Minimal Genus Problems in 4-Manifolds

Student Low-Dimensional Topology Seminar 28th October 2024

§1. Minimal Genus Problems

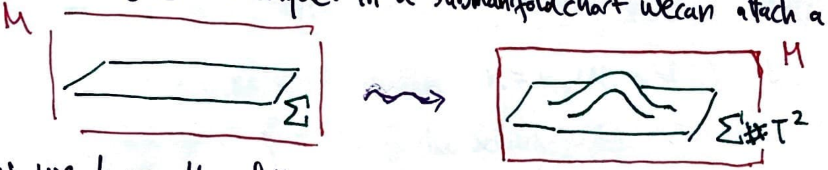
Let M be a smooth closed oriented 4-manifold (we assume this throughout). A classic problem in algebraic topology asks when homology classes are represented by manifolds (Steenrod's problem).
It turns out every $\beta \in H_2(M;\mathbb{Z})$ is represented by an embedded closed (connected) oriented surface Σ.

__Why?__ Let $c = PD(\beta) \in H^2(M;\mathbb{Z})$. Recall,

$$\{\mathbb{C}\text{-line bundles on } M\} \cong [M, \mathbb{CP}^\infty] \cong H^2(M;\mathbb{Z}) \quad \text{given by } L \mapsto c_1(L).$$

So let L be a line bundle w/ $c = c_1(L)$. By obstruction theory, a generic section s of L has zero set $s^{-1}(0) = \Sigma$ with $[\Sigma] = \beta$. □

Note Σ is highly non-unique: In a submanifold chart we can attach a handle.



This will not alter the homology class.

Thus we have the following natural question.

__Q.__ For a given M and $\beta \in H_2(M;\mathbb{Z})$, what is the minimal genus of an embedded submanifold representing β?
The corresponding question for 3-mflds is well understood. The minimal genus defines a seminorm on homology "Thurston norm" and Gabai shows minimal genus surfaces correspond to leaves of taut foliations.
In 4-manifold theory, this question is hard and little was understood prior to gauge theory.
For example, the following basic problem was out of reach w/ classical methods.

__Thm:__ (Thom Conjecture, Kron.-Mrow. '94) A degree d algebraic curve in $\mathbb{CP}^2$ is genus-minimizing in its homology class. I.e. if $[\Sigma] = d[\mathbb{CP}^1]$ then,

$$\text{genus}(\Sigma) \geq \tfrac{1}{2}(|d|-1)(|d|-2) \quad \text{(here we use degree-genus formula).}$$

§2. Kervaire-Milnor Theorem

We now prove this conjecture for $d=3$. I.e. $3[\mathbb{CP}^1]$ is not represented by an embedded sphere.

__Proof:__ Suppose it were, $[\Sigma]^2 = 9$. We may blowup 8 times to obtain $\tilde{\Sigma} \subset \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$ w/ $\tilde{\Sigma}^2 = 1$.
Let E_i be exceptional divisor in ith blowup. $[\tilde{\Sigma}] = 3[\mathbb{CP}^1] - \sum [E_i]$.
Let ν be a tubular nbhd of $\tilde{\Sigma}$. $\partial \nu$ is a circle bundle on S^2 of Euler class one, thus the Hopf bundle w/ total space S^3. We can remove ν and glue in D^4 to obtain Y w/ $Y \# \mathbb{CP}^2 = \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$.
Note Σ was PD to $w_2(\mathbb{CP}^2)$, hence $\tilde{\Sigma}$ PD to $w_2(\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2})$. Thus $w_2(Y) = 0$.
Since Y is simply-connected and spin, Rokhlin's theorem implies $\sigma(Y) \equiv 0 \pmod{16}$.
But $\sigma(Y) = 8\sigma(\overline{\mathbb{CP}^2}) = -8$. We have a contradiction. □

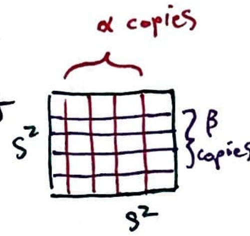
We used very little about $\Sigma' \subseteq \mathbb{CP}^2$ here. We say $\Sigma \subseteq M$ is a <u>characteristic surface</u> if $[\Sigma] = PD(w_2(M)) \pmod 2$. We have the following.

Thm: (Kervaire-Milnor, '61) Let Σ be a smoothly embedded characteristic sphere in M simply connected. Then, $\sigma(M) \equiv \Sigma \cdot \Sigma \pmod{16}$.

Another corollary of this is the following.

Corollary: The class $2[S^2 \times pt] + 2[pt \times S^2]$ in $S^2 \times S^2$ is not represented by a sphere.

Note: From SW theory we know for $|\alpha\beta|>0$, $(\alpha,\beta) \in H_2(S^2 \times S^2)$ is minimally represented by surface of genus $(|\alpha|-1)(|\beta|-1)$. This can be seen in following picture.
(after resolving crossings)



The classical theory can only disallow certain spheres. For further progress, we turn to SW theory.

§3. The Adjunction Inequality and Seiberg-Witten Theory

Let $K \in H^2(M;\mathbb{Z})$ be characteristic ($K = w_2(M) \bmod 2$). This determines a spin^c-structure and hence a moduli space of sol$^{\text{n}}$s of the Seiberg-Witten eq$^{\text{n}}$s $\mathcal{M}_K$ (for chosen metric and perturbation).

For $b_2^+(M)>0$, we can define <u>Seiberg-Witten invariants</u> $SW_M(K)$ by pairing natural cohomology class with $[\mathcal{M}_K]$. This is independent of metric if $b_2^+>1$ and for $b_2^+=1$ the dependence on metric can be understood using wall crossing formula.

Def$^{\text{n}}$:
- K is called a <u>basic class</u> if $SW_M(K) \neq 0$.
- M is <u>simple type</u> if $\dim \mathcal{M}_K = 0$ for every basic class.

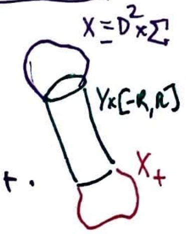
Thm: (Taubes, "SW $\Rightarrow$ Gr") Symplectic 4-manifolds with $b_2^+>1$ are of simple type.

Now we come to the central result.

Thm: (Generalized Adjunction Inequality) Let M be a simply connected 4-manifold with $b_2^+(M)>1$. Let Σ_g be an embedded surface with $\Sigma^2 \geq 0$. Then for every basic class K,

$$\boxed{2g-2 \geq |K\cdot\Sigma| + \Sigma^2}$$

We need a lemma. Suppose $\Sigma \subseteq M$ an embedded surface of positive genus with trivial normal bundle. Let X_- be a tubular nbhd of Σ, X_+ its complement, so that $M = X_- \cup_Y X_+$ w/ $Y = S^1 \times \Sigma$. Define a family of metrics on M by "stretching the neck" along Y to form $X_- \cup [-R,R]\times Y \cup X_+$.



Lemma: Suppose the Seiberg-Witten moduli space of M is non-empty for arbitrarily large R. Then,

$$|c_1(L)[\Sigma]| \leq 2g-2.$$

Proof: Kronheimer-Mrowka show in this situation there is a translation invariant SW sol$^{\text{n}}$ on $\mathbb{R}\times Y$. Normalize so area$(\Sigma)=1$. Weitzenböck formula and translation invariance imply:

$|F_A| \leq 2\pi(2g-2)$. So, $|c_1(L)[\Sigma]| = \frac{1}{2\pi}\left|\int_\Sigma F_A\right| \leq \frac{1}{2\pi}\sup|F_A| \cdot \text{area}(\Sigma) \leq 2g-2$. $\square$

Proof: (Adjunction) Say $\Sigma^2 = n \geq 0$. Blow up n-times to form $\tilde{\Sigma}$ inside $\tilde{M}$ with trivial normal bundle. Let $\tilde{K}$ be K minus the hyperplane classes of exceptional divisors. Kronheimer-Mrowka prove blowup preserves SW invariants, so $\tilde{K}$ is basic.
By lemma,
$$2g-2 \geq c_1(\tilde{L})[\tilde{K}] = \tilde{K}\cdot\tilde{\Sigma} = K\cdot\Sigma + \Sigma^2.$$
This gives the result for $g \geq 1$. If $g=0$, there are slightly more technical details. $\square$

Proof: (Thom Conjecture) The same argument as the adjunction inequality works. The only problem is $b_2^+(\mathbb{CP}^2)=1$ so we don't know there are always SW sol'ns as we neck stretch.
But wall crossing formula implies no. of sol'ns mod 2 given by sign of $c_1(L)\cdot[\omega_g]$. self-dual harmonic form assoc. g
For Kähler metric of positive scalar curvature, $SW=0$ and $c_1(L)\cdot[\omega_g] > 0$.
For long neck $c_1(L)\cdot[\omega_g] \to 3-d$ where $[\Sigma] = d[\mathbb{CP}^1]$.
Thus for $d > 3$, we can apply lemma to conclude. $\square$

Ozsváth and Szabo in 2000, generalized adjunction ~~formula~~ inequality to negative self-intersection assuming simple type. Note symplectic 4-manifolds all have $b_2^+ > 0$. By Taubes' result, we obtain same adjunction inequality for all symplectic 4-mflds.

Note if Σ is a symplectic surface (equivalently embedded J-hol. curve) the adjunction formula says
$$\chi(\Sigma) = -\Sigma^2 + c_1(M)\cdot\Sigma.$$ This is just equality in adjunction ineq.
Thus we have:

Thm: (Symplectic Thom Conjecture) In a symplectic 4-manifold, embedded J-holomorphic curves are genus-minimizing in their homology class.

Cor: In a Kähler surface, complex curves are genus minimizing.

Note: This is fundamentally a symplectic/complex result. Consider example of Mikhalkin '97.
$M = \#3\,\mathbb{CP}^2$ w/ almost complex str. giving $c_1 = (3,3,1) \in H^2(M)$.
Consider J-hol Σ representing class $(4,0,0)$. By adjunction,
$$\chi(\Sigma) = -\Sigma^2 + c_1\cdot\Sigma = -(4,0,0)^2 + (3,3,1)\cdot(4,0,0) = 12-16 = -4.$$
So, Σ is genus 3.
But one can do surgery twice on a quartic in 1st factor of $\mathbb{CP}^2$ to get a torus representing the same class.

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