Bredon Cohomology

Berkeley GRT Seminar
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Recall: To a group G, we associated its orbit category.

$\mathcal{O}_G$: Obj = Spaces of cosets G/H. $\mathrm{Mor}(G/H, G/K) \cong \mathrm{Maps}^G(G/H, G/K)$.

Elmendorf's theorem said $G\mathrm{Top}$ is in some sense equivalent to the category $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ of Top-valued presheaves on $\mathcal{O}_G$. I.e. a G-space X is the same as a "coherent" assignment of fixed point spaces $G/H \mapsto X^H$.

Today: We want to define a cohomology theory for G-spaces. This will essentially amount to performing cellular cohomology for G-CW complexes.

Note: normally cohomology has coeff. in some abelian group. But a G-space is really a coherent assemblage of spaces and so our coeffs. should be a coherent assemblage of abelian groups.

Defn: Let G be finite. A coefficient system is a presheaf $X \in \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Ab})$.

Note: For Lie groups, replace $\mathcal{O}_G$ with homotopy category $h\mathcal{O}_G$.

E.g. (i) Const. coeff. $\underline{\mathbb{Z}}$ sending $G/H \mapsto \mathbb{Z}$. Could use any abelian group here.

(ii) $\underline{\pi}_n(X): G/H \mapsto \pi_n(X^H)$

(iii) $\underline{H}_n(X): G/H \mapsto H_n(X^H; \mathbb{Z})$.

Now consider a G-CW complex X with n-skeleton X_n. We can define a chain complex of coefficient systems $\underline{C}_*(X) := \underline{H}_*(X_n, X_{n-1}; \mathbb{Z})$ (i.e. send G/H to the CW complex of X^H: $C_*^{CW}(X^H)$)

Defn: The Bredon cohomology of X for a coefficient system M is

$$H^n_G(X; M) := H^n(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Ab})}(\underline{C}_*(X), M)).$$

($\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Ab})$ is abelian cat. For each i, $\mathrm{Hom}(\underline{C}_i(X), M)$ is abelian gp and these assemble into cochain complex)

If $M \in \mathrm{Fun}(\mathcal{O}_G, \mathrm{Ab})$, can also define Bredon homology $H_n^G(X; M) := H_n(\underline{C}_*(X) \underbrace{\otimes_{\mathcal{O}_G}}_{\text{tensor here means coend}} M)$

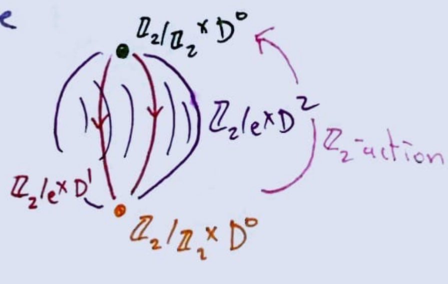
We now focus on several examples to come to grips with this definition.

Examples:

1. Let $\mathbb{Z}_2 \circlearrowright S^2$ by rotation by π. Let's compute $H^*_{\mathbb{Z}_2}(S^2; \underline{\mathbb{Z}})$.

Recall we have a CW structure

- Two 0-cells $\mathbb{Z}_2/\mathbb{Z}_2 \times D^0$
- One 1-cell $\mathbb{Z}_2/e \times D^1$
- One 2-cell $\mathbb{Z}_2/e \times D^2$



Our orbit category $\mathcal{O}_{\mathbb{Z}_2}$ is

$$\begin{tikzcd} \mathbb{Z}_2/e \arrow[loop above, "\text{swap}"] \arrow[d] \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{tikzcd}$$

The chain complex looks like:

$$\begin{tikzcd}
\mathbb{Z}_2/e : \left(\substack{\text{cells fixed}\\ \text{by } e}\right) & 0 \arrow[r] & \mathbb{Z}^2 \arrow[loop above, "\text{swap}"] \arrow[r, "d_2"] & \mathbb{Z}^2 \arrow[loop above, "\text{swap}"] \arrow[r, "d_1"] & \mathbb{Z}^2 \arrow[loop above, "\text{triv}"] \arrow[r] & 0 & & \mathbb{Z}^2 \arrow[loop above, "\text{triv}"] \\
\mathbb{Z}_2/\mathbb{Z}_2 : \left(\substack{\text{cells fixed}\\ \text{by } \mathbb{Z}_2}\right) & 0 \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \mathbb{Z}^2 \arrow[u, "\text{id}"] \arrow[r] & 0 \arrow[rr] & & \mathbb{Z} \arrow[u, "\text{id}"]
\end{tikzcd} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \underline{\mathbb{Z}}$$

From the usual cellular boundary, $d_2(x,y) = (x-y, y-x)$, $d_1(u,v) = (u+v, -u-v)$.

Note there are $\mathbb{Z}^2$ worth of maps $\begin{matrix}\mathbb{Z}^2\\ \uparrow \text{id}\\ \mathbb{Z}^2\end{matrix} \to \begin{matrix}\mathbb{Z}\\ \uparrow \text{id}\\ \mathbb{Z}\end{matrix}$ but only $\mathbb{Z}$ worth of maps equiv. under swap $\begin{matrix}\mathbb{Z}^2 \circlearrowleft \text{swap}\\ \uparrow\\ 0\end{matrix} \to \begin{matrix}\mathbb{Z}\\ \uparrow\\ \mathbb{Z}\end{matrix}$.

So cochain complex is $\quad 0 \leftarrow \mathbb{Z} \xleftarrow{\delta_2} \mathbb{Z} \xleftarrow{\delta_1} \mathbb{Z}^2 \leftarrow 0$

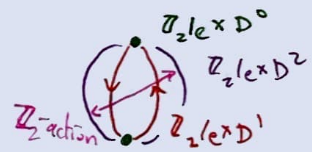
Let $\varphi(u,v) = u+v$ generate C^1. Then

$$\begin{aligned}(\delta_1(a,b))(u,v) &= (a,b)(d_1(u,v))\\ &= (a,b)(u+v, -u-v)\\ &= (a-b)u + (a-b)v = (a-b)\varphi.\end{aligned}$$

So, $\delta_1(a,b) = a-b$.

Let $\varphi(x,y) = x+y$ generate C^2. $\delta_2\varphi(x,y) = \varphi d_2(x,y) = \varphi(x-y, y-x) = 0$. So $\delta_2 = 0$.

Thus, $H^*_{\mathbb{Z}_2}(S^2; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & * = 0,2 \\ 0 & \text{else}\end{cases}$.



2. Now let $\mathbb{Z}_2 \circlearrowright S^2$ by antipodal map, use coeffs. $\underline{\mathbb{Z}}$.

Now have CW-structure with one 0-cell, 1-cell, and 2-cell $\mathbb{Z}_2/e\times D^0$, $\mathbb{Z}_2/e \times D^1$, $\mathbb{Z}_2/e\times D^2$.

Chain complex is

$$\begin{tikzcd}
\mathbb{Z}_2/e : & 0 \arrow[r] & \mathbb{Z}^2 \arrow[loop above, "\text{swap}"] \arrow[r, "d_2"] & \mathbb{Z}^2 \arrow[loop above, "\text{swap}"] \arrow[r, "d_1"] & \mathbb{Z}^2 \arrow[loop above, "\text{swap}"] \arrow[r] & 0 & & \mathbb{Z} \\
\mathbb{Z}_2/\mathbb{Z}_2 : & 0 \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[rr] & & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

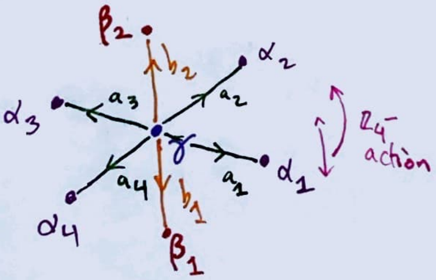
$d_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}$ and $d_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}$.

Cochain complex is $\quad 0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$

Note: We've really just rebuilt cochain complex of $\mathbb{RP}^2$. $\quad H^*_{\mathbb{Z}_2}(S^2; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \text{else}\end{cases}$.

In general, for
- free action $\quad H^*_G(X; M) \cong H^*(X/G; M(G/e))$
- triv action $\quad H^*_G(X; M) \cong H^*(X; M(e))$.

3. Now we graduate to $\mathbb{Z}_4$. Consider the graph w/ edges $\pm e_x, \pm e_y, \pm e_z$ in $\mathbb{R}^3$. And let $\mathbb{Z}_4$ act with generator $\sigma = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1\end{bmatrix}$.



We have cell structure:

0-cells	1-cells
$\mathbb{Z}_4/\mathbb{Z}_4 \times D^0$ $\quad \gamma$	$\mathbb{Z}_4/\mathbb{Z}_2 \times D^1$ $\quad \{b_1, b_2\}$
$\mathbb{Z}_4/\mathbb{Z}_2 \times D^0$ $\quad \{\beta_1, \beta_2\}$	$\mathbb{Z}_4/e \times D^1$ $\quad \{a_i\}$
$\mathbb{Z}_4/e \times D^0$ $\quad \{\alpha_i\}$	

The orbit category is:

$$\begin{array}{c} \mathbb{Z}_4/e \circlearrowleft \mathbb{Z}_4 \\ \downarrow \\ \mathbb{Z}_4/\mathbb{Z}_2 \circlearrowleft \mathbb{Z}_2 \\ \downarrow \\ \mathbb{Z}_4/\mathbb{Z}_4 \end{array}$$

Our chain complex is

$$\begin{tikzcd} \mathbb{Z}^{2\circlearrowleft \text{swap}} \oplus \mathbb{Z}^{4\circlearrowleft \text{rot.}} \arrow[r, "d"] & \mathbb{Z} \oplus \mathbb{Z}^{2\circlearrowleft \text{swap}} \oplus \mathbb{Z}^{4\circlearrowleft \text{rot.}} \\ \mathbb{Z}^{2\circlearrowleft \text{swap}} \arrow[u, hook] \arrow[r, "d'"] & \mathbb{Z} \oplus \mathbb{Z}^{2\circlearrowleft \text{swap}} \arrow[u, hook] \\ 0 \arrow[u] \arrow[r] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

We have $d = \begin{pmatrix} -1 & \cdots & -1 \\ \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & & 0 \\ 0 & & I \end{pmatrix}$ and $d' = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then consider Homs to $\underline{\mathbb{Z}}$.

The cochain complex is:

$$\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}^2, \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}^4, \mathbb{Z}) \longleftarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}^2, \mathbb{Z}) \oplus \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}^4, \mathbb{Z})$$

i.e.

$\mathbb{Z}^2 \longleftarrow \mathbb{Z}^3$ with differential $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. So, $H^*_{\mathbb{Z}_4} = \begin{cases} \mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$.

4. We now return to $\mathbb{Z}_2$ but with non-const. coeffs!

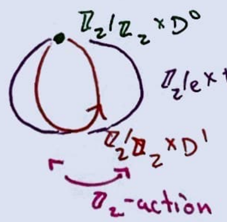
Let $\mathbb{Z}_2 \circlearrowright S^2$ by reflection across equator. Use coeff. system $M: \mathbb{Z} \longrightarrow \mathbb{Z}[i] \circlearrowleft$ complex conj.

Have CW-structure

0-cell $\quad \mathbb{Z}_2/\mathbb{Z}_2 \times D^0$

1-cell $\quad \mathbb{Z}_2/\mathbb{Z}_2 \times D^1$

2-cell $\quad \mathbb{Z}_2/e \times D^2$



Chain Complex

$$\begin{tikzcd} \mathbb{Z}^{2\circlearrowleft \text{swap}} \arrow[r, "(1\ 1)"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} & & \mathbb{Z}[i]^{\circlearrowleft \text{conj.}} \\ 0 \arrow[u] \arrow[r] & \mathbb{Z} \arrow[u, "\mathrm{id}"] \arrow[r, "0"] & \mathbb{Z} \arrow[u, "\mathrm{id}"] \arrow[r, phantom, "\longrightarrow"] & & \mathbb{Z} \arrow[u] \end{tikzcd}$$

In degree zero/one, cochains are $\mathrm{Hom}(\underline{\mathbb{Z}}, M) \simeq \mathbb{Z}$

In degree two, cochains are $\mathrm{Hom}(\mathbb{Z}^{2\circlearrowleft \text{swap}}, \mathbb{Z}[i]^{\circlearrowleft \text{conj.}}) \simeq \mathbb{Z}^2$

Complex is $\mathbb{Z}^2 \xleftarrow{\text{rk } 1} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$. So, $H^*_{\mathbb{Z}_2}(S^2; M) = \begin{cases} \mathbb{Z} & *=0,2 \\ 0 & \text{else} \end{cases}$ (c.f. $H^*_{\mathbb{Z}_2}(S^2; \underline{\mathbb{Z}}) = H^*(pt)$)

Bredon cohomology is uniquely determined by Eilenberg–Steenrod axioms.

(i) H^*_G invariant under weak G-equivalences. (ii) H^*_G satisfies usual LES of a pair.

(iii) H^*_G satisfies excision: $X = A \cup B$, $H^n_G(X/A; M) \cong H^n_G(B/A\cap B; M)$. (iv) $H^n_G(\coprod X_i; M) \cong \prod H^n_G(X_i; M)$.

(v) $H^*_G(G/H; M)$ concentrated in degree zero: $H^0_G(G/H; M) = M(G/H)$.

Notes: (i) Relaxing last axiom gives generalized G-equiv. cohomology theories, e.g. equiv. K-theory, cobordism, stable cohomotopy. These will correspond to G-spectra.

(ii) We have in some sense developed the "wrong" theory. The right one should be graded by the representation ring $RO(G)$ instead of $\mathbb{Z}$. This requires upgrading coefficient systems to Mackey functors. In this upgraded theory one can recover e.g. Poincaré duality.

(iii) We would also like a universal coefficient theorem. We can't guarantee two-step projective resolutions in $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Ab})$. So instead of SES, an injective resolution of M gives spectral sequence:

$$\mathrm{Ext}^{p,q}(\underline{C}_*(X); M) \Rightarrow H^{p+q}_G(X; M).$$

Smith Theory

We now discuss Smith theory, which shows how Bredon cohomology can be applied to understand the topology of the fixed point set X^G of a G-space X.

Thm: (Smith '41) Let G be a finite p-group and X a finite G-CW complex which is an $\mathbb{F}_p$-cohomology n-sphere: $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$. Then X^G is either empty or an $\mathbb{F}_p$-cohomology m-sphere for $m \leq n$. Moreover, for p odd, $n-m$ is even and if n is even X^G is non-empty.

Note: Our proof will follow a useful general technique of designing coefficient systems whose Bredon cohomology detects various parts of the equivariant topology, and exploiting algebraic relations between these coeff. systems. This is in line w/ Smith's original approach which predated Bredon cohom. There's also a quite different proof using Borel cohomology and equivariant localization.

Proof: Note for $H \trianglelefteq G$, $X^G \cong (X^H)^{G/H}$, so by induction it suffices to consider $G = \mathbb{Z}/p$.

We need coefficient systems L, M, N such that:

$$H^*_G(X;L) \cong \tilde{H}^*((X/X^G)/G;\mathbb{F}_p),\quad H^*_G(X;M) \cong H^*(X;\mathbb{F}_p),\quad H^*_G(X;N) \cong H^*(X^G;\mathbb{F}_p).$$

By the Eilenberg–Steenrod axioms, we can determine the coeff. systems by taking $X = G/H \times *$ a 0-cell.

We find $L = \mathbb{F}_p \leftarrow 0$, $M = \overset{\circlearrowleft}{\mathbb{F}_p}[G] \leftarrow \mathbb{F}_p$, $N = 0 \leftarrow \mathbb{F}_p$.
$$\textstyle\sum 1\cdot g \longleftarrow\!\shortmid 1$$

There is an augmentation ideal $I = \ker(\mathbb{F}_p[G] \to \mathbb{F}_p)$ and we can define coefficient systems $I^n = \overset{\circlearrowleft}{I^n} \to 0$.

We now have two SES's of coeff. systems:

$$0 \to I \to M \to L \oplus N \to 0 \quad \text{and} \quad 0 \to L \to M \to I \oplus N \to 0.$$

The first is clear. For the second, note $\mathbb{F}_p[G] \cong \mathbb{F}_p[t]/(t^p)$ and $I^n \cong (t-1)^n$.

Then $L(G) \cong \mathbb{F}_p \cong I^{p-1} = (t-1)^{p-1}/(t^p)$. And $\mathbb{F}_p[t]/(t^p,(t-1)^{p-1}) \xrightarrow{\cong} (t-1)$ by mult. by $(t-1)$.

From the SES's, we obtain LES's on homology:

$$\cdots \to H^q_G(X;I) \to H^q(X) \to \tilde{H}^q((X/X^G)/G;\mathbb{F}_p) \oplus H^q(X^G) \to H^{q+1}_G(X;I) \to \cdots$$

$$\cdots \to \tilde{H}^q((X/X^G)/G;\mathbb{F}_p) \to H^q(X) \to H^q_G(X;I) \oplus H^q(X^G) \to \tilde{H}^{q+1}((X/X^G)/G;\mathbb{F}_p) \to \cdots$$

Let $a_q = \dim \tilde{H}^q((X/X^G)/G)$, $\bar{a}_q = \dim H^q_G(X;I)$, $b_q = \dim H^q(X)$, $c_q = \dim H^q(X^G)$.

From the LES's we obtain: $a_q + c_q \leq b_q + \bar{a}_{q+1}$ and $\bar{a}_q + c_q \leq b_q + a_{q+1}$. Iterating gives

$$(\star)\quad a_q + c_q + c_{q+1} + \cdots + c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1} \quad \text{for } q, r \geq 0,\ r \text{ odd}.$$

Taking $q=0$, $r \gg 0$ gives $\sum_q c_q \leq \sum_q b_q$. i.e. we have for a general finite $G = \mathbb{Z}/p$-space:

$$\sum_i \operatorname{rk} H^i(X^G;\mathbb{F}_p) \leq \sum_i \operatorname{rk} H^i(X;\mathbb{F}_p). \quad \text{(Smith Inequality)}$$

In our case, X is a $\mathbb{Z}/p$ cohomology sphere. So, $\sum_i^{!} \operatorname{rk} H^i(X^G;\mathbb{F}_p) \leq 2$.

We need to show the sum is not 1 and the top non-zero rank is at most n.

For this, note $I^n/I^{n+1} = \frac{(t-1)^n}{(t-1)^{n+1}} \cong \mathbb{F}_p$ for $n \leq p-1$. So we have SES:

$$0 \to I^{n+1} \to I^n \to L \to 0. \quad \text{Hence } \chi(X;I^n) = \chi(X;I^{n+1}) + \chi(X;L).$$

Summing these for $n \leq p-1$ gives:

$$\sum_{i=1}^{p-1} \chi(X;I^i) = \sum_{i=1}^{p-1} \left[\chi(X;I^{i+1}) + \chi(X;L)\right] \quad \text{This telescopes.}$$

$$\Rightarrow \quad \chi(X;I) = \chi(X;I^p = 0) + (p-1)\cdot\chi(X;L)$$

Note from previous SES, $\chi(M) = \chi(I) + \chi(L) + \chi(N)$. Hence,

$$\chi(X;M) = \chi(X;N) + p\cdot\chi(X;L).$$

Or equivalently,

$$\chi(X) = \chi(X^G) + p\cdot\tilde{\chi}((X/X^G)/G).$$

In particular, $\chi(X) \equiv \chi(X^G) \pmod p$ and so $\sum^{!} \operatorname{rk} H(X^G;\mathbb{F}_p) \neq 1$.

If p is odd, we also deduce $n-m$ even and $\sum^{!} \operatorname{rk} H(X^G) = 2$ when n is even.

To show $m \leq n$, recall $(\ast)$ with $q = n+1$:

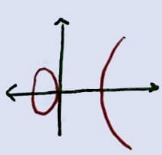
$$a_{n+1} + \sum_{n+1}^{\infty} \operatorname{rk} H^*(X^G) \leq \sum_{n+1}^{\infty} \operatorname{rk} H^*(X) = 0. \qquad \square$$

Here's a nice classical application of the Smith inequality we obtained earlier.

<u>Thm</u>: (Harnack) Let $X_{\mathbb{R}}$ be the real locus of a smooth projective algebraic curve of degree d.

Then, $\#\binom{\text{connected components}}{\text{of } X} \leq \frac{(d-1)(d-2)}{2} + 1 = \operatorname{genus}(X_{\mathbb{C}}) + 1$.

E.g. Elliptic curves are degree 3, genus 1. Real locus looks like



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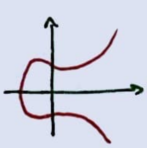
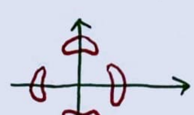


. So, # components $\leq 2 = g+1$.

Can construct quartics with 4 components



"Trott curve" / Plücker quartic
Has 28 bitangents.

<u>Proof</u>: $X_{\mathbb{R}} = X_{\mathbb{C}}^{\mathbb{Z}/2}$ under $\mathbb{Z}/2$ action by conjugation. By Smith inequality

$$\underbrace{\sum_i \operatorname{rk} H^i(X_{\mathbb{R}};\mathbb{F}_2)}_{= 2\cdot\#\text{ components}} \leq \underbrace{\sum_i \operatorname{rk} H^i(X_{\mathbb{C}};\mathbb{F}_2)}_{= 1 + 2\cdot\operatorname{genus}(X_{\mathbb{C}}) + 1}.$$

$\square$