The PSS Isomorphisms and Maps in Floer theory

Symplectic and Contact Geometry Seminar, 10^{th} April 2024

The talk is based on:

- Symplectic Floer–Donaldson Theory and Quantum Cohomology – S. Piunikhin, D. Salamon, and M. Schwarz
- Floer Trajectories with Immersed Nodes – Y.-G. Oh and K. Zhu.

I. Review of Hamiltonian Floer Theory

Let (M^{2n},ω) be a semi-positive compact symplectic manifold, i.e. one of ↗ holds
$$\begin{cases} \langle [\omega],\pi_2(M)\rangle = \lambda\langle c_1,\pi_2(M)\rangle \quad \lambda>0 \\ \langle c_1,\pi_2(M)\rangle = 0 \quad (N=\infty) \\ \langle c_1,\pi_2(M)\rangle = N\mathbb{Z},\ N\geq n-2 \end{cases}$$
↳ minimal Chern number

Let $\Gamma = \mathrm{im}(\pi_2(M)) \subseteq H_2(M) = H_2(M;\mathbb{Z})/\text{torsion}$

Let $\tilde{\mathcal{L}} = \{\tilde{x}=[x,u] : x \text{ contractible loop in } M, u \text{ a choice of contraction}\}/[x,u]\sim[x,v]$ if $u\#-v=0\in\Gamma$.

$\tilde{\mathcal{L}}$ covers contractible loop space $\mathcal{L}$ w/ Γ as gp. of deck transformations $(A,[x,u])\mapsto[x,A\#u]$.

Define Novikov Ring $\Lambda = \{\sum_{A\in\Gamma}\lambda_A e^{2\pi i A},\ \lambda_A\in\mathbb{Q},\ \#\{\lambda_A\neq 0, \omega(A)\leq c\}<\infty\ \forall c\}$

This is graded by $\deg(e^{2\pi i A}) = 2c_1(A)$.

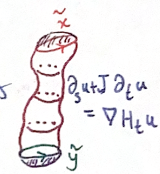
(I'm going to be very sloppy keeping track of Λ, but it's always implicitly there in homology.)

The quantum cohomology of M is $QH^k(M) = \bigoplus_j H^j(M)\otimes\Lambda_{k-j}$, module over Λ.

Fix a non-degenerate 1-periodic Hamiltonian $H: S^1\times M\to\mathbb{R}$ and let $\tilde{\mathcal{P}}(H)\subseteq\tilde{\mathcal{L}}$ be its periodic contractible orbits.

Have the Conley-Zehnder index $\mu:\tilde{\mathcal{P}}(H)\to\mathbb{Z}$, $\mu(A\#\tilde{x}) = \mu(\tilde{x}) - 2c_1(A)$.

Obtain $\mathbb{Z}/2N\mathbb{Z}$-graded chain complex over Λ,
$$CF_k(H) = \Big\{\sum_{\substack{\tilde{x}\in\tilde{\mathcal{P}}(H)\\ \mu(\tilde{x})=k}} \xi_{\tilde{x}}\langle\tilde{x}\rangle : \xi_{\tilde{x}}\in\mathbb{Q},\ \#\{\xi_{\tilde{x}}\neq 0, \mathcal{A}_H(\tilde{x})\geq c\}<\infty\ \forall c\Big\}$$



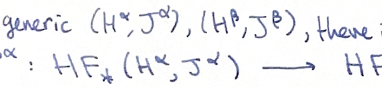
We get differential $\partial: CF_k\to CF_{k-1}$ by counting sol^ns of Floer eq^n (for generic choice of compatible almost complex structure) connecting $\tilde{x}$ to $\tilde{y}$ w/ $\mu(\tilde{x})-\mu(\tilde{y})=1$.

The homology of this is well defined: $HF_*(H,J)$ the Hamiltonian Floer Homology.

Thm: Given generic $(H^\alpha,J^\alpha),(H^\beta,J^\beta)$, there is an isomorphism of graded Λ-modules
$$\Phi^{\beta\alpha}: HF_*(H^\alpha,J^\alpha)\longrightarrow HF_*(H^\beta,J^\beta),$$
called the continuation map.

These satisfy $\Phi^{\gamma\beta}\circ\Phi^{\beta\alpha} = \Phi^{\gamma\alpha}$, $\Phi^{\alpha\alpha}=\mathrm{Id}$.

Proof: Pick generic homotopy $(H^{\lambda}_t, J^{\lambda}_t)$ from $\lambda_0=\alpha$ to $\lambda_1=\beta$. Count solutions of time-dependent Hamiltonian Floer-type equation connecting $\tilde{x}\in\tilde{\mathcal{P}}(H^\alpha)$ to $\tilde{y}\in\tilde{\mathcal{P}}(H^\beta)$ to get coefficient $\langle\Phi^{\beta\alpha}(\tilde{x}),\tilde{y}\rangle$.



$$\partial_s u + J^{\lambda_s}\partial_t u = \nabla H^{\lambda_s}_t u$$

□

Floer showed Floer homology agrees with singular homology by picking a C^2-small autonomous Hamiltonian whose Floer homology is just its Morse homology and then applying above theorem.

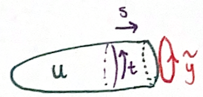
We would like a more direct isomorphism. II. The PSS Isomorphism

Such an approach was outlined by Piunikhin, Salamon, and Schwarz in 1996, which is now known as the <u>PSS Isomorphism</u>. Pick a Morse-Smale pair (f,g), a Hamiltonian H_0 w/ generic compatible almost complex structure J_0. We wish to define a map $\Phi : CM_*(M,f,g) \to CF_*(M, H_0, J_0)$.
Let Σ be the punctured Riemann sphere with cylindrical coordinates $(0,\infty)\times S^1$ near the puncture.
Fix a Hamiltonian $H(s,t,x): \mathbb{R}^{\geq 0}\times S^1\times M \to \mathbb{R}$ vanishing for $s\leq 1$ and agrees with H_0 for large s.
Fix an almost complex structure J which similarly depends on (s,t), vanishes for $s\leq 1$ and agrees with J_0 for large s. For $\tilde{y}\in \tilde{\mathcal{P}}(H_0)$, we define $\mathcal{M}(\tilde{y})$ as the space of smooth maps $u: \Sigma \to M$ so that:

(i) u is J-holomorphic off the cylindrical end.

(ii) On the cylindrical end with coordinates (s,t), u satisfies a Floer-type equation:

$$\partial_s u + J(u)\partial_t u - \nabla H(s,t,u) = 0, \quad y = \lim_{s\to\infty} u(s,\cdot)$$



(iii) If $\tilde{y} = [y, v]$, then $u \amalg_y v$ is torsion in $H^2(M;\mathbb{Z})$

Now suppose $x_0 \in \mathrm{Crit}(f)$. We consider moduli space of "spiked disks" ↗ i.e. $u\in\mathcal{M}(\tilde{y})$ intersecting unstable manifold of x_0 at marked point.

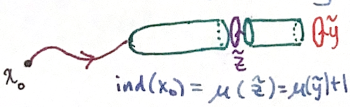
$$\mathcal{M}^{PSS}(x_0,\tilde{y}) = \{(u,\gamma) \mid u\in\mathcal{M}(\tilde{y}),\ \gamma\in C^\infty(\mathbb{R},M),\ \dot{\gamma} = -\nabla_g f(\gamma),\ \lim_{t\to-\infty}\gamma(t) = x_0,\ \gamma(0) = u(0)\}$$

If $\mathrm{ind}(x_0) = \mu(\tilde{y})$, this is generically a finite set (cpct 0-dim'l mfld) and we define

$$\Phi(x_0) = \sum_{\substack{\tilde{y}\in\tilde{\mathcal{P}}(H_0)\\ \mu(\tilde{y}) = \mathrm{ind}(x_0)}} \#\mathcal{M}^{PSS}(x_0,\tilde{y})\langle\tilde{y}\rangle .$$

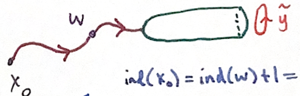
<u>Claim:</u> This is a chain map.

<u>Proof:</u> Take $\mathrm{ind}(x_0) = \mu(\tilde{y})+1$. Now $\mathcal{M}^{PSS}(x_0,\tilde{y})$ is (generically) one dimensional manifold w/ a natural compactification, by breaking spiked disks in one of two ways:



$\mathrm{ind}(x_0) = \mu(\tilde{z}) = \mu(\tilde{y})+1$

<u>OR</u>



$\mathrm{ind}(x_0) = \mathrm{ind}(w)+1 = \mu(\tilde{y})+1$

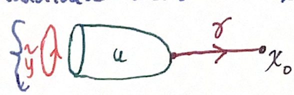
$\#\{\nearrow\} = \langle \partial_{Floer}\circ\Phi(x_0), \tilde{y}\rangle$ $\qquad$ $\#\{\nearrow\} = \langle \Phi\circ\partial_{Morse}(x_0), \tilde{y}\rangle$.

The oriented boundary of a 1-mfld is zero, hence, counting orientations,

$$\partial_{Floer}\circ\Phi(x_0) - \Phi\circ\partial_{Morse}(x_0) = 0 \Rightarrow \Phi \text{ is chain map.} \qquad \square$$

<u>Thm:</u> (PSS) $\Phi : (Q)HM_*(M,f,g) \to HF_*(M,H_0,J_0)$ is an isomorphism.

<u>Proof:</u> First we need a candidate inverse. Let us build a moduli space of backwards spiked disks:

$\mathcal{M}^{SSP}(x_0,\tilde{y}) = \{$  $\}$. Let $\Psi(\tilde{y}) = \sum_{\substack{x_0\in\mathrm{Crit}(f)\\ \mu(\tilde{y}) = \mathrm{ind}(x_0)}} \#\mathcal{M}^{SSP}(x_0,\tilde{y})\langle x_0\rangle$

For same reason as above $\Psi : CF_*(M,H_0,J_0) \to CM_*(M,f,g)$ is chain map.

We want to show $\Psi\circ\Phi = \mathrm{Id}_{Morse}$ and $\Phi\circ\Psi = \mathrm{Id}_{Floer}$ in homology.

III. $\Psi \circ \Phi : HM_* \to HF_* \to HM_*$ For $x_0, z_0 \in \mathrm{Crit}(f)$, $\mathrm{ind}(x_0) = \mathrm{ind}(z_0)$,

$\langle \Psi \circ \Phi(x_0), z_0 \rangle = \# \Big\{$ 

$\Big| \tilde{y} \in \tilde{\mathcal{P}}(H_0),\ \mu(y) = \mathrm{ind}(x_0) \Big\}$

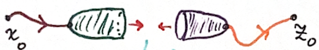
$=: \# \mathcal{M}^{\Psi\Phi}(x_0, z_0)$

Pick generic smooth homotopies: j^R of complex strs. defined on Riemann sphere for $R \in [0,\infty)$ so that j^R gives the sphere the structure of two disks glued by a standard complex cylinder of length $2R$. H^R should agree with H_0 for $|s| \leq 2R$ and vanish outside a neighbourhood of that cylinder with $H^{R=0}$ constantly zero. Now set: similarly for J^R a.c. str.

$\mathcal{M}_R(x_0, z_0) = \Big\{$ 

↳ satisfy (H^R, J^R)-Floer-type eqn $\Big\}$ and set $\mathcal{M}(x_0, z_0) = \bigcup_{R \in (0,\infty)} \mathcal{M}_R(x_0, z_0)$.

This glues for generic choices to give a one-dimensional manifold with compact 0-dim boundary given by:

(A) $\mathcal{M}_\infty(x_0, z_0) = \Big\{$ 

$\Big\} = \mathcal{M}^{\Psi\Phi}(x_0, z_0)$

↳ finite energy means must limit to common orbit.

(B) $\mathcal{M}_0(x_0, z_0) = \Big\{$ 

↳ J-hol. sphere $\Big\}$

If there were a sol^n w/ a non-const. J-hol. sphere, there would be a 1-dim^l subgp of $PGL(2,\mathbb{C})$ worth of solutions. Since this is finite set, any such sphere is constant.

$= \Big\{$ 

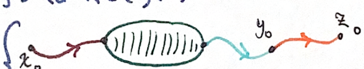
$\Big\}$

Again any non-const. sol^n would give 1-dim^l family of solutions by reparameterization.

$\# \mathcal{M}_0 = \langle \mathrm{Id}(x_0), z_0 \rangle$

(C) Solutions may "split" in two ways for $R \in (0,\infty)$:

$\Big\{$ 

$\mathrm{ind}(x_0) = \mathrm{ind}(y_0) + 1 = \mathrm{ind}(z_0)$ $\Big\} \cup \Big\{$ 

$\mathrm{ind}(x_0) = \mathrm{ind}(y_0) - 1 = \mathrm{ind}(z_0)$ $\Big\} = \mathcal{M}^R_{\mathrm{broken}}(x_0, z_0)$

Define $h : CM_* \to CM_{*+1}$ by: $h(x_0) = \sum_{\mathrm{ind}(y_0) = \mathrm{ind}(x_0)+1} \# \Big\{$ 

$\Big| R \in (0,\infty),\ \mathrm{ind}(y_0) = \mathrm{ind}(x_0) + 1 \Big\} \langle y_0 \rangle$

For generic R such objects do not exist, but in 1-param. family have finitely many solutions.

We see: $\bigcup_{R \in (0,\infty)} \mathcal{M}^R_{\mathrm{broken}}(x_0, z_0) = \langle h \circ \partial_{\mathrm{Morse}}(x_0) + \partial_{\mathrm{Morse}} \circ h(x_0), z_0 \rangle$.

Boundary of 1-dim^l mfld should be empty w/ orientations counted:

$$0 = \# \partial \mathcal{M}(x_0, z_0) = \# \Big[\mathcal{M}_0(x_0, z_0) \cup (-\mathcal{M}_\infty(x_0, z_0)) \cup \bigcup_{R \in (0,\infty)} \mathcal{M}^R_{\mathrm{broken}}(x_0, z_0) \Big]$$

$$= \langle \mathrm{Id}(x_0) - \Psi \circ \Phi(x_0) + h \circ \partial_{\mathrm{Morse}}(x_0) + \partial_{\mathrm{Morse}} \circ h(x_0), z_0 \rangle$$

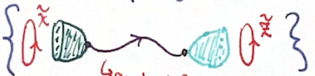
$\Rightarrow \Psi \circ \Phi - \mathrm{Id}_{\mathrm{Morse}} = \partial_{\mathrm{Morse}} \circ h + h \circ \partial_{\mathrm{Morse}}$, so $\Psi \circ \Phi$ is chain homotopic to the identity.

IV. $\Phi\circ\Psi: HF_* \to HM_* \to HF_*$

For $\tilde{x},\tilde{z}\in\tilde{\mathcal{P}}(H_0)$, $\mu(\tilde{x})=\mu(\tilde{z})$,

$\langle \Phi\circ\Psi(\tilde{x}),\tilde{z}\rangle = \#\{$  $\mid y\in \mathrm{Crit}(f),\ \mathrm{ind}(y)=\mu(\tilde{x})\} =: \#\mathcal{M}^{\Phi\Psi}(\tilde{x},\tilde{z})$.

We now define one-parameter family of moduli spaces $\mathcal{M}_\lambda(\tilde{x},\tilde{z})$ for $\lambda\in(-\infty,1)$:

(i) $\lambda\in(-\infty,0)$ $\quad \mathcal{M}_\lambda = \{$  $\}$

↳ gradient flow for time $2|\lambda|$

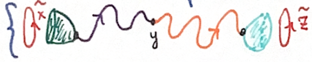
(ii) $\lambda=0$, $\quad \mathcal{M}_0 = \{$  $\}$

(iii) $\lambda\in(0,1)$ $\quad \mathcal{M}_\lambda = \{$  $\}$

↳ $(\tilde{H}_\lambda,\tilde{J}_\lambda)$-Floer-type

We pick specific homotopy from $(\tilde{H}_0,\tilde{J}_0)$ agreeing w/ $\mathcal{M}_0$ to $(\tilde{H}_1,\tilde{J}_1)$ constantly equal to (H_0,J_0) so that for small λ we introduce a perturbation $\varepsilon(\lambda)f$ to $\tilde{H}_\lambda$ and $\varepsilon(\lambda)J_g$ to $\tilde{J}_\lambda$ w/ $\varepsilon\to 0$ as $\lambda\to 0$ near the nodal point.

Now set $\mathcal{M}(\tilde{x},\tilde{z}) = \bigcup_{\lambda\in(-\infty,1)}\mathcal{M}_\lambda(\tilde{x},\tilde{z})$. It is a highly non-trivial fact this glues to one-dimensional manifold with compact boundary given by:

(A) $\mathcal{M}_{-\infty}(\tilde{x},\tilde{z}) = \{$  $\} = \mathcal{M}^{\Phi\Psi}(\tilde{x},\tilde{z})$

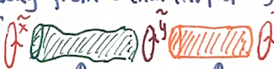
(B) $\mathcal{M}_1(\tilde{x},\tilde{z}) = \{$  $\}$

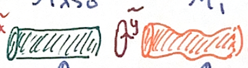
↳ (H_0,J_0)-Floer

$= \langle \mathrm{Id}(\tilde{x}),\tilde{z}\rangle$.

If there were a non-constant sol^n, there would be 1-dim^l family by reparameterization. Since this is finite set, only sol^n is const.

(C) Solutions may "split" in four ways for λ bounded away from 0 (non-triv property of gluing choice):

$\{$  $: \mu(\tilde{y})=\mu(\tilde{x})+1\}$ ($\mathcal{M}_{\lambda<0}$, $\mathcal{M}_{\lambda=1}$) $\cup \{$  $: \mu(\tilde{y})=\mu(\tilde{x})+1\}$ ($\mathcal{M}_{\lambda>0}$, $\mathcal{M}_1$)

$\cup\{$  $: \mu(\tilde{y})=\mu(\tilde{x})-1\}$ ($\mathcal{M}_1$, $\mathcal{M}_{\lambda<0}$) $\cup\{$  $: \mu(\tilde{y})=\mu(\tilde{x})-1\}$ ($\mathcal{M}_1$, $\mathcal{M}_{\lambda>0}$) $= \mathcal{M}_\lambda^{\text{broken}}(\tilde{x},\tilde{z})$

Over one-parameter family of λ, have finitely many elements. Define,

$g: CF_*\to CF_{*+1}$ by: $g(\tilde{x}) = \sum_{\mu(\tilde{y})=\mu(\tilde{x})+1}\big(\#\{$  $\mid \lambda\in(-\infty,0)\} + \#\{$  $\mid\lambda\in(0,1)\}\big)\langle\tilde{y}\rangle$ ($\mathcal{M}_{\lambda<0}$, $\mathcal{M}_{\lambda>0}$)

We see that $\bigcup_{\lambda\in(-1,\infty)}\mathcal{M}_\lambda^{\text{broken}}(\tilde{x},\tilde{z}) = \langle g\circ\partial_{\text{Floer}}(\tilde{x}) + \partial_{\text{Floer}}\circ g(\tilde{x}),\tilde{z}\rangle$.

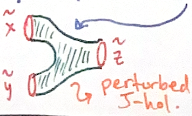
Count of oriented boundary of 1-dim^l manifold should be zero:

$$0=\#\partial\mathcal{M}(\tilde{x},\tilde{z}) = \#\Big[\mathcal{M}_1(\tilde{x},\tilde{z})\cup(-\mathcal{M}_{-\infty}(\tilde{x},\tilde{z}))\cup\bigcup_{\lambda\in(-1,\infty)}\mathcal{M}_\lambda^{\text{broken}}(\tilde{x},\tilde{z})\Big]$$
$$=\langle \mathrm{Id}(\tilde{x}) - \Phi\circ\Psi(\tilde{x}) + g\circ\partial_{\text{Floer}}(\tilde{x}) + \partial_{\text{Floer}}\circ g(\tilde{x}),\tilde{z}\rangle$$

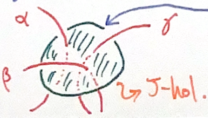
$\Rightarrow \Phi\circ\Psi - \mathrm{Id}_{\text{Floer}} = \partial_{\text{Floer}}\circ g + g\circ\partial_{\text{Floer}}$, so $\Phi\circ\Psi$ chain homotopic to the identity. $\square$

Remark: The PSS isomorphism is actually a ring isomorphism (in cohomology) if we use the pair of pants product on Floer cohomology and the quantum cup product on quantum cohomology.

The pair of pants product counts



The quantum cup product counts



The PSS isomorphism interpolates these by gluing.

V. More General Maps

In their paper, PSS extend this to a more general construction. I'm not sure if this is rigourously defined in the literature, but still holds some interest.

Let Σ' be a genus g Riemann surface with d marked points and l punctures w/ cylindrical ends $Z_i = (0,\infty)\times S^1$.

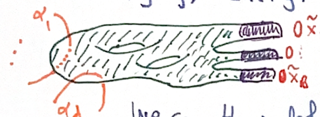
Fix Hamiltonians $H_i(s,t,x): \mathbb{R}\times S^1\times M \to \mathbb{R}$ all vanishing near $s=0$ and constant in s for large $s\geq 1$. Fix for almost complex str. J_t $\tilde{x}_i \in \tilde{\mathcal{P}}(H_i(1,\cdot))$ and classes $\alpha_1,\dots,\alpha_d \in H_*(M)$. Now define $\mathcal{M}_\Sigma(\alpha_1,\dots,\alpha_d,\tilde{x}_1,\dots,\tilde{x}_l)$ as the space of smooth maps $u:\Sigma\to M$ so that:

(i) u is J-hol on $\Sigma\setminus\bigcup Z_i$. (ii) On Z_i, u satisfies $\partial_s u + J\partial_t u = \nabla H_i(s,t,u)$, $x_i = \lim_{s\to\infty} u_i(s,t)$.

(iii) After fixing generic representatives of the homology classes, u intersects α_j at the j^{th} marked point.

(iv) If $\tilde{x}_i = [x_i, v_i]$ then u capped off by the v_i is torsion in $H_2(M;\mathbb{Z})$.

If $\sum 2n - \deg(\alpha_j) = 2n(1-g) - \sum \mu(\tilde{x}_i)$, $\mathcal{M}_\Sigma$ is (in theory) a finite set of objects looking like:



↳ need to worry about virtual vs. real dimension and Deligne-Mumford compactification

We can thus define $\Psi_\Sigma(\alpha_1,\dots,\alpha_d) = \sum_{\tilde{x}_i\in\tilde{\mathcal{P}}(H_i)} \#\mathcal{M}(\alpha_1,\dots,\alpha_d,\tilde{x}_1,\dots,\tilde{x}_l)\langle\tilde{x}_1,\dots,\tilde{x}_l\rangle \in CF_*(H_1)\otimes\cdots\otimes CF_*(H_l)$

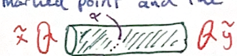
Under suitable assumptions this should descend to a map on homology:

$$\Psi_\Sigma: H_*(M)\otimes\cdots\otimes H_*(M)\to HF_*(H_1)\otimes\cdots\otimes HF_*(H_l),$$ where $*$ satisfy dim formula above.

PSS call these "symplectic relative Donaldson invariants" as they mirror role of instanton Floer homology in Donaldson theory.

Ex. (i) Take $\Sigma' = \mathbb{C}$ w/ one marked point. We get back PSS iso. recalling that spiked disk is like intersecting disk w/ unstable manifold of x_0, which is generic cell of dimension $\mathrm{ind}(x_0)$.

(ii) Recall $\bar{H}_t := -H_{-t}$ satisfies $HF^k(H)\cong HF_{2n-k}(\bar{H})$. Take $\Sigma' = \mathbb{R}\times S^1$ w/ one marked point and the Hamiltonians $H,\bar{H}$ at the ends. Dualizing, this gives map:

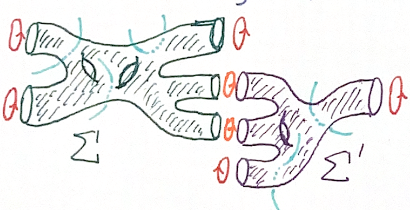


$\Phi: H^j(M)\otimes HF_k(H)\to HF_{k-j}(H)$. This is like a cap product. $\Phi([M]^{PD},\cdot)$ is continuation isomorphism.

(iii) Take Σ a pair of pants with no marked points and Hamiltonians $H,H,\bar{H}$ at ends. Get map, after dualizing,

$$\Psi_\Sigma: HF^j(H)\otimes HF^k(H)\to HF^{j+k}(H).$$ This is pair of pants product described above.

We can build any marked, punctured surface out of these pieces. The maps Ψ_Σ should compose in the obvious way when gluing surfaces, to give functoriality property:



$$\Psi_{\Sigma'}\circ\Psi_\Sigma = \Psi_{\Sigma\#\Sigma'}.$$

This gives our theory a structure similar to a 1D "T"QFT. The Hilbert space of a closed 1-manifold (loop) w/ associated Hamiltonian H is the Floer homology $HF_*(H)$. For a closed 2-manifold the TQFT should produce a numerical invariant which in this case is the Gromov-Witten invariant.
↳ for chosen homology classes α_i

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