The Seiberg-Witten Equations and Gradient Flow

Monopole Floer homology learning seminar Oct. 2nd 2024

- Based on chapters 4 & 5 of Kronheimer & Mrowka [KM]

§1. The Floer Story

Let us recall the general procedure of constructing a Floer theory.

1. Define a functional $\mathcal{L}$ on a ∞-dim^l configuration space $\mathcal{C}$ associated to mfld M.
2. Consider set Γ of critical points of $\mathcal{L}$. These solve some PDE.
3. Wrt metric, consider gradient flow of $\mathcal{L}$. Defining some elliptic PDE $\mathfrak{F}u = 0$.
4. "Finite energy" solutions of $\mathfrak{F}u=0$ define gradient flow lines asymptotic to solutions in Γ at ends. For $x,y\in\Gamma$, we can form space $\mathcal{M}(x,y)$ of solutions of $\mathfrak{F}$ with given asymptotics.
5. Do lots of analysis & $\mathcal{M}(x,y)$. Show, after perturbation of $\mathfrak{F}$, it is a ~~Banach~~ manifold with a natural stratified compactification and orientation.
6. Do Morse theory: consider chain complex generated by Γ with ∂ counting 0-dim^l components of $\coprod_{x,y\in\Gamma}\mathcal{M}(x,y)$.

The monopole version has additional complications, most notably we need to blow up configuration space, mod out by gauge group and study S^1-equivariant Morse theory.

Today we will go through steps ①-③ of monopole Floer homology and begin to discuss energy and compactness.

§2. The Chern-Simons-Dirac Functional

Let Y be a closed connected Riemannian 3-mfld w/ spin^c-structure $\mathfrak{s}=(S,\rho)$.
Pick a reference spin^c-connection B_0 on S, which determines connection B_0^t on $\Lambda^2 S$.
We define a configuration space,

$$\mathcal{C}(Y,\mathfrak{s}) = \mathcal{A}(Y,\mathfrak{s})\times\Gamma(S) = \{(B,\Psi)\mid B \text{ a spin}^c\text{-connection on } S,\ \Psi \text{ a section of } S\}$$

We can then define the Chern-Simons-Dirac functional $\mathcal{L}:\mathcal{C}(Y,\mathfrak{s})\to\mathbb{R}$ by:

$$\mathcal{L}(B,\Psi) = \underbrace{-\frac{1}{8}\int_Y (B^t-B_0^t)\wedge(F_{B^t}+F_{B_0^t})}_{U(1)\text{-Chern-Simons functional}} + \underbrace{\frac{1}{2}\int_Y \langle D_B\Psi,\Psi\rangle d\mathrm{vol}}_{\text{Massless Dirac Lagrangian from QED}}$$

Remark: Where does this functional come from?
- The $SU(2)$ Chern-Simons functional leads to instanton Floer homology
- 4 dim^l SW theory is "formally dual" to 4 dim^l Donaldson theory: they are opposite asymptotic limits of the same SUSY $SU(2)$ YM theory.

The configuration space is an affine space. We can identify:

$$T_{(B,\Psi)}\mathcal{C}(Y,\mathfrak{s}) = \Gamma(Y, iT^*Y\oplus S).$$

here we use the natural L^2 metric on $\mathcal{C}$.

By making a small variation in $\mathcal{L}$ and integrating by parts we can compute $\mathrm{grad}\,\mathcal{L}:\mathcal{C}(Y,\mathfrak{s})\to T\mathcal{C}(Y,\mathfrak{s})$.
It is given by, $\mathrm{grad}\,\mathcal{L} = \left(\left(\frac{1}{2}\star F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0\right)\otimes 1_S,\ D_B\Psi\right)$.
Since $\rho(\star\alpha) = -\rho(\alpha)$, we find the critical points of $\mathcal{L}$ are pairs (B,Ψ) satisfying:

$$\begin{cases} \frac{1}{2}\rho(F_{B^t}) - (\Psi\Psi^*)_0 = 0 \\ D_B\Psi = 0 \end{cases}$$

These are the 3-dimensional Seiberg-Witten equations. //

Recall we have the gauge group $\mathcal{G}(Y) = C^\infty(Y, S^1)$ which acts by
$u \cdot (B, \Psi) = (B - u^{-1}du \otimes 1_S, u\Psi)$.
The components of $\mathcal{G}(Y)$ are indexed by $H^1(Y, \mathbb{Z})$ and we write $[u]$ for the homotopy class of a gauge transformation u.

<u>Lemma:</u> For $u \in \mathcal{G}(Y)$, $\mathcal{L}(u\cdot(B,\Psi)) - \mathcal{L}(B,\Psi) = 2\pi^2([u] \smile c_1(S))[Y]$.

We conclude, $\mathcal{L}$ descends to a functional $\tilde{\mathcal{L}} : \mathcal{C}(Y, \mathfrak{s}) / \mathcal{G}(Y) \to \mathbb{R}/2\pi^2\mathbb{Z}$.
If c_1 is torsion, $\tilde{\mathcal{L}}$ will be single-valued in $\mathbb{R}$.
When c_1 is torsion one will have reducible solutions $(B, 0)$ on which $\mathcal{G}$ acts non-freely.

§3. Gradient Flow of $\mathcal{L}$

The negative L^2-gradient flow of $\mathcal{L}$ has the form:

$$\begin{cases} \frac{\partial}{\partial t} B^t = -* F_{B^t} - 2\rho^{-1}(\Psi\Psi^*)_0 \\ \frac{\partial}{\partial t}\Psi = -D_B \Psi \end{cases}$$

for (B^t, Ψ) a time-dependent family in $\mathcal{C}(Y, \mathfrak{s})$.
We can instead regard this as a pair on the cylinder.

We can put a spin^c-structure on $Z = \mathbb{R}\times Y$. Take $S_Z = S^+ \oplus S^- = S_Y \oplus S_Y$. The Clifford multiplication is:
$\rho_Z(\partial_t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\rho_Z(v) = \begin{pmatrix} 0 & -\rho(v)^* \\ \rho(v) & 0 \end{pmatrix}$ for $v \in TY$.
There is an embedding $\mathcal{A}(Y, \mathfrak{s}) \to \mathcal{A}(Z, \mathfrak{s}_Z)$, $B \mapsto A$. We send the covariant derivative ∇_B to $\nabla_A = {}^d/_{dt} + \nabla_B$. The image of this map gives connections in "temporal gauge", i.e. connections whose one-form component in the t-direction is trivial.

Have Dirac operator $D_A^+ : \Gamma(S^+) \to \Gamma(S^-)$ given by $D_A^+ = \frac{d}{dt} + D_B$.
We have curvature $F_{A^t} = dt \wedge ({}^d/_{dt} B^t) + F_{B^t}$. On $\Lambda^2 S^+$, this gives its self-dual part:
$F_{A^t}^+ = \frac{1}{2}(* \frac{d}{dt} B^t + F_{B^t} + dt \wedge ({}^d/_{dt} B^t + *F_{B^t}))$.
Under Clifford multiplication, $\rho_Z(F_{A^t}^+) = -\rho(\frac{d}{dt} B^t + * F_{B^t})$ in $\mathfrak{sl}(S^+)$.
Note also a section Φ of S^+ is just a time-dependent spinor Φ on Y.
Thus the gradient flow for (B^t, Ψ) can be translated into equations for the pair (A^t, Φ):

$$\begin{cases} \frac{1}{2}\rho_Z(F_{A^t}^+) - (\Phi\Phi^*)_0 = 0 \\ D_A^+ \Phi = 0 \end{cases}$$

These are the 4D Seiberg-Witten equations (where we take A in temporal gauge $(A^t)_{time} = 0$).

Define for a 4-mfld X, the operator
$\mathcal{F} : \mathcal{C}(X, \mathfrak{s}_X) \to \Gamma(i\,\mathfrak{su}(S^+) \oplus S^-)$
$\mathcal{F}(A, \Phi) = (\frac{1}{2}\rho(F_{A^t}^+) - (\Phi\Phi^*)_0, D_A^+\Phi)$

The sol^{n}s of the SW eq^{n}s on X are configurations (A, Φ) with $\mathcal{F}(A, \Phi) = 0$.

§4. Energy and Compactness

Def^n: Let X be a compact Riemannian 4-mfld w/ orientable boundary Y. A configuration $(A,\Phi)\in \mathcal{C}(X,\mathfrak{s}_X)$ has <u>topological</u> and <u>analytic</u> <u>energies</u>:

$$\mathcal{E}^{top}(A,\Phi) = \frac{1}{4}\int_X F_{A^t}\wedge F_{A^t} - \int_Y \langle \Phi|_Y, D_B\Phi|_Y\rangle + \int_Y \frac{H}{2}|\Phi|^2$$

mean curvature of Y.

$$\mathcal{E}^{an}(A,\Phi) = \frac{1}{4}\int_X |F_{A^t}|^2 + \int_X |\nabla_A\Phi|^2 + \frac{1}{4}\int_X \left(|\Phi|^2 + \frac{s}{2}\right)^2 - \int_X \frac{s^2}{16}.$$

scalar curvature of X

Remark. If $\partial X = \emptyset$, $\mathcal{E}^{top}$ is purely topological ((Chern-Weil theory). In general it depends only on ∂X. One can see it as measuring the intrinsic energy needed to excite a field on the bundle. It could even be described as a particle count in the closed case. $\mathcal{E}^{an}$ measures the actual energy of the field. The following prop. shows SW sol^ns minimize the active energy of the field. This is analogous to the energies $\int \mathrm{Tr}(F_A^2)$ and $\int |F_A|^2 d\mu$ in Donaldson theory.

Prop: $\mathcal{E}^{an}(A,\Phi) = \mathcal{E}^{top}(A,\Phi) + \|\mathfrak{F}(A,\Phi)\|^2$.

Hence $\mathcal{E}^{an} \geq \mathcal{E}^{top}$ w/ equality iff (A,Φ) solves the SW eq^ns.

Given $(A,\Phi)\in\mathcal{C}(Y\times I,\mathfrak{s})$ write $(\check{A},\check{\Phi})$ for associated 1-parameter family in $\mathcal{C}(Y,\mathfrak{s}_Y)$.
The pair $(\check{A},\check{\Phi})$ lose the information of the t-component of (A,Φ).
In this case $X = Y\times I$, we have: $(I=[t_1,t_2])$

$$\mathcal{E}^{top}(A,\Phi) = 2\mathcal{L}(\check{A}(t_1),\check{\Phi}(t_1)) - 2\mathcal{L}(\check{A}(t_2),\check{\Phi}(t_2))$$

If (A,Φ) in temporal gauge, $\mathcal{E}^{an}(A,\Phi) = \int_{t_1}^{t_2}\left(\|\tfrac{d}{dt}(\check{A}(t),\check{\Phi}(t))\|^2 + \|\mathrm{grad}\,\mathcal{L}\|^2\right)dt.$

We can now state the basic result giving compactness for $\mathcal{M} = \{\mathfrak{F}u = 0\}/\mathcal{G}$.

Thm: (5.1.1 [KM], see also Ch. 5 [Morgan]) Let X be a compact Riemannian 4-mfld.

(i) For any C, there are only finitely many spin^c-structures $\mathfrak{s}_X$ so that the SW eq^ns have a sol^n (A,Φ) w/ $\mathcal{E}^{top}(A,\Phi)\leq C$.

(ii) Suppose (A_n,Φ_n) a seq. of smooth sol^ns satisfying above energy bound. Then there is seq. $\{u_n\}\subseteq\mathcal{G}(X)$ so that:
(a) On a subseq. $u_n\cdot(A_n,\Phi_n) \longrightarrow_{L_1^2} (A,\Phi)$ an L_1^2 configuration.
(b) If $\limsup\limits_{n\to\infty}\mathcal{E}^{top}(A_n,\Phi_n) = \mathcal{E}^{top}(A,\Phi)$, then the above is strong L_1^2 convergence.
(c) The subseq. from (a) converges strongly in C^∞ on $X'\subset\subset X$.

This will imply convergence to a SW sol^n.

Corollary: Let $\gamma_n\in\mathcal{C}(Z,\mathfrak{s}_Z)$ be a seq. of SW sol^ns on cylinder $Z=[t_1,t_2]\times Y$.
Suppose $\mathcal{L}(\check{\gamma}_n(t_1)) - \mathcal{L}(\check{\gamma}_n(t_2))\leq C$ for all n.
Then there is a seq. $\{u_n\}\subseteq\mathcal{G}(Z)$ so that on a subseq. $u_n(\gamma_n)\to u(\gamma)$ in C^∞ on $[t_1',t_2']\times Y$ for $t_1<t_1'<t_2'<t_2$. With this bound, sol^ns only exist for finitely many spin^c-structures.

Proof: The SW eq^ns are elliptic in "Coloumb gauge". After fixing this gauge, we can apply standard elliptic bootstrapping.