# The Bauer-Furuta Invariant and 10/8

Student Low-Dimensional Topology Seminar 11th April 2025

## §1. Finite Dimensional Approximation

Consider a 4-mfld $M$ with $b_1(M)=0$ and a spin$^c$-structure $\mathfrak{s}$. Fix some spin$^c$ connection $A_0$.
The Seiberg-Witten equations can be regarded as a map of Hilbertspaces: (sometimes called "monopole map")

$$SW: \Omega^1(M)\oplus\Gamma(S^+)\to\Gamma(S^-)\oplus\Omega^2_+(M),\ (a,\varphi)\mapsto(\not{D}_{A_0+a}\varphi, F^+_{A_0+a}-\sigma(\varphi)).$$

Implicitly, we can pass to $\widetilde{SW}$ the equations modulo the pointed gauge group $\mathcal{G}_0$ by Coloumb gauge fixing.
These equations decompose as $\widetilde{SW}=l+c$ where $l(a,\varphi)=(\not{D}_{A_0}\varphi, d^+a)$ is linear and $c(a,\varphi)=(\frac{1}{2}a\cdot\varphi, F^+_{A_0}-\sigma(\varphi))$ is compact.
Note $\widetilde{SW}$ is equivariant w.r.t. the residual $S^1$-action.
We now execute finite dimensional approximation. Let $V_2$ be an $S^1$-invariant subspace of codomain of $\widetilde{SW}$ transverse to $\mathrm{im}(l)$. Let $V_1=l^{-1}(V_2)$. $\widetilde{SW}$ descends to a map,

$FD(\widetilde{SW})=l+\pi_{V_2}\circ c: V_1\to V_2$. By elliptic bootstrapping, $\widetilde{SW}^{-1}(\text{bounded})$ is bounded. Hence this map extends to one point compactifications (formally work on $\varepsilon$-balls): $FD(\widetilde{SW})^\vee: V_1^+\to V_2^+$.
This is a map of spheres and so defines an element of stable homotopy:
$[FD(\widetilde{SW})^\vee]\in\pi_{\mathrm{ind}(\widetilde{SW})}(\mathbb{S})$. (Note $\dim V_2=\dim\mathrm{coke}(l)+i$, $\dim V_1=\dim\ker l+i$).

## §2. Bauer-Furuta Invariant

The map $FD(\widetilde{SW})^\vee$ is $S^1$-equivariant, so it actually defines an element of $S^1$-equivariant stable homotopy of $\mathbb{S}$.
What this means is a little complicated, but elements represented by $S^1$-equivariant maps between "representation spheres" and so groups indexed by "relative $U(1)$-representation" instead of relative dimension.

**Def$^n$**: The Bauer-Furuta invariant $BF(M,\mathfrak{s})$ is the class $[FD(\widetilde{SW})^\vee]_{U(1)}\in\pi^{U(1)}_{\mathbb{C}^{\mathrm{ind}(\not{D})}-\mathbb{R}^{b_2^+}}(\mathbb{S})$.
ind($\not{D}$) copies of std. rep$^n$ from spinor
$b_2^+$ copies of triv. rep$^n$ from connection

**Thm**: (BF '04) Studying the invariant in families, $BF(M,\mathfrak{s})$ is independent of choices.

How to extract SW invt from BF? We can consider BF as $S^1$-equiv. map $V^+\to W^+$ where $V=\mathbb{R}^m\oplus\mathbb{C}^n$, $W=\mathbb{R}^{m'}\oplus\mathbb{C}^{n'}$ and $m'-m=b_2^+$, $n-n'=\mathrm{ind}_{\mathbb{C}}(\not{D})$. Say $b_2^+>1$. The $S^1$-fixed pts of $V^+$ map to codim $\geq 2$ subspace of $W^+$.
The preimage of a generic point in the complement is a manifold $\mathcal{M}$ of dimension $\mathrm{ind}_{\mathbb{R}}(\not{D})-b_2^+$ with a free $U(1)$ action.
When $\dim\mathcal{M}=2k+1$ is odd, $SW(M,\mathfrak{s})=e(\mathcal{M}\times\mathbb{C}^k/S^1)$ (to define this we need homology orientation).
This is basically Pontryagin-Thom

Another perspective: For $b_2^+>1$, we can dualize and ~~destabilize~~ unequivariantize:

$$\pi^{U(1)}_{\mathbb{C}^{\mathrm{ind}(\not{D})}-\mathbb{R}^{b_2^+}}(\mathbb{S})\cong\pi^{\mathbb{R}^{b_2^+}}_{U(1)}(\mathbb{C}^{\mathrm{ind}(\not{D})})\cong\pi_{st}^{b_2^+-1}(\mathbb{CP}^{\mathrm{ind}(\not{D})-1}).$$

There is a Hurewicz map, $\pi_{st}^{b_2^+-1}(\mathbb{CP}^{\mathrm{ind}(\not{D})-1})\to\tilde{H}^{b_2^+-1}(\mathbb{CP}^{\mathrm{ind}(\not{D})-1})$.
When $b_2^+$ odd, we can identify this cohomology group with $\mathbb{Z}$ and so BF defines an integer invariant, again the SW invariant.

Note: The BF invt may contain more information than SW invt, in particular if $b_2^+$ is even or BF lives in kernel of Hurewicz map (this can be analyzed explicitly with the AHSS).

Here's an example of how BF contains more information. Recall the SW invts of a connected sum are always trivial.

**Thm:** (Bauer, '04) $BF(M_0 \# M_1, \mathfrak{s}_0 \# \mathfrak{s}_1) = BF(M_0, \mathfrak{s}_0) \wedge BF(M_1, \mathfrak{s}_1)$. → map of equivariant spectra induced by smash product on spaces.

**E.g.** Forgetting the equivariant structure $BF(K3 \text{ surface})$ defines an element of $\pi_1(\mathbb{S}) \simeq \mathbb{Z}/2$. It turns out this is the non-trivial element $\eta$ the Hopf map. One has $\eta^3 \neq 0$ in the ring $\pi_*(\mathbb{S})$, so $K3^{\#2}$ and $K3^{\#3}$ have non-trivial BF invariants.

**However**, the BF invts are nilpotent. This follows from Devinatz-Hopkins-Smith thm. which generalizes Nishida theorem that $\pi_*(\mathbb{S})$ is nilpotent.

## §3. Classifying 4-Manifolds

From now on take the $M$ to be a closed simply-connected 4-manifold.

All the homological information about $M$ is contained in $H_2(M)$ and its intersection form $Q$, an unimodular symmetric bilinear quadratic form over $\mathbb{Z}$.

Such forms can be definite or indefinite and even ($Q(x,x) \in 2\mathbb{Z}$) or odd.

The indefinite forms are all of form $m[+1] \oplus n[-1]$ (odd) or $\pm m E_8 \oplus n\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (even).

Definite forms have no classification: $E_8$, Leech lattice etc.

**Thm:** (Freedman '82) Up to homeomorphism, there is one $M$ with a given even $Q$ and two with a given odd $Q$ (distinguished by KS invariant).

**What about smoothly?**

(i) [Geography] Which $Q$ are smoothable? (ii) [Botany] How many smoothings up to diffeo.?

(ii) is too hard, but (i) is mostly understood.

**Thm:** (Rokhlin '52) If $M$ smooth and $Q$ even, $\sigma(Q) \equiv 0 \pmod{16}$ (as opposed to mod 8) ~~$E_8$~~ not smoothable

**Thm:** (Donaldson '83) If $M$ smooth and $Q$ definite, then $Q$ is diagonal. ~~Leech lattice etc.~~ not smoothable

Diagonal definite forms and odd indefinite forms always smoothable by connect sums of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}}^2$'s.

Only remains to deal w/ even definite forms.

**Q:** When is $Q = 2m E_8 \oplus n\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ smoothable? (Need $2m$ by Rokhlin)

**Note:** If $n \geq 3|m|$, $Q$ is realized by $\# m K3 \# (n - 3|m|) S^2 \times S^2$.

**Conjecture:** (11/8) Matsumoto '81 There is a smooth $M$ with $Q$ as above iff $n \geq 3|m|$. Equivalently $b_2(M) \geq \frac{11}{8}|\sigma(M)|$.

**Corollary:** Every smooth simply connected 4-mfld is homeo. to $\# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ or $\# \pm m K3 \# n S^2 \times S^2$.

**Thm:** (Furuta '01 (10/8)) smooth $M$ with $Q$ as above has $n \geq 2|m| + 1$.

Equivalently, $b_2(M) \geq 10/8 |\sigma(M)| + 2$ whenever $Q$ is even.

There is a recent extension due to Hopkins, Lin, Shi, and Xu.

**Thm:** (HLSX, '22) Excepts for things homeo. to $S^4$, $S^2 \times S^2$, $K3$, $b_2(M) \geq \frac{10}{8}|\sigma(M)| + 4$ when $Q$ is even.

In fact $n \geq 2|m| + \begin{cases} 2 & : m \equiv 1,2,5,6 \\ 3 & : m \equiv 3,4,7 \\ 4 & : m \equiv 0 \end{cases} \pmod 8$.

**Note:** HLSX show this is the optimal result obtainable from studying the topology of the monopole map.

# §4. Proving 10/8

We conclude with a sketch of Furuta's (quite technical) proof of the 10/8 theorem. The proof combines the BF invariant with input from Pin(2)-equivariant K-theory.

Suppose $M$ has even $Q$. By Wu's formula, $M$ is spin. The monopole map extends to have a $Pin(2) = S^1 \cup jS^1 \subseteq \mathbb{H}$ symmetry and the BF invariant lives in Pin(2)-equivariant stable homotopy.

Thus we get Pin(2)-equiv. map of rep$^n$ spheres: $(\mathbb{H}^{k+m} \oplus \mathbb{R}^n)^+ \longrightarrow (\mathbb{H}^m \oplus \mathbb{R}^{b_2^+ + n})^+$ where $k = \mathrm{ind}_{\mathbb{H}}(\not{D}) = |\sigma(M)|/16$.

$\mathbb{H}$: std. Pin(2) representation; $\mathbb{R}$: $j \in Pin(2)$ acts by a sign. (possibly after changing orientation)

It suffices to show if this map exists then $b_2^+ \geq 2k+1$.

**Def$^n$:** For a G-space $X$, its <u>G-equivariant K-theory</u> $K^0_G(X)$ is the Grothendieck group of iso. classes of complex G-vector bundles on $X$. $K^0_G(*) = R(G)$ the ring of G-representations.

Given a G-representation $V$, the Thom isomorphism gives, $\tilde{K}^0_G(V^+) \cong \beta_V \cdot R(G)$ — $\beta_V$: Thom class / "Bott element"

**Def$^n$:** Given a map $f: V^+ \to W^+$, one has $f^* \beta_W = \alpha(f) \beta_V$ where $\alpha(f) \in R(G)$ is called the G-equivariant K-theory <u>degree</u> of $f$.

More straightforwardly, $\sum (-1)^d [\Lambda^d W] = \alpha(f) \underbrace{\sum (-1)^d [\Lambda^d V]}$. We call this the Euler characteristic $\chi(V)$

In our case $R(Pin(2)) = \mathbb{Z}[\mathbb{C}, \mathbb{H}] / (\mathbb{C}^2 - 1, \mathbb{C}\mathbb{H} - \mathbb{H})$.

We can complexify the BF map, $\mathbb{C}BF : (\mathbb{H}^{2k+2m} \oplus \mathbb{C}^n)^+ \longrightarrow (\mathbb{H}^{2m} \oplus \mathbb{C}^{b_2^+ + n})^+$.

This has a K-theory degree $\alpha(\mathbb{C}BF) = a_0 + b_0 \mathbb{C} + \sum_n a_n \mathbb{H}^n \in R(Pin(2))$.

First one shows $\alpha(\mathbb{C}BF)$ is in the kernel of the forgetful map $R(Pin(2)) \to R(S^1)$.

This is because we may pass to the map on $S^1$ fixed points $(\mathbb{C}^n)^+ \to (\mathbb{C}^{b_2^+ + n})^+$ which is trivial for $b_2^+ > 0$. (For trivial spheres, K-theory degree is just homological degree which vanishes when spheres not equidimensional)

We conclude $\alpha(\mathbb{C}BF) = a_0(1 - \mathbb{C})$ for some $a_0 \in \mathbb{Z}$.

Now apply forgetful map $R(Pin(2)) \to R(\langle j \rangle = \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}[\sigma]/(\sigma^4 - 1)$.

We compute $\chi(\mathbb{C}) = 1 - \sigma^2$, $\chi(\mathbb{H}) = 1 - (\sigma + \sigma^3) + 1$.

So by definition of degree, $a_0(1-\sigma^2)(2 - \sigma - \sigma^3)^{2k+2m}(1-\sigma^2)^n = (2 - \sigma - \sigma^3)^{2m}(1-\sigma^2)^{b_2^+ + n}$.

Taking the trace/character at $j$: $a_0 \cdot 2^{2k+1} = 2^{b_2^+}$.

Since $a_0 \in \mathbb{Z}$, $b_2^+ \geq 2k+1$ as desired!