}(F))$.

If F is $(k-1)$-connected, there will be a "first obstruction" $c_E \in H^{k+1}(E; \pi_k(F))$ depending only on the fibration.

Thus to understand lifts/obstructions in our case, we need to know homotopy gps of Top/PL, $PL/Diff$.

Thm: (Kirby–Siebenmann '69) Top/PL is the Eilenberg–MacLane space $K(\mathbb{Z}/2, 3)$.

Thus for a topological mfld M of dim ≥ 5, there is a single "characteristic class" the Kirby–Siebenmann invariant $ks(M) \in H^4(M; \mathbb{Z}/2)$ which vanishes if and only if M has a PL structure.

Thm: (Munkres, Hirsch–Mazur '74) $\pi_n(PL/Diff) = \Theta_n := \{\text{homotopy } n\text{-spheres}\}/h\text{-cobordism} =$ "Kervaire–Milnor groups" ↳ also group of exotic spheres for $n \geq 5$.

One has $\Theta_n = 0$ for $n < 7$, $\Theta_7 = \mathbb{Z}/28$, $\Theta_8 = \mathbb{Z}/2$, $\Theta_9 = \mathbb{Z}/2^2$, $\Theta_{10} = \mathbb{Z}/6$, $\Theta_{11} = \mathbb{Z}/992$, etc.

So, PL manifolds are smoothable in dim < 8, and in higher dimensions smoothness equivalent to the vanishing of a sequence of obstructions in $H^k(X, \Theta_{k-1})$.

§3. The Triangulation Conjecture

Let $\Theta_3^H = \{\text{homology 3-spheres}\}/\text{homology cobordism}$. We can define the Rokhlin invariant $\mu: \Theta_3^H \to \mathbb{Z}/2$ by $\mu(Y) = \sigma(W)/8 \pmod 2$ where W is a spin null-bordism of Y and σ is its signature.

This is a grouphom. under the connect sum operation on Θ_3^H. Hence we have short exact sequence
$$(*) \quad 0 \to \ker \mu \to \Theta_3^H \to \mathbb{Z}/2 \to 0.$$

Thm: (Galewski–Stern '80, Matumoto '78) There are non-triangulable manifolds in dim ≥ 5 iff $(*)$ doesn't split.

Why? There is a classifying space $BTri$ as with $BTop$, BPL, $BDiff$. It turns out Top/Tri is $K(\ker\mu, 4)$. Hence there is a single obstruction to triangulation in dim ≥ 5 given in $H^5(M; \ker\mu)$.

By basic algebraic topology, this must arise as $\beta(ks(M))$ where $\beta: H^4(M; \mathbb{Z}/2) \to H^5(M; \ker\mu)$ is the Bockstein from $(*)$. If $(*)$ splits, then β must be trivial.

If $(*)$ doesn't split (as turns out to be the case), GS show $Sq^1 ks(M) \neq 0 \Rightarrow \beta(ks(M)) \neq 0$.

One can find M so that $Sq^1 ks(M) \neq 0$ as follows, which then is not triangulable.

Let X be the fake $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. X is homeomorphic to $\overline{X}$; let M be the mapping torus of this homeomorphism. Then, by the Wu formula,
$$Sq^1 ks(M) = \underbrace{ks(M)}_{\text{↳ dual to section}} \smile \underbrace{w_1(M)}_{\text{↳ dual to fibre}} \neq 0.$$
In higher dimensions, we can take $M \times T^{n-5}$. $\square$

To conclude, we just need to show $(*)$ does not split. Equivalently, we want to show there does not exist a homology 3-sphere Y with $\mu(Y) = 1$ and $Y \# Y = S^3$.

To do this, we lift μ to an integral invariant β such that $\beta(-Y) = -\beta(Y)$. (note: β will not be a homomorphism)
Then w/ Y as above, $Y = -Y \Rightarrow \beta(Y) = -\beta(Y) \Rightarrow \beta(Y) = 0 \equiv \mu(Y) \pmod 2$.
We will construct β using Floer homotopy theory; it has similar properties to Casson and Frøyshov invariants.
As a warmup, we discuss the Frøyshov invariant.
Recall Kronheimer–Mrowka construct monopole Floer homology $\check{HM}(Y)$ as Floer theory of 3D/4D SW eqn's.
This has a U-map of degree -2, and hence is a module over $\mathbb{Z}[U]$. The chain complex looks as follows (we will see why shortly):
(Note: ∂ may interact between 2 parts of complex)

$$\mathbb{Z} \xleftarrow{U} 0 \quad \mathbb{Z} \xleftarrow{U} 0 \quad \mathbb{Z} \xleftarrow{U} \cdots \qquad \mathbb{Z}^N \circlearrowleft U \qquad \check{CM}(Y)$$

Infinite "U-tower" — Finite dimensional part

Define the Frøyshov invariant $\delta(Y) = -\frac{1}{2}$(min. degree of element in U-tower surviving in homology).
This has the kernel of what we want for β, but does not satisfy $\delta(Y) \equiv \mu(Y) \pmod 2$.
We need Floer homotopy theory as envisioned by Cohen–Jones–Segal. The idea is to enhance a Floer homology $HF(Y)$ to a stable homotopy type (suspension spectrum) Y_{FH} whose homology is $HF(Y)$. Then we can define more exotic algebro-topological invariants (e.g. Floer K-theory) by applying those theories to Y_{FH}.
Manolescu executes this for monopole Floer theory using ideas of Floer and Bauer–Furuta: we approximate SW eq'ns in finite dimensions and use the concept of a Conley index from dynamical systems to build stable homotopy type Y_{FH} so that $\tilde{H}_*(Y_{FH}) = \widetilde{HM}(Y)$. Constant gauge transformations induce an S^1-action on Y_{FH}.
We can thus define S^1-equivariant homology and $\tilde{H}^{S^1}_{*+2n(Y,g)}(Y_{FH}) = \check{HM}_*(Y)$.
Here, $n(Y,g) = -2(\mathrm{ind}(\partial\!\!\!/_W) + \frac{1}{8}\sigma(W))$ where $\partial W = Y$ comes from spectral flow and is needed for metric-independence.
S^1-equivariant homology is a module over $H^*(BS^1) = \mathbb{Z}[U]$ which recovers U-map.
The infinite U-tower comes from the (unique) reducible SW sol'n, while finite part is the S^1-orbits of irreducibles.
Why is this useful? SW eq'ns actually have a $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{H}$ action which extends to Y_{FH} (by conjugating spin^c structure).
So we may define the Pin(2)-equivariant Seiberg-Witten Floer homology $SWFH^{\mathrm{Pin}(2)}_{*-2n(Y,g)}(Y) := H^{\mathrm{Pin}(2)}_*(Y_{FH}; \mathbb{Z}/2)$.
This is a module over $H^*(B\mathrm{Pin}(2); \mathbb{Z}/2) \cong \mathbb{Z}/2[q,v]/(q^3)$, $|v| = 4, |q| = 1$. The Floer complex looks like:

$$\mathbb{Z}/2 \xleftarrow{q} \mathbb{Z}/2 \xleftarrow{q} \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \xleftarrow{q} \mathbb{Z}/2 \xleftarrow{q} \mathbb{Z}/2 \quad 0 \quad \cdots \qquad (\mathbb{Z}/2)^N \circlearrowleft q, \circlearrowleft v$$

grading mod 4: 2μ, $2\mu+1$, $2\mu+2$, $2\mu+3$, 2μ, $2\mu+1$, $2\mu+2$, $2\mu+3$

Infinite "(q,v)-tower" of reducibles — Finite dimensional complex of irreducibles

Define $\beta(Y) = \frac{1}{2}$(min. degree of element in (q,v)-tower with grading $2\mu+1 \pmod 4$ surviving in homology) $- \frac{1}{2}$.

- $\beta \equiv \mu \pmod 2$ from the 4-periodic structure + grading.
- This is well defined on Θ^H_3 by functoriality of Seiberg-Witten Floer theory with respect to cobordisms.
- $\beta(-Y) = -\beta(Y)$ since swapping orientations amounts to swapping $\check{HM}_*$ with $\hat{HM}_{-*}$; these are related by long exact sequence.

E.g. Consider Brieskorn sphere $\Sigma(2,3,11)$. This has two S^1-orbits of irreducibles (one Pin(2)-orbit).
$SWFC^{\mathrm{Pin}(2)}_*$ looks like:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 \;\cdots \end{array}$$

$$\mathbb{Z}/2 \xrightarrow{\partial} \mathbb{Z}/2$$

Hence $\beta = \frac{1}{2}(1-1) = 0$.