The Casson Invariant and Instanton Floer Homology for Homology 3-Spheres

Student 3-Manifold Seminar

Wed. Feb. 28th 2024

Talk Based On:
- Saveliev "Invariants for Homology 3-Spheres"
- Floer "An Instanton-Invariant for 3-Manifolds"
- Donaldson "On the Work of Andreas Floer"
- Taubes "Casson's Invariant and Gauge Theory"

I. Homology 3-Spheres

Def^n: A homology 3-sphere is a closed 3-manifold whose integral homology agrees with that of S^3.

Remarks: (i) It is a good exercise to check that condition $H_1(X;\mathbb{Z})=0$ is sufficient. Equivalently, $\pi_1(X)$ is perfect, i.e. $[\pi_1(X),\pi_1(X)] = \pi_1(X)$.

(ii) Prior to proof of Poincaré conjecture, these spaces of interest as source of potential counterexample. Still an interesting class of manifolds w/ for example relations to knot theory.

Ex. (i) Let $I \subseteq SO(3)$ be the symmetries of the icosahedron. Lift to $I^* \subseteq SU(2) \cong S^3$. I^* is binary icosahedral group and $P = S^3/I^*$ is the Poincaré 3-sphere, a hom. 3 sphere since I^* perfect.

(ii) Recall given knot $K \subset \Sigma$ 3mfld, we can perform Dehn surgery. Cut out tubular nbhd of K and glue back in via homeo. $T^2 \to T^2$ of the boundary.

Any such homeo. induced by map on homology $[m] \mapsto p\cdot[m] + q\cdot[\ell]$, $p,q\in\mathbb{Z}$.
Resulting space is p/q-Dehn surgery on $K \subseteq \Sigma$, denoted $\Sigma + p/q\, K$.

From Mayer-Vietoris, if Σ is hom. 3 sphere, then so is $\Sigma + 1/q\, K$ for any knot K.

Eg. $S^3 - L = P$ where L is left handed trefoil.

II. The Casson Invariant

In 1985 at MSRI, Andrew Casson introduced the Casson Invariant associating to a homology 3-sphere Σ, $\lambda(\Sigma)\in\mathbb{Z}$.

Prop:
- $\lambda(S^3)=0$, • $\lambda(P)=-1$, • $\mathrm{Im}(\lambda)$ not contained in proper subgroup of $\mathbb{Z}$,
- $\lambda(-\Sigma) = -\lambda(\Sigma)$, • $\lambda(\Sigma \# \Sigma') = \lambda(\Sigma)+\lambda(\Sigma')$
- $\lambda(\Sigma + \frac{1}{q+1}K) - \lambda(\Sigma + \frac{1}{q}K)$ indep. of q and hence a knot invariant (related to Alexander polynomial)

We now give a definition of the Casson Invariant, essentially by counting representations of $\pi_1(\Sigma)$.
Given a group π, let $R^*(\pi)$ denote the set of irreducible rep^ns $\rho: \pi \to SU(2)$. Note the reducible representations are precisely those with image in $U(1)$. Let $\mathcal{R}^*(\pi) = R^*(\pi)/SO(3)$ where $SO(3)$ acts by conjugation. If $\pi = \pi_1(M)$ we write $\mathcal{R}^*(M)$ for $\mathcal{R}^*(\pi_1(M))$.

Prop: If S_g is surface of genus g, then $\mathcal{R}^*(S_g)$ is smooth mfld of dim $6g-6$.
If M is handle body of genus g, then $\mathcal{R}^*(M)$ is smooth mfld of dim $3g-3$.

↳ i.e. a filled in genus g surface

Proof: $\pi_1(M) = \mathbb{Z}^{*g}$, so $R(\pi) = SU(2)^g$, $\dim = 3g$.
$SO(3)$ acts freely on open subset of irred. Hence get $3g-3$ dim^l mfld. $\pi_1(S_g \setminus D) = \mathbb{Z}^{*2g}$. $R(S_g\setminus D) = SU(2)^{2g}$.
Killing ∂D gives $6g-3$ dim^l space which is smooth after quotienting by $SO(3)$ on irreducibles. □

Now consider Σ a homology 3-sphere. Like any 3-mfld, it admits a Heegaard splitting, i.e. it is two handlebodies M_1, M_2 glued along boundary S_g, surface of genus g.

By Seifert-Van Kampen have diagram of surjections:

$$\begin{tikzcd} & \pi_1(M_1) \arrow[dr] & \\ \pi_1(S_g) \arrow[ur] \arrow[dr] & & \pi_1(\Sigma) \\ & \pi_1(M_2) \arrow[ur] & \end{tikzcd}$$

Applying $\mathrm{Hom}(-, SU(2))$, removing irreps, and modding by $SO(3)$ gives diagram of injections:

$$\begin{tikzcd} & R^*(M_1) \arrow[dl, "j_1"'] & \\ R^*(S_g) & & R^*(\Sigma) \arrow[ul] \arrow[dl] \arrow[ll, "i"'] \\ & R^*(M_2) \arrow[ul, "j_2"] & \end{tikzcd}$$

A rep'n of Σ is one of M_1 and M_2 agreeing on S_g,

hence $i(R^*(\Sigma)) = j_1(R^*(M_1)) \cap j_2(R^*(M_2))$.

Claim: $i(R^*(\Sigma))$ is compact.

Proof: $R(\pi_1(\Sigma))$ is variety cut out by polynomial equations in $(S^3)^{|\pi_1|}$, hence compact.

So, $R(\Sigma) = R(\Sigma)/SO(3)$ compact. We still need to remove irreducibles.

Any red. rep. factors through $U(1)$ (abelian) and hence through $H_1(\Sigma, \mathbb{Z})$:

$$\begin{tikzcd} \pi_1(\Sigma) \arrow[r] \arrow[d] & U(1) \arrow[r] & SU(2) \\ H_1(\Sigma;\mathbb{Z}) \arrow[ur, dashed] & & \end{tikzcd}$$

Since we have homology 3-sphere, only irrep is trivial. Can check trivial rep^n is isolated point in image, so $i(R^*(\Sigma))$ compact. $\square$

This is where we need hom. 3-sphere restriction.

After isotopy of $R^*(S_g)$, $R^*(M_1)$ and $R^*(M_2)$ are perturbed to intersect transversely. The perturbed copy of $i(R^*(\Sigma))$ is compact 0-mfld. (i.e. finite set) It also has natural orientation, as discussed by Taubes.

Defn: The Casson invariant is $\lambda(\Sigma, M_1, M_2) = \frac{(-1)^g}{2} \#(j_1(R^*(M_1)) \cap j_2(R^*(M_2)))$. (oriented count of points)

Prop: λ is integral and independent of Heegaard splittings and perturbation.

III. Instanton Floer Homology

We now consider a homology theory of hom. 3-spheres introduced by Andreas Floer in a 1988 paper.

We need to recall some gauge theory. Let Σ be a closed oriented 3-mfld and $E = \Sigma \times SU(2)$ a (necessarily) trivial $SU(2)$-bundle. It has a space of connections $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{su}(2))$, i.e. Lie-algebra-valued 1-forms.

$A \in \mathcal{A}$ has a curvature $F_A = dA + A \wedge A \in \Omega^2(\Sigma, \mathfrak{su}(2))$. The automorphisms of this bundle form the gauge group $\mathcal{G} = C^\infty(\Sigma, SU(2))$. This acts by $g^*A = g^{-1}dg + g^{-1}Ag$ and $F_{g^*(A)} = gF_Ag^{-1}$.

The Chern-Simons functional on $\mathcal{A}$ given by

$$CS(A) = \frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(A \wedge F_A - \tfrac{1}{3} A \wedge A \wedge A).$$

Modulo $\mathbb{Z}$ this is indep. of $\mathcal{G}$-action.

So we get, $CS: \mathcal{A}^*/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$. Here we restrict to irreducible connections $\mathcal{A}^* \subset \mathcal{A}$, i.e. those w/ trivial stabilizer under $\mathcal{G}$-action, so that $\mathcal{A}^*/\mathcal{G}$ is ∞-dim^l manifold.

From now on implicitly work modulo $\mathcal{G}$.

Now fix a metric g on Σ. The L^2-gradient of CS is:

$\nabla CS(A) = - \underbrace{*}_{\text{Hodge star wrt } g \text{ and Killing form.}} F_A$. The critical points of CS are <u>flat connections</u>: $F_A = 0$.

Given two flat connections A_0, A_1, have relative Floer index $\mu(A_0, A_1) \in \mathbb{Z}$. This is defined modulo 8 on gauge equiv. classes. This is defined in terms of "spectral flow" (c.f. Floer's paper)

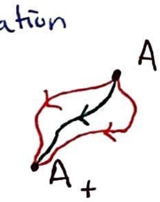
Now we can do <u>Morse theory</u>. We can study the gradient flow ~~of~~ the CS functional. i.e. we look for families of connections $A(t)$ on $\mathbb{R}_t \times \Sigma$ ~~s~~atisfying the PDE:

$$\boxed{\frac{\partial A(t)}{\partial t} = \nabla CS(A(t)) = - * F_{A(t)}}$$

If A has finite Dirichlet energy, then $\lim_{t \to \pm\infty} A(t) = A_\pm$ are two flat connections.
(think of ODE where gradient flow connects crit pts)

$\hookrightarrow$ This is the ASD eq^n on the cylinder $\mathbb{R} \times \Sigma$ (in temporal gauge $A_t = 0$). Its solutions are well studied by Uhlenbeck, Taubes, and Donaldson, and were the basis for Donaldson theory.
Its solutions are called instantons; they are absolute minima of Yang Mills functional.

One can show if $\mu(A_+, A_-) = 1$, then the instantons asymptotic to $A_\pm$ modulo reparameterization form a 0-dim^l compact manifold, i.e. a finite set of points w/ natural orientation.
(mod 8)
$\hookrightarrow$ after perturbation of ASD eq^n. (ignoring higher dim^l components)



Now we can define circular chain complex. The chain groups are:

$IC_n(\Sigma) = \mathbb{Z}\langle [A] \in \mathcal{A}^*/\mathcal{G} : F_A = 0, \mu(A, 0) = n \rangle$. Complex is

$$\cdots \to IC_0(\Sigma) \to IC_7(\Sigma) \to \cdots \to IC_1(\Sigma) \to IC_0(\Sigma) \to \cdots$$

Here, the boundary given by $\langle \partial A_1, A_2 \rangle = \# \{\text{Gradient Flow Lines from } A_1 \text{ to } A_2\}$ — Again, signed count.
(instantons mod reparam.)

If Σ a homology 3-sphere, then $\partial^2 = 0$. (As before, this is because no reducible rep^{n}s causing problems)

The resulting homology groups $I_*(\Sigma) = H_*(IC_*(\Sigma), \partial)$ are the <u>instanton Floer homology</u> of Σ.

<u>Prop</u>: $I_*(\Sigma)$ is independent of choice of metric and of our perturbation of ASD equation.

<u>Thm</u>: (Taubes) The Euler characteristic of $I_*(\Sigma)$ is (up to a sign) twice the Casson invariant of Σ.

<u>Why?</u> From chain groups, Euler char. is signed count of gauge equiv. classes of flat irred. connections. Any flat connection gives uniquely an $SU(2)$ rep^n of $\pi_1(\Sigma)$ via its holonomy. Irreducible connections precisely those inducing irred. rep^n. Conjugation of connection by $\mathcal{G}$ equiv. to adjoint action of $SO(3)$ on holonomy.

So have bijection

$$\{\text{Flat Irred } SU(2) \text{ Connections}\}/\mathcal{G} \xrightarrow{\cong} \{\text{Irreps } \pi_1(\Sigma) \to SU(2)\}/SO(3)$$

Only remains to show signs agree; this is difficult result of Taubes.

<u>Note</u>: At its simplest, Donaldson theory produces an integer invariant of closed 4-manifolds by counting instantons.
If M is 4-mfld w/ $\partial M = \Sigma$, can instead count instantons asymptotic to a certain flat connection on Σ. This defines invariant of M as element of instanton Floer homology.
One deduces this theory has the structure of a <u>topological quantum field theory</u>!