The Open-Closed and Closed-Open Maps

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Our goal for the next few talks will be to show under some assumption that symplectic cohomology agrees with the Hochschild homology of the wrapped Fukaya category.

We study two maps, the <u>open-closed</u> and <u>closed-open</u> maps

$$[\mathcal{OC}]: HH_{*-n}(\mathcal{W},\mathcal{W}) \to SH^*(M), \quad [\mathcal{CO}]: SH^*(M) \to HH^*(\mathcal{W},\mathcal{W}).$$

note this is more general than what we've seen before..

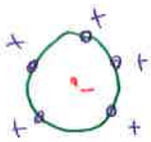
Defn: M is <u>non-degenerate</u> if there is a finite set of Lagrangians $\{L_i\}$ so that $e \in SH^*(M)$ is in the image of $\mathcal{OC}$ applied to HH_* of the wrapped Fukaya category generated by $\{L_i\}$.

In this talk we prove:

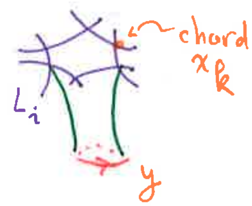
Thm: If M is non-degenerate, $[\mathcal{OC}]$ is surjective and $[\mathcal{CO}]$ is injective.

How to define maps?

$\mathcal{R}^1_d$ moduli space of disks w/ d positive boundary punctures 1 negative interior puncture



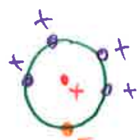
$\xrightarrow{\text{pick Floer data}}$ $\mathcal{R}^1_d(y;\vec{x})$ maps from element of $\mathcal{R}^1_d$ satisfying Floer-type eqn asymp. to Ham. orbit y & chords $x_1,\dots,x_d$



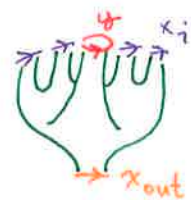
Counting $\mathcal{R}^1_d(y;\vec{x})$ yields $\mathcal{OC}_d: \hom_{\mathcal{W}}(L_d,L_1)\otimes\cdots\otimes\hom_{\mathcal{W}}(L_1,L_2) \to SC^*(M)$.

Summing gives, $\mathcal{OC}: CC_*(\mathcal{W},\mathcal{W}) \to SC^*(M)$.

$\mathcal{R}^{1,1}_d$ moduli space of disks w/ d positive and 1 negative boundary punctures 1 positive interior puncture



$\xrightarrow{\text{Floer data}}$ $\mathcal{R}^{1,1}_d(x_{out};y,\vec{x})$ moduli space of curves

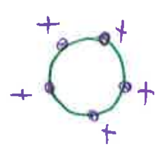


Counting $\mathcal{R}^{1,1}_d(x_{out};y,\vec{x})$ yields $\mathcal{CO}_d: SC^*(M) \to \hom(\mathcal{W}^{\otimes d},\mathcal{W})$

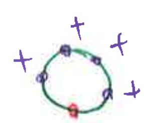
Summing gives, $\mathcal{CO}: SC^*(M) \to CC^*(\mathcal{W},\mathcal{W})$.

To understand these maps better, we need to know more about Hochschild (co)homology. It will help to use an "operadic" framework instead of writing formulas.

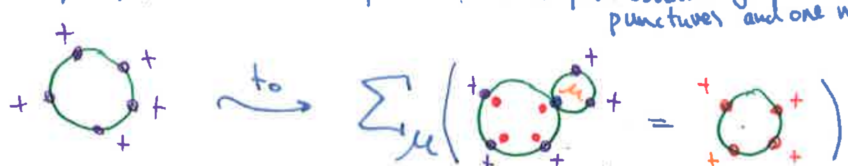
- A Hochschild chain in $CC_*(\mathcal{W},\mathcal{W})$ is a cyclic tensor product of homs. ~~thought~~ We think of as a disk w/ pos. boundary punctures



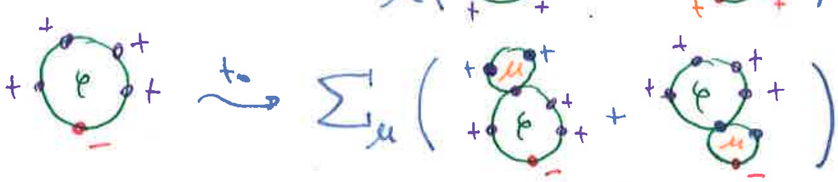
- A Hochschild cochain in $CC^*(\mathcal{W},\mathcal{W})$ is an element of $\hom(\mathcal{W}^{\otimes d},\mathcal{W})$. We think of as a disk w/ pos boundary punctures and one neg. puncture



- Hochschild differential sends



- Hochschild codifferential sends



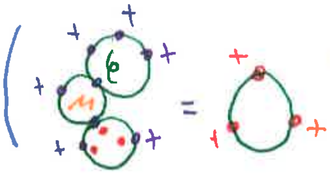
Note: $\mathcal{R}^1_d(-)$ and $\mathcal{R}^{1,1}_d(-)$ can break either via a Floer cylinder or by bubbling as in these pictures. This implies $\mathcal{OC}$ and $\mathcal{CO}$ are chain maps.

- Hochschild cohomology has a product, the Yoneda product: $*: HH^*(\mathcal{W},\mathcal{W}) \otimes HH^*(\mathcal{W},\mathcal{W}) \to HH^*(\mathcal{W},\mathcal{W})$

Given $\varphi,\psi \in CC^*(\mathcal{W},\mathcal{W})$, define $\varphi * \psi(x_1,\dots,x_k) = \sum (-1)^{*} \mu^k(x_1,\dots,\varphi(x_{i+1},\dots,x_{i+r}),\dots,\psi(x_{j+1},\dots,x_{j+l}),\dots,x_k)$.

In terms of pictures $\varphi * \psi$ is the sum $\sum_{\mu,\varphi,\psi}\Big($  $\Big)$.

- There is a cap product $\cap: HH^*(\mathcal{W},\mathcal{W}) \otimes HH_*(\mathcal{W},\mathcal{W}) \to HH_*(\mathcal{W},\mathcal{W})$.

Given $\varphi \in CC^*(\mathcal{W},\mathcal{W})$, it sends a Hochschild chain  to the sum $\sum_{\varphi,\mu}\Big($  $\Big)$.

Now we can state two important properties of our $\mathcal{CO}$ and $\mathcal{OC}$ maps.

Prop: $[\mathcal{CO}]$ is a ring homomorphism with respect to pair of pants and Yoneda products.

Assuming this, $\mathcal{CO}$ composed with cap product makes $HH_*(\mathcal{W},\mathcal{W})$ a $SH^*(M)$-module.

$SH^*(M)$ is also a $SH^*(M)$-module via the pair of pants product.

Prop: $[\mathcal{OC}]$ is a map of $SH^*(M)$-modules.

Let's see how these two results give our main theorem.

Proof: (of Thm.) Since M is non-degenerate, there is $\sigma \in HH_*(\mathcal{W},\mathcal{W})$ with $\mathcal{OC}(\sigma) = e \in SH^*(M)$.

Consider any $s \in SH^*(M)$. Since $\mathcal{OC}$ is a map of SH^*-modules:

$$\mathcal{OC}(\mathcal{CO}(s) \cap \sigma) = \mathcal{OC}(s\cdot\sigma) = s\cdot\mathcal{OC}(\sigma) = s\cdot e = s.$$

Thus $\mathcal{OC}$ is surjective. If $s \neq 0$, then $\mathcal{CO}(s) \neq 0$, so $\mathcal{CO}$ is injective. $\square$

With the time that remains, we show $\mathcal{OC}$ is SH^*-module map. Equivalently, we show the diagram

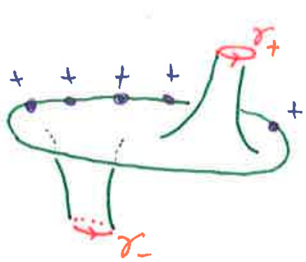
$$\begin{array}{ccc} CC_*(\mathcal{W}) \times SC^*(M) & \xrightarrow{\mathcal{OC}} & SC^*(M)\times SC^*(M) \\ \downarrow \mathcal{CO} & & \downarrow PP \\ CC_*(\mathcal{W}) \times CC^*(\mathcal{W}) & \xrightarrow{\mathcal{OC}\circ\cap} & SC^*(M) \end{array}$$

homotopy commutes.

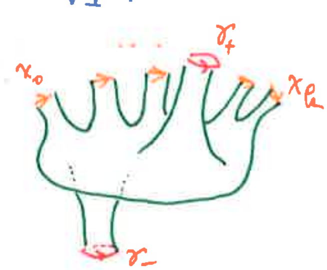
To do this, define $\mathcal{P}^2_d(r)$ the moduli space of unit disks in $\mathbb{C}$ with $d+1$ positive boundary punctures, one z_0 distinguished, and two interior punctures $K_\pm$ one positive, one negative, so that after automorphism, z_0, K_+, K_- lie at $-i, r, -r$ respectively.

We let $\mathcal{P}^2_d = \bigcup_{r\in(0,1)} \mathcal{P}^2_d(r)$.

After picking Floer data, we can consider moduli space $\mathcal{P}^2_d(\gamma_-;\gamma_+,\vec{x})$ of Floer-type curves asymp. to Ham. chords $\vec{x}$ and orbits $\gamma_\pm$.



or



When it is one-dimensional, how can this moduli space degenerate?

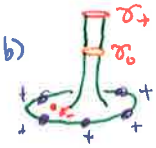
(i) Boundary Bubbling:

$\mathcal{R}^{d_1}(-) \times \mathcal{P}^2_{d_2}(\gamma_-; \gamma_+, -)$.

(ii) Floer Breaking in Interior:

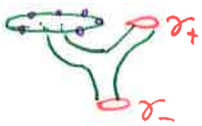
(a)

$\mathcal{P}^2_d(-) \times \mathcal{M}(\gamma_-; \gamma_0)$ OR (b)

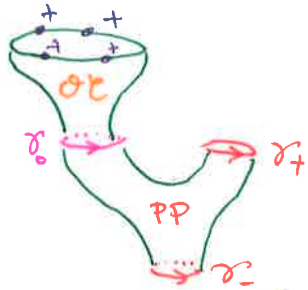


$\mathcal{M}(\gamma_0, \gamma_+) \times \mathcal{P}^2_d(-)$

(iii) $r \to 0$: For r small,



in limit

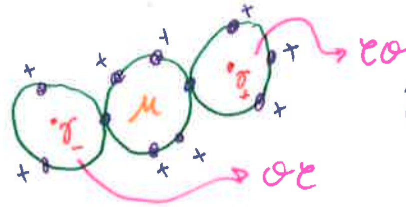


$\mathcal{R}^1_d(-) \times \mathcal{S}_2(\gamma_-; \gamma_0, \gamma_+)$

↳ pair of pants moduli space

(iv) $r \to 1$: For r big,

in limit



$\mathcal{R}^{1,1}_{d_3}(-) \times \mathcal{R}_{d_2}(-) \times \mathcal{R}^1_{d_1}(-)$

Let $\mathcal{H}: SC^*(M) \times CC_*(\mathcal{W},\mathcal{W}) \to SC^*(M)$ denote the map counting points of the moduli spaces $\mathcal{P}^2_d(-)$ for all d when they are zero dimensional.

Our breaking analysis implies the algebraic relation: for $s \in SC^*(M)$, $\alpha \in CC_*(\mathcal{W},\mathcal{W})$,

$$\underbrace{\partial_{SH} \circ \mathcal{H}(s,\alpha)}_{(ii)(a)} \pm \underbrace{\mathcal{H}(s, \partial_{HH}\alpha)}_{(i)} \pm \underbrace{\mathcal{H}(\partial_{SH}s, \alpha)}_{(ii)(b)} = \underbrace{\mathcal{OC}(\alpha) *_{PP} s}_{(iii)} - \underbrace{\mathcal{OC}(\mathcal{CO}(s) \cap \alpha)}_{(iv)}.$$

This precisely asserts $\mathcal{H}$ is a chain homotopy for our earlier commutative diagram. This proves we have a SH^*-module map. □

<u>Exercise</u>: Study $\mathcal{P}^2_d$ but now with both interior punctures positive and one exterior puncture negative to show $\mathcal{CO}$ is ring map.