Generating Functions and Singular Support

I was asked to explain p.35 from R. Casals "A microlocal introduction to Legendrian submanifolds".

Context:

Pseudo-holomorphic Curve Theory (Gromov, Floer, Eliashberg, Hofer, ...)

* Non-squeezing phenomena
* Symplectic rigidity
* Symplectic & contact dynamics
* Topology of Lagrangians
* Floer homology & homotopy
* Fukaya categories & mirror symmetry

$\longrightarrow$ All these can be accessed through generating functions and/or sheaves?

Generating Functions (+ μloc Sheaves) (Arnold, Eliashberg, Laudenbach, Sikorav, Chekanov, Viterbo, ...)

Def^n: Let M be a (compact) smooth manifold. A generating function on M is $f: M_x \times \mathbb{R}^k_\xi \to \mathbb{R}$ a smooth function so that 0 is a regular value of $\partial_\xi f$.

Let $f_x = f(x, \cdot)$. Usually assume f_x is linear or quadratic at infinity.

Associated to f is an (immersed) Lagrangian in T^*M:

$$L_f = \{ (x, d f|_x(\cdot, \xi)) \mid \xi \text{ is a crit. pt. of } f_x \}.$$

E.g. If $k=0$, L_f is just the graph of the exact 1-form df.

Although not every Lagrangian admits a generating function, many do.

Thm: (Sikorav) Given $\varphi \in \mathrm{Ham}(T^*M, \omega_{can})$, $\varphi(L_f)$ admits a generating function.

We can now do a "Legendrian lift" of this theory.

Recall the 1-jet bundle $J^1M = T^*M_{(q,p)} \times \mathbb{R}_z$ is contact w/ contact 1-form $\lambda = \vec{p} \cdot d\vec{q} + dz$.

E.g. $f: M \to \mathbb{R}$ defines a Legendrian $\{(x, df_x, f(x)) \in J^1M\}$.

More generally, a generating function f gives an (immersed) Legendrian

$$\Lambda_f = \{ (x, df|_x(\cdot, \xi), f(x,\xi)) \mid \xi \text{ is a crit pt of } f_x \}.$$

There is a front projection $\Pi: J^1M \to M \times \mathbb{R}$.

$$\Pi(\Lambda_f) = \{ (x,z) \in M \times \mathbb{R} \mid z \text{ is a crit. value of } f_x \}.$$

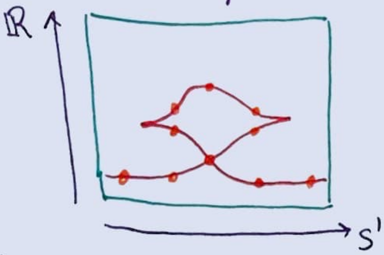
E.g. If $M = \{pt\}$, $\Pi(\Lambda_f)$ is just the set of crit. vals. of a Morse function $f: \mathbb{R}^k \to \mathbb{R}$.

We can think of what we're doing as parametrized Morse theory, aka "Cerf theory".

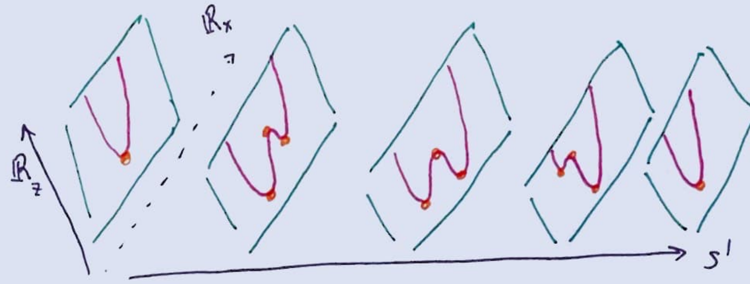
$\Pi(\Lambda_f)$ is a Cerf diagram showing caustic/wavefront where topology changes.

Introducing sheaves will allow us to do some kind of parametrized Morse homology.

E.g. $M = S^1$. Have Legendrian in $J^1 S^1$ in front projection:



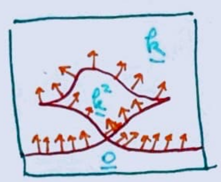
This is $\pi(\Lambda_f)$ for generating f^n $f: S^1 \times \mathbb{R}_x \to \mathbb{R}_z$ given as



$\pi(\Lambda_f)$ is "Cerf diagram" records crit. vals of S^1 family of functions $f(\theta, \cdot)$.
Because $\dim S^1 = 1$, only three local singularities: birth, death, and crossing change / handleslide.

Consider overgraph $S_f = \{(x, \xi, z) \in M \times \mathbb{R}^k \times \mathbb{R} \mid f(x, \xi) \leq z\} \xhookrightarrow{i} M \times \mathbb{R}^k \times \mathbb{R}$.
Let $k_{S_f} = i_* \underline{k}$ constant sheaf. Have $p: M \times \mathbb{R}^k \times \mathbb{R} \to M \times \mathbb{R} = \pi(J^1 M)$ projection.
Now define $\mathcal{F}_f = p_* k_{S_f}$, a sheaf on $M \times \mathbb{R}$.

E.g. Return to above example and consider $\mathcal{F}_f$



Stalks are shown. Constant in the complementary components to $\pi(\Lambda_f)$.
Singular support shown in orange, lives along $\pi(\Lambda_f)$, in upwards perpindicular direction.

Key general fact: $\boxed{\mathrm{Sing\,Supp}(\mathcal{F}_f) = \Lambda_f}$

What does this mean? Clearly (from Morse theory), the singular support lives along the front $\pi(\Lambda_f)$. Just need to determine correct codirections. Intuitively, codirection should be orthogonal to the front itself, oriented positively wrt z-axis.
Locally, away from cusps, can represent front as graph of $z(q_1, \dots, q_n)$ w/ $\{q_i\}$ loc. coords on M.
Recall contact form is $\lambda = \vec{p} \cdot d\vec{q} + dz$, so can recover full Legendrian as $\vec{p} = -\frac{\partial z}{\partial \vec{q}}$.
Since Legendrian is tangent to contact plane $\ker \lambda$, the codirection is exactly λ (projected to front).
So at a point $(q_1, \dots, q_n, z(q_1, \dots, q_n))$, the singular support is the ray spanned by $\lambda_{(q_1, \dots, q_n, z)} = (\vec{p}, 1)$ in coordinates $\{d\vec{q}, dz\}$.
I.e. singular support is exactly encoded by the Legendrian lift of $\pi(\Lambda_f)$!