GETZLER RESCALING AND SYMBOL CALCULUS IN LOCAL INDEX THEORY VIA WEIGHTINGS ON VECTOR BUNDLES

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Contents

1. Introduction

The Atiyah-Singer index theorem is one of the most significant results in differential geometry, uniting the Gauss-Bonnet-Chern Formula, the Hierzebruch-Riemann-Roch Theorem, and the Signature Theorem into one formula and initiating the mathematical study of index theory. Of several diverse proof methods, one popular analytic technique in the case for Dirac operators uses a study of heat kernels (this often goes by the name of local index theory).

A particularly nice approach to the heat equation proof of the Theorem, which we outline in the next section, involves Ezra Getzler's trick of Getzler rescaling, in which we introduce a parameter λ to an asymptotic expansion of a heat kernel. Sending $\lambda \to 0$ isolates the term of interest in our expansion. Additionally, the rescaling is compatible with a Getzler symbol calculus on differential operators which sends the generalized Laplacian of the heat kernel to a harmonic oscillator which we can solve for exactly and complete the proof of the index theorem.

To better understand the geometry of this rescaling, Nigel Higson and Zelin Yi introduced a rescaled spinor bundle in [2] which is a vector bundle over the tangent groupoid that naturally encorporates Getzler's fibrewise-rescaling. We will further study this construction and clarify it using the notions of weightings on manifolds, linearly weighted vector bundles, and weighted deformation spaces as studied by Yiannis Loizides, Eckhard Meinrenken, and Dan Hudson (see [4]).

The use of weightings can also extend to studying other symbol calculi, including one developed by Pavol Ševera [9] of a similar form to the construction of Higson and Yi. We show that the Ševera construction also takes the form of a weighted deformation space and explore the compatibility of its algebraic structures with weightings.

Since the initial time of writing, some of these ideas have been refined and further studied by Dan Hudson in [3].

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2. Clifford Modules and Dirac Operators

To begin we outline some of the necessary algebraic and geometric preliminaries underlying the index theorem. The definitions and results of this section are all standard and can be found in [5].

2.1. **Clifford Algebras.** Consider a vector space with a symmetric bilinear form $(V, \langle\cdot,\cdot\rangle)$. Its corresponding *Clifford algebra* $\mathrm{Cl}(V)$ is defined as the quotient of the tensor algebra of V by the ideal generated by elements of the form,

$$vw + wv - 2\langle v, w\rangle 1.$$

One defines the *symbol map* $\sigma : \mathrm{Cl}(V) \to \bigwedge V$ so that for $e_1, \dots e_k$ orthogonal, $\sigma(e_1 \cdots e_k) = e_1 \wedge \cdots \wedge e_k$. One can check this is a well defined vector-space isomorphism. $\mathrm{Cl}(V)$ has a filtered algebra structure given by

$$\mathrm{Cl}_k(V) = \{v_1 \cdots v_k | v_i \in V,\ \forall i \in \{1, \dots, k\}\}.$$

Note $\bigwedge V$ is naturally a graded algebra and $\mathrm{Cl}(V)$ inherits a graded vector space structure via $\mathrm{Cl}^k(V) = \sigma(\bigwedge^k V)$. In fact, $\bigwedge V$ is canonically isomorphic to the associated graded algebra of $\mathrm{Cl}(V)$ as given by the symbol map. If $\mathrm{Cl}(V)$ acts by endomorphisms on a vector space W, then W has a module structure over $\mathrm{Cl}(V)$. In particular, if V is even-dimensional, $\mathrm{Cl}(V)$ has a unique (up to isomorphism) irreducible representation on a vector space S called the space of *spinors* and one has the relation $\mathrm{End}(S) \cong \mathbb{C}\mathrm{l}(V) = \mathrm{Cl}(V) \otimes \mathbb{C}$. For a general module W, we have $\mathrm{End}(W) \cong \mathbb{C}\mathrm{l}(V) \otimes \mathrm{End}_{\mathrm{Cl}}(W)$, where $\mathrm{End}_{\mathrm{Cl}}$ refers to Clifford-commuting endomorphisms.

Example 2.1. As an important case of a module, consider $\bigwedge V$. For $v \in V$, let v act on $\bigwedge V$ by,

$$c(v) = \varepsilon(v) - \iota(v) \in \mathrm{End}\Big(\bigwedge V\Big),$$

where ε is exterior multiplication and ι is interior multiplication by the dual vector as determined canonically by the metric. On $w \in V \subset \bigwedge V$, we have,

$$c(v)w = v \wedge w - \langle v, w\rangle.$$

This naturally extends to a module action of the tensor algebra on $\bigwedge V$; one finds,

$$\begin{aligned} c(v \otimes w) + c(w \otimes v) &= c(v)c(w) + c(w)c(v) \\ &= \varepsilon(v)\varepsilon(w) + \varepsilon(w)\varepsilon(w) + \iota(v)\iota(w) + \iota(w)\iota(v) \\ &\quad - \varepsilon(v)\iota(w) - \iota(v)\varepsilon(w) - \varepsilon(w)\iota(v) - \iota(w)\varepsilon(v) \\ &= -2\langle v, w\rangle \end{aligned}$$

and so the tensor algebra module descends to a Clifford algebra module.

We also need to construct an important Lie group contained in $\mathrm{Cl}(V)$. For V with its Clifford algebra $\mathrm{Cl}(V)$, we define the *Pin group* as a Lie group,

$$\mathrm{Pin}(V) = \{v_{i_1} \cdots v_{i_k} \in \mathrm{Cl}(V) | \forall\, 1 \leq j \leq k : \, v_{i_j} \in V \text{ and } \|v_{i_j}\|^2 = 1\}.$$

Now, one constructs the *Spin group* as the Lie subgroup

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \left(\bigoplus_{i=0,\, i \text{ even}}^{n} \mathrm{Cl}^i(V) \right).$$

If we take $V = \mathbb{R}^n$ with the usual inner product we write $\mathrm{Spin}(n)$ for $\mathrm{Spin}(\mathbb{R}^n)$. In turns out that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$, in particular $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$.

2.2. **Clifford Bundles.** Now consider a Riemannian manifold (M, g); since its tangent spaces define smoothly varying inner product spaces, we can define the *Clifford bundle* $\mathrm{Cl}(TM)$ over M as the union of Clifford algebras, $\mathrm{Cl}(T_pM)$, with its natural smooth structure. A vector bundle $\mathcal{E}$ is a *Clifford module* if $\Gamma(\mathrm{Cl}(TM))$ acts on it by sections of $\mathrm{End}(\mathcal{E})$. Such a Clifford bundle whose fibres are irreducible representations of $\mathrm{Cl}(T_pM)$ is called a *spinor bundle*; only some manifolds can carry a spinor bundle, in which case we call them a *spin manifold*. For a spinor bundle $\mathscr{S}$, $\mathrm{End}(\mathscr{S}) \cong \mathbb{C}\mathrm{l}(TM)$. Given a n-dimensional manifold, one can consider a $\mathrm{GL}(n, \mathbb{R})$-principal bundle $F_{\mathrm{GL}(n)}$, called the *frame bundle*, whose fibre over $p \in M$ consists of the automorphisms of T_pM. The tangent bundle can be described as an associated bundle $TM = F_{\mathrm{GL}(n)} \times_{\mathrm{GL}(n)} \mathbb{R}^n$. A *$G$-structure* on the frame bundle for G a Lie subgroup of $\mathrm{GL}(n)$ is a reduction of $F_{\mathrm{GL}(n)}$ to a principal G-bundle F_G such that we may write $TM = F_G \times_G \mathbb{R}^n$. For example, an orientation is determined by an $\mathrm{SL}(n)$-structure $F_{\mathrm{SL}(n)}$ and a Riemannian structure is determined by an $\mathrm{O}(n)$-structure $F_{\mathrm{O}(n)}$. Given an orientable Riemannian manifold with a $\mathrm{SO}(n)$-structure $F_{\mathrm{SO}(n)}$, if this bundle admits a lift to a $\mathrm{Spin}(n)$-principal bundle $F_{\mathrm{Spin}(n)}$, we obtain a *spin structure* and an identification,

$$TM = F_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} \mathbb{R}^n.$$

If M has a spin structure, then it carries a spinor bundle which we may describe as,

$$\mathscr{S} = F_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} S$$

for S the n-dimensional spinors.

Remark 2.2. A manifold carrying a spinor bundle is equivalent to the existence of a more general structure called a *spin$^\mathbf{c}$-structure*. Necessary and sufficient conditions for spin structures and spin$^\mathbf{c}$-structures to exist on a manifold M can be described in terms of characteristic classes.

From now on we assume our manifold has even dimension n and that our Clifford modules are $\mathbb{Z}_2$-graded bundles (also called *super vector bundles*) with a compatible $\mathbb{Z}_2$ grading of the Clifford action. A complex Clifford module over an oriented manifold is given a natural grading by splitting into $\pm$ subspaces on which the *chirality operator* $\Gamma = i^{n/2} e_1 \cdots e_n$ acts by ± 1 for any local oriented orthonormal frame $e_1, \ldots, e_n$. Given a Clifford module $\mathcal{E}$ with a connection $\nabla^{\mathcal{E}}$, we say the connection is a *Clifford connection* if,

$$[\nabla^{\mathcal{E}}_X, c(a)] = c(\nabla^{LC}_X a)$$

for all $a \in \Gamma(\mathrm{Cl}(TM))$ and $X \in \Gamma(TM)$. Here, $c : \Gamma(\mathrm{Cl}(TM)) \to \Gamma(\mathrm{End}(\mathcal{E}))$ is the smoothly varying family of $\mathrm{Cl}(T_pM)$-representations on module fibres and ∇^{LC} is the Levi-Civita connection on TM extended to its Clifford algebra. From this, we can define a first order operator D in the following manner:

$$D : \Gamma(\mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \Gamma(\mathcal{E} \otimes T^*M) \xrightarrow{g^\sharp} \Gamma(\mathcal{E} \otimes TM) \xrightarrow{c} \Gamma(\mathcal{E})$$

or locally,

$$D = \sum_i c(g^\sharp(\mathrm{d}x_i))\nabla^{\mathcal{E}}_{\partial_i}.$$

This will be odd with respect to our $\mathbb{Z}_2$-grading and D^2 will be a *generalized Laplacian* in the sense that, locally,

$$D^2 = \sum_{ij} g^{ij}\partial_i\partial_j + \text{ lower order terms }.$$

We call D the *Dirac operator* associated to a Clifford module with Clifford connection.

3. The Index Theorem

Now we will apply the concepts discussed above to sketch the proof of the index theorem using the heat kernel method and the trick of Getzler rescaling. Explanations of some of the important results contained in the index theorem are also given. Again, these results are all standard and exposition of them can be found in chapters 1-4 of [1] or [7].

3.1. **The Heat Kernel.** Given a Dirac operator D, one can exponentiate its square to obtain a bounded operator e^{-tD^2}. The action of this operator on $s \in \Gamma(\mathcal{E})$ is given by its *heat kernel* $k_t(x, y) \in \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*, M^2)$, i.e.

$$(e^{-tD^2}s)(x) = \int_{y\in M} k_t(x, y)s(y).$$

Here and in all subsequent discussions we make implicit the presence of half-densities in sections which we integrate as they don't impact on subsequent calculations. This heat kernel has several properties: first it is suitably differentiable in all of its entries, second it satisfies the generalized heat equation,

$$(\partial_t + D^2)k_t(x, y) = 0,$$

and third it has the boundary condition $\lim_{t\to 0} e^{-tD^2} = \mathrm{id}$.

One shows that such a heat kernel exists and has asymptotic expansion,

$$k_t(x, y) = \frac{e^{-\|x\|^2/4t}}{(4\pi t)^{n/2}} \sum_{i=0}^{\infty} t^i \phi_i(x, y). \tag{3.1}$$

Since we are working with a super bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, if we expand an operator T in terms of our grading as

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}, \quad \text{then} \quad \mathrm{Str}(T) = \mathrm{Tr}(T_{00}) - \mathrm{Tr}(T_{11})$$

is the *supertrace* of T. We have the relationship, for P an operator with smoothing kernel p,

$$\mathrm{Str}(P) = \int_{m\in M} \mathrm{Str}(p(m, m)).$$

If we consider p as locally taking values in $\mathrm{End}(\mathcal{E}_q) \simeq \mathbb{C}\mathrm{l}(T_qM) \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}}(\mathcal{E}_q)$, then the supertrace can be evaluated on $\mathbb{C}\mathrm{l}(T_qM)$ under the symbol map as $(2/i)^{n/2}$ times the coefficient on $e_1 \wedge \cdots \wedge e_n$ for $e_1 \ldots, e_n$ any oriented orthonormal basis of T_qM. If we write $\overline{\mathrm{Str}}$ for the supertrace on the Clifford-commuting portion and note that integrating a form over M implicitly only integrates the n-form part we may write,

$$\mathrm{Str}(P) = (-2i)^{n/2} \int_{m \in M} \sigma \otimes \overline{\mathrm{Str}}(p(m,m)). \tag{3.2}$$

Since our Dirac operator is odd, we write $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ and define the *index* of D as,

$$\mathrm{ind}(D) = \dim(\ker(D^+)) - \dim(\ker(D^-)).$$

One can think of this as a $\mathbb{Z}_2$-graded or *super dimension* of the kernel of D. Alternatively, one can show it is equal to the regular Fredholm index of D^+. We now can state an important partial result towards the index theorem, the central theorem of local index theory.

Theorem 3.1 (McKean-Singer Formula)**.** *For D a Dirac operator associated to a Clifford Module, and k_t its heat kernel,*

$$\mathrm{ind}(D) = \mathrm{Str}(e^{-tD^2}) = \int_{m \in M} \mathrm{Str}(k_t(m,m)). \tag{3.3}$$

Since the left hand side of (3.3) is time independent, we may expand the right side using (3.1) and retain only the time independent term to obtain,

$$\mathrm{ind}(D) = \frac{1}{(4\pi)^{n/2}} \int_{m \in M} \mathrm{Str}(\phi_{n/2}(m,m)).$$

In light of (3.2), this becomes,

$$\mathrm{ind}(D) = \frac{1}{(2\pi i)^{n/2}} \int_{m \in M} \sigma \otimes \overline{\mathrm{Str}}(k_t(m,m)) = \frac{1}{(2\pi i)^{n/2}} \int_{m \in M} \sigma \otimes \overline{\mathrm{Str}}(\phi_{n/2}(m,m)). \tag{3.4}$$

3.2. **Getzler Rescaling and the Index Theorem.** To obtain the index theorem, we need only obtain an explicit value for $\mathrm{Str}(\phi_{n/2}(m,m))$. This is a priori not easy to compute, so we introduce our Getzler rescaling to isolate for the desired term in the power series. Fix $q \in M$ and consider $k_t(\mathbf{x}, q)$ where $\mathbf{x}$ is given in normal coordinates for M centred on q. In small enough coordinates we can trivialize our bundle so that our kernel takes values in $\mathrm{End}(\mathcal{E}_q) \cong \mathbb{C}\mathrm{l}(T_qM) \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}}(\mathcal{E}_q)$. We introduce a parameter λ and a map δ_λ on operators which sends $\mathbf{x} \mapsto \lambda\mathbf{x}$, $t \mapsto \lambda^2 t$, and $c \in \mathbb{C}\mathrm{l}^j(T_pM) \mapsto \lambda^{-j}c$. That is,

$$\delta_\lambda k(\mathbf{x}, q, t) = \sum_{i=0}^{n} \lambda^{-i} k(\lambda\mathbf{x}, q, \lambda^2 t)_{[i]} \tag{3.5}$$

for $k_{[i]} \in \mathbb{C}\mathrm{l}^i(TM)$. Consider $r_\lambda(\mathbf{x}, t) = \lambda^n \delta_\lambda k_t(\mathbf{x}, q)$; note that the term $\phi_{n/2}$ is not changed by this transformation. Additionally one proves that when $\lambda \to 0$, the rescaled kernel r_λ has a well defined limit which under the symbol map can be thought of as a local section of $TM \times \bigwedge_{\mathbb{C}} T_qM \otimes \mathrm{End}_{\mathbb{C}\mathrm{l}}(\mathcal{E}_q) \to TM$ whose sole term with a component in $\bigwedge^n$ comes from $\phi_{n/2}$. (Actually we will define things in terms of the dual bundle $\bigwedge_{\mathbb{C}} T_q^*M$ instead, but it is immaterial since we can identify them using the metric). We see that r_0 satisfies the rescaled equation,

$$(\partial_t + \lambda^2 \delta_\lambda D^2 \delta_\lambda^{-1}) r_0 = 0.$$

We can compute this rescaled operator $\lambda^2\delta_\lambda D^2\delta_\lambda^{-1}$ in the limit $\lambda \to 0$ and we will obtain a type of harmonic oscillator on TM which given an orthonormal basis $\{x^i\}_i$ of T_pM has the form,

$$L = -\sum_i \left(\partial_{x^i} + \frac{1}{4}\sum_j R_{ij}x^j\right)^2 + F. \tag{3.6}$$

Here R is the Riemann curvature as a matrix of 2-forms and F is the "twisted curvature," which is the portion of the connection's curvature commuting with the Clifford action. This kernel associated with (3.6) can be computed explicitly with Mehler's formula to obtain,

$$r_0(\mathbf{x}, q, t) = (4\pi t)^{-n/2}\det{}^{1/2}\Big(\tfrac{tR/2}{\sinh(tR/2)}\Big)\exp\big(-\tfrac{1}{4t}\left\langle\tfrac{tR}{2}\coth\big(\tfrac{tR}{2}\big)\mathbf{x}, \mathbf{x}\right\rangle\big)\exp(-tF). \tag{3.7}$$

Evaluating this at $\mathbf{x} = 0, t = 1$, we obtain,

$$r_0(0, q, 1) = (4\pi)^{-n/2}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp(-F).$$

Note by what we've found, the supertrace of this must be equal to that of $k_1(q, q)$ and so this can be plugged into (3.4). We have,

$$\operatorname{ind}(D) = (2\pi i)^{-n/2}\int_M \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\overline{\operatorname{Str}}(\exp(-F)). \tag{3.8}$$

Characteristic classes are certain cohomology classes of a vector bundle's base space that describe "twisting" of the bundle and help determine certain classification questions and the existence of compatible G-structures. We will look at a couple such classes defined by a differential form given in terms of the curvature of a connection; the de Rham cohomology of these forms turns out to be an invariant of the choice of connection and so defines a characteristic class. For a vector bundle $\mathcal{E} \to M$ with connection ∇ and curvature $F_\nabla(X, Y) = [\nabla_X, \nabla_y] - \nabla_{[X,Y]}$, the two characteristic classes of interest to us are the *Chern character* and the *A-hat genus* defined by,

$$\operatorname{ch}(\mathcal{E}) = \operatorname{Str}\big(e^{-F_\nabla}\big) \quad \text{and} \quad \hat{A}(\mathcal{E}) = \det{}^{1/2}\left(\frac{F_\nabla/2}{\sinh(F_\nabla/2)}\right).$$

Note that in (3.8), we can see the formula for the $\hat{\mathcal{A}}$-genus appear where we take the connection to be the Levi-Civita on the tangent bundle. The formula for the Chern character also appears but where we use the twisted curvature F; we refer to the characteristic class this gives as the *twisted Chern character* and denote it $\operatorname{ch}(\mathcal{E}/\mathscr{S})$, in reference to the fact it comes from "factoring out" the spinor bundle and its curvature from the class computation. Repackaging (3.8) with our new found notation, we obtain the central theorem.

Theorem 3.2 (Atiyah-Singer Index Theorem)**.** *Let M be an n-dimensional oriented Riemannian compact manifold, for n even. For D a Dirac operator associated to a Clifford module $\mathcal{E} \to M$,*

$$\operatorname{ind}(D) = (2\pi i)^{-n/2}\int_M \hat{\mathcal{A}}(TM)\operatorname{ch}(\mathcal{E}/\mathscr{S}). \tag{3.9}$$

An alternate equivalent approach to obtain the final result, as exposited in [7], is by setting a symbol calculus on the differential operators of $\mathcal{E}$, i.e. fixing a certain filtration and studying the associated graded space. We define the Getzler filtration by demanding for any vector field

X, ∇_X and $c(X)$ have degree one and elements of $\text{End}_{\text{Cl}}(\mathcal{E})$ have degree zero; these elements generate the space of differential operators and so fully determine our filtration. One can check both D and D^2 have degree two in this filtration. Corresponding to this filtration is an associated graded algebra which is isomorphic to sections of the bundle $\mathscr{P}(TM) \otimes \bigwedge TM \otimes \text{End}_{\text{Cl}}(\mathcal{E}) \to TM$, here $\mathscr{P}(TM)$ is the space of polynomial coefficient differential operators on TM. We have a symbol map σ. taking differential operators to the corresponding element in the associated graded space. One finds that $\sigma_2(D^2)$ is precisely given by (3.6). Indeed, the rescaling applied to any differential operator will give its symbol. Thus the Getzler rescaling provides a way to obtain a certain useful symbol of differential operators while preserving relevant aspects of our kernel. This relationship will be explored further below.

3.3. **Some Examples.** To understand the broad generality of the index theorem, it will be insightful to see a few of the important theorems which appear as special cases of the result. The computations of indices and characteristic classes associated with the various choices of Dirac operator are given in [1].

Let M be a Riemannian manifold and consider the bundle $\bigwedge_{\mathbb{C}} TM \to M$. $\mathrm{d} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is a differential operator on this space which has a *formal adjoint* $\mathrm{d}^* : \Omega^{\bullet+1}(M) \to \Omega^\bullet(M)$ defined by the relation,

$$\int_M g(\mathrm{d}\alpha, \beta)\omega = \int_M g(\alpha, \mathrm{d}^*\beta)\omega$$

for all $\alpha, \beta \in \Omega(M)$ and ω the Riemannian volume form. The *Hodge de Rham operator* is the Dirac operator $D = \mathrm{d} + \mathrm{d}^*$. If the $\mathbb{Z}_2$-grading of $\bigwedge_{\mathbb{C}} TM$ is given by even and odd degree forms then the index of D can be shown to be the Euler characteristic of M, $\chi(M)$, which is the alternating sum of its Betti numbers. We obtain the following.

Theorem 3.3 (Chern-Gauss-Bonnet Theorem)**.** *Let M be an oriented compact Riemannian manifold with even dimension n and curvature R. Then,*

$$\chi(M) = \frac{1}{2\pi}\int_M \det{}^{1/2}(-R).$$

If $n = 2$, then $\det^{1/2}(-R)$ is *half* the scalar curvature κ and we obtain the classical Gauss-Bonnet Theorem,

$$\chi(M) = \frac{1}{4\pi}\int_M \kappa.$$

Consider $\bigwedge_{\mathbb{C}} TM$ again equipped with the Hodge de Rham operator $D = d + d^*$, however this time define its $\mathbb{Z}_2$-grading based on the ± 1 eigenspaces of the action $c(\Gamma) = \varepsilon(\Gamma) - \iota(\Gamma)$ of the chirality operator $\Gamma = i^{n/2} e_1 \cdots e_n$. Consider a quadratic form Q on a vector space which in a basis $x_1, \ldots, x_n$ is given by $Q(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$. Then the *signature* of Q is the integer $p - q$. If M has dimension n divisible by 4 then the $n/2$ de Rham cohomology class of M inherits a symmetric bilinear form,

$$(\alpha, \beta) = \int_M \alpha \wedge \beta.$$

The signature of the corresponding quadratic form is called the *signature of* M and is denoted $\sigma(M)$. It turns out that with our given grading on $\bigwedge_{\mathbb{C}} TM$, if the dimension of M divides four, then the index of D will be $\sigma(M)$. We obtain the following.

Theorem 3.4 (Hirzebruch Signature Theorem)**.** *Let M be an n-dimensional oriented compact Riemannian manifold with curvature R, where n divides four. Then,*

$$\sigma(M) = (\pi i)^{-n/2} \int_M \det{}^{1/2}\left(\frac{R/2}{\tanh R/2}\right).$$

Let M be a *Kähler manifold*, i.e. a symplectic, complex, Riemannian manifold whose three structures are mutually compatible. One obtains a decomposition of the complexified cotangent bundle, $T^*_{\mathbb{C}}M = (T^*_{\mathbb{C}}M)^{1,0} \oplus (T^*_{\mathbb{C}}M)^{0,1}$ where sections of the two components are spanned by the differentials of holomorphic and anti-holomorphic sections respectively. The exterior derivative d on $\bigwedge(T^*_{\mathbb{C}}M)^{1,0} \oplus \bigwedge(T^*_{\mathbb{C}}M)^{0,1}$ can be decomposed as $d = \partial + \overline{\partial}$ where ∂ raises the degree on $\bigwedge(T^*_{\mathbb{C}}M)^{0,1}$ by one and $\overline{\partial}$ does the same on the other factor. These operators square to zero and anti-commute. One finds that complex manifolds always have a spin structure and the spinor bundle is given by $\mathscr{S} = \bigwedge(T^*_{\mathbb{C}}M)^{0,1}$. Thus, every complex Clifford module over M splits as $\mathcal{E} = \bigwedge(T^*_{\mathbb{C}}M)^{0,1} \otimes \mathcal{W}$ for some Hermitian holomorphic vector bundle $\mathcal{W}$. One finds that $\mathcal{W}$ inherits a unique *holomorphic connection* $\nabla^{\mathcal{W}}$, i.e. a connection whose splitting $\nabla = \nabla^{1,0} + \nabla^{0,1}$ satisfies $\nabla^{0,1} = \overline{\partial}$. This is called the *Chern connection* of $\mathcal{W}$. The Clifford action on $\mathcal{E} = \bigwedge(T^*_{\mathbb{C}}M)^{0,1} \otimes \mathcal{W}$ for $\omega = \omega^{1,0} + \omega^{0,1} \in \Gamma(\bigwedge^1 T^*_{\mathbb{C}}M)$ is given by,

$$c(\omega) = \sqrt{2}(\varepsilon(\omega^{1,0}) - \iota(\omega^{0,1})).$$

Further, $\mathcal{E}$ has a canonical Clifford connection given by the tensor of the Levi-Civita connection extended to $\bigwedge(T^*_{\mathbb{C}}M)^{0,1}$ and the Chern connection on $\mathcal{W}$. Corresponding to this is a Dirac operator $D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$.

Associated to the holomorphic bundle $\mathcal{W}$ is a certain set of cohomology groups $H^i(M, \mathcal{W})$ called the *Dolbeault cohomology.* Similar to the definition of the Euler characteristic, one defines the *Euler number of* $\mathcal{W}$, $\mathrm{Eul}(\mathcal{W})$, as the alternating sum of the dimensions of the Dolbeault cohomology groups. One finds that the index of D is given by $\mathrm{Eul}(\mathcal{W})$. An important characteristic class is the *Todd genus*, $\mathrm{Td}(M)$, which is given for a complex manifold by,

$$\mathrm{Td}(M) = \det\left(\frac{R^{1,0}}{e^{R^{1,0}} - 1}\right).$$

Here $R^{1,0}$ is the curvature restricted to $TM^{1,0}$. One obtains the following theorem.

Theorem 3.5 (Hirzebruch-Riemann-Roch Theorem)**.** *Let M be an n-dimensional compact Kähler manifold with holomorphic vector bundle $\mathcal{W}$. Then,*

$$\mathrm{Eul}(\mathcal{W}) = (2\pi i)^{-n/2} \int_M \mathrm{Td}(M)\mathrm{ch}(\mathcal{W}).$$

If M is a Riemann surface and $\mathcal{L}$ is a holomorphic complex line bundle with curvature F then one finds (from Taylor expansion),

$$\mathrm{Td}(M) = 1 - \frac{R^{1,0}}{2} \quad \text{and} \quad \mathrm{ch}(\mathcal{L}) = 1 - F.$$

So the theorem above reduces to,

$$\mathrm{dim} H^0(M, \mathcal{L}) - \mathrm{dim} H^1(M, \mathcal{L}) = -\frac{1}{4\pi i} \int_M R^{1,0} + 2F.$$

One defines the *degree* of $\mathcal{L}$ to be the integer,

$$\deg(\mathcal{L}) = -\frac{1}{2\pi i}\int_M F.$$

While we note from the theorem applied to the zero bundle,

$$\dim H^0(M) - \dim H^1(M) = -\frac{1}{4\pi i}\int_M R^{1,0}.$$

The dimension of the zeroth cohomology group is always one, since the space is connected, while the dimension of the first is the genus g of M, i.e. the number of holes in the surface. Hence we obtain,

$$\mathrm{Eul}(\mathcal{L}) = 1 - g + \deg(\mathcal{L})$$

which is the classical Riemann-Roch Theorem.

4. The Higson and Yi Approach to Getzler Rescaling

Nigel Higson and Zhelin Yi have recently studied Getzler rescaling by incorporating it into a vector bundle which they call the rescaled spinor bundle [2]. We briefly summarize the geometry of this space and how the index theorem follows from it.

4.1. **Deformation Spaces and the Tangent Groupoid.** The construction of the bundle begins by considering the tangent groupoid $\mathbb{T}M$. The tangent groupoid is a special case of the more general deformation to the normal cone construction initially from algebraic geometry. Consider a manifold M with submanifold N. Note the normal bundle $\nu(M, N)$ is a vector bundle N of dimension equal to that of M. We construct the *deformation space* $\delta(M, N)$ as a manifold of dimension $\dim(M) + 1$ with a submersion $\pi : \delta(M, N) \to \mathbb{R}$ whose fibres are,

$$\delta(M,N)|_{\pi^{-1}(\lambda)} = \begin{cases} M & t \neq 0 \\ \nu(M,N) & t = 0. \end{cases}$$

This has a natural smooth structure: given submanifold coordinates $x_1, \ldots, x_n, y_1, \ldots, y_{m-n}$ for $N \subset M$, we take coordinates on $\delta(M, N)$ to be $\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{y}_1, \ldots, \widetilde{y}_{m-n}, \lambda$. Here λ is given by projection under π and the other coordinates are given on π-fibres by,

$$\widetilde{x}_i = x_i \quad \text{and} \quad \widetilde{y}_j = \lambda^{-1} y_j.$$

Example 4.1. A visualizable example to keep in mind is the following. Let p be a point on the circle. Then we can consider $\delta(S^1, \{p\})$ as an embedded submanifold of $\mathbb{R}^3$. Take a coordinate axis L in $\mathbb{R}^3$ parameterized by $\lambda \in \mathbb{R}$. For each $\lambda \neq 0$, embed a scaled copy of S^1 with $\{p\}$ intersecting L at λ, lying in the plane normal to L at λ, and with radius $1/|\lambda|$. Taking the union of these circles, we get on either side of $\lambda = 0$ smooth funnels which shrink to a point as $|\lambda| \to \infty$ and expand to arbitrarily large size as $|\lambda| \to 0$. This structure smoothly extends to a connected manifold by taking the closure as a subspace of $\mathbb{R}^3$; it's clear that this closure is obtained by inserting a line through $\lambda = 0$ which is parallel to the tangent spaces T_pS^1 for all the embedded copies of the circle. Noting that as manifolds $U(1) \cong S^1, \mathfrak{u}(1) \cong \mathbb{R}$, we can also see this example as a specific case of the construction for Lie groups; the space $\delta(G, \{e\})$ determines a smooth deformation of a Lie group G to its Lie algebra $\mathfrak{g}$.

Of central importance to us is the *tangent groupoid* $\mathbb{T}M$, as initiated and applied to index theory by Alain Connes. For any manifold M, $\mathbb{T}M$ is defined as the deformation space $\nu(M^2, M_\Delta)$ (here M_Δ is the diagonally embedded copy of M inside M^2). This is a Lie groupoid which is given as a family of subgroupoids over the submersion π: it is the Pair groupoid $\mathrm{Pair}(M)$ over $\lambda \neq 0$ and the additive groupoid TM over $\lambda = 0$. The groupoid structure is not terribly important to us, but it underlies the K-theory approach to the index theorem and appears in the work of Higson-Yi and Ševera; as such, sometimes we will prefer the notation $\mathrm{Pair}(M)$ to M^2.

Example 4.2. In line with our construction of $\delta(S^1, \{p\})$, one can almost visualize an example of the tangent groupoid construction. Consider again our circle deformation space, but replace each circle with a torus (whose thickness is contained in some fourth dimension). These tori (projected down to $\mathbb{R}^3$) are like hollow annuli of increasing radius as $|\lambda| \to 0$. At $\lambda = 0$, these family of tori flatten out to become a cylinder. As such, one obtains $\mathbb{T}S^1 = \delta(T^2, S^1)$ as an embedded submanifold of $\mathbb{R}^5$.

4.2. **The Rescaled Spinor Bundle.** Assume M is a spin manifold of even dimension n with a spinor bundle $S \to M$. Higson and Yi define their *rescaled spinor bundle* $\varpi : \mathbb{S} \to \mathbb{T}M$ by its fibres over $\pi \circ \varpi$,

$$\mathbb{S}|_{\varpi^{-1}(\pi^{-1}(\lambda))} = \begin{cases} S \boxtimes S^* \to \mathrm{Pair}(M) & \lambda \neq 0 \\ \bigwedge TM \to TM & \lambda = 0. \end{cases}$$

Much of Higson and Yi's paper concerns describing a smooth structure on this space in terms of its sheaf of sections. Later we will explain how this same structure can be arrived at through the concept of weighted deformation spaces.

Higson and Yi go on to show ([2, Theorem 3.28]) that for a differential operator T acting on the first factor of $S \boxtimes S^*$ of filtration degree k in the sense described at the end of the last section, we get an operator on $S \boxtimes S^*$,

$$\widetilde{T}_{[k]}|_{\pi^{-1}(\lambda)} = \begin{cases} \lambda^k T & \lambda \neq 0 \\ \sigma_k(T) & \lambda = 0. \end{cases}$$

Here $\sigma_\bullet$ is the symbol from the end of the last section. In particular, our generalized Laplacian D^2 on S (once we pick a connection) will be upgraded to $\widetilde{D}^2_{[2]}$ which over the 0-fibre is the familiar formula from (3.6).

Given an operator $\widetilde{T}_{[k]}$ as above, and its associated family of kernels $\widetilde{t}_{[k]} \in \Gamma(\mathbb{S}, \mathbb{T}M)$, obviously the family of supertraces,

$$\mathrm{Str}_\lambda(\widetilde{T}_{[k]}) = \mathrm{Str}(\widetilde{T}_{[k]}|_{\pi^{-1}(\lambda)})$$

is smooth for $\lambda \neq 0$ and Higson and Yi show ([2, Theorem 5.6]) it smoothly extends to zero by,

$$\mathrm{Str}_0(\widetilde{T}_{[k]}) = (-2i)^{n/2} \int_{m \in M} \widetilde{t}_{[k]}(0_m, 0).$$

I.e. we integrate the top form part of the kernel along the zero section of TM.

In particular, applying this to $e^{-\widetilde{D}^2_{[2]}}$ and using (3.3) we observe that, for $\lambda \neq 0$,

$$\mathrm{Str}_\lambda(e^{-\widetilde{D}^2_{[2]}}) = \mathrm{Str}(e^{-\lambda^2 D^2}) = \mathrm{ind}(D).$$

This is a constant valued function of λ which must extend to give,

$$\mathrm{ind}(D) = (-2i)^{n/2} \int_{m \in M} \widetilde{k}_{[2]}(0_m, 0).$$

But $\widetilde{k}_{[2]}(0_m, 0)$ is precisely the kernel of (3.6) at m in the spin case evaluated at $\mathbf{x} = 0, t = 1$. So from this we immediately obtain the formula (3.8), and hence the index theorem (at least for spin manifolds). The results easily extend to the full setting as we will outline with the simpler picture of weightings.

5. Weighted Manifolds

We now outline Loizides and Meinrenken's theory of weightings on manifolds. This is a generalization of "order of vanishing" for functions in $C^\infty(M)$ which allows us to assign integer weights to different directions by which we multiply the vanishing order. The content of the first subsection up until Lemma 5.1 is taken from [4].

5.1. **Weightings, Weighted Normal Bundles, and Weighted Deformation Spaces.**

A *weighting* on a manifold M is a filtration of the sheaf of sections,

$$C^\infty(M) = C^\infty(M)_{(0)} \supset C^\infty(M)_{(1)} \supset C^\infty(M)_{(2)} \supset \cdots$$

satisfying the following local property: around any $m \in M$ there should be a coordinate patch U with local coordinates $x_1, \dots, x_n$ and a weight sequence $(w_1, \dots, w_n) \in \mathbb{N}_0^n$ such that the restriction of $C^\infty(M)_{(i)}$ to U is the ideal generated by the set,

$$\{x_1^{s_1} \cdots x_n^{s_n} | s_1 \cdot w_1 + \cdots s_n \cdot w_n \geq i\}.$$

Given weighted manifolds M_1, M_2, a smooth map $\phi : M_1 \to M_2$ is *weighted* if it is compatible with the filtrations, i.e. $\phi^*(C^\infty(M_2)_{(i)}) \subset C^\infty(M_1)_{(i)}$. We call our local coordinates from the definition *weighted coordinates* and the maximum of the weights w_i over all patches and weighted coordinates is called the *order* of the weighting. It is simple to show that $C^\infty(M)_{(1)}$ must be the vanishing ideal of a closed submanifold $N \subset M$; in this setting we will sometimes refer to the *weighting of M along N*.

As a most basic example, the only weighting of M over N of order 1 is given by letting $C^\infty(M)_{(i)} = \mathcal{I}_N^i$ which we call the *trivial weighting* along N. Higher order weightings can be more complicated to characterize; refer to Loizides and Meinrenken for some more details. Given weighted manifolds M_1, M_2 we can construct the *product weighting* on $M_1 \times M_2$ in the obvious way: simply take as weighted coordinates the coordinates of the two factors, or in terms of filtrations,

$$C^\infty(M_1 \times M_2)_{(i)} \quad \text{is generated by} \quad \sum_{j=0}^{i} C^\infty(M_1)_{(j)} \otimes C^\infty(M_2)_{(i-j)}.$$

A filtration of M along N of order r also induces a filtration on the normal bundle,

$$\nu(M, N) = F_{-r} \supset F_{-r+1} \supset \cdots \supset F_0 = N,$$

given by,

$$\Gamma(\mathrm{ann}(F_{-i+1})) = C^\infty(M)_{(i)}/(C^\infty(M)_{(i)} \cap \mathcal{I}_N^2).$$

The normal bundle of $N \subset M$ allows us to consider Taylor approximations of functions; we are tempted to generalize this to the weighted context. The *weighted normal bundle* $\nu_{\mathcal{W}}(M, N)$ of a weighted manifold M along N is defined algebraically as,

$$\nu_{\mathcal{W}}(M, N) = \mathrm{Hom}_{\mathrm{alg}}(\mathrm{gr}(C^\infty(M)), \mathbb{R}).$$

Here the associated graded algebra is the obvious one induced by the weighting filtration of $C^\infty(M)$. This is given a unique smooth structure by stipulating that the weighted-homogeneous approximation $f^{[i]}$ of functions $f \in C^\infty(M)_{(i)}$ are smooth functions on $\nu_{\mathcal{W}}(M,N)$ for all i. In general, the weighted normal bundle is a graded bundle over N but need not be a vector bundle. In the case of a trivial weighting, this reduces to the definition of the usual normal bundle. Given a weighted morphism $\phi : M_1 \to M_2$, we obtain a corresponding morphism $\nu_{\mathcal{W}}\phi : \nu_{\mathcal{W}}M_1 \to \nu_{\mathcal{W}}M_2$ which in coordinates will just be the homogeneous approximation of ϕ in our homogeneous coordinates. As such, we obtain a *normal bundle functor* from weighted manifolds to graded manifolds.

In light of our interest in deformation spaces, we are interested again in a weighted version. Given a weighting of M along N, the *weighted deformation space* $\delta_{\mathcal{W}}(M,N)$ is defined algebraically as,

$$\delta_{\mathcal{W}}(M,N) = \mathrm{Hom}_{\mathrm{alg}}(\mathrm{Rees}(C^\infty(M)), \mathbb{R})$$

where the *Rees algebra* of $C^\infty(M)$ is,

$$\mathrm{Rees}(C^\infty(M)) = \left\{ \sum_{i\in\mathbb{Z}} z^{-i} f_i \middle| f_i \in C^\infty(M)_{(i)} \right\} \subset C^\infty(M)[z, z^{-1}].$$

More concretely, there is a submersion $\pi : \delta_{\mathcal{W}}(M) \to \mathbb{R}$ for which,

$$\pi^{-1}(\lambda) = \begin{cases} M & \lambda \neq 0 \\ \nu_{\mathcal{W}}(M,N) & \lambda = 0. \end{cases}$$

This is made smooth in such a way that for any $f \in C^\infty(M)_{(i)}$, the real-valued function $\widetilde{f}^{[i]}$ on the deformation space given by,

$$\widetilde{f}^{[i]}\big|_{\pi^{-1}(\lambda)} = \begin{cases} \lambda^{-i} f & \lambda \neq 0 \\ f^{[i]} & \lambda = 0 \end{cases}$$

is smooth. In particular, weighted coordinates $x_1, \dots, x_m$ of M define weighted coordinates $\lambda, \widetilde{x}_1^{[w_1]}, \dots, \widetilde{x}_m^{[w_m]}$ for $\delta_{\mathcal{M}}(M,N)$. There is a "zoom action" κ_u for $u \in \mathbb{R}^*$ on $\delta_{\mathcal{W}}(M,N)$. On λ-fibres for $\lambda \neq 0$, $\kappa_u(\pi(x)) = \pi(x)/u$ and κ_u acts trivally within the fibre. For $\lambda = 0$, κ_u scales along the ith degree component of the graded bundle structure by u^i. The action is such that $\kappa_u^*(\widetilde{f}^{[i]}) = u^i \widetilde{f}^{[i]}$. Thus, we may say that a function $f \in C^\infty(M)_{(i)}$ is uniquely extended to a function $\widetilde{f}^{[i]}$ on the deformation space which is homogeneous of degree i under κ.

We will need one simple result about weightings later.

Lemma 5.1. *For M a weighted manifold and any $f \in C^\infty(M)_{(i)} \setminus C^\infty(M)_{(i+1)}$ there exists a smooth weighted path $\gamma_f : \mathbb{R} \to M$ for which $\gamma_f^*(f) \in C^\infty(\mathbb{R})_{(i)} \setminus C^\infty(\mathbb{R})_{(i+1)}$. Here $\mathbb{R}$ is given the trivial weighting along* 0*.*

Proof. It is enough to show this locally. Consider a coordinate patch $U \subset M$ with weighted coordinates $x_1, \dots, x_n$ of weights $w_1, \dots, w_n$. Consider a smooth path which near zero has the form $\gamma(t) = (\alpha_1 t^{w_1}, \cdots, \alpha_n t^{w_n})$ in our weighted coordinates where the coefficients α are arbitrary. Note that, for a multi-index $I = (i_1, \dots, i_\ell) \in \{1, \dots, n\}^\ell$,

$$\gamma^*(x^I) = \alpha_{i_1} \cdots \alpha_{i_\ell} t^{w_{i_1} + \cdots + w_{i_\ell}}.$$

Expanding $f \in C^\infty(M)_{(i)} \setminus C^\infty(M)_{(i+1)}$ in our coordinates, we see,

$$\gamma^*(f) = q(\alpha_1, \dots, \alpha_n)t^i + \mathcal{O}(t^{i+1})$$

for some non-zero real polynomial q in n variables. Now define a specific γ_f by choosing $\alpha_1, \dots, \alpha_n$ for which $q(\alpha_1, \dots, \alpha_n) \neq 0$. With this choice, $\gamma_f^*(f)$ vanishes to exactly degree i at 0 and hence $\gamma_f^*(f) \in C^\infty(\mathbb{R})_{(i)} \setminus C^\infty(\mathbb{R})_{(i+1)}$ as desired. For a general function $g \in C^\infty(M)_{(j)}$ we see by above that $\gamma_f^*(g) \in \mathcal{O}(t^j)$ hence $\gamma_f^*(g) \in C^\infty(\mathbb{R})_{(j)}$ and so γ_f is weighted. □

Proposition 5.2. *A smooth map $\phi : M_1 \to M_2$ between weighted manifolds is weighted if and only if for all weighted paths $\gamma : \mathbb{R} \to M_1$ the path $\phi \circ \gamma : \mathbb{R} \to M_2$ is weighted.*

Proof. The forward direction follows immediately from the definition of weighted maps. For the backwards direction assume that a path γ being weighted implies $\phi \circ \gamma$ is. Consider $g \in C^\infty(M_2)_{(i)}$. Let $\phi^*(g) \in C^\infty(M_1)_{(j)} \setminus C^\infty(M_1)_{(j+1)}$. By Lemma 5.1, we can find a weighted smooth path $\gamma : \mathbb{R} \to M_1$ for which,

$$(\phi \circ \gamma)^*(g) = \gamma^* \phi^*(g) \in C^\infty(\mathbb{R})_{(j)} \setminus C^\infty(\mathbb{R})_{(j+1)}.$$

By assumption, $\phi \circ \gamma$ is weighted and so $i \leq j$ since otherwise $(\phi \circ \gamma)^*(g) \in C^\infty(\mathbb{R})_{(i)}$ would violate the fact $(\phi \circ \gamma)^*(g) \notin C^\infty(\mathbb{R})_{(j+1)}$. Thus, $\phi^*(g) \in C^\infty(M_1)_{(j)}$ means $\phi^*(g) \in C^\infty(M_1)_{(i)}$. This holds for general g meaning ϕ^* preserves filtrations and thus ϕ is weighted. □

5.2. **Linear Weightings on Vector Bundles.** Of interest to us is putting a weighting on a vector bundle. Of course a vector bundle is a manifold and so we may put a weighting on it like any other manifold. However, we may hope that the weighting is in some way compatible with the linearity of a vector bundle. Additionally, it is more natural to think about the space of sections of a vector bundle than about its smooth functions and we may hope for an equivalent formulation of weightings in terms of a filtration of its sections.

Given a weighted manifold M and a vector bundle $\mathcal{E} \to M$, we say that $\mathcal{E}$ is a *linearly weighted vector bundle over* M if it is weighted as a manifold and if we may take $x_1, \dots, x_n, \omega_1, \dots, \omega_n$ as its weighted coordinates at a point where $x_1, \dots x_n$ are the projection to the zero section composed with weighted coordinates for M of the same weight and $\omega_1, \dots, \omega_n$ are a local frame of $\mathcal{E}^*$. The maximum of the weights assigned to a local frame ω_i over all sets of weighted coordinates will be called the *order* of the weighting.

Note that the space of sections of $\mathcal{E}^*$ is a subset of $C^\infty(\mathcal{E})$ and hence it obtains a filtration from our weighting. Automatically this filtration is compatible with module structure of the space of sections,

$$C^\infty(M)_{(i)} \Gamma(\mathcal{E}^*)_{(j)} \subset \Gamma(\mathcal{E}^*)_{(i+j)}.$$

In local coordinates on $U \subset M$, if $\omega_1, \dots, \omega_n \in \Gamma(\mathcal{E}^*, U)$ have weights $w_1, \dots, w_n$ then $\Gamma(\mathcal{E}^*, U)_{(i)}$ consists of section,

$$\sum_j f_j \omega_j$$

where each $f_j \in C^\infty(U)_{(i-w_j)}$. We can now obtain a filtration of the space of section of $\mathcal{E}$ in such a way that the pairing,

$$\langle \cdot, \cdot \rangle : \Gamma(\mathcal{E}^*) \times \Gamma(\mathcal{E}) \to C^\infty(M) \tag{5.1}$$

respects the filtration. I.e. $\eta \in \Gamma(\mathcal{E})_{(j)}$ if and only if $\omega \in \Gamma(\mathcal{E}^*)_{(i)}$ implies $\langle \omega, \eta \rangle \in C^\infty(M)_{(i+j)}$. From now on we will refer to such maps that respect the filtration as *filtered morphisms*. This filtration is different from what we've seen in that it starts in negative numbers; if the weighting of $\mathcal{E}$ has order n, then the filtration of $\Gamma(\mathcal{E})$ is,

$$\Gamma(\mathcal{E}) = \Gamma(\mathcal{E})_{(-n)} \supset \cdots \supset \Gamma(\mathcal{E})_{(0)} \supset \cdots .$$

Again this is compatible with the module structure. In local coordinates on $U \subset M$, if $\omega_1, \ldots, \omega_n \in \Gamma(\mathcal{E}^*, U)$ are weighted coordinates with weights $w_1, \ldots, w_n$ and $\xi_1, \ldots, \xi_n \in \Gamma(\mathcal{E}, U)$ are the dual sections defined by $\omega_i(\xi_j) = \delta_{ij}$ then,

$$\Gamma(\mathcal{E}, U)_{(p)} = \left\{ \sum_k f_k \xi_k | f_k \in C^\infty(U)_{(p+w_k)} \right\}. \tag{5.2}$$

Remark. This construction in terms of a dual basis is obviously non-canonical unless we have a metric. Additionally, one must be careful as linearity does not ensure the characterization in (5.2) works for general bases, only in our weighted frame. As an example, take a 2-dimensional vector space V with basis $\{v_1, v_2\}$ and dual basis $\{v^1, v^2\}$. Make this space a weighted vector bundle over $\{0\}$ by setting v^1, v^2 as weighted coordinates with weights 1 and 2. Using the above characterization, the induced filtration on sections (in this case they are sections over $\{0\}$, so just vectors) is,

$$\Gamma(V)_{(-2)} = V \supset \Gamma(V)_{(-1)} = \operatorname{span}(v_1) \supset \Gamma(V)_{(0)} = \{0\}.$$

If we had instead begun with coordinates $v^1 + v^2, v^1 - v^2$ both of weight 1, then naively following (5.2) would tell us the dual basis vectors $v_1 + v_2, v_1 - v_2$ were sections of filtration degree minus one when we know from above they both have filtration degree minus two. The inconsistency obviously comes from the fact $\{v^1 + v^2, v_1 - v_2\}$ are *not* a set of weighted coordinates.

The construction of a filtration on sections from a weighted bundle also works in reverse. Consider a weighted manifold and a filtration of $\Gamma(\mathcal{E})$ which is compatible with the module structure and such that in local coordinates of M the filtration is given by assigning non-positive weights to local sections in a local frame. Asking that the map (5.1) again is a filtered morphism, we get a non-negative filtration of the space of dual sections. The space $C^\infty(\mathcal{E})$ is generated as an algebra by $C^\infty(M)$ and $\Gamma(\mathcal{E}^*)$ from which we get a filtration on the total space of real valued functions that is compatible with the existing filtrations. By considering these filtrations in local coordinates we see that we obtain a weighting and the induced filtration on the space of sections is the one we started with.

Example 5.3. At this point it is helpful to keep in mind a fundamental example. Consider a weighted manifold M and define a filtration on $\Gamma(TM)$ so that the derivation operation of vector fields on $C^\infty(M)$ is a filtered morphism. Locally, if $x_1, \ldots, x_n$ are weighted coordinates of M with weights $w_1, \ldots, w_n$ then $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ have filtration degrees $-w_1, \ldots, -w_n$. The corresponding filtration on the dual bundle $\Omega^1(M)$ is locally given by $dx_1, \ldots, dx_n$ having filtration degrees $w_1, \ldots, w_n$; equivalently this is the filtration for which the exterior derivative is a filtered morphism which in turn extends our filtration to one on $\Omega(M)$. Similarly we can extend this to filtrations on multi-vector fields and other tensor fields. With these filtrations the usual operations on tensor fields like tensor products, contraction, and the Lie derivative are all filtered morphisms.

Remark 5.4. It is natural to consider filtrations of sections for which the assigned weights are allowed to be positive. In this context, we won't obtain a weighting but a more general filtration of smooth functions that begins in the negative integers. Within this more general setting, many of the constructions, like weighted normal bundles, should still work. And we might conjecture that we obtain isomorphisms between these negative weighted spaces and the true weighted space given by shifting the filtration indices to begin at 0. We don't address these "negative weightings" further as they don't appear in our main settings of interest, however they are relevant to other considerations such as considering the bundle of (p, q)-tensors as weighted.

One other filtration we can induce is on the space of differential operators $\mathscr{D}(\mathcal{E})$ by asking that the action of operators on sections,

$$\mathscr{D}(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}) \tag{5.3}$$

forms a filtered morphism. By now one should see that the presence of a weighted vector bundle induces a plethora of filtrations relevant to the structures of the bundle. There is certainly more to be gained by further studying the relationships between these filtrations beyond what has been described here. The interplay between these filtrations will be key to the constructions in the following section.

Because of our linear structure, the resulting weighted normal bundles and weighted deformation spaces will be particularly nice. We now prove some results to this effect.

Proposition 5.5. *Given a linearly weighted vector bundle $\mathcal{E}$ over M weighted along N the weighted normal bundle $\nu_{\mathcal{W}}(\mathcal{E}, N)$ is a vector bundle over $\nu_{\mathcal{W}}(M, N)$ whose space of polynomial sections is isomorphic to* $\mathrm{gr}(\Gamma(\mathcal{E}))$*. Furthermore, there is a natural inclusion of* $\mathrm{gr}(\mathscr{D}(\mathcal{E}))$ *into the differential operators of $\nu_{\mathcal{W}}(\mathcal{E}, N)$.*

Proof. First we show that $\nu_{\mathcal{W}}(\mathcal{E}, N) \to \nu_{\mathcal{W}}(M, N)$ is a vector bundle. Consider local weighted coordinates $x_1, \ldots, x_n$ with weights $w_1, \ldots, w_n$ for M and over it a local weighted frame $s_1, \ldots, s_m$ with weights $v_1, \ldots, v_m$ for $\mathcal{E}$. We also get a dual frame $s^1, \ldots, s^m$ for $\mathcal{E}^*$. This defines local coordinates $x_1^{[w_1]}, \ldots, x_n^{[w_n]}, s^{1,[-v_1]}, \ldots, s^{m,[-v_m]}$ for $\nu_{\mathcal{W}}(\mathcal{E}, N)$.

There is a well defined submersion $\pi : \nu_{\mathcal{W}}(\mathcal{E}, N) \to \nu_{\mathcal{W}}(M, N)$ that projects to the values in the coordinates $x_1^{[w_1]}, \ldots, x_n^{[w_n]}$. Given x in our local coordinate neighbourhood of $\nu_{\mathcal{W}}(\mathcal{E}, N)$, $s^{1,[-v_1]}(x), \ldots, s^{m,[-v_m]}(x)$ are the basis of a m-dimensional vector space. Dualizing this, endows the fibre $\pi^{-1}(x)$ with the structure of an m-dimensional vector space as well. Dualizing the fibre coordinates $s^{i,[-v_i]}$ using our vector space structures gives local "sections" over our submersion $s_1^{[-v_1]}, \ldots, s_m^{[-v_m]}$. These clearly determine a local trivialization of the bundle which is linear on the fibres. Hence we have a vector bundle.

Now from the fact that $\mathrm{gr}(C^\infty(\mathcal{E})) = C^\infty_{\mathrm{pol}}(\nu_{\mathcal{W}}(\mathcal{E}, N))$, we obtain by restriction and compatibility of filtrations the same identification for sections of the dual bundle. Dualizing, which again is compatible with filtrations, we obtain,

$$\mathrm{gr}(\Gamma(\mathcal{E})) = \Gamma_{\mathrm{pol}}(\nu_{\mathcal{W}}(\mathcal{E}, N)).$$

Here by polynomial section, we mean sections whose coefficients lie in $C^\infty_{\mathrm{pol}}(\nu_{\mathcal{W}}(\mathcal{E}, N))$ when expanded locally in terms of the frame $s_1^{[-v_1]}, \ldots, s_m^{[-v_m]}$. Because of compatibility, the space of graded differential operators acts naturally on graded sections and under the isomorphism above we get an identification of graded operators with certain operators on the normal bundle that send polynomial sections to polynomial sections. □

Proposition 5.6. *Given a linearly weighted vector bundle $\mathcal{E}$ over M weighted along N the weighted deformation space $\delta_{\mathcal{W}}(\mathcal{E}, N)$ is a vector bundle over $\delta_{\mathcal{W}}(M, N)$.*

Proof. The submersion $\pi : \delta_{\mathcal{W}}(\mathcal{E}, N) \to \mathbb{R}$ determines a foliation of the weighted deformation space into constant rank vector bundles by Proposition 5.5. To show this is a vector bundle, we only need to show that the bundle is locally trivial transverse to the leaves of our foliation. Given a local trivialization of $\mathcal{E}$ over $U \subset M$, we obtain a weighted frame $s_1, \ldots, s_n$ dual to a frame of weighted coordinates $s^1, \ldots, s^n$. These extend to smooth varying coordinates on the fibres over $U \times \mathbb{R}$, $\widetilde{s}^{1,[v_1]}, \ldots, \widetilde{s}^{n,[v_n]}$ that form a local frame when restricted to each leaf of the foliation. Dualizing this frame on every fibre, we obtain a smoothly varying local frame $\widetilde{s}_1^{[v_1]}, \ldots, \widetilde{s}_n^{[v_n]}$ on the leaves of the foliation. But then these varying frames determine a local linear trivialization of the deformation space over $U \times \mathbb{R}$. Hence the deformation space $\delta_{\mathcal{W}}(\mathcal{E}, N)$ is a vector bundle over $\delta_{\mathcal{W}}(M, N)$. □

Note that if $\mathcal{E}$ is linearly weighted, then the zoom action κ_u acts by automorphisms on $\delta_{\mathcal{W}}(\mathcal{E})$. Furthermore, the action is compatible with the zoom action on the base in the sense that $\kappa_u^M \circ \pi = \pi \circ \kappa_u^{\mathcal{E}}$, where π is projection to the base. Thus the zoom action makes $\delta_{\mathcal{W}}(\mathcal{E})$ an $\mathbb{R}^*$-*equivariant bundle.* Given such a G-equivariant bundle, we always obtain a G-action on sections. In this context, we are interested in the pullback of the action which takes the form,

$$(\kappa_u^* s)(x) = \kappa_{1/u}^{\mathcal{E}}(s(\kappa_u^M(x))) \tag{5.4}$$

for $s \in \Gamma(\delta_{\mathcal{W}}(\mathcal{E}))$ and $x \in M$. In turn, we obtain a pullback action on differential operators in the following way,

$$(\kappa_u^* D)(s) = \kappa_u^* D(\kappa_{1/u}^* s) \tag{5.5}$$

for $s \in \Gamma(\delta_{\mathcal{W}}(\mathcal{E}))$ and $D \in \mathscr{D}(\delta_{\mathcal{W}}(\mathcal{E}))$.

The rescaled sections employed in Proposition 5.6 now have a familiar characterization in terms of our zoom action. We can define for $s \in \Gamma(\mathcal{E})_{(i)}$ a corresponding rescaled section on $\delta_{\mathcal{W}}(\mathcal{E}, N)$, $\widetilde{s}^{[i]}$, as the unique smooth section which is homogeneous of degree i under the $\mathbb{R}^*$ action of κ_u^* and which restricts to s on the $\lambda = 1$ fibre of the space's submersion over $\mathbb{R}$. The compatibility of rescaling operations immediately tell us that given a local frame for $\Gamma(\mathcal{E})$, the rescaled dual frame is the same as the dual of the rescaled frame. Similarly, for $D \in \mathscr{D}(\mathcal{E})$, we uniquely obtain a smooth differential operator $\widetilde{D}^{[i]}$ on the deformation space which is homogeneous of degree i under κ_u^* and which agrees with D on $\lambda = 1$. We now have the following,

Proposition 5.7. *Let $\mathcal{E}$ be a linearly weighted vector bundle over M weighted along N. For $s \in \Gamma(\mathcal{E})_{(i)}$ and $D \in \mathscr{D}(\mathcal{E})_{(j)}$, the restriction to the $\lambda = 0$ fibre of the section $\widetilde{s}^{[i]}$ and operator $\widetilde{D}^{[j]}$ on the deformation space will be the symbols $\sigma_i(s)$ and $\sigma_j(D)$ in the associated graded spaces as given by Proposition 5.5.*

Proof. Note that by compatibility of our filtrations and our scaling, $\widetilde{s}^{[i]}$ will have as its coefficients in local coordinates rescaled functions $\widetilde{f}^{[k]}$. Thus, on the zero section, $\widetilde{s}^{[i]}$ will lie in the polynomial sections of $\nu_{\mathcal{W}}(\mathcal{E}, N)$. Hence by Proposition 5.5, $\widetilde{s}^{[i]}$ is given by an element of the associated graded algebra relative to the section space filtration. We additionally see by compatibility that an operator $\widetilde{D}^{[j]}$ acts on these graded sections. Since our section and operator are homogeneous of degree i and j respectively under our scaling maps κ_u, we see that we may consider,

$$(\star) \qquad \widetilde{s}^{[i]}|_{\pi^{-1}(0)} \in \Gamma(\mathcal{E})_{(i)}/\Gamma(\mathcal{E})_{(i+1)} \quad \text{and} \quad \widetilde{D}^{[j]}|_{\pi^{-1}(0)} \in \mathscr{D}(\mathcal{E})_{(j)}/\mathscr{D}(\mathcal{E})_{(j+1)}.$$

Consider r, a coordinate in a weighted frame for $\Gamma(\mathcal{E}^*)$ of weight $-i$. Thinking of r as an element of $C^\infty(\mathcal{E})_{(i)}$ determines a dual section $\widetilde{r}^{[-i]}$ of the deformation space. We know, by considerations from the usual weighted normal bundle theory, that $\widetilde{r}^{[-i]}|_{\pi^{-1}(0)}$ considered as an element of $\mathrm{gr}(C^\infty(\mathcal{E}))$ is the quotient of r by $C^\infty(\mathcal{E})_{(-i+1)}$. Since the filtrations of $C^\infty((E))$ and $\Gamma(\mathcal{E}^*)$ are compatible, we conclude that $\widetilde{r}^{[-i]}|_{\pi^{-1}(0)}$ is the $-i$th symbol of r relative to our $C^\infty(M)$-module filtration.

Relative to a weighted coordinate frame $r_1, \ldots, r_n$ of weights $-w_1, \ldots, -w_n$ we obtain a dual frame $s_1, \ldots, s_n$ of local sections of $\mathcal{E}$ with filtration degrees $w_1, \ldots, w_n$. We then obtain our rescaled sections $s_i^{[w_i]}$. Note the compatible scaling actions κ_u on sections and dual sections determine $s_i^{[w_i]}$ and $r_i^{[-w_i]}$ on the fibre $\pi^{-1}(u)$ given their value on $\pi^{-1}(1)$. Since $\{r_i\}, \{s_i\}$ are dual local frames, we conclude $\{r_i^{[-w_i]}, s_i^{[w_i]}\}$ are dual local frames on all fibres $\pi^{-1}(u)$ where $u \neq 0$. Since our sections are smooth, we conclude that the duality extends to 0. Thus, since $\widetilde{r_i}^{[-w_]}|_{\pi^{-1}(0)}$ is the symbol of r and the symbol maps are compatible with the duality, $\widetilde{s}^{[w_i]}|_{\pi^{-1}(0)}$ must be the symbol of s with our identification of $(\star)$. Since the symbol map is linear and compatible with the $C^\infty(M)$-module structure, this identification extends from the basis $\{s_i\}$ to all local sections.

Consider $D \in \mathscr{D}(\mathcal{E})_{(j)}$. By compatibility of filtrations, we should have for any section $s \in \Gamma(\mathcal{E})_{(i)}$,

$$\widetilde{Ds}^{[i+j]} = \widetilde{D}^{[j]}(\widetilde{s}^{[i]}).$$

In particular, under the identifications of $(\star)$,

$$(\diamond) \qquad \widetilde{D}^{[j]}_{\pi^{-1}(0)}(\sigma_i(s)) = \widetilde{Ds}^{[i+j]}_{\pi^{-1}(0)}$$

for all $i \in \mathbb{Z}$ and $s \in \Gamma(\mathcal{E})_{(i)}$. Note that,

$$\Gamma_{\mathrm{pol}}(\nu_{\mathcal{W}}(\mathcal{E}, N)) = \bigoplus_{i \in \mathbb{Z}} \sigma_i(\Gamma(\mathcal{E})_{(i)}).$$

In particular, by linearity, $\widetilde{D}^{[j]}_{\pi^{-1}(0)}$ is totally and uniquely determined by how it acts on the images of the symbol maps, i.e. by the relation $(\diamond)$. But since our filtrations are by definition compatible one has,

$$\sigma_j(D) \cdot \sigma_i(s) = \sigma_{i+j}(Ds)$$

for all $s \in \Gamma(\mathcal{E})_{(i)}$. Thus $\sigma_j(D)$ always satisfies the role of $\widetilde{D}^{[j]}_{\pi^{-1}(0)}$ in $(\diamond)$, from which we conclude,

$$\sigma_j(D) = \widetilde{D}^{[j]}_{\pi^{-1}(0)}$$

for all $j \in \mathbb{Z}$ and $D \in \mathscr{D}(\mathcal{E})_{(j)}$. □

6. Weightings in the Index Theorem

We would like to apply our theory of weightings to the Getzler rescaling technique and the apparatus of Higson and Yi. More precisely, we are looking for a vector bundle weighting on $S \boxtimes S^*$ for which the rescaled spinor bundle $\mathbb{S}$ of Higson and Yi is the corresponding weighted deformation space.

6.1. The Getzler Weighting.

Consider our usual setting of a bundle $\mathcal{E} \to M$ with a Clifford connection. Recall our Getzler filtration on the algebra of differential operators $\mathscr{D}(\mathcal{E})$ that assigns order one to ∇_X and $c(X)$ for $X \in \Gamma(TM)$ and order zero to elements of $\mathrm{End}_{\mathrm{Cl}}(\mathcal{E})$. We will reverse the filtration so that an order i operator has filtration degree $-i$; this will mean that our filtration, like the others we consider, is decreasing and fits with our usual notion of an order i differential operator lowering a function's order of vanishing by at least i. This filtration extends indefinitely only in the negative direction,

$$\cdots \supset \mathscr{D}(\mathcal{E})_{(-i)} \supset \cdots \supset \mathscr{D}(\mathcal{E})_{(-1)} \supset \mathscr{D}(\mathcal{E})_{(0)} = \mathrm{End}_{\mathrm{Cl}}(\mathcal{E}).$$

We are also have a sensible decreasing algebra filtration on $\Gamma(\mathbb{C}\mathrm{l}(TM))$,

$$\Gamma(\mathbb{C}\mathrm{l}(TM)) = \Gamma(\mathbb{C}\mathrm{l}(TM))_{(-n)} \supset \Gamma(\mathbb{C}\mathrm{l}(TM))_{(1-n)} \supset \cdots \supset \Gamma(\mathbb{C}\mathrm{l}(TM))_{(0)}$$

where we set,

$$s \in \Gamma(\mathbb{C}\mathrm{l}(TM))_{(i)} \iff s \in \Gamma(\mathbb{C}\mathrm{l}_{-i}(TM)).$$

This defines a weighted vector bundle if we give M the zero weighting (i.e. M is given the trivial weighting along itself where $C^\infty(M)_{(1)} = \{0\}$); but, this structure is not really significant. Note given the bundle $\mathcal{E}$, locally over small enough $U \subset M$ we may trivialize sections:

$$\Gamma(\mathrm{End}(\mathcal{E})|_U, U) \cong \Gamma(\mathbb{C}\mathrm{l}(TU) \otimes \mathrm{End}_{\mathrm{Cl}}(\mathcal{E}|_U)) \cong \Gamma(\mathbb{C}\mathrm{l}(TU)) \otimes \Gamma(\mathrm{End}_{\mathrm{Cl}}(\mathcal{E}|_U)).$$

We can define a filtration of these trivialized sections as tensor product filtration given by the filtration of $\Gamma(\mathbb{C}\mathrm{l}(TU))$ defined above and the 0 filtration on our second factor. Patching these together, we get a filtration of the total sheaf $\Gamma(\mathrm{End}(\mathcal{E}))$,

$$\Gamma(\mathrm{End}(\mathcal{E}))_{(-n)} \supset \cdots \supset \Gamma(\mathrm{End}(\mathcal{E}))_{(0)}.$$

Note that $\mathscr{D}(\mathcal{E})$ acts on $\Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$ by operating on the first factor of $\mathcal{E} \boxtimes \mathcal{E}^*$. Also, from the inclusion $\iota : M_\Delta \to M^2$, we obtain a pullback on sections of $\mathcal{E} \boxtimes \mathcal{E}^*$ to sections of $\mathrm{End}(\mathcal{E})$. Now we are ready for our filtration of $\Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$.

Definition 6.1. Given the filtrations on $\mathscr{D}(\mathcal{E})$ and $\Gamma(\mathrm{End}(\mathcal{E}))$, we get a filtration of $\Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$ so that σ defined by the diagram,

$$\begin{array}{ccc} \mathscr{D}(\mathcal{E}) \times \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*) & \longrightarrow & \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*) \\ & \overset{\sigma}{\searrow} & \downarrow \iota^* \\ & & \Gamma(\mathrm{End}(\mathcal{E}), M_\Delta) \end{array} \tag{6.1}$$

is a filtered morphism. We say that a section in $\Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)_{(i)}$ has *scaling order* i. This is the same as the notion of scaling order in [2, Definition 3.6].

A simple but important consequence of this as proved in [2, Lemmas 3.9 and 3.10] is the following.

Proposition 6.2. *Let $C^\infty(M^2)$ be filtered by the trivial weighting of M^2 along M_Δ and let $\mathscr{D}(\mathcal{E})$ and $\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)$ be given the Getzler order and scaling order filtrations respectively. Then, the module operations of function multiplication and differential action in the first factor,*

$$C^\infty(M^2)\times\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)\to\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)$$
$$\mathscr{D}(\mathcal{E})\times\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)\to\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)$$

are both filtered morphisms.

Additionally, [8, Proposition 4.3.1] proves an equivalent description of this filtration in local coordinates. If $\{e_1,\dots,e_n\}$ is a local orthonormal frame of TM, it determines a local orthonormal frame $\{e_I\}$ of $\mathbb{C}\mathrm{l}(TM)$. This becomes a local frame $\{e_I\otimes f_j\}$ of $\mathrm{End}(\mathcal{E})$, extending by any local frame $\{f_j\}$ of the Clifford-commuting portion of the bundle. We can identify a neighbourhood of M_Δ with a neighbourhood of M by projecting to the second factor of M^2 and we can extend to a neighbourhood of M^2 be extending along the first factor in normal coordinates at each point of the diagonal. We can thus uniquely extend $\{e_I\otimes f_j\}$ to a local frame of $\mathcal{E}\boxtimes\mathcal{E}^*$ in a neighbourhood of $M_\Delta\subset M^2$ by parallel transporting along radial geodesics in the first factor at every point of the diagonal. We then have that locally a section,

$$\sum_{I,j}\eta_{I,j}e_I\otimes f_j \tag{6.2}$$

is in $\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)_{(i)}$ if and only if $\eta_{I,j}\in C^\infty(M^2)$ vanishes to order $|I|+i$ along the diagonal for each multi-index I and index j. Thus, our filtration is defined locally by assigning local sections $\{e_I\otimes f_j\}$ weights $-|I|$. This proves the following:

Proposition 6.3. *The scaling order filtration of $\Gamma(\mathcal{E}\boxtimes\mathcal{E}^*)$ as defined by* (6.1) *determines a linear weighting of order n on $\mathcal{E}\boxtimes\mathcal{E}^*$ over M^2 weighted trivially along M_Δ.*

We will refer to this linear weighting of $\mathcal{E}\boxtimes\mathcal{E}^*$ as the *Getzler weighting.*

6.2. **Sections as Taylor Expansions.** If we consider our local characterization of scaling order in (6.2) and Taylor expand the terms, we obtain an alternate characterization which will be useful. If we consider $x_1,\dots,x_n$ as our normal coordinates off of the diagonal of M^2, then we can consider a local Taylor expansion of a section η,

$$\eta=\sum_I x^I s_I \tag{6.3}$$

where s_I are local sections of $\mathrm{End}(\mathcal{E})$ that are uniquely extended to local sections of $\mathcal{E}\boxtimes\mathcal{E}^*$ by parallel transport along radial geodesics. We will have that η locally has scaling order i if and only if each section s_I lies in $\mathbb{C}\mathrm{l}_{|I|-i}(TM)\otimes\mathrm{End}_{\mathrm{Cl}}(\mathcal{E})$. So, we may locally identify sections $\mathcal{E}\boxtimes\mathcal{E}^*$ with sections of $\mathbb{C}[[TM]]\otimes\mathrm{End}(\mathcal{E})$; taking the tensor product filtration of the usual filtration of $\mathbb{C}[[TM]]$ and the filtration we have defined of $\mathrm{End}(\mathcal{E})$, we recover our scaling order filtration.

Given this filtered algebra, we should have an associated graded algebra. It will locally be the tensor product of $\mathbb{C}[[TM]]$ and $\mathrm{End}_{\mathrm{Cl}}(\mathcal{E})$, which are already graded, with the associated graded algebra $\bigwedge_{\mathbb{C}}TM$ of $\mathbb{C}\mathrm{l}(TM)$. So, we have a symbol map,

$$\sigma_\bullet:\Gamma(\mathcal{E}\otimes\mathcal{E})\to\Gamma(\mathbb{C}[[TM]]\otimes\mathrm{gr}(\mathrm{End}(\mathcal{E})),TM). \tag{6.4}$$

The differential operators also have a filtration and a symbol map to the associated graded space. This associated graded space will be the one naturally acting on the space of sections obtained above. We obtain a symbol map,

$$\sigma_\bullet : \mathscr{D}(\mathcal{E}) \to \Gamma(\mathscr{P}(TM) \otimes \mathrm{gr}(\mathrm{End}(\mathcal{E})), TM) \tag{6.5}$$

where we remember $\mathscr{P}(TM)$ as the space of polynomial coefficient differential operators, graded in the obvious way. There is an action,

$$\Gamma(\mathscr{P}(TM)\otimes\mathrm{gr}(\mathrm{End}(\mathcal{E})), TM)\times\Gamma(\mathbb{C}[[TM]]\otimes\mathrm{gr}(\mathrm{End}(\mathcal{E})), TM) \to \Gamma(\mathbb{C}[[TM]]\otimes\mathrm{gr}(\mathrm{End}(\mathcal{E})), TM)$$

where we let $\mathscr{P}(TM)$ act on power series and compose the endomorphisms. As a consequence of Proposition 6.2 this action is "homorphism-like compatible" with the symbols in the sense that $\sigma_a(D)\sigma_b(s) = \sigma_{a+b}(Ds)$.

Consider a section η as in equation (6.3) which has scaling order p. In analogy with the Taylor expansion from multivariable calculus, we can explicitly determine the value of the sections s_I at $m \in M_\Delta$ as,

$$s_I(m) = \frac{1}{I!}\nabla^I(s)(m,m)$$

where we define multi-index factorials and derivatives in the usual way and let $\nabla^a = \nabla_{\partial_a}$. If we are interested in evaluating the image of s under the symbol map, i.e. quotienting it by sections of scaling order $p-1$, we should select for the component of Clifford degree $|I|-p$ in each Taylor series term s_I. This can be done by taking the $(|I|-p)$-th symbol of the Taylor term. Summing over these top degree parts of the Taylor series terms, we obtain our symbol in coordinates,

$$\sigma_p(s)(X_m) = \sum_{i=0}^{\infty} \frac{1}{i!}\sigma_{i-p}(\nabla^i_{X_m}(s)(m,m)) \tag{6.6}$$

where σ on the right side is our Clifford symbol. In light of Proposition 5.7 we upgrade $s \in \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$ to a smooth section,

$$\widetilde{s}^{[p]} \in \Gamma(\delta(\mathcal{E} \boxtimes \mathcal{E}^*, M_\Delta)) \quad \text{where} \quad \widetilde{s}^{[p]}(\gamma) = \begin{cases} \lambda^{-p}s(x,y) & \gamma = (x,y,\lambda) \in M^2 \times \mathbb{R}^* \\ \sigma_p(s)(X_m) & \gamma = (X_m, 0) \in TM \times \{0\} \end{cases}$$

which will be homogeneous of degree p under the induced $\mathbb{R}^*$ scaling map κ on sections.

6.3. **Scaling and Symbols.** We now do some work to build up the framework of the actual Getzler rescaling by comparing coordinate expressions of rescaled functions with those in terms of our zoom action κ. Consider a general weighted manifold M and a linearly weighted bundle $\mathcal{E} \to M$. Locally, we have weighted coordinates $x_1, \ldots, x_n$ of weights $w_1, \ldots, w_n$ for M. This determines extensions to coordinates $z_1, \ldots, z_n, \lambda$ for $\delta_{\mathcal{W}}(M, N)$ where z_i agrees with z_i at $\lambda = 1$ and is homogeneous of degree i under κ. Given $f \in C^\infty(M)_{(i)}$, we have our usual homogeneous extension to the deformation space which we now may write as,

$$\widetilde{f}^{[i]}(z_1, \ldots, z_n, \lambda) = \begin{cases} \lambda^{-i} f(\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n) & \lambda \neq 0 \\ f^{[i]}(z_1, \ldots, z_n) & \lambda = 0. \end{cases}$$

This serves as a sort of proof of the smooth structure of our deformation space, as these functions are clearly smooth in coordinates. Equivalently, given a function $f \in C^\infty(M)_{(i)}$, which we may consider as the restriction to the $\lambda = 1$ fibre of a smooth function, we may describe its homogeneous extension away from $\lambda = 0$ as,

$$\widetilde{f}^{[i]}|_{\pi^{-1}(\lambda)} = \lambda^i \kappa_{1/\lambda}^* f.$$

Now, consider our vector bundle and pick a weighted local frame of $\mathcal{E}$, $\{\omega_i\}$, with weights $-v_i$. These extend to a local frame $\widetilde{\omega_i}$ on the deformation space, again homogeneous under κ. We have for $s \in \Gamma(\mathcal{E})_{(i)}$ with $s = \sum f_j \omega_j$ and $f_j \in C^\infty(M)_{(w_j)}$,

$$\widetilde{s}^{[i]}(z_1, \ldots, z_n, \lambda) = \begin{cases} \lambda^{-i} \sum \lambda^{-v_j} f(\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n) \omega_j & \lambda \neq 0 \\ s^{[i]}(z_1, \ldots, z_n) & \lambda = 0. \end{cases} \tag{6.7}$$

Analogously to above we can describe this extension away from $\lambda = 0$ in terms of the pullback action of κ on sections in (5.4). Thinking of $s \in \Gamma(\mathcal{E})_{(i)}$ as the $\lambda = 1$ fibre of a deformation space section,

$$\widetilde{s}^{[i]}|_{\pi^{-1}(\lambda)} = \lambda^i \kappa_{1/\lambda}^* s.$$

Since κ and our sections are smooth and by Proposition 5.7,

$$\lim_{\lambda \to 0} \lambda^i \kappa_{1/\lambda}^* s = \sigma_i(s). \tag{6.8}$$

Now, our natural next step is to apply this to differential operators. Based on the compatibility given in (5.3), we know that for $D \in \mathscr{D}(\mathcal{E})_{(-q)}$ and $\eta \in \Gamma(\mathcal{E})_{(p)}$,

$$\widetilde{(D(\eta))}^{[p-q]} = \widetilde{D}(\widetilde{\eta}^{[p]})^{[-q]}.$$

In terms of κ, extended to operators by (5.5), if $D \in \mathscr{D}(\mathcal{E})_{(-q)}$ is considered as an operator on the deformation space restricted to $\lambda = 1$,

$$\widetilde{D}^{[-q]}|_{\pi^{-1}(\lambda)} = \lambda^q \kappa_{1/\lambda}^* D$$

away from $\lambda = 0$. Again using smoothness and Proposition 5.7,

$$\lim_{\lambda \to 0} \lambda^q \kappa_{1/\lambda}^* D = \sigma_{-q}(D). \tag{6.9}$$

6.4. **The Getzler Symbol.** We now transition back to the specific case of our Getzler weighting. We already have an explicit formulation of the symbols of our sections as given in (6.6). The results of the last subsection allow us to do the same for operators. For example, consider the differential operator of Clifford multiplication $c(X)$. We know, that our space of endomorphisms is locally in isomorphism with $\bigwedge TM \otimes \mathrm{End}_{\mathrm{Cl}}(\mathcal{E})$ on which the Clifford action is $c(X) = \varepsilon(X) - \iota(X)$, i.e. exterior multiplication by X minus interior multiplication by its dual. We know that exterior multiplication raises the graded degree by one, while interior multiplication lowers it. Hence, under the local isomorphism to $\bigwedge TM \otimes \mathrm{End}_{\mathrm{Cl}}(\mathcal{E})$,

$$\begin{aligned} \lambda \kappa_{1/\lambda}^* c(X) &= \lambda \kappa_{1/\lambda}^* \circ c(X) \circ \kappa_\lambda^* \\ &= \lambda(\lambda^{-1} \varepsilon(X) - \lambda \iota(X)) \end{aligned}$$

Taking λ to zero, we conclude from (6.9) that $\sigma_{-1}(c(X)) = \varepsilon(X)$. We could similarly derive, using [1, Lemma 4.15], that,

$$\sigma_{-1}(\nabla_{e_i}) = \frac{\partial}{\partial e_i} + \frac{1}{2} \varepsilon(\gamma \circ R(\cdot, e_i))$$

where γ is the isomorphism,

$$\gamma : \mathfrak{so}(TM) \xrightarrow{\cong} \bigwedge^2 TM \quad \text{such that in a basis } \{e_i\}, \quad \gamma(A) = \frac{1}{4}\sum_i A(e_i) \wedge e_i$$

and R is the curvature endomorphism as a skew symmetric matrix valued two-form. Since these two forms of operators along with Clifford-commuting endomorphisms (which are preserved by the symbol) generate $\mathscr{D}(\mathcal{E})$, this is enough to determine the symbol map on any differential operator. It is already presented in [1, Proposition 4.20], but we show how we can deduce from this the symbol of our generalized Laplacian D^2.

First we use the *Lichnerowicz formula*, proved in [1, Theorem 3.52], to write D^2 in terms of the Bochner Laplacian, the twisted curvature F given as an element of $\mathrm{Cl}^2(TM)$, and the scalar curvature,

$$D^2 = \Delta^{\mathcal{E}} + \sigma_{\mathrm{Cl}}^{-1}(F) + \frac{\kappa}{4}$$

In a local orthonormal $\{e_1, \ldots, e_n\}$,

$$= -\sum_i (\nabla^{\mathcal{E}}_{e_i} \nabla^{\mathcal{E}}_{e_i} - \nabla^{\mathcal{E}}_{\nabla^{LC}_{e_i} e_i}) + \sum_{i<j} F(e_i, e_j) c(e_i) c(e_j) + \frac{\kappa}{4}.$$

Now we take the symbol by only keeping the terms of Getzler order two and taking their symbol,

$$(\dagger) \qquad \sigma_2(D^2) = -\sum_i \sigma_1(\nabla^{\mathcal{E}}_{e_i}) \sigma_1(\nabla^{\mathcal{E}}_{e_i}) + \sum_{i<j} F(e_i, e_j) \sigma_1(c(e_i)) \sigma_1(c(e_j)).$$

Now we look at the connection's symbol in coordinates. For $\omega \in \Gamma(\bigwedge T_q M,) \otimes \mathrm{End}_{\mathrm{Cl}}(TM)$ and η a tangent vector we know,

$$\begin{aligned} \sigma_1(\nabla^{\mathcal{E}}_{e_i})\omega(\eta) &= \frac{\partial}{\partial e_i} + \frac{1}{2}\gamma \circ R(\eta, e_i) \wedge \omega \\ &= \frac{\partial}{\partial e_i}\omega(\eta) + \frac{1}{8}\sum_\ell R(\eta, e_i)e_\ell \wedge e_\ell \wedge \omega(\eta) \\ &= \frac{\partial}{\partial e_i}\omega(\eta) + \frac{1}{8}\sum_{\ell k} \langle R(\eta, e_i)e_\ell, e_\ell \rangle e_k \wedge e_\ell \wedge \omega(\eta). \end{aligned}$$

Writing $\eta = \sum_j x^j e_j$ for $x^j = \langle \eta, e_j l \rangle$ and using the Riemannian curvature 4-tensor,

$$= \frac{\partial}{\partial e_i}\omega(\eta) + \frac{1}{8}\sum_{j\ell k} R_{ji\ell k} x^j e_k \wedge e_\ell \wedge \omega(\eta).$$

By skew symmetry,

$$= \left(\frac{\partial}{\partial e_i} + \frac{1}{4}\sum_{j,k<\ell} R_{ijk\ell} x^j \varepsilon^k \varepsilon^\ell \right) \omega(\eta).$$

Applying this result to $(\dagger)$ gives,

$$\sigma_2(D^2) = -\sum_i \left(\frac{\partial}{\partial e_i} + \frac{1}{4}\sum_{j,k<\ell} R_{ijk\ell} x^j \varepsilon^k \varepsilon^\ell \right)^2 + \sum_{i<j} F(e_i, e_j) \varepsilon^i \varepsilon^j$$

which is precisely the same as the formula from (3.6).

Remark 6.4. In the above computation we derive the symbol from the equality,

$$\sigma_{-2}(D) = \lim_{\lambda\to 0} \lambda^2 \kappa^*_{1/\lambda} D^2$$
$$= \lim_{\lambda\to 0} \lambda^2 \kappa^*_{1/\lambda} \circ D^2 \circ \kappa^*_{\lambda}.$$

Meanwhile, in the Getzler Rescaling approach of section 3.2 we derived the symbol from,

$$\sigma_{-2}(D) = \lim_{\lambda\to 0} \lambda^2 \delta_\lambda \circ D^2 \circ \delta_{1/\lambda}.$$

We immediately, notice a striking similarity. Indeed, when acting on *time-independent* sections $s \in \Gamma(\mathcal{E})$, we have an equality of scalings

$$\delta_\lambda s = \kappa^*_{1/\lambda} s|_{\pi^{-1}(\lambda)}$$

This can be checked by the coordinate description in (6.7). The problem of dealing with time-dependent sections and scaling, as with our heat kernel, is addressed in the next subsection.

6.5. **Getzler Rescaling and Time-Dependent Sections.** Now we are interested in time-dependent sections, i.e. we consider a section smoothly dependent on a parameter $t \in \mathbb{R}^+$. More specifically, we are interested in these in the context of our generalized heat equation. Recall that associated to the heat equation $\partial_t + D^2$, we have a smoothing operator e^{-tD^2}. On the deformation space under the Getzler weighting, we can extend this to a heat equation $\partial_t + \widetilde{D^2}^{[2]}$ which has a smoothing operator $e^{-t\widetilde{D^2}^{[2]}}$. Evaluating this at time $t = 1$ gives,

$$e^{-\widetilde{D^2}^{[2]}} = e^{-\lambda^2 \kappa^*_{1/\lambda} D^2}. \tag{6.10}$$

As $\lambda \to 0$, this equality admits two interpretations of the result: either as the smoothing kernel of $\sigma_{-2}(D^2)$ evaluated at $t = 1$, or as the smoothing kernel of the rescaled D^2 at $t = \lambda^2$. If k_t is the associated smoothing kernel to e^{-tD^2} given in (3.1), the second interpretation tells us we should study the rescaled kernel $\kappa^*_{1/\lambda} k_{\lambda^2}$ whose time we let vary with the fibre over $\mathbb{R}$ in the deformation space.

Like any section, $k_1 \in \Gamma(\mathcal{E})_{(-n)}$ and there is no reason why it need have scaling order any greater than $-n$. Hence, we may obtain a rescaling of the time-dependent section,

$$\widetilde{k_{\lambda^2}}^{[-n]} = \lambda^n \kappa^*_{1/\lambda} k_{\lambda^2}.$$

We conclude from (6.10) and (6.8) that,

$$\widetilde{k_{\lambda^2}}^{[-n]}|_{\pi^{-1}(0)} = \lim_{\lambda\to 0} \lambda^n \kappa^*_{1/\lambda} k_{\lambda^2}$$

is the kernel associated to $\sigma_{-2}(D^2)$. Using, (6.7), we may describe,

$$\widetilde{k_{\lambda^2}}^{[-n]} = \lambda^n \sum_{i=0}^{n} \lambda^{-i} k(\lambda \mathbf{x}, q, \lambda^2 t)_{[i]}.$$

One should notice this is precisely the Getzler rescaling of (3.5) and $\widetilde{k_{\lambda^2}}^{[-n]}$ restricted to $\pi^{-1}(\lambda)$ is exactly equal to the rescaled kernel $r(x, 1) = \lambda^n \delta_\lambda k_1$ we gave in Section 3.2. The computations of [1, Theorem 4.1] can be used to show that with our additional scaling of t everything is still analytically well-defined. Once this is verified, the construction of our deformation space guarantees the desired property that the limit of this rescaled kernel will be the kernel associated to $e^{-\sigma_{-2}(D^2)}$. As we observed in Section 3.2, the rescaling has no effect on the top degree part of $\phi_{n/2}$ (in our new formalism this amounts to the observation that $(\phi_{n/2})_{[n]}$ has scaling order $-n$ and is independent of time in the asymptotic expansion of k_t). Thus,

$$\mathrm{Str}(k_1) = \mathrm{Str}(\widetilde{k_{\lambda^2}}^{[-n]}|_{\pi^{-1}(\lambda)}) = \mathrm{Str}(\lim_{\lambda\to 0} \lambda^n \kappa^*_{1/\lambda} k_{\lambda^2}) = \mathrm{Str}(e^{-\sigma_{-2}(D^2)}).$$

Because $\sigma_{-2}(D^2)$ is given by (3.6) which has kernel given by (3.7), we once again arrive at a complete proof of the index theorem.

7. The Symbol Calculus of Ševera

Now we move on to some study of a construction of Pavol Ševera given in an unpublished letter written to Alan Weinstein [9, Letter 6]. First we review the structure of this construction and then consider how it may be obtained as a weighted deformation space.

7.1. **Clifford Algebroids.** Ševera defines an *associative algebroid* to be a vector bundle $\mathcal{A}$ over a Lie groupoid G with smoothly varying associative multiplication maps between fibres: $\mathcal{A}_g \otimes \mathcal{A}_h \to \mathcal{A}_{gh}$ (the same concept is considered by Higson and Yi under the name of a multiplicative structure). Let A be a vector bundle with a metric and fibres of dimension n over a manifold M. Consider the $\mathrm{SO}(n)$*-frame bundle of* A, a $\mathrm{SO}(n)$-principal bundle whose points are the special orthogonal isomorphisms from $\mathbb{R}^n$ to a fibre of A, which we denote $F_{\mathrm{SO}(n)}(A)$. If this admits a lift to a $\mathrm{Spin}(n)$-principal bundle, we obtain a bundle $F_{\mathrm{Spin}(n)}(A)$. We define the *Clifford algebroid* as the associated bundle over $\mathrm{Pair}(M)$,

$$\mathscr{C}\!\ell(A) = \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times_{\mathrm{Pair}(\mathrm{Spin}(n))} \mathbb{C}\mathrm{l}(\mathbb{R}^n)$$

(note this is slightly different from Ševera who uses $\mathrm{Cl}(\mathbb{R}^n)$ instead). Here $(g_1, g_2) \in$ acts on $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A))$ by standard right action in each fibre and acts on $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ by $a \mapsto g_1 a g_2^{-1}$. This becomes an associative algebroid over the pair groupoid with the multiplication on $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n)$,

$$(p_1, p_2, a) \cdot (p_2, p_3, b) = (p_1, p_3, ab)$$

which descends to a well defined multiplication on $\mathscr{C}\!\ell(A)$ under the quotient by $\mathrm{Pair}(\mathrm{Spin}(n))$. Since the adjoint action of $\mathrm{Spin}(n)$ on $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ preserves Clifford order, one can see that the restriction of $\mathscr{C}\!\ell(A)$ to M_Δ is the bundle of Clifford algebras $\mathbb{C}\mathrm{l}(A)$. This algebroid reduces to something familiar in the case $A = TM$. In this case, we are asking that M have a spin structure and the bundle $F_{\mathrm{Spin}(n)}(TM)$ is just the $\mathrm{Spin}(n)$-frame bundle $F_{\mathrm{Spin}(n)}$. Let $\mathscr{S}$ be the spinor bundle of M which recall can be described $F_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} S$. We obtain the following equalities of associated bundles, which can be checked explicitly in terms of the relevant group actions,

$$\begin{aligned}
\mathscr{S} \boxtimes \mathscr{S}^* &= (F_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} S) \boxtimes (F_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} S) \\
&= \mathrm{Pair}(F_{\mathrm{Spin}(n)}) \times_{\mathrm{Pair}(\mathrm{Spin}(n))} S \otimes S^* \\
&= \mathrm{Pair}(F_{\mathrm{Spin}(n)}(TM)) \times_{\mathrm{Pair}(\mathrm{Spin}(n))} \mathbb{C}\mathrm{l}(\mathbb{R}^n) \\
&= \mathscr{C}\!\ell(TM).
\end{aligned}$$

We define a second algebroid over the groupoid TM which I call the *graded algebroid of* A as an associated bundle given by,

$$\mathscr{G}\!\mathscr{r}(A) = TF_{\mathrm{Spin}(n)}(A) \times_{\mathfrak{so}(n) \rtimes \mathrm{Spin}(n)} \bigwedge\nolimits_{\mathbb{C}} \mathbb{R}^n.$$

Here $\mathrm{Spin}(n)$ acts on $TF_{\mathrm{Spin}(n)}$ by its right action and $\mathfrak{so}(n)$ acts by addition of vectors. Also, $\mathrm{Spin}(n)$ acts on $\bigwedge_{\mathbb{C}}\mathbb{R}^n$ as $\mathrm{SO}(n)$ under the double cover with its usual representation on $\mathbb{R}^n$ extended to the exterior algebra and $\mathfrak{so}(n)$ acts by exterior multiplication of its exponential as a two-form. This odd action is explained in terms of weightings below. This is made a algebroid through the multiplication on $TF_{\mathrm{Spin}(n)}(A)\times\bigwedge_{\mathbb{C}}\mathbb{R}^n$, $(v_1,c_1)\cdot(v_2,c_2)=(v_1+v_2,c_1\wedge c_2)$ which again can be shown to descend to the associated bundle. In the case of $A=TM$, we again obtain something familiar. First recall we may write TM as a frame bundle,

$$TM = F_{\mathrm{Spin}(n)}\times_{\mathrm{Spin}(n)}\mathbb{R}^n.$$

We can take the corresponding exterior algebra bundle and complexify,

$$\bigwedge_{\mathbb{C}} TM = F_{\mathrm{Spin}(n)}\times_{\mathrm{Spin}(n)}\bigwedge_{C}\mathbb{R}^n.$$

Applying the tangent bundle functor to both sides,

$$\begin{aligned}\bigwedge_{\mathbb{C}} TM \to TM &= TF_{\mathrm{Spin}(n)}(TM)\times_{T\mathrm{Spin}(n)}\bigwedge_{C}\mathbb{R}^n\\ &= \mathscr{G}\!\mathscr{r}(TM).\end{aligned}$$

We now can form an algebroid over the tangent groupoid $\mathbb{T}M$ given as a smooth family of algebroids sitting over a submersion π onto $\mathbb{R}$. We define the *rescaled Clifford algebroid* ${}^{\tau}\mathscr{C}\ell(A)$ in terms of its π-fibres as,

$${}^{\tau}\mathscr{C}\ell(A)|_{\pi^{-1}(\lambda)} = \begin{cases}\mathscr{C}\ell(A)\to\mathrm{Pair}(M) & \lambda\neq 0\\ \mathscr{G}\!\mathscr{r}(A)\to TM & \lambda = 0.\end{cases}$$

This is given a smooth structure in such a way that the $\mathbb{R}^*$ group action $\varkappa_t$ given by,

$$\varkappa_t(\gamma,\lambda)=\begin{cases}(\gamma,\lambda/t^2) & \lambda\neq 0,\gamma\in\mathscr{C}\ell(A)\\ ((\widetilde{p,\lambda\cdot v}),\lambda^2X_m,0) & \lambda=0,\gamma=((\widetilde{p,v}),\lambda^2X_m)\in\mathscr{G}\!\mathscr{r}(A)\end{cases}\tag{7.1}$$

where $\lambda\cdot v$ is the scalar multiplication on $\mathbb{R}^n$ extended to $\bigwedge\mathbb{R}^n$, acts by diffeomorphisms. Note that by our above computation, ${}^{\tau}\mathscr{C}\ell(TM)$ has the same fibres over its submersion onto $\mathbb{R}$ as the rescaled spinor bundle $\mathbb{S}$, however the way these fibres fit together is different. In particular, we will see our rescaled Clifford algebroid corresponds to a differently weighted deformation space.

The purpose of the construction of Ševera is for a certain symbol calculus. We can consider a certain family of differential operators on the Clifford algebroid which we write $\mathscr{D}_1(\mathscr{C}\ell(A))$. This will refer to differential operators on sections of $\mathscr{C}\ell(A)$ which only act on the first factor, in the sense that when pulled back under the quotient by the action of $\mathrm{Pair}(\mathrm{Spin}(n))$ to an operator on $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A))\times_{\mathrm{Pair}(\mathrm{Spin}(n))}\mathrm{Cl}(\mathbb{R}^n)$, they act as the identity on the second copy of $F_{\mathrm{Spin}(n)}(A)$. In a local trivialization, centred on the diagonal and fixing a value of the second coordinate, the Clifford algebroid just looks like a trivial bundle $U\times\mathrm{Cl}(\mathbb{R}^n)$. Locally, elements of $\mathscr{D}_1(\mathscr{C}\ell(A))$ look like usual differential operators on U with coefficients in $\mathrm{Cl}(\mathbb{R}^n)$ that act by multiplication. We can define locally a filtration on $\mathscr{D}_1(\mathscr{C}\ell(A))$ as the tensor product of the usual Clifford filtration on $\mathrm{Cl}(\mathbb{R}^n)$ and the doubled standard filtration on differential operators on U, i.e. a first order derivative has filtration degree -2. Gluing together local trivializations so that the transition functions on the bundle fibres are given by elements of $\mathrm{Spin}(n)$, we get a well defined filtration on $\mathscr{D}_1(\mathscr{C}\ell(A))$ which we call the *Ševera filtration.* This filtration has the property that the associated graded algebra is graded commutative. Ševera explains that the assocoated graded algebra of operators under this filtration can be considered as $\mathscr{G}\!\mathscr{r}(A)$. Furthermore, $P\in\mathscr{D}_1(\mathscr{C}\ell(A))$ is extended to an operator on ${}^{\tau}\mathscr{C}\ell(A)$

which is equivariant under $\varkappa$ and whose restriction to the zero fibre is the symbol of P. This talk of rescaling and symbols should all sound very familiar and we discuss the details in the next subsection. Some discussion of this filtration, including a coordinate independent definition and its applications to Dirac operators and Courant algebroids is given in a paper of Ševera and Fridrich Valach [10, Section 6].

7.2. **Ševera's Construction as a Weighted Deformation Space.** Note as usual, for any manifold M, $\mathrm{Pair}(M)$ has a standard trivial weighting along M_Δ which gives TM as the (weighted) normal bundle. In reference to the symbol calculus of Ševera, we will now be interested in the $\mathrm{Pair}(M)$ but with the doubled weighting, so that a function vanishing to order i along the diagonal has filtration degree $2i$. The resulting weighted normal bundle will still be isomorphic to TM, but when we consider the product weighting, we will get something different. From now on in this section when we write $\mathrm{Pair}(M)$ for any manifold M we assume it has this doubled weighting. We can also put a weighting on the vector space $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ considered as a manifold in the standard way. We take the standard basis $e_1, \dots, e_n$ for $\mathbb{R}^n$ and by multi-indexing extend to a basis e_I for $\mathrm{Cl}(\mathbb{R}^n)$ with canonical dual basis e^I; if we give e^I the weight $|I|$ then we get globally defined weighted coordinates. This extends to a weighting on $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ by tensoring with the zero-weighting on $\mathbb{C}$. The corresponding weighted normal bundle is clearly $\bigwedge_{\mathbb{C}} \mathbb{R}^n$.

Now we can give $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n)$ the product weighting and obtain by applying the weighted normal bundle functor,

$$\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n) \xrightarrow{\nu_{\mathcal{W}}} TF_{\mathrm{Spin}(n)}(A) \times \textstyle\bigwedge_{\mathbb{C}} \mathbb{R}^n.$$

Additionally, one has the product weighting on $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathrm{Pair}(\mathrm{Spin}(n)(A)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n)$. The group action α should take this space into $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n)$. We can hope to apply the weighted normal bundle functor to this morphism and obtain the following commutative diagram,

$$\begin{array}{ccc} \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathrm{Pair}(\mathrm{Spin}(n)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n) & \xrightarrow{\alpha} & \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n) \\ \downarrow \nu_{\mathcal{W}} & & \downarrow \nu_{\mathcal{W}} \\ TF_{\mathrm{Spin}(n)}(A) \times T\mathrm{Spin}(n) \times \bigwedge_{\mathbb{C}} \mathbb{R}^n & \xrightarrow{\beta} & TF_{\mathrm{Spin}(n)}(A) \times \bigwedge_{\mathbb{C}} \mathbb{R}^n. \end{array} \tag{7.2}$$

To check that this actually commutes, we need to show that α is a weighted morphism and that the corresponding action β of $T\mathrm{Spin}(n)$ is the same as what Ševera claimed. First, we need a lemma. Recall we have vector space isomorphisms $\gamma : \mathfrak{so}(n) \to \bigwedge \mathbb{R}^n$ and $q = \sigma^{-1} : \bigwedge \mathbb{R}^n \to \mathrm{Cl}^2(\mathbb{R}^n)$; let their composition be the map λ. The following result is proved in [5, Proposition 3.2]

Lemma 7.1. *The following diagram commutes,*

$$\begin{array}{ccc} \mathrm{Spin}(n) & \hookrightarrow & \mathrm{Cl}(\mathbb{R}^n) \\ \uparrow \exp & & \uparrow \exp \\ \mathfrak{so}(n) & \xrightarrow[\lambda]{} & \mathrm{Cl}^2(\mathbb{R}^n). \end{array}$$

Here exp *on the left is the Lie group exponential under the identification $\mathfrak{so}(n) \cong \mathfrak{spin}(n)$ and* exp *on the right consists of applying the usual exponential Taylor series to* $\mathrm{Cl}^2(\mathbb{R}^n)$.

Proposition 7.2. *The group action α is a weighted morphism.*

Proof. A full proof can be found in [3] Appendix A. One applies Proposition 5.2. A weighted path $\mathbb{R} \to \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathrm{Pair}(\mathrm{Spin}(n)) \times \mathbb{C}\mathrm{l}(\mathbb{R}^n)$ will be one which is tangent to $(F_{\mathrm{Spin}(n)})_\Delta$ and $\mathrm{Spin}(n)_\Delta$ to second order and which is tangent to $\mathbb{C}\mathrm{l}^i(\mathbb{R}^n)$ to ith order. As such, we may locally near $t = 0$ write an arbitrary weighted path into our space as,

$$\gamma(t) = \left(x + t^2 x_1(t), x + t^2 x_2(t), g \exp\big(\xi t + \eta_1(t) t^2\big), g \exp\big(\xi t + \eta_2(t) t^2\big), \sum_{i=1}^{\infty} t^i c_{[i]} \right)$$

where x is a local coordinate in $F_{\mathrm{Spin}(n)}(A)$, $x_1(t), x_2(t)$ are local paths in $F_{\mathrm{Spin}(n)}(A)$, $g \in \mathrm{Spin}(n)$, $\xi \in \mathfrak{spin}(n)$, $\eta(t)$ is a path in $\mathfrak{spin}(n)$, and $c_{[i]}$ has Clifford order at most i. One then applies the action and show this plays nicely with the filtrations involved. □

Now we compute the corresponding normal bundle map β. We will do it by looking at the action on each factor, first on $\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A))$. Given the action of $(g_1, g_2) \in \mathrm{Pair}(\mathrm{Spin}(n))$ we may take $g_2 = g$, $g_1 = \exp(\xi) g$ for some $\xi \in \mathfrak{spin}(n)$. We have,

$$(p_1, p_2) \cdot (\exp(\xi) g, g) = (p_1 \cdot \exp(\xi) g, p_2 \cdot g).$$

To understand the corresponding weighted normal bundle action, we can introduce a parameter t, which we then take to zero, that represents the infinitesimal directions (with appropriate weighted degrees) that are magnified by the transition to the normal bundle. Consider $(p, p) \in \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A))$. If we locally trivialize our frame bundle of A, we can identify fibres with copies of $\mathrm{Spin}(n)$ and the right action above will just be right multiplication. Under this identification of the fibres surrounding $p \in F_{\mathrm{Spin}(n)}(A)$, we can consider $(p \exp\big(\eta t^2\big), p) \in \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A))$ and $(g \exp\big(\xi t^2\big), g) \in \mathrm{Pair}(\mathrm{Spin}(n))$ and consider the resulting term which is second order in t. Multiplying,

$$(p \exp\big(\eta t^2\big), p) \cdot (g \exp\big(\xi t^2\big), g) = (p(1 + \eta t^2 + \cdots) g(1 + \xi t^2 + \cdots), pg) = (pg(1 + (\eta + \xi) t^2 + \mathcal{O}(t^4)), pg).$$

We conclude that the resulting action of $T\mathrm{Spin}(n)$ on $TF_{\mathrm{Spin}(n)}(A)$ will be,

$$(p, \eta) \cdot (g, \xi) = (pg, \eta + \xi).$$

Here the addition of tangent vectors makes sense under the identification of the space tangent to a fibre of $F_{\mathrm{Spin}(n)}(A)$ with an affine copy of $\mathfrak{spin}(n)$.

Now we study the action on $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$. Note that $(\exp(\xi) g, g) \in \mathrm{Pair}(\mathrm{Spin}(n))$ acts on $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ first by ad_g and then by left multiplication by $\exp(\xi)$; we study the resulting weighted normal bundle actions of each. Expanding $c \in \mathbb{C}\mathrm{l}(\mathbb{R}^n)$ in terms of products of an orthonormal basis $e_1, \ldots, e_{2^n}$, it's enough to consider the adjoint action on a multi-index $e_I = e_{i_1} \cdots e_{i_k}$. We also need to recall the simple algebraic fact that $g \in \mathrm{Spin}(n)$ acts on $v \in \mathbb{R}^n$ using Clifford multiplication,

$$\tau(g) v = \mathrm{ad}_g(v) = g v g^{-1}.$$

This is a representation and $\tau(g)$ always acts as by special orthogonal transformations. In fact,

$$\tau : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

is a double cover. We now compute,

$$\begin{aligned} \mathrm{ad}_g(e_I) &= g e_{i_1} \cdots e_{i_k} g^{-1} \\ &= g e_{i_1} g^{-1} g e_{i_2} g^{-1} \cdots g e_{i_k} g^{-1}. \end{aligned}$$

Here we are considering g as lying in the diagonal of $\mathrm{Pair}(\mathrm{Spin}(n))$ and so it will be unaffected by our normal bundle functor. On the other hand, $\mathbb{C}\mathrm{l}(\mathbb{R}^n)$ is filtered in the usual way and so the normal bundle functor will take it to its associated graded space via the symbol map. As such we obtain,

$$\nu_{\mathcal{W}}(\mathrm{ad}_g)(e_I) = \sigma \circ \mathrm{ad}_g \circ \sigma^{-1}(e_I).$$

Because $\{e_i\}$ is orthonormal, e_I as a k-fold Clifford product is sent to e_I as a k-form by the symbol map,

$$= \sigma(g e_{i_1} g^{-1} g e_{i_2} g^{-1} \cdots e_{i_k} g^{-1}).$$

As stated above, $\tau(g) = \mathrm{ad}_g \in \mathrm{SO}(n)$ and so $\{g e_i g^{-1}\}_i$ are still an orthonormal basis for $\mathbb{R}^n$. Hence,

$$\begin{aligned} &= g e_{i_1} g^{-1} \wedge \cdots \wedge g e_{i_k} g^{-1} \\ &= \tau(g) \cdot e_I \end{aligned}$$

where the last line represents extending the $\mathrm{SO}(n)$ group action on $\mathbb{R}^n$ to $\bigwedge \mathbb{R}^n$ in the usual way.

Second, we consider the action of left multiplication by $\exp(\xi)$. In analogy with what we did earlier, we add a parameter t and expand $\exp(\xi t^2)$ to consider terms of appropriate order. We should also expand our Clifford element as a series $c = \sum t^i c_i$ for $c_i \in \mathrm{Cl}^i(\mathbb{R}^n)$. We get,

$$\exp(t^2 \xi) \sum_i t^i c_i = \sum_{ij} \frac{1}{j!} t^{i+2j} \xi^j c_i.$$

To consider the resulting normal approximation we should take $i+2j$th symbol of the i,jth term of the series to effectively quotient by higher order parts,

$$\begin{aligned} \nu_{\mathcal{W}}(\exp(\xi) \cdot c) &= \sum_{ij} \frac{1}{j!} \sigma_{i+2j}(\xi^j c_i) \\ &= \sum_{ij} \frac{1}{j!} \sigma_2(\xi)^j \wedge \sigma_i(c_i). \end{aligned}$$

Here we have first identified $\xi \in \mathfrak{spin}(n)$ with an element of $\mathrm{Cl}^2(\mathbb{R}^n)$ and then taken its symbol. One can check the composition just gives our isomorphism γ of $\mathfrak{so}(n)$ and 2-forms. We have,

$$= \exp(\gamma(\xi)) \wedge \sigma(c).$$

So we see that the tangent functor applied to left multiplication by $\exp(\xi)$ gives us exterior multiplication by the exponential of ξ as a two form. In total, we conclude our action β is given by,

$$\beta(p, x, g, \xi, \omega) = (p \cdot g, x + \xi, \exp(\gamma(\xi)) \wedge \tau(g) \cdot \omega)$$

For $p \in F_{\mathrm{Spin}(n)}(A), x \in T_p F_{\mathrm{Spin}(n)}(A), g \in \mathrm{Spin}(n), \xi \in \mathfrak{spin}(n), \omega \in \bigwedge_{\mathbb{C}} \mathbb{R}^n$. After some parsing, one concludes this is precisely what Ševera suggests.

So, we now have that the diagram in (7.2) is commutative and the construction of Ševera is compatible with our weightings. We can now define the *Ševera weighting on $\mathscr{Cl}(A)$* in the obvious way. Let $\pi : \mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{Cl}(\mathbb{R}^n) \to \mathscr{C}\ell(A)$ be the quotient by our group action. Then, let a smooth function f on the Clifford algebroid satisfy,

$$f \in C^\infty(\mathscr{C}\ell(A))_{(i)} \iff \pi^* f \in C^\infty(\mathrm{Pair}(F_{\mathrm{Spin}(n)}(A)) \times \mathbb{Cl}(\mathbb{R}^n))_{(i)}.$$

One can check compatibility with the module structure and the form in trivialized local coordinates to conclude that this defines a weighted vector bundle structure of the Clifford algebroid over $\mathrm{Pair}(M)$ with the doubled weighting along M_Δ.

Note in the case of $A = TM$, this is clearly different from our Getzler weighting as the weighting on the base $\mathrm{Pair}(M)$ is different. Additionally, this weighting was defined without a choice of connection on A which was essential to the Getzler construction. We now have the following result.

Proposition 7.3. *Given the Ševera weighting on $\mathscr{C}\ell(A)$, the corresponding weighted normal bundle is $\mathscr{G}\!r(A)$ and the weighted deformation space is ${}^{\tau}\mathscr{C}\ell(A)$.*

Proof. The fact that $\nu_{\mathcal{W}}(\mathscr{C}\ell(A)) = \mathscr{G}\!r(A)$ is immediate from the diagram (7.2) and the functoriality of the weighted normal bundle. Thus, $\delta_{\mathcal{W}}(\mathscr{C}\ell(A))$ and ${}^{\tau}\mathscr{C}\ell(A)$ share the same fibre-wise construction over their submersions onto $\mathbb{R}$. It only remains to show that the smooth structure of the deformation space is the same as the one described by Ševera for the rescaled Clifford algebroid.

Recall from (7.1) that Ševera describes the smooth structure of ${}^{\tau}\mathscr{C}\ell(A)$ in terms of a scaling action $\varkappa_\lambda$ for $\lambda \in \mathbb{R}^*$. If π is the submersion from the rescaled algebroid to $\mathbb{R}$, the action sends the π-fibre t to the fibre t/λ^2. Additionally the action does not affect the geometry of the fibres $\pi^{-1}(t)$, except at $t = 0$ where it scales along the tangent spaces of TM by λ^2 and scales the algebroid A by λ.

Unpacking definitions, one concludes that $\varkappa_\lambda$, is the smooth "zoom action" κ_λ we defined for any weighted deformation space composed with the diffeomorphism $t \mapsto \sqrt{t}$ that acts on our coordinate t given by the submersion π onto $\mathbb{R}$. In particular, $\varkappa_\lambda$ describes a smooth action of $\mathbb{R}^*$ on $\delta_{\mathcal{W}}(\mathscr{C}\ell(A))$. We conclude that $\delta_{\mathcal{W}}(\mathscr{C}\ell(A)) \cong {}^{\tau}\mathscr{C}\ell(A)$. ☐

Now from Propositions 5.5, 5.6, and 5.7, we obtain a filtration on differential operators and an extension to the deformation space which internalizes its symbol calculus. This should restrict to the desired Ševera filtration of $\mathscr{D}_1(\mathscr{C}\ell(A))$. Indeed, given our filtration of functions, Clifford multiplication by a vector field should have filtration degree -1 while a derivative in a coordinate transverse to the diagonal should have filtration degree -2, which locally generates precisely the filtration Ševera describes. We don't outline it in detail, but just as we did in the Higson-Yi picture, $D \in \mathscr{D}_1(\mathscr{C}\ell(A))_{(i)}$ is promoted to an operator on the rescaled algebroid whose zero section is the desired symbol $\sigma_i(D)$ in the associated graded (and graded commutative) algebra. This rescaling can be defined through the zoom action as Ševera claims.

8. Concluding Thoughts

We review some ideas and constructions beyond what we've done above. This suggests interesting extensions and related concepts within our framework, which we present roughly in order from least to most speculative.

8.1. **General Rescaling Techniques in Index Theory.** The index theorem admits many other generalizations to various exotic bundles, many of which can be found in the later chapters of [1]. Some of these results admit proofs which naturally generalize our Getzler rescaling argument for the original theorem. We quickly review two examples given in [1, Chapters 8, 10].

Given a compact Lie group G, consider a G-manifold M with a G-equivariant Clifford module $\mathcal{E}$. Assume we have a Dirac operator D on $\mathcal{E}$ which is G-equivariant in the obvious sense. Then G must preserve the kernel of D and we can define an *equivariant index* for $g \in G$ as,

$$\mathrm{ind}_G(g, D) = \mathrm{Str}(g, \ker(D)) = \mathrm{Tr}(g, \ker D^+) - \mathrm{Tr}(g, \ker D^+).$$

For $X \in \mathfrak{g}$ sufficiently small, one obtains an integral formula, called the *Kirillov formula*, for $\mathrm{ind}_G(\exp(-X), D)$ almost like that for the usual index of D but using equivariant versions of the characteristic classes. In [1, Section 8.3], Berline, Getzler, and Vergne give a proof of this formula by rescaling the heat kernel of a certain generalized Laplacian using a novel *equivariant Clifford filtration.* The proof follows the structure of the original Getzler heat kernel proof.

Another result is the Bismut local family index theorem which is for a smoothly varying family of Dirac operators D_M on a bundle of manifolds $M \to B$. Given a super vector bundle $\mathcal{E}$, an odd parity first order differential operator $\mathbb{A}$ acting on E valued differential forms which satisfies the Leibniz rule,

$$\mathbb{A}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbb{A}(\beta)$$

for $\alpha \in \Gamma(\bigwedge^k T^*M)$ and $\beta \in \Gamma(E \otimes \bigwedge T^*M)$ is called a *superconnection.* Consider a superconnection $\mathbb{A}$ whose action on $\Gamma(E \otimes \bigwedge T^*M)$ composed with the projection $\Gamma(E \otimes \bigwedge T^*M) \to \Gamma(E)$ is given by a Dirac operator; a particularly nice example of such a superconnection on a bundle over $M \to B$ is the Bismut superconnection. There is a certain rescaling of the Bismut superconnection $\mathbb{A}$ by a parameter t to give a family of superconnections $\mathbb{A}_t$. The local family index theorem then asserts that $\lim_{t\to 0} \mathrm{ch}(\mathbb{A}_t)$ exists and is given by an expression similar to the usual Atiyah-Singer integrand but with a twisted version of the $\hat{\mathcal{A}}$-class and integrated over the quotient M/B. Berline, Getzler, and Vergne present a proof of the result in [1, Section 10.4] which proceeds by performing a Getzler-type rescaling of the heart kernel associated to the curvature $\mathbb{A}^2$. The proof is again similar to the Getzler heat kernel proof although slightly complicated by dealing with superconnections. In [1, Section 10.7], it is also shown that in a particular principal bundle setting the local family index theorem is equivalent to the Kirillov formula.

A third application of rescaling techniques appears from a result of Bismut that gives formulae in different contexts for the supertrace of a heat kernel of the *Bismut hypoelliptic Laplacian* shifted by a group action. Bismut develops this theory using concepts from Getzler's rescaling argument [11, Section 1.5]. In his PhD thesis, Zelin Yi outlines how his Higson-Yi rescaled bundle can be modified to a certain vector bundle over a new deformation space which he calls the *relative tangent groupoid*; as with the index theorem, the geometry of this bundle naturally incorporates the results of Bismut [11, Chapter 5].

In each of these cases, and likely more, Getzler rescaling provides a useful framework for proving index type results. In each such case, we should expect that there is linearly weighted vector bundle structure whose weighted deformation space naturally incorporates the rescaling through a zoom action. In specific cases we may also expect additional compatibilities with the structures of our bundle, for example notions of a weighted equivariant bundle, a weighted family of bundles, or a weighted Lie group.

8.2. **Atiyah-Patodi-Singer Index Theorem.** Consider a vector bundle $\mathcal{E}$ on a manifold M *with boundary*. We can study the index of a Dirac operator D on this bundle which we may expect depends on the usual Atiyah-Singer integrand for D as well as some boundary consideration. The Atiyah-Patodi-Singer index theorem gives one situation in which such a result holds; we give a brief exposition following [7, Chapter 13].

One way of working with a manifold with boundary is to extend the boundary off to infinity. We define a *manifold with a cylindrical end* N to be a manifold M which is the union of a compact manifold M_0 with boundary N and $N \times [0, \infty)$ glued together along ∂M_0 and $N \times \{0\}$. Associated to a Dirac operator D, we can define a function called the *eta function* given by,

$$\eta(s) = \sum_{\lambda_j \neq 0} \operatorname{sgn}(\lambda_j) |\lambda_j|^{-s}$$

where λ_j are the eigenvalues of D. It can be shown that η, at least on manifolds of positive scalar curvature, analytically extends to a meromorphic function on $\mathbb{C}$ with a finite value for $\eta(0)$ which we call the *eta-invariant*. We obtain the following,

Theorem 8.1 (Atiyah-Patodi-Singer Index Theorem). *Let M be an even n-dimensional spin manifold with a cylindrical end N of positive scalar curvature and an associated Dirac operator D. Then,*

$$\operatorname{ind}(D) = (2\pi i)^{-n/2} \int_M \hat{\mathcal{A}}(TM) - \frac{1}{2}\eta_N(0)$$

where η_N is the eta-function of D restricted to N.

This result can be proved with techniques similar to the Getzler rescaling for the original index theorem, as done in Richard Melrose's book [6]. In fact Melrose's framing of the proof is quite similar to ours and centres around the idea of rescaled bundles and compatible filtrations, see [6, Chapter 8]. As such, it should be reasonable to fit the "APS" theorem into the framework of the Getzler weighting which we developed above. More generally one might be interested in asking about weightings in the context of b-geometry, as developed in [6, Chapter 2].

8.3. **Linear Weightings on Associated Bundles.** In the case of the Ševera Clifford algebroid we had a nice description as a weighted bundle by thinking of it as an associated bundle. We may wonder how general this kind of description is.

Recall any rank n vector bundle $\mathcal{E}$ with fibres isomorphic to E can be viewed as an associated bundle,

$$\mathcal{E} = \mathrm{F}_{\mathrm{GL}(n)}(\mathcal{E}) \times_{\mathrm{GL}(n)} E.$$

One can ask: when does a linearly weighted vector bundle $\mathcal{E}$ admit a "trivialization" of its weighting, i.e. when is the weighting induced from the product of a vector space weighting on E and a bundle weighting on $\mathrm{F}_{\mathrm{GL}(n)}(\mathcal{E})$ (or some other subbundle for a given G-structure). Here by *bundle weighting* we just mean a weighting of a frame bundle which determines a weighting on the base under pullback by the submersion to the base. More generally, one could investigate the connection between weightings on frame bundles and principal bundles.

8.4. **Deformation Quantization.** Given a manifold M, the space $C^\infty(M)[[\hbar]]$ is called a *deformation quantization* if it is has an algebra structure $*$ that for $f, g \in C^\infty(M)$ is given by,

$$f * g = fg + \hbar F_1(f,g) + \hbar^2 F_2(f,g) + \cdots$$

for F_i bi-differential operators.

One can naturally view the Clifford algebra as a certain quantization of the exterior algebra by taking,

$$vw + wv = 2\hbar\langle v, w\rangle.$$

One recovers the exterior algebra under the "classical" limit $\hbar \to 0$, which is precisely the action of the symbol map.

Analogously, one can quantize the symmetric algebra $\mathrm{Sym}(V)$ and consider the quotient of the tensor algebra by the ideal generated by,

$$vw - wv - 2\hbar\langle v, w\rangle$$

for $v, w \in V$. The resulting space will be the *Weyl algebra* $W(V)$, which is isomorphic as a vector space to $\mathrm{Sym}(V)$ and hence infinite dimensional.

We've already implicitly seen how weighted deformation spaces can encapsulate some of this quantization process in the Clifford case. We can consider the submersion over $\mathbb{R}$ in our deformation space as parameterizing our variable $\hbar$. As we go to the zero section of the deformation space, we recover the classical picture. The Weyl algebra should be similarly described, potentially with some technicalities because of the infinite dimensionality.

One may look to find a more formal relationship between weighted deformation spaces and quantization. For example, describing Poisson manifolds in terms of this weighted deformation picture.

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