# Review Notes for Qualifying Exam

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CHAPTER 1

# Algebraic Topology

## 1. Basic Theory

We begin with some basic fundamental concepts. In particular we build familiarity with CW complexes, important types of spaces and maps between them, and the basic notion of equivalence in algebraic topology.

**1.1. CW Complexes, Elementary Spaces, and Operations.** Most of the spaces we deal with have the following description which allows for easier calculations and implies nice properties.

**Definition 1.1**

A *CW complex* $X$ is a space formed as the inductive limit of spaces $X_0 \subset X_1 \subset \cdots$, where $X_n$ is obtained from $X_{n-1}$by,

$$X_n = X_{n-1} \bigsqcup_{\alpha \in A, \varphi_\alpha} e_\alpha,$$

where $A$ is an indexing set, $e_\alpha$ are homeomorphic to $D^n$ and we glue $e_\alpha$ to $X_{n-1}$ by continuous maps $\varphi_\alpha : \partial e_\alpha \to X_{n-1}$, so that $X$ satisfies:

**Weak Topology:** $C \subset X$ is closed iff $C \cap X_i$ is closed for all $i$.

We say $X_n$ is the *n-skeleton* of $X$ and a choice of decomposition $\{X_n\}$ of $X$ is a *cell structure*. A Hausdorff space is a CW complex iff it can be partitioned into cells $e_\alpha$ homeomorphic to disks so that it satisfies the weak topology axiom and:

**Closure Finiteness:** The boundary of any cell is in the closure of a finite collection of lower dimensional cells.

Now we recall a number of fundamental spaces and describe their CW structure where applicable.

Important Spaces

**Euclidean space:** We have the basic spaces $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and their Cartesian products $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$. In the direct limit under inclusion, we obtain $\mathbb{R}^\infty, \mathbb{C}^\infty, \mathbb{H}^\infty$. These are homeomorphic but are each interesting for their distinct algebraic structures.

**Spheres:** One has the sphere $S^n$ which can be formed as a CW complex from an $n$-cell glued by its whole boundary to a 0-cell. Alternatively, $S^n$ can be formed inductively by gluing two $n$-cells by the identity to $S^{n-1}$. This second structure allows us to define $S^\infty$ as a direct limit, and a simple lemma proves $S^\infty$ is contractible.

**Projective Spaces:** The space of real (resp. complex or quaternionic lines) in $\mathbb{R}^{n+1}$ (resp. $\mathbb{C}^{n+1}$ or $\mathbb{H}^{n+1}$) defines the projective space $\mathbb{R}P^n$ (resp. $\mathbb{C}P^n$ or $\mathbb{H}P^n$). These spaces can also be formed from $S^n$ (resp. $S^{2n+1}$ or $S^{4n+3}$) by quotienting by $\mathbb{Z}_2$ (resp. $S^1$ or $S^3$) acting by unit scalar multiplication. We can form projective space as a CW structure inductively forming $P^n$ from $P^{n-1}$ by gluing on an $n$-cell (resp. $2n$ or $4n$-cell) via the quotient map just described. We obtain $\mathbb{R}P^\infty, \mathbb{C}P^\infty, \mathbb{H}P^\infty$ as a direct limit.

**Grassmannians:** One has the Grassmannians $G(n,k)$ defined as the space of $k$-dimensional subspaces of $\mathbb{R}^n$ (note $G(n+1,1) = \mathbb{R}P^n$). Analogously we may define complex and quaternionic Grassmannians $\mathbb{C}G(n,k)$, $\mathbb{H}G(n,k)$ and oriented Grassmannians $G_+(n,k)$.

There is a CW structure on $G(n,k)$, the Schubert decomposition, whose cells are in bijection with Young diagrams that fit in a $k \times (n-k)$ rectangle (we won't describe the gluing). The dimension of each cell is the number of squares in the diagram. We can do the same for $\mathbb{C}G(n,k)$ and $\mathbb{H}G(n,k)$ doubling (resp. quadrupling) the cells dimensions, and for $G_+(n,k)$ doubling the number of cells. In the direct limit one obtains $G(\infty,k)$ and taking another limit, $G(\infty,\infty)$ (analogously for the other versions).

**Stiefel Manifolds:** The Stiefel manifolds $V(n,k)$ (resp. $\mathbb{C}V(n,k)$, $\mathbb{H}V(n,k)$ are the spaces of orthonormal (resp. unitary, symplectic) $k$-frames in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$, $\mathbb{H}^n$). These have Bruhat cell partitions which we won't describe.

**Lie Groups:** Groups which have the structure of a smooth manifold with smooth group operations are called Lie groups. Important examples include $\mathrm{O}(n)$ and $\mathrm{U}(n)$ the orthogonal and unitary groups of $n \times n$ orthogonal and unitary matrices. These are compact analogues of the general linear groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, the space of invertible real and complex matrices. These groups have "special" subgroups where we restrict the determinant to be one, namely $\mathrm{SO}(n), \mathrm{SU}(n), \mathrm{SL}(n, \mathbb{R})$, and $\mathrm{SL}(n, \mathbb{C})$ respectively. There is also the group $\mathrm{GL}^+(n, \mathbb{R})$ of positive determinant real matrices. One can also define the symplectic groups $\mathrm{Sp}(2n, \mathbb{R}), \mathrm{Sp}(2n; \mathbb{C})$ of real and complex matrices preserving the standard symplectic form. The quaternionic unitary group can be described as,

$$\mathrm{Sp}(n) := \mathrm{Sp}(2n; \mathbb{C}) \cap \mathrm{U}(2n).$$

It's useful to remember the dimension of some of these as manifolds:

$$\begin{aligned}
\dim(\mathrm{O}(n)) = \dim(\mathrm{SO}(n)) &= \frac{n(n-1)}{2} \\
\dim(\mathrm{U}(n)) &= n^2 \\
\dim(\mathrm{SU}(n)) &= n^2 - 1 \\
\dim(\mathrm{Sp}(2n; \mathbb{R})) = \dim(\mathrm{Sp}(n)) &= n(2n+1) \\
\dim(\mathrm{Sp}(2n; \mathbb{C})) &= 2n(2n+1).
\end{aligned}$$

The tangent space to the identity of a Lie group defines its Lie algebra, there are several of these worth remembering:

$$\begin{aligned}
\mathfrak{gl}(n) &= \{n \times n \text{ matrices}\} \\
\mathfrak{sl}(n) &= \{\text{traceless matrices}\} \\
\mathfrak{o}(n) = \mathfrak{so}(n) &= \{\text{skew-symmetric matrices}\} \\
\mathfrak{u}(n) &= \{\text{skew-adjoint matrices}\} \\
\mathfrak{su}(n) &= \{\text{traceless skew-adjoint matrices}\} \\
\mathfrak{sp}(n) &= \{A : JA + A^* J = 0\}.
\end{aligned}$$

There are several "accidental isomorphisms" that occur in low dimensions.

$$S^1 \cong \mathrm{SO}(2) \cong \mathrm{U}(1), \quad S^3 \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1) \quad \text{and} \quad \mathrm{Sp}(1, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C}).$$

One also has $\mathrm{SO}(4) \cong \mathrm{SO}(3) \times \mathrm{SU}(2)$ but only as manifolds, not as groups. There are also some more general topological descriptions one can obtain. For example, $\mathrm{SU}(3)$ is an $S^3$ fibre bundle over $S^5$.

**Homogeneous Spaces:** Many of the spaces we have just seen can be described as homogeneous spaces, i.e. quotients of compact Lie groups by Lie subgroups. We have,

$$\begin{aligned} S^{n-1} &= \mathrm{O}(n)/\mathrm{O}(n-1) = \mathrm{SO}(n)/\mathrm{SO}(n-1) \\ S^{2n-1} &= \mathrm{U}(n)/\mathrm{U}(n-1) = \mathrm{SU}(n)/\mathrm{SU}(n-1) \\ S^{4n-1} &= \mathrm{Sp}(n)/\mathrm{Sp}(n-1) \\ G(n,k) &= \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k) \\ V(n,k) &= \mathrm{O}(n)/\mathrm{O}(n-k) \overset{\text{if } n>k}{=} \mathrm{SO}(n)/\mathrm{SO}(n-k). \end{aligned}$$

And similar can be inferred for oriented, complex, or quaternionic analogues.

With many spaces at our disposal, we now consider some basic operations to be performed on spaces.

Operations

**Products:** We can take the Cartesian product of two spaces $X \times Y$. For infinite products, we use the weak topology.

**Cylinders, Cones and Suspensions:** Given $X$, the cylinder over $X$ is $X \times I$. The cone over $X$ is $CX = X \times I/X \times \{1\}$. The suspension over $X$ is $\Sigma X$ formed as two copies of $CX$ glued along $X \times \{0\}$. Note $\Sigma S^n = S^{n+1}$. If $X$ is a pointed space, we instead consider the reduced suspension also denoted $\Sigma X$, which is the suspension quotient the line $\{x_0\} \times I$. For nice spaces this is homotopy equivalent to the usual suspension.

**Attachings:** Given $f : X \to Y$, the mapping cylinder of $f$ is,

$$M_f = X \times I \bigsqcup_{(x,1)\sim f(x)} Y.$$

The mapping cone of $f$ is,

$$C_f = CX \bigsqcup_{(x,0)\sim f(x)} Y.$$

If $X = Y$, the mapping torus of $f$ is,

$$T_f = X \times S^1/(x,1) \sim (f(x),0).$$

**Joins:** The join of $X$ and $Y$ is given by,

$$X * Y = X \times Y \times I/(x,y_1,0) \sim (x,y_2,0) \quad \text{and} \quad (x_1,y,1) \sim (x_2,y,1).$$

**Mapping Spaces:** Given $X, Y$ the space $C(X,Y)$ of continuous maps $X \to Y$ can be equipped with the compact-open topology, i.e. it has a basis $\mathcal{B} = \{B(K,U)\}$ indexed by compacts sets $K \subset X$ and open sets $U \subset Y$ so that $B(K,U)$ consists of all continuous maps mapping $K$ inside $U$. In particular, we have the path space $C(I,X) = \mathcal{P}X$ and the loop space $C(S^1,X) = \Lambda X$. We often restrict to paths or loops with fixed endpoints to obtain $EX$ and $\Omega X$ respectively.

**Wedge Sum:** Given pointed spaces $X, Y$ we can glue their base points together to get a wedge sum $X \vee Y$.

**Smash Product:** Given pointed spaces, we have the smash product,

$$X \wedge Y = X \times Y/X \vee Y.$$

One has $X \wedge S^k = \Sigma^k X$.

**1.2. Homotopy.** We discuss now the central equivalence notion for the study of algebraic topology.

**Definition 1.2**

Two maps $f, g : X \to Y$ are *homotopic* if there is a map $F : X \times I \to Y$ so that if $F_t := F(\cdot,t)$ then $F_0 = f, F_1 = g$. This is an equivalence relation on $C(X,Y)$. We denote the set of homotopy equivalence class as $[X,Y]$. We can similarly define homotopy relative to a subset $A \subset X$ by demanding $F_t|_A$ is independent of $A$. For the particular case of a point, we have the set of pointed homotopy classes $[(X,x_0),(Y,y_0)]$. When $X$ and $Y$ are pointed we will write $[X,Y]$ to mean the homotopy classes relative to the basepoints.

Two spaces $X$ and $Y$ are *homotopy equivalent* if there are maps $f : X \to Y$, $g : Y \to X$ called *homotopy equivalences* so that $g \circ f$ and $f \circ g$ are homotopic to $\mathrm{id}_X$ and $\mathrm{id}_Y$ respectively.

A space is *contractible* if it is homotopy equivalent to a point.

PROPOSITION 1.3. *The following are equivalent:*

*(i) $X$ and $Y$ are homotopy equivalent.*

*(ii) For any $Z$, there is a natural bijection $[X, Z] \to [Y, Z]$.*

*(iii) For any $Z$, there is a natural bijection $[Z, X] \to [Z, Y]$.*

PROOF. $(i) \implies (iii)$ : Suppose $f : X \to Y$ and $g : Y \to X$ provide a homotopy equivalence. Then $f_*$ gives a natural map $[Z, X] \to [Z, Y]$ with inverse $g_*$.

$(iii) \implies (i)$ : Take $Z = X$, giving a bijection $[X, X] \to [X, Y]$. Let $f$ be the image of the identity. Taking $Z = Y$, gives bijection $[Y, X] \to [Y, Y]$. Let $g$ be the inverse image of the identity. Naturality implies $f \circ g$ and $g \circ f$ are homotopic to the identities.

A similar argument gives $(i) \iff (ii)$. □

**Definition 1.4**

We say $X$ and $Y$ are *weak homotopy equivalent* if the conditions $(ii)$ or $(iii)$ in the above proposition hold just for any CW complex $Z$.

REMARK 1.5. In the special case where a map $f : X \to Y$ induces the natural bijection, we will later see this is the same, and more commonly defined, as saying $f$ induces an isomorphism in homotopy or homology. △

### 1.3. Cofibrations and Fibrations.

There are two special types of maps in topology that generalize the ideas of embeddings and submersions/fibre bundles from topology.

**Definition 1.6**

A map $i : A \to X$ is a *cofibration* if for every homotopy $F : A \times I \to Y$ so that $F_0$ extends to a map $X \to Y$, the entire homotopy extends. Equivalently any commutative square of the following form admits the following lift

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{P}Y \arrow[d, "\pi_0"] \\ X \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

If $A \to X$ is a fibration, $X/i(A)$ is called the *homotopy cofibre* of $i$.

A map $\pi : E \to B$ is a *Hurewicz* or a *strong Serre fibration* if for any homotopy $F : X \times I \to B$ so that $F_0$ lifts to a map $X \to E$, the entire homotopy lifts. Equivalently any commutative square of the following form admits the following lift

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dotted] & B. \end{tikzcd}$$

A map $\pi : E \to B$ is a *Serre fibration* if the above is known to hold only for $X$ a CW complex (or equivalently a disk). If $\pi : E \to B$ is a fibration with path connected base, and $b \in B$, then $\pi^{-1}(b)$ is called the *homotopy fibre* of $\pi$.

To show these generalize geometric notions, we have the following pair of results.

**Theorem 1.7: Borsuk's Theorem**

For every CW pair $A \subset X$, i.e. a CW complex $X$ and subcomplex $A$, the inclusion $i : A \hookrightarrow X$ is a cofibration.

Proof. Let $F : A \times I \to Y$ so that $F_0$ extends to $f : X \to Y$. We wish to extend $F$ to $X \times I$. We can work inductively on the dimension of cells in $X \setminus A$. On the 0 skeleton, we just define $F$ to constantly agree with $f$. Now suppose $F$ has been extended to the $n$ skeleton of $X$. On a given $n+1$ cell $e^{n+1}$, we wish to define $F$ on the cylinder $e^{n+1} \times I$ given that we know it on the "vase" $\partial e^{n+1} \times I \cup e^{n+1} \times \{0\}$. Note that there is an obvious deformation retraction map from the cylinder to the vase and we can extend $F$ on the cylinder by composing this retraction with $F$ defined on the vase. Doing this for all $n+1$ cells gives a continuous extension of $F$ to $X_{n+1} \times I$. We conclude by induction. □

**Theorem 1.8**

Every fibre bundle $\pi : E \to B$ is a Serre fibration. If $B$ is paracompact, $\pi$ is a Hurewicz fibration.

Proof. We only deal with the Serre case. If the fibre bundle is trivial, this follows from Borsuk's theorem. For a non-trivial bundle $E$ consider the case where we have a relative homotopy from the disk $(D^n, \partial D^n)$ we wish to extend. Then we can pullback the bundle and it will be trivial by Feldbau's lemma below. Hence we may apply the above result. For homotopies from general CW complexes, we may just glue relative homotopies on each cell. $\square$

A couple more important results about fibrations.

**Theorem 1.9: Feldbau's Lemma**

Every Hurewicz fibration $\pi : E \to B$ with contractible CW base $B$ and homotopy fibre $F$ is homotopy equivalent to the projection $B \times F \to B$. If $\pi$ is only Serre, there is still a weak homotopy equivalence.

PROOF. We prove only for a fibre bundle. By naturality of bundles, if $f : B \to *$ and $g : * \to B$ so that $g \circ f \sim \mathrm{id}_B$ we have that $E = (g \circ f)^* E = f^* g^* E$. But $g^* E$ is a bundle over a point and hence trivial, so its pullback by $f$ is too. $\square$

The following justifies the definition of homotopy fibre.

**Theorem 1.10**

The fibres of a Hurewicz fibration over a path connected base are homotopy equivalent. The fibres of Serre fibration over a path connected base are weak homotopy equivalent.

PROOF. Consider a path $\gamma$ between two points $x_0, x_1$ in the base with fibres $F_0, F_1$. Given a map $Z \to F_0$, we can use $\gamma$ and the definition of a fibration to obtain a map $G : Z \times I \to E$ so that $G_0$ lands in $F_0$ and $G_1$ lands in $F_1$. This gives a map from $[Z, F_0]$ to $[Z, F_1]$. We may compose paths to compose maps and homotopic paths induce homotopic maps. Taking the inverse path we conclude our map is a bijection and it is easily seen to be natural in $Z$. Thus by Proposition 1.3, $F_0, F_1$ are homotopy equivalent. If we only have a Serre fibration, we are restricted to $Z$ a CW complex from which we conclude we have a weak homotopy equivalence. $\square$

A final interesting result is as follows.

**Theorem 1.11**

Every map $f : X \to Y$ is homotopic (by which we mean equivalent in the homotopy category) to a fibration and to a cofibration.

Proof. $Y$ is homotopic to the mapping cylinder $M_f$ and so our map is equivalent in the homotopy category to the embedding $X \hookrightarrow M_f$, which is a cofibration. The homotopy cofibre of this map is the mapping cone $C_f$

For the fibration case, first by above, we may assume $f : X \to Y$ is an embedding. Now consider the space $\mathcal{P}_X Y$ of continuous paths in $Y$ that begin in $f(X)$. Truncating the paths gives a homotopy equivalence with $X$. Hence our map is equivalent in the homotopy category to the map $\mathcal{P}_X Y \to Y$ sending a path to its endpoint. This is naturally a fibration with homotopy fibre $E_X Y$, the space of paths in $Y$ beginning in $X$ and ending at $y_0$. □

Remark 1.12. Let $f : X \to Y$ be a map which we homotope by above to be a cofibration and inclusion. We can extend to a coexact sequence of cofibrations,

$$X \xrightarrow{f} Y \to C_f \to \Sigma X \to \Sigma Y \to \Sigma C_f \to \Sigma^2 X \to \cdots .$$

We can see what's going on here with a simple picture seen in Figure 1 (noting if we take $X \subset Y$, then we can identify $C_f$ with gluing to $Y$ a cone over $X \subset Y$).

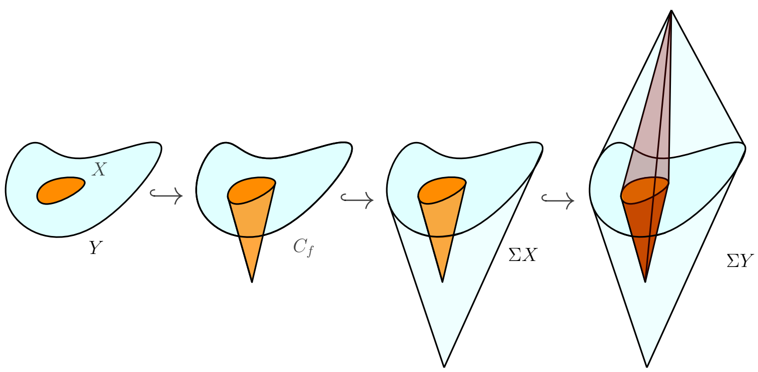



Figure 1. Puppe sequence

Similarly, there is an exact sequence associated to a fibration,

$$\cdots \to \Omega^2 Y \to \Omega(E_X Y) \to \Omega X \to \Omega Y \to E_X Y \to X \to Y.$$

These are both called the *Puppe sequence*. △

**1.4. Cellular Approximation.** We say a map $X \to Y$ of CW complexes is *cellular* if $X_n$ maps into $Y_n$ for each $n$.

**Theorem 1.13: Cellular Approximation Theorem** [FF, pp. 52]

A continuous map $f : X \to Y$ of CW complexes is homotopic to a cellular map. More generally, if $A \subset X$ is a subcomplex and $f|_A$ is cellular, than $f$ is homotopic relative to $A$ to a cellular map.

Proof. (Sketch) We work by induction. Suppose $f : X \to Y$ is cellular on $X_{n-1} \cup A$ and let $e^n$ be an $n$-cell of $X \setminus A$. $f$ maps the closure of $e^n$ to a compact set of $Y$. Thus it can only meet finitely many cells of $Y$. Let $\varepsilon$ be one of these cells of $Y$ of maximal dimension $m$.

There is a technical lemma that we skip which implies, for $m > n$, $f|_{A\cup X^n\cup e^n}$ is homotopic to a map $f'$, cellular on $A \cup X^n$, so $f'(e^n)$ intersects the same cells as $f(e^n)$ and is not surjective onto $\varepsilon$. By Borsuk's theorem $f'$ extends to $X$. On $e^n$, we may define a homotopy of $f'$ pushing its image off of $\varepsilon$ by radial projection away from a point not in $f'(e^n)$. By Borsuk's theorem, this homotopy can be extended to $X$. Inductively, we can ensure $f(e^n)$ lands in no cells of dimension greater than $n$. Doing this for every cell, we obtain our result inductively. $\square$

## 2. The Fundamental Group and Coverings

We keep this section short and sweet. Results about fundamental groups can usually be reproved as exercises with just a little creativity.

### 2.1. Fundamental Group.

We give a very abbreviated treatment.

**Definition 2.1**

The *fundamental group* $\pi_1(X, x_0)$ of a space is the set $[(S^1, *), (X, x_0)]$ endowed with the operation of loop composition. Equivalently, it is the path components of $\Omega X$.
Up to isomorphism, the fundamental group only depends on the path component of $x_0$ and so we often abbreviate to $\pi_1(X)$.
If $X$ is connected and $\pi_1(X)$ is trivial, we say $X$ is *simply connected*.

A continuous map $f : X \to Y$ induces a group homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ by post-composition with a loop. This is natural with respect to composition of maps and depends only on the homotopy class of our maps.

Corollary 2.2. *Homotopy equivalent spaces have the same fundamental group.*

### 2.2. Covering Spaces.

We again only sketch the basics.

**Definition 2.3**

A *covering map* $p : T \to X$ is a fibre bundle with discrete fibre; we call $T$ a *covering space* of $X$. Equivalently, $X$ is covered by neighbourhoods

$V_i$ whose preimage under $p$ is a disjoint collection of open sets $U_{ij}$ so that $p : U_{ij} \to V_i$ is a homeomorphism for each $i$ and $j$.

**Theorem 2.4: Map Lifting**

Let $p : T \to X$ be a covering and $Z$ path connected and locally path connected. A map $f : Z \to X$ lifts to $F : Z \to T$ if and only if $f_*\pi_1(Z, z_0) \subset p_*\pi_1(T, t_0)$. If we specify, $F(z_0) = t_0$ with $p(t_0) = f(z_0)$, the map is unique.

In particular, a path $\gamma : I \to X$ always lifts to $\widetilde{\gamma} : I \to T$ and the lift is unique if we specify $\widetilde{\gamma}(0)$. By a relative version of the map lifting, if $F_t : Z \to X$ is a homotopy and $F_0$ lifts to $\widetilde{F_0} : Z \to T$, then there is a unique lift $\widetilde{F_t} : Z \to T$ of $F_t$ agreeing with $\widetilde{F_0}$ for $t = 0$.

**Theorem 2.5: Classification of Covering Spaces**

If $X$ is path connected, locally path connected, and semi-locally simply connected, then for every subgroup $H \subset \pi_1(X)$, there is a covering space $p : X_H \to X$ with $p_*(\pi_1(X_H)) = H$ (and hence $\pi_1(X_H) \cong H$). Moreover there is a bijection between isomorphism classes of coverings and conjugacy classes of subgroups of $\pi_1(X)$.

Proof. We only construct the *universal cover* $\widetilde{X}$, which is the simply-connected cover. Other covering spaces are realized as quotients of $\widetilde{X}$ in a somewhat obvious way. Fix a basepoint $x_0 \in X$. We define $\widetilde{X}$ as the homotopy classes of paths in $X$ with fixed endpoints that begin at $x_0$. There is an obvious map $p : \widetilde{X} \to X$ by projecting to the end of a path which is clearly a covering. Moreover, given a loop $\gamma$ in $X$ that lifts to a loop in $\widetilde{X}$, we see that the lifted loop $[\gamma_t]$ represents the homotopy class of the loop $\gamma$ truncated at time $t$. But the fact it lifts to a loop means $[\gamma] = [\gamma_0]$ is null-homotopic. Hence, $p_*(\pi_1(\widetilde{X})) = 0$. But uniqueness in Theorem 2.4 implies that $p_*$ is injective, and so $\pi_1(X) = 0$.

To see the correspondence, note that by changing the base point we obtain conjugate subgroups of $\pi_1(X)$ and isomorphic coverings. ☐

**Definition 2.6**

Given a covering space $p : T \to X$, an isomorphism $f : T \to T$ is called a *deck transformation* if it commutes with $p$. Note a deck transformation is fully determined by the image of one point.

**Definition 2.7**

A covering $p : T \to X$ is *normal* (or sometimes called *regular*) if $p_*\pi_1(T)$ is a normal subgroup of $\pi_1(X)$.

PROPOSITION 2.8. *A covering $p : T \to X$ is normal if and only if the group of deck transformations acts transitively on the set $\{p^{-1}(x_0)\}$, where $x_0$ is a point of $X$.*

PROPOSITION 2.9. *The group of deck transformations $D$ of a covering $p : T \to X$ is isomorphic to the quotient of the normalizer,*

$$D = \pi_1(X)/N_{\pi_1(X)}(p_*\pi_1(T)).$$

We can finally make some computations.

COROLLARY 2.10. *$\pi_1(S^1) = \mathbb{Z}$.*

PROOF. Note $\mathbb{R} \to S^1$ given by $z \mapsto e^{2\pi i z}$ is the universal covering of $S^1$. We know that a subgroup of the group of deck transformations isomorphic to $\mathbb{Z}$ is given by translations by $2\pi i n$ for $n \in \mathbb{Z}$. But since these translations act transitively on the preimages of the basepoint, this is in fact all of the deck transformations. By Proposition 2.9, we conclude $\pi_1(S^1) = D = \mathbb{Z}$. $\square$

COROLLARY 2.11. *$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for $n > 1$.*

PROOF. We will prove momentarily that $\pi_1(S^n) = 0$ for $n > 1$. Thus the universal cover of $\mathbb{R}P^n$ is $S^n$ under the double cover quotient map identifying antipodal points. The map $p : S^n \to S^n$ sending a point to its antipode is a deck transformation which swaps the two preimages of the basepoint. Hence the covering is normal, $p$ is the only non-trivial deck transformation, and $\pi_1(S^n) = D = \mathbb{Z}_2$. $\square$

COROLLARY 2.12. *$\pi_1(T^2) = \mathbb{Z}^2$.*

PROOF. The universal cover is $\mathbb{R}^2 \to T^2$ given by the quotient map by the integer lattice. There are deck transformations $\mathbb{R}^2 \to \mathbb{R}^2$ given by translations by $(n,m)$ for $(n,m) \in \mathbb{Z}^2$. These act transitively on the preimages of the base-point and so the covering is normal. We have $\pi_1(T^2) = D = \mathbb{Z}^2$. Alternatively, we could use that $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. $\square$

**2.3. Seifert–Van Kampen Theorem and $\pi_1$ of CW Complexes.** The following is very useful in more advanced computations where there is no clear candidate for universal cover (or we want to show the space itself is simply connected).

**Theorem 2.13: Seifert–Van Kampen**

Let $X = U_1 \cup U_2$ with $U_1, U_2$ open, path connected and $U_1 \cap U_2$ path connected. Then the maps on $\pi_1$ induced by inclusions of $U_1 \cap U_2, U_1, U_2$ and $X$ define a pushout square, i.e. $\varphi$ is an isomorphism in the following colimit diagram:

$$\begin{tikzcd}
& \pi_1(U_1) \arrow[dr] \arrow[rrd, dashed, bend left, "j_1"] & & \\
\pi_1(U_1 \cap U_2) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \arrow[r, dashed, "\varphi"] & \pi_1(X). \\
& \pi_1(U_2) \arrow[ur] \arrow[rru, dashed, bend right, "j_2"'] & &
\end{tikzcd}$$

Equivalently, if $\pi_1(U_2 \cap U_2)$ has generator set $G_0$, $\pi_1(U_1)$ has presentation $\langle G_1 | R_2 \rangle$, and $\pi_1(U_2)$ has presentation $\langle G_2 | R_2 \rangle$, then $\pi_1(X)$ has presentation,

$$\left\langle G_1, G_2 \middle| R_1, R_2, i_1 G_0 = i_2 G_0 \right\rangle.$$

Corollary 2.14. *For $n > 1$, the sphere $S^n$ is simply connected.*

Proof. Let $U_1, U_2$ be open neighbourhoods of the two hemispheres of the sphere. They are each simply connected and have connected intersection a thickened $n-1$ sphere. The pushout of the maps $\mathbb{Z} \to 0, \mathbb{Z} \to 0$ is clearly $0$ and so by Theorem 2.13, $\pi_1(S^n) = 0$. $\square$

Note by Seifert–Van Kampen, if we have a nice space $X$ and we attach a 2-cell $D$ to it via $\varphi : \partial D \to X$, the effect on $\pi_1$ will be to kill the element of $\pi_1(X)$

represented by $\varphi(\partial D)$. Furthermore, if we attach an $n$-cell to $X$ for $n > 2$, it will not change its fundamental group.

Now consider a one-dimensional path connected CW complex $X$. This is the same thing as a graph. By contracting edges joining distinct vertices of $X$, one sees that $X$ is homotopy equivalent to a (possibly infinite) bouquet of circles. We now find the fundamental group of $X$.

PROPOSITION 2.15. *If $X, Y$ are pointed spaces with base point contractible neighbourhoods, then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.*

PROOF. This directly follows from Theorem 2.13 applied to the complements of these contractible neighbourhoods. □

COROLLARY 2.16. *The fundamental group of a bouquet of circles is a free group with generators corresponding to the circles.*

PROOF. For finite bouquets this follows inductively from the above proposition. For infinite bouquets, note any loop lies in only finitely many circles by compactness, and hence belongs in $\pi_1$ to the free group generated by those circles. □

Combining our findings, we can deduce the following theorem.

**Theorem 2.17: The Fundamental Group of a CW Complex**

Suppose $X$ is a connected CW complex with a single zero cell. Then $\pi_1(X)$ has a presentation whose generators correspond to the one cells of $X$ and whose relations impose that the loop class $[\partial e^2 \to X_1]$ as a product of generators is trivial for any two cell $e^2$ of $X$.

PROOF. The fundamental group of the one skeleton $X_1$ is a free group generated by the one-cells, since $X_1$ is a bouquet of circles, one circle per one-cell. Attaching a two-cell has the effect of killing its parameterized boundary and hence imposing a relation of the form described above. Doing this for all two cells inductively, we conclude $\pi_1(X_2)$ is as described in the theorem statement. Attaching a cell of dimension three or greater has no effect on $\pi_1$ and hence $\pi_1(X) = \pi_1(X_2)$ is as claimed. □

EXERCISE 2.18. Our assumption $X$ has a single zero cell is no restriction, since a connected CW complex $X$ is always homotopy equivalent to one with a single zero cell. ▲

We can now easily find the fundamental groups of all our favourite spaces.

**Some Fundamental Groups**

- $\pi_1(\Sigma_g) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g | [a_1, b_1] \cdots [a_g, b_g] = e \rangle$.
- $\pi_1((\mathbb{R}P^2)^{\# g}) = \langle c_1 \ldots, c_g | c_1^2 \cdots c_g^2 \rangle$.
- $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for $2 \leq n \leq \infty$ with universal cover $S^n$.
- $\pi_1(G(n,k)) = \mathbb{Z}_2$ for $1 \leq k \leq n-1$ and $n > 2$ with universal cover $G_+(n,k)$.
- The complex and quaternionic projective spaces, Grassmannians, and Stiefel manifolds are all simply connected.
- $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$. Its double cover is called $\mathrm{Spin}(n)$, the *Spin group*. There are exceptional isomorphisms,
$$\begin{aligned} &\mathrm{Spin}(2) \cong S^1, && \mathrm{Spin}(3) \cong S^3 \cong \mathrm{SU}(2), \\ &\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2), && \mathrm{Spin}(5) \cong \mathrm{Sp}(2), \quad \mathrm{Spin}(6) \cong \mathrm{SU}(4). \end{aligned}$$
- $\pi_1(\mathrm{SU}(n)) = \pi_1(\mathrm{Sp}(n)) = 0$ and $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$. The universal cover of $\mathrm{U}(n)$ is $\mathbb{R} \times \mathrm{SU}(n)$ given by $(\theta, A) \mapsto e^{2\pi i \theta} A$.
- $\pi_1(\mathrm{Sp}(2n, \mathbb{R})) = \mathbb{Z}$ and $\pi_1(\mathrm{Sp}(2n, \mathbb{C})) = 0$. The double cover of $\mathrm{Sp}(2n, \mathbb{R})$ is the *metaplectic group* $\mathrm{Mp}(2n, \mathbb{R})$. It is *not* a matrix group.

## 3. Homology and Cohomology

We try to stick to the fundamentals, emphasizing a range of versions of homology, and focusing on algebraic tools and geometric intuition with minimal actual examples.

**3.1. Singular Homology and Its Many Imitators.** We begin in total generality. Take $R$ to be a commutative ring (actually any abelian group would be fine); we almost exclusively use $\mathbb{Z}, \mathbb{Z}_2$, $\mathbb{Q}$ or $\mathbb{R}$. If unspecified, $R$ should be taken to be $\mathbb{Z}$.

**Definition 3.1: Homology**

A *chain complex* $(A_i, \partial)$ is a collection of $R$-modules $A_i$ indexed by $\mathbb{Z}$ and boundary maps $\partial : A_i \to A_{i-1}$ so that $\partial^2 = 0$. The failure of the complex,
$$\cdots \xrightarrow{\partial} A_{i+1} \xrightarrow{\partial} A_i \xrightarrow{\partial} A_{i-1} \xrightarrow{\partial} \cdots$$

to be an exact sequence is measured by its *homology groups*,

$$H_i(A) := \ker(\partial : A_i \to A_{i-1}) / \operatorname{im}(\partial : A_{i+1} \to A_i).$$

We call elements in this kernel *cycles* and elements in this image *boundaries*, so homology is the group of cycles modulo the boundaries.

A morphism of chain complexes $f : (A_*, \partial_A) \to (B_*, \partial_B)$, or a *chain map*, is a collection of module homomorphisms $f_i : A_i \to B_i$ so that $f_{i-1} \circ \partial_A = \partial_B \circ f_i$. Equivalently, the following commutes,

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r, "\partial_A"] \arrow[d, "f_{i+1}"] & A_i \arrow[r] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r, "\partial_B"] & B_i \arrow[r] & B_{i-1} \arrow[r] & \cdots .
\end{tikzcd}$$

A chain map $f : A_i \to B_i$ can be easily seen to induce a map on homology $f_* : H_i(A) \to H_i(B)$.

A *chain homotopy* between chain maps $f, g : (A_*, \partial_A) \to (B_*, \partial_B)$ is a collection of maps $h_i : A_i \to B_{i+1}$ so that $h \circ \partial_A + \partial_B \circ h = f - g$.

If there is a chain map $f : (A_*, \partial_A) \to (B_*, \partial_B)$ and a chain map $g$ in the other direction so that $f \circ g$ and $g \circ f$ are chain homotopic to the identities, then we say $f$ is a *quasi-isomorphism*.

PROPOSITION 3.2. *If $f, g : (A_*, \partial_A) \to (B_*, \partial_B)$ are chain homotopic, then they induce the same map on homology. If $f : (A_*, \partial_A) \to (B_*, \partial_B)$ is a quasi-isomorphism, then $(A_*, \partial_A)$ and $(B_*, \partial_B)$ have isomorphic homology.*

PROOF. Let $[x] \in H_i(A)$ be represented by $x \in \ker(\partial_A) \subset A_i$. Then, as an element of $H_i(B)$,

$$[f(x) - g(x)] = [h \circ \partial_A x] + [\partial_B \circ h(x)].$$

Since $x \in \ker \partial_A$,

$$= [\partial_B \circ h(x)].$$

Since we quotient by the image of $\partial_B$,

$$= 0.$$

So $[f(x)] = [g(x)]$ for any $[x] \in H_i(A)$ and the induced maps on homology thus agree. $\square$

**Definition 3.3**

A *cochain complex* $(A_i, \mathrm{d})$ is like a chain complex but with coboundary maps $\delta : A_i \to A_{i+1}$ so that $\delta^2 = 0$. We extract from the complex its *cohomology groups*,

$$H^i(A) := \ker(\delta : A_i \to A_{i+1}) / \operatorname{im}(\mathrm{d} : A_{i-1} \to A_i),$$

which is the group of *cocycles* modulo *coboundaries*. We define *cochain maps* and *cochain homotopies* in the obvious way.

*Singular Homology.* Let $X$ be a topology space and let $C_n(X; R)$ be the (gigantic) free $R$-module generated by continuous maps $\Delta^n \to X$, for $\Delta^n$ the standard $n$-dimensional simplex. Elements of $C_n(X; R)$ are called *singular $n$-chains.*

We will orient our simplices in the sense that we keep track of an ordering of their vertices. We will give the boundary strata of a simplex compatible orientations induced from the ordering of their subcollection of vertices. Let $\Delta_i^n$ be the face of $\Delta^n$ excluding the $i$th vertex of $\Delta^n$. Given a map $\sigma : \Delta^n \to X$, we define its boundary,

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{\Delta_i^n}.$$

This extends by linearity to a map $C_n(X; R) \to C_{n-1}(X; R)$. Clearly $\partial^2 = 0$ and so we have a chain complex. We define the *singular homology of* $X$ as the homology of the singular chain complex,

$$H_n(X; R) := H_n(C_*(X; R)).$$

Given a continuous map $f : X \to Y$, we obtain a pushforward map $f_* : C_n(X; R) \to C_n(Y; R)$, given by post-composing a singular simplex $\Delta^n \to X$ with $f$ and extending linearly. It is almost obvious that $f_*$ is a chain map.

**Theorem 3.4**

If $f, g : X \to Y$ are homotopic, they induce the same same maps on homology $f_*, g_* : H_n(X; R) \to H_n(Y; R)$. Hence homology is an invariant of the homotopy type of a topological space.

Proof. By our previous proposition, it suffices to exhibit a chain homotopy between $f_*, g_* : C_n(X; R) \to C_n(Y; R)$. Let $F : X \times I \to Y$ be a homotopy between $f$ and $g$. Given a singular simplex $\sigma : \Delta^n \to X$, we can define a map,

$$H(\sigma) : \Delta^n \times I \xrightarrow{\sigma\times\mathrm{id}} X \times I \xrightarrow{F} Y.$$

We may split $\Delta^n \times I$ along diagonals into $n+1$ copies of $\Delta^{n+1}$, and interpret $H(\sigma)$ as a sum of $n+1$ singular $n+1$ simplices in $Y$. Extending by linearity, we thus obtain a map $H : C_n(X; R) \to C_{n+1}(X; R)$.

Note that $\partial H(\sigma)$ consists of several faces. There are the bottom and top faces coming from restricting $H(\sigma)$ to $\Delta^n \times \partial I$ which correspond to $f_*(\sigma)$ and $g_*(\sigma)$ respectively. And there are the side faces coming from restricting $H(\sigma)$ to $(\partial\Delta^n) \times I$, which correspond to $H(\partial\sigma)$. Analyzing orientations we conclude,

$$\partial H(\sigma) + H(\partial\sigma) = f_*(\sigma) - g_*(\sigma).$$

Hence, $H$ is a chain homotopy of $f_*$ and $g_*$. $\square$

*Singular Cohomology.* Define a *singular cochain* as an $R$-valued functional on the space of singular chains. The $R$-module of singular cochains is the dual group to the singular chains: $C^n(X; R) = \mathrm{Hom}(C_n(X), R)$. We define $\delta : C^n(X; R) \to C^{n+1}(X; R)$ dual to $\partial$, so that given $\varphi \in C^n(X; R)$ and $\sigma \in C_{n+1}(X)$, $\delta(\varphi)(\sigma) = \varphi(\partial\sigma)$. Clearly $\delta^2 = 0$ and so $C^n(X; R)$ becomes a cochain complex. We define the *singular cohomology* of $X$ as the cohomology of the singular cochain complex,

$$H^n(X; R) := H^n(C^*(X; R)).$$

*Simplicial Homology.* We say a space $X$ has a *simplicial structure*, if it can be decomposed into simplices in the following sense:

(1) There is a collection of maps $\sigma : \Delta^n \to X$, where $n$ may vary, so that each map is injective on the interior of $\Delta^n$.

(2) Every point of $X$ is in the image of one of the maps, and no point is in the image of two of the maps restricted to the interior of $\Delta^n$ for the same $n$.

(3) Each map $\sigma : \Delta^n \to X$ when restricted to a face of $\Delta^n$ agrees with one of the maps $\sigma' : \Delta^{n-1} \to X$.

Remark 3.5. This definition is not quite correct (every map should be uniquely determined by its face maps) but any structure of the above form can be refined (by twice applying barycentric subdivision) to obtain a true simplicial structure. $\triangle$

Let $X$ have a simplicial structure, and let $C_n^\Delta(X; R)$ be the free $R$-module generated by the structure maps $\Delta^n \to X$. This is a much smaller subcomplex of $C_n(X; R)$. By property (3), the boundary restricts to a map on this subcomplex. And we may thus define the *simplicial homology* of $X$ as,

$$H_n^{\Delta}(X;R) := H_n(C_*^{\Delta}(X;R)).$$

The advantage of this theory is we can actually compute this homology, and moreover we may do it purely combinatorially. It turns out this homology is independent of our triangulation and in fact $H_n^{\Delta}(X;R) = H_n(X;R)$ for any simplicial complex; we will not prove this directly, but instead observe simplicial homology as a special case of the following construction.

*Cellular Homology.* Let $X$ be a CW complex with a specified cell structure. We define $\mathcal{C}_n(X;R)$ as the free $R$-module generated by the $n$-cells of $X$. We define a boundary map $\partial : \mathcal{C}_n(X;R) \to \mathcal{C}_{n-1}(X;R)$ as follows. If $e$ is an $n$-cell,

$$\partial e = \sum_{f \subset X_{n-1}} d_{ef} \cdot f,$$

where the sum is over $(n-1)$-cells $f$ in $X$, and then we extend by linearity. Here the coefficient $d_{ef}$ is the degree of the map $S^{n-1} \to S^{n-1}$ given by composing the attaching map $\partial D^n \to X_{n-1}$ of $e$ with the quotient map $X_{n-1} \to S^{n-1}$ given by identifying the complement of $f$ to a point. Recall the *degree* of a map $S^{n-1} \to S^{n-1}$ can be computed as the image of the unit under the induced map on $H^{n-1}(S^{n-1};\mathbb{Z})$. Alternatively, one can homotope the map to be smooth and make an oriented count of a generic point's preimage.

It's not too hard to see $\partial^2 = 0$ and hence conclude $\mathcal{C}_*(X;R)$ form a chain complex. We define the *cellular homology* of $X$ as the homology of the cellular chain complex,

$$H_n(X;R) := H_n(\mathcal{C}_*(X;R)).$$

It's easy to check this coincides with simplicial homology for a simplicial complex. We will give an alternate definition of cellular homology later that makes it manifestly equal to singular homology.

*De Rham Cohomology.* Let $X$ be a smooth manifold. It has a tangent bundle $TX$ and its dual the cotangent bundle $T^*X$. Taking exterior powers of this bundle gives the bundles $\bigwedge^n T^*X$, whose fibres are wedge products of covectors.

Sections of the bundle $\bigwedge^n T^*X$ are called *differential $n$-forms*; the space of $n$-forms is denoted $\Omega^n(X)$. If $x^1, \ldots, x^m$ are local coordinates on $X$, then $TX$ has a local basis of sections $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$ and we denote the dual sections $\mathrm{d}x^1, \ldots, \mathrm{d}x^m$. We define the *exterior derivative* $\mathrm{d} : \Omega^n(X) \to \Omega^{n+1}(X)$ to be the unique $\mathbb{R}$-linear map satisfying the following properties:

(1) $\mathrm{d}f\,(V) = V(f)$ for any $f \in C^\infty(X)$ and $V \in \Gamma(TX)$.

(2) d satisfies a super-Leibniz rule,

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \mathrm{d}\beta.$$

(3) $\mathrm{d}^2 = 0$.

Locally, d is specified on an $n$-form by,

$$\mathrm{d}\left(\sum_{|I|=n} f_I \,\mathrm{d}x^I\right) = \sum_{i=1}^{m} \sum_{|I|=n} \frac{\partial f_I}{\partial x^i} \,\mathrm{d}x^i \wedge \mathrm{d}x^I \,.$$

The groups $\Omega^n(X)$ thus define a cochain complex over $\mathbb{R}$ with differential the exterior derivative d, called the de Rham complex. The *deRham cohomology* of $X$ is the cohomology of the de Rham complex,

$$H^n_{dR}(X) := H^n(\Omega^*(X); \mathbb{R}).$$

A consequence of the Whitney approximation theorem is that we if consider not all singular chains, but only those that are regular enough to be integrated over, we still obtain the same singular cohomology theory. One then has a map,

$$F : \Omega^n(X; R) \to \mathrm{Hom}(C_n(X), \mathbb{R}) = C^n(X; \mathbb{R}) \quad \text{by} \quad \omega \mapsto F_\omega(\sigma) = \int_{\Delta^n} \sigma^*\omega.$$

By computing locally and patching together with the Mayer-Vietoris theorem (discussed later) one concludes that $F$ is an isomorphism. Hence, for any smooth manifold, the de Rham cohomology coincides with the singular cohomology; this is de Rham's theorem.

*Morse Homology.* One final (more esoteric but dear to my heart and research) cousin of singular homology for compact manifolds is defined as follows. Let $f$ be a *Morse function* on a smooth closed manifold $X$. That is, the critical points of $f$ are non-degenerate, or equivalently the Hessian of $f$ at its critical points is non-singular. The *index* of a critical point $x$ of $f$ is the dimension of the negative eigenspace of the Hessian of $f$ at $x$.

If we pick a Riemannian metric $g$ on $X$, it defines a gradient vector field $\nabla f$ by the condition $g(\nabla f, V) = V(f)$ for any $V \in \Gamma(TX)$. This defines a complete gradient flow $\varphi_t$. We define the *unstable and stable manifolds* of a critical point $x$ respectively as,

$$W^u(x; f, g) = \{p \in X : \lim_{t \to -\infty} \varphi_t(p) = x\}$$
$$W^s(x; f, g) = \{p \in X : \lim_{t \to \infty} \varphi_t(p) = x\}.$$

These are open disks of dimension $\mathrm{ind}(x)$ and $\dim(X) - \mathrm{ind}(x)$ respectively by the Hartman–Grobman theorem using the flow to provide an explicit diffeomorphism.

We say that the pair $(f, g)$ is *Morse–Smale* if $f$ is Morse and for any pair of critical points, $x, y$ the intersection of $W^u(x; f, g)$ and $W^s(y, f; g)$ is transverse (i.e. the tangent spaces of the submanifolds span $TX$ at any point of intersection); it is a consequence of the Sard–Smale theorem that generically pairs $(f, g)$ are Morse–Smale. In this case, the intersection is a manifold of dimension $\mathrm{ind}(x) - \mathrm{ind}(y)$. Note the intersection also has a free $\mathbb{R}$-action from the flow. Quotienting by this action, we obtain a manifold of dimension $\mathrm{ind}(x) - \mathrm{ind}(y) - 1$, which we denote, $\mathcal{M}(x, y)$. Moreover, $\mathcal{M}(x, y)$ has a natural compactification as a manifold with corners, whose $k$th boundary stratum consists of $k$-times "broken flow lines" connecting $x$ to $y$ and stopping at $k$ intermediate critical points.

We now define the Morse complex for $(f, g)$ a Morse–Smale pair, with chain group $CM_n(X; f, g)$ the free $\mathbb{Z}_2$-module generated by the critical points of $f$ of index $n$. The set of such critical points is finite by the Morse condition. We define a boundary operator $\partial : CM_n(X) \to CM_{n-1}(X)$ by the linear extension of,

$$\partial x = \sum_{\mathrm{ind}(y)=n-1} \#\mathcal{M}(x, y) \cdot y.$$

Counting dimensions, $\mathcal{M}(x, y)$ is a zero dimensional compact manifold, i.e. a finite set of points, and so we can count the set modulo 2. We can also define this complex over $\mathbb{Z}$ using the observation that $W^u$ and $W^s$ are orientable, and the $\mathbb{R}$-action direction carries a natural orientation; we would then count $\mathcal{M}(x, y)$ as an oriented set of points.

To see that $\partial^2 = 0$, we note that the $\langle \partial^2 x, y \rangle$ consists of broken flow lines connecting $x$ to $y$ and stopping at one intermediate critical point. But by our description of the compactification, this is precisely the boundary of the 1-manifold $\mathcal{M}(x, y)$. A 1-manifold has no boundary points modulo 2 or counting orientations, and so $\langle \partial^2 x, y \rangle = 0$ for any $x$ and $y$.

Thus we obtain the *Morse homology* of $X$ as the homology of the Morse complex,

$$HM_n(X; f, g) := H_n(CM_*(X; f, g); \mathbb{Z}_2).$$

This again agrees with singular homology, which we now show. We may define a map $F : CM_n(X) \to C_n(X; \mathbb{Z}_2)$ by sending a critical point $x$ to the singular map extending the inclusion of the descending manifold of $X$ by compactifying (strictly we should be dealing with some kind of current instead of singular chains). This is a chain map because the boundary of the descending manifold consists of the descending manifolds of critical points of index one less (counted mod 2 or with appropriate signs). We define $G : C_n(X;\mathbb{Z}_2) \to CM_n(X)$ that sends a generic singular $n$-simplex $\sigma$ to a sum of index $n$ critical points $x$ with coefficients counting mod 2 or with orientations the flow lines passing through the image of $\sigma$ at $t = 0$ and terminating at $x$. This is a chain map because the space of flow lines from $\sigma$ terminating at $q$ of index $n-1$ is a one-manifold with boundary given by flow lines of $\partial\sigma$ terminating at $q$ (i.e. $\langle G(\partial\sigma), q\rangle$) and flow lines of $\sigma$ terminating at some $p$ of index $n$ followed by flow lines from $p$ to $q$ (i.e. $\langle \partial(G\sigma), q\rangle$).

To a singular $n$-simplex $\sigma$, we associated $H(\sigma)$ which is the $n+1$-simplex given by compactifying the full forward gradient flow of $\sigma$. This extends to a map $H : C_n(X;\mathbb{Z}_2) \to C_{n+1}(X;\mathbb{Z}_2)$. Note that $\partial H(\sigma)$ consists of $\sigma$ itself, the forward gradient flow of $\partial\sigma$, and the descending manifolds of the critical points $\sigma$ flows to at infinity. That is, keeping track of signs,

$$\partial H(\sigma) + H(\partial\sigma) = F \circ G(\sigma) - \sigma,$$

hence $F \circ G$ is chain homotopic to the identity. On the other hand, $G \circ F$ is verbatim the identity on chain groups. Hence $F$ is a quasi-isomorphism and so our chain complexes have the same homology.

It is possible to extend to the case of compact manifolds $X$ with boundary. We can set up a Morse–Smale pair as usual under the additional assumption that $\nabla f$ points inward along all of $\partial X$. We can define the Morse complex like usual and obtain homology $HM_*(X)$ that again agrees with singular homology. If we instead assume that $\nabla f$ points outward along $\partial X$, we will define again Morse homology in the same way. However in this case the Morse homology computes something called the relative homology $H_*(X, \partial X)$ that we will shortly study more.

For more details on our construction of Morse homology and many more aspects of the theory, one can read the lecture notes of Hutchings [**Hutchings**].

Remark 3.6. Note, by studying the unstable manifolds which are homeomorphic to open balls (i.e. cells) and how they glue (i.e. to lower dimensional cells), we see that Morse homology endows every compact $n$-manifold with a CW structure. We will use Morse theory for convenient proofs of many fundamental results down the road. △

**3.2. Geometric Constructions.** We will proceed to consider a number of extensions of the basic homology theory and a couple relevant theorems, all based on some geometric ideas. Throughout we deal with singular homology over $\mathbb{Z}$ acknowledging that other theories and coefficients work identically.

*Reduced Homology.* What is $H_0(X)$? We know a singular 0-chain is a linear combination of maps $* \to X$. This is always a cycle since $\partial C_0(X) = 0$. A 0-chain is a boundary if there is a sum of maps $I \to X$ whose boundary is our 0-chain. Equivalently, two maps $* \to X$, representing points $x_1, x_2 \in X$ (their image) are the same in homology if and only if $x_1$ and $x_2$ are joined by a path.

We conclude $H_0(X)$ is a free $\mathbb{Z}$-module generated by the path components of $X$. This can be a little awkward since we might hope that $H_n(X) = 0$ for $X$ contractible; indeed this is true for $n > 1$ since $H_n(X) = H_n(*) = 0$. But, $H_0(*) = \mathbb{Z}$. To deal with this, we will sometimes want to reduce the dimension of the zeroth homology by one. Thus we define the *reduced homology* $\widetilde{H}_n(X)$ of $X$ as,

$$\widetilde{H}_n(X) = \begin{cases} H_n(X) & n \neq 0 \\ H_0(X)/[\sum x_i] & n = 0, \end{cases}$$

where the $x_i$ are maps $* \to X$ with image $x_i$, having picked one point $x_i$ in each path component of $X$. More formally, we could modify the chain complex $C_n(X)$ by setting $C_{-1}(X) = \mathbb{Z}$. And we could set $\partial : C_0(X) \to C_{-1}(X)$ to send any singular 0-simplex $* \to X$ to 1 and extend linearly. The homology of this extended complex will be exactly $\widetilde{H}_n(X)$.

*Homology of a Pair.* One has the following extension of the usual definition of homology. Given a subspace $A \subset X$, let $C_n(X, A)$ denote the quotient $C_n(X)/C_n(A)$. These are *relative singular $n$-chains.* This is a free $\mathbb{Z}$-module generated by simplices that don't entirely land within $A$. The boundary map clearly restricts to this subcomplex and the resulting homology $H_n(X, A)$ is called the *relative homology* of $(X, A)$.

EXERCISE 3.7. Show $H_n(X, *) = \widetilde{H}_n(X)$. ▲

**Theorem 3.8: Long Exact Sequence of a Pair and a Triple**

Given a topological pair $(X, A)$, its homology forms a long exact sequence,

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots .$$

The maps $i_*, j_*$ are induced by the inclusions $A \to X$ and $(X, \varnothing) \to (X, A)$ respectively. The map $\partial_*$ sends a relative $n$-cycle to its boundary, which is an $(n-1)$-cycle in $A$.

More generally, for a triple $B \subset A \subset X$, there is a long exact sequence,

$$\cdots \to H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A,B) \to \cdots.$$

PROOF. By definition of relative chains, there is a short exact sequence,

$$0 \to C_n(A) \xrightarrow{i_*} C_n(X) \xrightarrow{j_*} C_n(X,A) \to 0.$$

By the snake lemma, this becomes a long exact sequence in homology, with the maps as described. The triple case is the same. □

To simplify the computation of homology of a pair, one can apply the following result.

**Theorem 3.9**

Given a topological pair $(X, A)$ so that the inclusion $A \to X$ has homotopy cofibre $B = X \cup_A CA$, then $H_n(X, A) \cong \widetilde{H}_n(B)$. In particular, if $A \to X$ is a cofibration, for example a CW pair, then $H_n(X, A) = \widetilde{H}_n(X/A)$.

REMARK 3.10. The general proof is a little involved, using the concept of transformators. Fortunately for the case of CW pairs, which is what we care most about, this equality follows quite readily from the construction of cellular homology. △

We use the following important lemma.

LEMMA 3.11 (Refinement Lemma). *Let $X$ be a topological space with a collection $\mathcal{U}$ of sets whose interiors cover $X$. Let $C_n^{\mathcal{U}}(X)$ be the subcomplex of $C_n(X)$ generated by singular simplices which land inside sets of $\mathcal{U}$. Then the inclusion $C_n^{\mathcal{U}}(X) \to C_n(X)$ is a quasi-isomorphism. I.e. cycles in $X$ are homologous to ones subordinate to $\mathcal{U}$ and homologous cycles subordinate $\mathcal{U}$ differ by the boundary of a chain subordinate to $\mathcal{U}$.*

PROOF. Given a singular simplex $\sigma : \Delta^n \to X$, it can be decomposed via multiple applications of barycentric subdivision into a sum of simplices subordinate to $\mathcal{U}$. It's geometrically clear this subdivided chain is homologous to $\sigma$. Moreover, its geometrically clear that if two cycles subordinate to $\mathcal{U}$ are cohomologous, we can subdivide their difference to be the boundary of a chain subordinate to $\mathcal{U}$. □

Proof. (of Thm. 3.9) Cover $B$ by $CA$ and $X \cup C'A$, the complement of a neighbourhood of the cone point; call this two set open cover $\mathcal{U}$. By the refinement lemma, the homology can be computed subordinate to $\mathcal{U}$. We have,

$$C_n^{\mathcal{U}}(B, CA) = C_n^{\mathcal{U}}(B)/C_n^{\mathcal{U}}(CA) = C_n(X \cup C'A)/C_n(C'A) = C_n(X \cup C'A, C'A).$$

We conclude the homology of $(B, CA)$ is the same as that of $(X \cup C'A, C'A)$, which by homotopy invariance is also that of $(X, A)$. Hence,

$$\widetilde{H}_n(B) = H_n(B, \mathrm{pt}) = H_n(B, CA) = H_n(X, A).$$

If $(X, A)$ is a cofibrant pair, $B \cong X/A$ giving the second statement. □

*Excision.* This is a crucial result for many geometric arguments.

**Theorem 3.12: The Excision Theorem**

Let $(X, A)$ be a topological pair. Suppose $B \subset A$ is such that the closure of $B$ is contained in the interior of $A$. Then the inclusion map induces an isomorphism in homology,

$$H_n(X \setminus B, A \setminus B) \cong H_n(X, A).$$

Proof. Consider the collection $\mathcal{U}$ consisting of $A$ and $B^c$. By our assumption, the interiors of this collection cover $X$. We have,

$$C_n^{\mathcal{U}}(X, A) = C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A) = C_n(X \setminus B)/C_n(A \setminus B).$$

By the refinement lemma, we obtain our result. □

*Mayer–Vietoris.* Homology possesses an analogue of the Seifert–Van Kampen theorem that allows for ease of computations, by splitting a space into simpler pieces.

**Theorem 3.13: Mayer–Vietoris Theorem**

Suppose $X$ is the union of the interiors of subspaces $A$ and $B$. Then there is a long exact sequence,

$$\cdots \to H_n(A \cap B) \xrightarrow{i_*} H_n(A) \oplus H_n(B) \xrightarrow{j_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \to \cdots .$$

Here $i_*$ denotes the difference of the maps induced by the two inclusions, and $j_*$ is the sum of the maps induced by the two inclusions. The map $\partial_*$ can described as follows. If $\sigma \in H_n(X)$, it is homologous to a sum $x + y$ of chains in $A$ and $B$. We have $\partial x = -\partial y \in H_{n-1}(A \cap B)$. Then, we have that $\partial_* \sigma = \partial x$.

PROOF. Take $\mathcal{U}$ to be the collection $A$ and $B$. Then we have the short exact sequence,

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n^{\mathcal{U}}(X) \to 0,$$

where the first map sends $\sigma$ to $(\sigma, -\sigma)$ and the second sends $(x, y)$ to $x + y$. By the refinement lemma, the homology of $C_n^{\mathcal{U}}(X)$ is $H_n(X)$. Thus by the snake lemma, we obtain the desired sequence. We easily deduce the maps are as described. □

*A Rigourous Construction of Cellular Homology.* First we need to actually know some homology.

PROPOSITION 3.14. *The homology of the $n$-sphere is,*

$$H_m(S^n) = \begin{cases} \mathbb{Z} & m = 0 \text{ or } n \\ 0 & \text{else.} \end{cases}$$

PROOF. We may cover the sphere by open neighbourhoods of the two hemispheres. These are contractible and their intersection is homeomorphic to the $n-1$ sphere. We then proceed by induction. Since we know the homology of a pair of points, we may apply Mayer–Vietoris to find $H_m(S^1)$. Inductively, the computation of $H_m(S^n)$ determines from Mayer–Vietoris the homology of $H_m(S^{n+1})$. □

PROPOSITION 3.15.

$$H_m(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & m = n \\ 0 & \text{else.} \end{cases}$$

PROOF. By Theorem 3.9,

$$H_m(D^n, S^{n-1}) = \widetilde{H}_m(D^n/S^{n-1}) = \widetilde{H}_m(S^n).$$

□

PROPOSITION 3.16. *The reduced homology of a bouquet of $n$-sphere is,*

$$\widetilde{H}_m\left(\bigvee_{\alpha \in A} S^n_\alpha\right) = \begin{cases} \bigoplus_{\alpha \in A} \mathbb{Z} & m = n \\ 0 & \text{else.} \end{cases}$$

PROOF. In fact, for any space $X_\alpha$,

$$\widetilde{H}_m\left(\bigvee_{\alpha \in A} X_\alpha\right) = \bigoplus_{\alpha \in A} \widetilde{H}_m(X_\alpha).$$

To see this, the bouquet of spaces is the disjoint union quotiented by identifying base points. Thus by Theorem 3.9, our homology is the homology of $\bigsqcup_{\alpha \in A} X_\alpha$ relative to the union of base points. It is easy to see, the homology of the disjoint union of the $X_\alpha$ is the direct sum of the homologies of the $X_\alpha$. Thus our result follows from the long exact sequence of a pair. $\square$

Now given a CW complex $X$, note that $X_n/X_{n-1}$ is a bouquet of $n$ spheres indexed by the cells of $X_n$. Hence we know the relative homology $H_*(X_n, X_{n-1})$.

Note also that $H_k(X_n) = 0$ for $k > n$. This follows from the fact $H_k(X_0) = 0$ for $k > 0$ and applying the long exact sequence of a pair inductively. Lastly, note that the map $H_k(X_n) \to H_k(X)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$. If $X$ is finite dimensional, this follows directly from the long exact sequence of a pair. For infinite dimensional spaces, note each singular simplex must lie in a finite subcomplex of $X$, from which the conclusion again follows.

Now for the CW complex $X$, we can construct the following commutative diagram, where the diagonal sequences are truncations of the long exact sequence of a pair, and the horizontal maps are defined to make the diagram commute.

$$
\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X_{n+1}) = H_n(X) \arrow[ur] & & & \\
 & & H_n(X_n) \arrow[ur, "i_*"] \arrow[dr, "j_*"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X_{n+1}, X_n) \arrow[ur, "\partial_*"] \arrow[rr, "\partial_{n+1}"] & & H_n(X_n, X_{n-1}) \arrow[rr, "\partial_n"] \arrow[dr, "\partial_*"] & & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X_{n-1}) \arrow[ur, "j_*"] \arrow[dr] & & \\
 & & & 0 \arrow[ur] & & H_{n-1}(X_n) = H_{n-1}(X) \arrow[dr] & \\
 & & & & & & 0
\end{tikzcd}
$$

Note the composition $\partial_n \circ \partial_{n+1}$ includes the segment $\partial_* \circ j_*$ of the short exact sequence of a pair. And hence $\partial^2 = 0$. Thus, the horizontal sequence forms a chain complex. Note from our description of the relative homology groups that $H_n(X_n, X_{n-1}) \cong \mathcal{C}_n(X; \mathbb{Z})$. By following the arrows in the above diagram, it can be seen that $\partial$ is given by the same description as our boundary in cellular homology. Hence we have recovered the cellular chain complex and the cellular homology. We can also show this description of cellular homology coincides with singular homology.

**Theorem 3.17**

If $X$ is a CW complex, its cellular and singular homologies coincide.

PROOF. We know from the diagram, $H_n(X)$ is the quotient of $H_n(X_n)$ by the image of the boundary map $\partial_*$. Since $j_*$ is injective, $\operatorname{im}(\partial_*) = \operatorname{im}(\partial_{n+1})$. On the other hand,

$$H_n(X_n) = \operatorname{im}(j_*) = \ker(\partial_*) = \ker(j_* \circ \partial_*) = \ker(\partial_n).$$

Thus,

$$H_n(X) = H_n(X_n)/\operatorname{im}(\partial_*) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}) = H_n(\mathcal{C}_*(X); \mathbb{Z}).$$

$\square$

*Eilenberg–Steenrod Axioms.* There is an axiomatic approach to homology which we now describe.

**Definition 3.18: Eilenberg–Steenrod Axioms**

A sequence of functors,

$$H_q(\cdot, \cdot; \pi) : \mathbf{hTopPair} \to \mathbf{AbGp},$$

from the homotopy category of topological pairs to the category of abelian groups together with natural transformations,

$$\partial : H_q(X, A; \pi) \to H_{q-1}(A; \pi) := H_{q-1}(A, \varnothing; \pi),$$

is called a *homology theory* if it satisfies the following axioms.

**Dimension:** If $X$ is a point, $H_0(X; \pi) = \pi$ and $H_q(X; \pi) = 0$ for all $q \neq 0$.

**Exactness:** Inclusion maps $A \to X$ and $(X, \varnothing) \to (X, A)$ induce a long exact sequence,

$$\cdots \to H_q(A; \pi) \to H_q(X; \pi) \to H_q(X, A; \pi) \xrightarrow{\partial} H_{q-1}(A; \pi) \to \cdots .$$

**Excision:** If $X$ is the union of the interiors of $A$ and $B$ then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism,

$$H_*(A, A \cap B; \pi) \to H_*(X, B; \pi).$$

**Additivity:** If $(X, A)$ is the disjoint union of pairs $(X_i, A_i)$, then the inclusions $(X_i, A_i) \to (X, A)$ induce an isomorphism,

$$\bigoplus_i H_*(X_i, A_i; \pi) \to H_*(X, A; \pi).$$

**Weak Equivalence:** If $f : (X, A) \to (Y, B)$ is a weak equivalence (see Theorem 4.25) then it induces an isomorphism,

$$f_* : H_*(X, A; \pi) \to H_*(Y, B; \pi).$$

**Theorem 3.19**

The Eilenberg–Steenrod axioms characterize a unique homology theory. Hence any sequence of functors satisfying these axioms agrees with the singular homology.

PROOF. We have showed that singular homology satisfies all of these properties, with the exception of Weak Equivalence which we prove in Proposition 4.33. We will prove later in Theorem 4.31 that every space $X$ has a weak equivalence $f : X^{CW} \to X$ from a CW complex and so it suffices to prove that any homology theory for CW complexes agrees with cellular and hence singular homology.

One can deduce that all the geometric constructions of this section, like reduced homology and Mayer–Vietoris, can be obtained just from our axioms. In particular, the homology of the sphere and the relative homology of the disk to its boundary follow from the axioms. This allows us to construct the cellular chain complex as a sequence of relative homology groups of successive skeleta $H_*(X_n, X_{n-1})$ for any homology theory. The same arguments we applied before show the resulting cellular homology agrees with the underlying homology theory and has the usual geometric interpretation of its differential. Hence the cellular homology (and consequently the singular homology) of any CW complex $X$ is equal to its homology in our arbitrary homology theory. □

Of course totally analogous constructions hold for cohomology.

This argument gives us an alternate route to proving a certain homology theory agrees with singular homology: we just need to verify these axioms. This is a reasonable way to show that simplicial and Morse homology, and deRham cohomology all reproduce singular (co)homology.

If one wants to obtain something like homology that gives new algebraic homotopy invariants of space with similar geometric properties, we need to weaken these axioms. What turns out to be the best way to do this may seem surprising because it involves removing the most trivial of the axioms.

**Definition 3.20**

A *generalized* or *extraordinary homology theory* is a sequence of functors $E_q(\cdot,\cdot) : \mathbf{hTopPair} \to \mathbf{AbGp}$ that satisfy all of Eilenberg–Steenrod axioms, except possibly the dimension axiom.

Much of modern algebraic topology is focused on studying these theories and we will meet a few of the most important ones later.

### 3.3. Algebraic Constructions.

*Algebraic Preliminaries: Tor and Ext.* We will consider coefficients in the category of $\mathbb{Z}$-modules, i.e. abelian groups. Tensoring with a group $B$ is a *right exact functor*, meaning that if,

$$M \to N \to A \to 0 \quad \text{is exact, then so is} \quad M \otimes B \to N \otimes B \to A \otimes B \to 0.$$

However, this need not be true on the left. Conversely, the functor $\mathrm{Hom}(\cdot, B)$ is *left exact*, so that given an exact sequence, $M \to N \to A \to 0$, the sequence,

$$\mathrm{Hom}(M, B) \leftarrow \mathrm{Hom}(N, B) \leftarrow \mathrm{Hom}(A, B) \leftarrow 0$$

is also exact. The reverse statement need not be true. The Tor and Ext functors measure this failure to be exact on the other sides.

**Definition 3.21: Tor and Ext**

Given an $\mathbb{Z}$-module $A$, we may always find a *free resolution* of $A$, which is a short exact sequence,

$$0 \to F_1 \to F_0 \to A \to 0,$$

for $F_1, F_0$ free abelian groups (just use a group presentation).

Given $\mathbb{Z}$-modules $A$ and $B$, the *Tor functor* $\mathrm{Tor}(A, B)$ is the unique $\mathbb{Z}$-module so that, for a given free resolution $F_1 \to F_0 \to A$ of $A$,

$$0 \to \mathrm{Tor}(A, B) \to F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0,$$

is exact.

The *Ext functor* $\mathrm{Ext}(A, B)$ is the unique $\mathbb{Z}$-module so that,

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(F_0, B) \to \mathrm{Hom}(F_1, B) \to \mathrm{Ext}(A, B) \to 0$$

is exact. Both definitions are independent of the choice of free resolution.

Proposition 3.22. *Tor is symmetric:* $\operatorname{Tor}(A, B) = \operatorname{Tor}(B, A)$.

Proof. We prove this with spectral sequences which we will define in a few sections. Suppose we have free resolutions,

$$0 \to F_1 \to F_0 \to A \to 0 \quad \text{and} \quad 0 \to G_1 \to G_0 \to B \to 0.$$

Tensoring together the $F_i$ and $G_j$ gives a bicomplex. We may extract a spectral sequence from each of the two gradings. In one direction, we have the following first three pages (we use $\operatorname{Tor}(A, F) = 0$ when $F$ is free, which we prove shortly).

$$
\begin{tikzcd}
1 & F_1 \otimes G_0 \arrow[d] & F_1 \otimes G_1 \arrow[d] \\
0 & F_0 \otimes G_0 & F_0 \otimes G_1 \\
E^0 & 0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 0 & 0 \\
0 & A \otimes G_0 & A \otimes G_1 \arrow[l] \\
E^1 & 0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 0 & 0 \\
0 & A \otimes B & \operatorname{Tor}(B, A) \\
E^2 & 0 & 1
\end{tikzcd}
$$

At which point the spectral sequence terminates. If we apply the spectral sequence in the other direction, we have the following first three pages.

$$
\begin{tikzcd}
1 & F_0 \otimes G_1 \arrow[d] & F_1 \otimes G_1 \arrow[d] \\
0 & F_0 \otimes G_0 & F_1 \otimes G_0 \\
E^0 & 0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 0 & 0 \\
0 & F_0 \otimes B & F_1 \otimes B \arrow[l] \\
E^1 & 0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 0 & 0 \\
0 & A \otimes B & \operatorname{Tor}(A, B) \\
E^2 & 0 & 1
\end{tikzcd}
$$

At which point the spectral sequence terminates. In either case, the spectral sequence computes the homology of the total bigraded complex. We conclude that $\operatorname{Tor}(A, B) = \operatorname{Tor}(B, A)$. □

Proposition 3.23. *The following hold.*

*(1)* $\operatorname{Tor}(A, F) = 0$ *for* $F$ *free.*

*(2)* $\operatorname{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}$.

*(3) For* $A, B$ *finitely generated,* $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(A) \otimes \operatorname{Tor}(B)$.

*(4)* $\mathrm{Tor}(A, \mathbb{Q}) = 0$ *for any* $A$.

*(5)* $\mathrm{Ext}(F, B) = 0$ *for* $F$ *free.*

*(6)* $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}$.

*(7)* $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$.

*(8)* $\mathrm{Ext}(A, \mathbb{Q}) = 0$ *for any* $A$.

PROOF. (1) Given a free resolution $0 \to F_1 \to F_0 \to A \to 0$, we get an injective morphism of free abelian groups $F_1 \to F_0$. But then the induced map $F_1 \otimes F \to F_0 \otimes F$ is clearly injective too if $F$ is free. Hence,

$$\mathrm{Tor}(A, B) = \ker(F_1 \otimes F \to F_0 \otimes F) = 0.$$

(2) $\mathbb{Z}_m$ has the free resolution,

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}_m \to 0.$$

Tensoring with $\mathbb{Z}_n$ gives an exact sequence,

$$0 \to \mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \to \mathbb{Z}_n \xrightarrow{\times m} \mathbb{Z}_n \to \mathbb{Z}_m \otimes \mathbb{Z}_n \to 0.$$

Thus,

$$\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) = \ker(\mathbb{Z}_n \xrightarrow{\times m} \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}.$$

(3) This follows from (1) and (2), plus the fundamental theorem of finitely generated abelian groups and the fact Tor distributes over direct sums.

(4) Given a free resolution $0 \to F_1 \to F_0 \to A$, $F_1 \to F_0$ is injective. Tensoring a free group $F$, i.e. no torsion, with $\mathbb{Q}$ has the consequence that $f \otimes (p/q) = f' \otimes (p'/q')$ if and only if $pq'f = p'qf'$. In particular, if $f \otimes (1/q) = e \otimes 0$ then $f = 0$ and hence $F_1 \otimes \mathbb{Q} \to F_0 \otimes \mathbb{Q}$ is also injective.

(5) We may take the free resolution $0 \to 0 \to F \to F \to 0$, from which the conclusion is obvious.

(6) Again take the free resolution $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}_m \to 0$. Applying $\mathrm{Hom}(\cdot, \mathbb{Z}_n)$ gives,

$$0 \to \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \xrightarrow{\circ \times m} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \to \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \to 0.$$

Recall that $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(n,m)}$ and $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) \cong \mathbb{Z}_n$, both being determined by where we send the unit. So the above becomes,

$$0 \to \mathbb{Z}_{(n,m)} \to \mathbb{Z}_n \xrightarrow{\times m} \mathbb{Z}_n \to \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \to 0.$$

Thus,

$$\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathrm{coker}(\mathbb{Z}_n \xrightarrow{\times m} \mathbb{Z}_n) = \mathbb{Z}_n / m\mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}.$$

(7) Using the same free resolution and Applying $\mathrm{Hom}(\cdot, \mathbb{Z})$ gives,

$$0 \to \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\circ \times m} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \to 0.$$

We have $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ while $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$. So this becomes,

$$0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \to 0.$$

Hence $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m$ as claimed.

(8) It is a classic fact that $\mathbb{Q}$, and any divisible group, is an injective $\mathbb{Z}$-module, so that given an injective map $F_1 \to F_0$, any element of $\mathrm{Hom}(F_1, \mathbb{Q})$ lifts to an element of $\mathrm{Hom}(F_0, \mathbb{Q})$. Hence, $\mathrm{Hom}(\cdot, \mathbb{Q})$ is right exact and $\mathrm{Ext}(A, \mathbb{Q}) = 0$ for any $A$. $\square$

*Universal Coefficient Theorem.* Recall that for a group $G$, $C_*(X; G) = C_*(X; \mathbb{Z}) \otimes G$ and $C^*(X; G) = \mathrm{Hom}(C_*(X; \mathbb{Z}); G)$. This need not be true on the level of homology, since tensoring and Hom fail to be exact. However, it's not so surprising that we can relate the homology between different coefficients as well as relate the homology and the cohomology using the Ext and Tor.

First, we have a simple way to relate homology with different coefficients.

Proposition 3.24 (Coefficient Long Exact Sequence). *Suppose $C_*(X)$ is a chain complex of free abelian groups. Suppose there is an exact sequence of abelian groups $0 \to A \to B \to C \to 0$. Then there is a long exact sequence in homology,*

$$\cdots \to H_n(X; A) \to H_n(X; B) \to H_n(X; C) \xrightarrow{\beta} H_{n-1}(X; A) \to \cdots.$$

*And in cohomology,*

$$\cdots \to H^n(X; A) \to H^n(X; B) \to H^n(X; C) \xrightarrow{\beta} H^{n+1}(X; A) \to \cdots.$$

*The maps $\beta$ are called the* Bockstein homomorphisms.

Proof. Since the chain complex is free, we have that $\mathrm{Tor}(G, C_n(X)) = \mathrm{Ext}(C_n(X), G) = 0$. Hence $\cdot \otimes C_n(X)$ and $\mathrm{Hom}(C_n(X), \cdot)$ are exact functors and we obtain short exact sequences,

$$0 \to C_n(X; A) \to C_n(X; B) \to C_n(X; C) \to 0,$$

and,

$$0 \to C^n(X; A) \to C^n(X; B) \to C^n(X; C) \to 0.$$

The result follows from the snake lemma. $\square$

**Theorem 3.25: Universal Coefficient Theorem**

Suppose $C_*(X)$ is a chain complex of free abelian groups. There are short exact sequences for any $n \in \mathbb{Z}$ and any abelian group $G$,

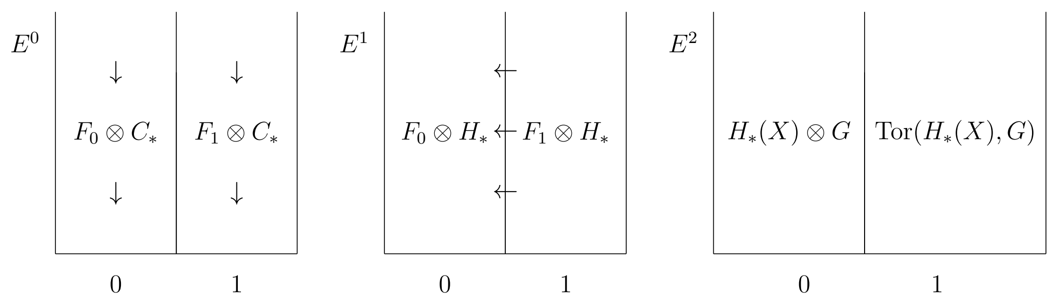
(i) $$0 \to H_n(X) \otimes G \to H_n(X;G) \to \mathrm{Tor}(H_{n-1}(X), G) \to 0,$$

(ii) $$0 \to H^n(X) \otimes G \to H^n(X;G) \to \mathrm{Tor}(H^{n+1}(X), G) \to 0,$$

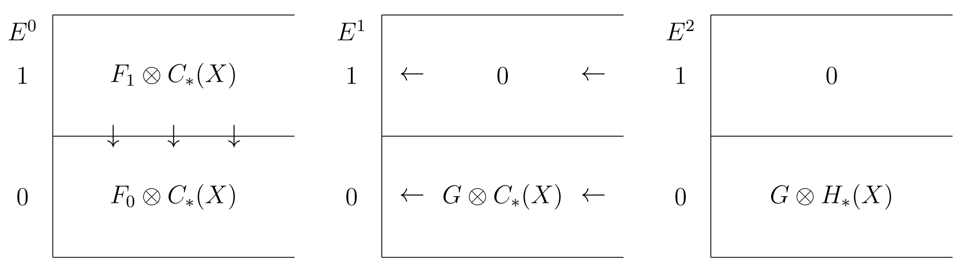
(iii) $$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X), G) \to 0.$$

These sequences are canonical. They split non-canonically.

Proof. (i) Let $0 \to F_1 \to F_0 \to G \to 0$ be a free resolution of $G$. Then $0 \to F_1 \to F_0 \to 0$ defines a complex and tensoring with $C_*(X)$, we obtain a bicomplex. We can look at the spectral sequence associated to each grading. In one direction, we degenerate after the following three pages,




While in the direction, we degenerate after the following three pages,




In both spectral sequences, the $E^\infty$ page should give the homology of the total bicomplex. Thus we have,

$$\bigoplus_{p+q=n} E^\infty_{p,q} = (H_n(X) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(X); G) = H_n(X; G).$$

(ii) This follows from the same argument applied to the bicomplex given by tensoring with $C^*(X)$.

(iii) The full chain complex $C_*(X)$ decomposes as a direct sum over $n$ of short complexes,

$$0 \to B_n \xrightarrow{\partial_{n+1}} Z_n \to 0,$$

of boundaries and cycles. This short sequence is a resolution for $H_n(X)$. Applying the $\mathrm{Hom}(\cdot, G)$ functor gives,

$$0 \to \mathrm{Hom}(H_n(X), G) \to \mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G) \to \mathrm{Ext}(H_n(X), G) \to 0.$$

Thus the short co-complex,

$$0 \to \mathrm{Hom}(Z_n, G) \xrightarrow{\delta_{n+1}} \mathrm{Hom}(B_n, G) \to 0,$$

has $(n+1)$st homology $\mathrm{Ext}(H_n(X), G)$ and $n$th homology $\mathrm{Hom}(H_n(X), G)$.

The direct sum of the homologies of all these short co-complexes must be the homology of $H^*(X; G)$. Taking the relevant homology from the $Z_n, B_n$ and $Z_{n-1}, B_{n-1})$ complexes, we have,

$$H^n(X; G) = \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}, G).$$

□

The universal coefficient theorem has the following corollary which is simpler to remember and is used all time.

**Theorem 3.26**

Suppose the homology groups $H_n(X)$ of a space are all finitely generated. Then,

$$\boxed{H^n(X) = H_n^{\text{free}}(X) \oplus H_{n-1}^{\text{tor}}(X),}$$

where free and tor denote the free and torsion parts of the homology.

Proof. This is immediate from the universal coefficient theorem if we remember the simple description of torsion for finitely generated abelian groups.

□

*Künneth Formula.* What can we say about the homology of a product of spaces? If we have two CW complexes $X, Y$, we can obtain a cell structure on $X \times Y$ whose cells are products of pairs of cells from $X$ and $Y$ with attaching maps given in the obvious way from attaching maps for $X$ and $Y$.

This means that $X \times Y$ has a cellular complex which is the tensor product of the complexes of $X$ and $Y$ with boundary operator the sum of the boundary operators on $X$ and $Y$. While this tensor product description doesn't descend to homology, it does after some correction from Tor groups. This relationship extends from CW complexes to all spaces by the technique of CW approximation.

**Theorem 3.27: The Künneth Formula**

Given two spaces $X$ and $Y$, the homology of their product is,

$$H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \bigoplus \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(X)).$$

Proof. As we show later, every space $X$ is weak homotopy equivalent to a CW complex. As we show later, weak homotopy equivalent spaces have isomorphic homology. Hence, without loss of generality $X$ and $Y$ are CW complexes.

By our comments above, the cellular complexes satisfy $\mathcal{C}_*(X \times Y) = \mathcal{C}_*(X) \otimes \mathcal{C}_*(Y)$.

Consider the short complexes $0 \to B_p \to Z_p \to 0$ and $0 \to B_q \to Z_q \to 0$. For $B, Z$ boundaries and cycles. The tensor product of these is a graded filtered complex. The associated spectral sequence has the following three pages after which it degenerates.

$$\begin{tikzcd}[row sep=large]
q+1 & Z_p \otimes B_q \arrow[d] & B_p \otimes B_q \arrow[d] \\
q & Z_p \otimes Z_q & B_p \otimes Z_q \\
E^0 & p & p+1
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=large]
q+1 & 0 & 0 \\
q & Z_p \otimes H_q & B_p \otimes H_q \arrow[l] \\
E^1 & p & p+1
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=large]
q+1 & 0 & 0 \\
q & H_p \otimes H_q & \mathrm{Tor}(H_p, H_q) \\
E^2 & p & p+1
\end{tikzcd}$$

Taking the direct sum over all pairs of complexes $B_p \to Z_p, B_q \to Z_q$ gives $\mathcal{C}_*(X \times Y)$. And the direct sum of these spectral sequences should compute the cohomology of the product $H_*(X \times Y)$. Checking the indexing of our entries in the spectral sequence, we obtain the claimed formula. $\square$

*Multiplications.* While the construction of cohomology may seem superfluous given the universal coefficient theorem, cohomology has the benefit over homology of having the structure of a ring. Throughout, suppose $R$ is a commutative ring.

**Definition 3.28: Cup Product**

Let $\varphi \in C^n(X;R)$ and $\psi \in C^m(X;R)$. The *cup product* $\varphi \smile \psi \in C^{n+m}(X;R)$ is defined so that for a singular simplex $\sigma : \Delta^{n+m} \to X$,

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[0,\dots,n]}) \cdot \psi(\sigma|_{[n,\dots,m]}).$$

Here, $[0,\dots,n]$ denotes the face of $\Delta^{n+m}$ containing the first $n+1$ vertices and $[n,\dots,m]$ denotes the face of $\Delta^{n+m}$ containing the last $m+1$ vertices. Extending by linearity, we obtain a module homomorphism,

$$\smile : C^n(X;R) \times C^m(X;R) \to C^{n+m}(X;R).$$

Exercise 3.29. Show the cup product has the following properties:

(1) $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$.

(2) $\delta(\alpha \smile \beta) = \delta\alpha \smile \beta + (-1)^{|\alpha|}\alpha \smile \delta\beta$.

(3) Given $f : X \to Y$, $f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta$.

(4) The cup product is super-commutative (i.e. commutative with a sign $(-1)^{|\alpha|\cdot|\beta|}$) up to some quasi-isomorphism permuting the vertices of simplices. ▲

This implies the cup product descends to a multiplication on cohomology,

$$\smile : H^n(X;R) \otimes_R H^m(X;R) \to H^{n+m}(X;R),$$

which is associative, super-commutative, and natural with respect to pullback by continuous maps. This makes the direct sum of the cohomology groups, which we denote $H^*(X;R)$, a graded ring.

There are two more operations on (co)homology which we sometimes use.

**Definition 3.30**

Suppose $X_1, X_2$ are two spaces, and let $\pi_i : X_1 \times X_2 \to X_i$ be the two projections. Given $\alpha \in H^n(X_1;R)$ and $\beta \in H^m(X_2;R)$, we can define the *cross product*,

$$\times : H^n(X_1;R) \times H^m(X_2;R) \to H^{n+m}(X_1 \times X_2;R),$$

by $\alpha \times \beta = \pi_1^*\alpha \smile \pi_2^*\beta$.

Given $\alpha \in H^n(X;R)$ and $\sigma : \Delta^{n+m} \to X$, we can define the *cap product* $a \frown \alpha \in H_m(X;R)$ by,

$$\sigma \frown \alpha = \alpha(\sigma|_{[0,\dots,n]})\sigma|_{[n,\dots,n+m]}.$$

Extending by linearity, we obtain a module homomorphism,

$$\frown : C_{n+m}(X;R) \times H^n(X;R) \to C_m(X;R).$$

Exercise 3.31. Show the cap product has the following properties:

(1) $(\partial\sigma) \frown \alpha = \sigma \frown \delta\alpha + (-1)^{|\alpha|}\partial(\sigma \frown \alpha)$.

(2) $\sigma \frown (\alpha \smile \beta) = (a \frown \alpha) \frown \beta$. ▲

Thus the cap product descends to an operation on homology,

$$\frown : H_{n+m}(X;R) \otimes H^n(X;R) \to H_m(X;R).$$

We may also perform a relative version of these operations. Suppose $A, B \subset X$ are sufficiently nice, for example they have an open neighbourhood deformation retracting to them. Then we may define relative versions of the cup and cap product:

$$\smile : H^*(X,A) \otimes H^*(X,B) \to H^*(X, A \cup B)$$
$$\frown : H_*(X,A) \otimes H^*(X,B) \to H_*(X, A \setminus B).$$

*Homology with Twisted Coefficients.* For a couple of our constructions later, it will be useful to study homology where the coefficients depend on some local data.

**Definition 3.32**

A *local coefficient system* $\{G_x\}$ on a space $X$ is a choice of abelian group $G_x$ for each $x \in X$ and a homomorphism $\Phi_\gamma : G_{\gamma(0)} \to G_{\gamma(1)}$ for each $\gamma : [0,1] \to X$ so that:

(i) $\Phi_\gamma$ depends only on the homotopy class of $\gamma$ with fixed end-points.

(ii) If $\gamma_1, \gamma_2$ are composable, $\Phi_{\gamma_1 * \gamma_2} = \Phi_{\gamma_1} \circ \Phi_{\gamma_2}$.

(iii) For $\gamma$ a constant path, $\Phi_\gamma = \mathrm{id}$.

Example 3.33. An important example for later is that if $E \to B$ is a Serre fibration, then the homology of its fibres $\{H_*(E_x)\}$ defines a local coefficient system on the base $B$. ▲

Consider a local coefficient system $\mathscr{G} = \{G_x\}$ on $X$. Let $\sigma : \Delta^k \to X$ be a singular simplex. Note since $\Delta^k$ is contractible that $G_\sigma(t)$ are canonically isomorphic for all $t \in \Delta^k$ and we may denote the local coefficients on the simplex by $G_\sigma$. Let $C_*(X;\mathscr{G})$ be the free $\mathbb{Z}$-module generated by pairs $(\sigma, g)$ where $\sigma$ is a singular simplex and $g \in G_\sigma$ modulo the relation,

$$(\sigma, g_1) + (\sigma, g_2) = (\sigma, g_1 + g_2).$$

We can extend the differential $\partial$ from the usual singular complex to this complex by letting it act linearly on the $\mathscr{G}$ coefficients. This still satisfies $\partial^2$ and so $C_*(X;\mathscr{G})$ is a chain complex.

**Definition 3.34**

The homology of $X$ with *local* or *twisted coefficients* in a local coefficient system $\mathscr{G}$ is the homology of the above chain complex,

$$H_*(X;\mathscr{G}) = H(C_*(X;\mathscr{G})).$$

The cohomology with local coefficients is defined analogously.

**3.4. Homology on Manifolds.** The well behaved structure of smooth manifolds allows for important additional properties and a better geometric understanding of the homology. We will assume throughout that our manifolds are compact (possibly with boundary) and oriented. The same conclusions hold for unoriented manifolds over $\mathbb{Z}_2$ instead of $\mathbb{Z}$.

*Fundamental Class.* For manifolds, we should think of elements of $H_*(X)$ as representing something like submanifolds of $X$. Our intuition is that if $\dim(X) = n$, then $H_n(X)$ should be generated by a single element representing $X$ itself. This is essentially true, and there are a few paths to a proof.

**Definition 3.35**

A simplical structure for $X$ is an $n$-dimensional *pseudomanifold* if,

(1) The maximum dimension of a simplex in the simplicial structure is $n$.
(2) Any two $n$-simplices of $X$ is connected by a finite chain of $n$-simplices which pairwise share an $n-1$ dimensional face.
(3) Every $n-1$ dimensional simplex is the face of precisely two $n$-simplices of $X$ (if we allow for boundaries, they may be the face of just one $n$-simplex as well).

It is a classical theorem of topology that every smooth $n$-manifold is homeomorphic to a $n$-dimensional triangulated subspace of Euclidean space, so that the homeomorphism is smooth on the $n$-simplices. And moreover, this triangulation is a $n$-dimensional pseudomanifold.

**Theorem 3.36: Fundamental Class**

Given a smooth compact oriented $n$-manifold $X$, we have $H_n(X, \partial X) = \mathbb{Z}$ if $X$ is orientable, and equals zero otherwise. For any smooth compact $n$-manifold $H_n(X, \partial X; \mathbb{Z}_2) = \mathbb{Z}_2$.

Proof. We can use our pseudomanifold structure to study simplicial homology. Let $a \in H_n(X, \partial X)$ be non-zero and represented by a linear combination of $n$-simplices with no boundary. Since every $(n-1)$-simplex in $X/\partial X$ is a face of two $n$-simplices, and the $n$-simplices are all connected, we immediately deduce that every $n$-simplex of $X$ appears in $a$, and furthermore all have the same coefficient up to a sign. If we work modulo 2 then we are done, giving a proof of the second part.

Working over $\mathbb{Z}$, we see that $H_n(X, \partial X)$ is generated by at most one element. Such a generator $a$ exists if and only if we can compatibly orient all the $n$-simplices of $X$ to match signs in our linear combination. If we have such compatible orientations, then the orientations of each simplex patch together to give an orientation of $X$. Conversely, if we cannot compatibly orient all the simplices, then there is a path through a sequence of $n$-simplices connecting an $(n-1)$ simplex to itself which reverses orientation, and so $X$ is not orientable. □

Proof. (Take Two) I have come up with another admittedly quite long and arduous proof, which avoids the need for pseudomanifolds (which I dislike) and instead uses Morse homology (which I like very much). The first part borrows an argument from here to show we can find a Morse function on $X$ with a unique local maximum. We'll instead find a function with a unique local minimum, and by flipping a sign we get what we want. Pick any Morse function $f$. Let $p_1, \ldots, p_a$ be the local minima, i.e. index zero critical points and $q_1, \ldots, q_b$ the index one critical points ($b$ may equal zero but $a \geq 1$ by the extreme value theorem). By decreasing $f$ adding a negative bump function near the $p_i$ we can ensure $f(p_i) < f(q_j)$ for any pair $i, j$.

Pick a compatible metric $g$. The network of gradient flow lines between the $p_i$ and the $q_j$ will form an finite embedded graph in $X$. Note that this embedded graph is the one-skeleton of a cell structure for $X$. Since $X$ is connected, the graph must be connected (by cell approximation any path connecting 0-cells in $X$ lands in the 1-skeleton). Now consider a spanning tree $T$ for the subcollection of vertices $p_1, \ldots, p_a$, which exists by basic graph theory. Let $U$ be a tubular neighbourhood of $T$ inside $X$. By adding to $f$ a large negative bump function subordinate to $U$ we may ensure that $f(q_i) < C < f(q_j)$ for any pair of index 1 critical points with $q_i$ in our graph and $q_j$ outside the graph and some constant $C$. Now consider $N = f^{-1}((-\infty, C])$. This is an open submanifold with boundary of $X$ containing $T$ and no other critical points of $f$. Applying the gradient flow of $f$ on $N \setminus T$ determines a smooth deformation retract of $N$ to $T$, which is itself contractible. Hence $N$ is diffeomorphic to a closed $n$-ball in $X$. We may alter $f$ on $N$, replacing it by some radial distance function smoothed at the boundary. This altered $f$ has a unique local minimum on $N$ and hence on all of $X$.

Considering the Morse complex for a Morse function $f$ with a unique local maximum $p$, we have $CM_n(X) \cong \mathbb{Z}$ generated by $p$. It just remains to show $\partial p = 0$ iff $X$ is oriented and always $\partial p = 0 \pmod 2$. Fix any critical point $q$ of index $n-1$. The Hessian of $f$ is symmetric so its eigenvalues are real and non-zero by our Morse assumption. Hence we can apply the Hartman–Grobman theorem which allows us to locally linearize the gradient flow of $f$ near $q$. We conclude from the local linear picture and the fact $\mathrm{Hess}_q(f)$ has only one positive eigenvalue, that there are exactly two gradient flow lines that stably converge to $q$. This immediately gives us that $\langle \partial p, q \rangle = 0 \pmod 2$.

The final (difficult) ingredient is to show these two flow lines converging to $q$ have opposite orientation precisely when $X$ is oriented. While orientations in Morse theory are hard, they are not so bad for the case of codimension one. We endow all the unstable manifolds of critical points with orientations of our choosing (this is possible because they are diffeomorphic to contractible disks). If $\gamma$ is a flow line from $p$ of index $n$ to $q$ of index $n-1$, then the tangent space to $X$ at $q$ has an orientation from the orientation of the unstable manifold at $q$ direct summed with the asymptotic direction of the gradient field along $\gamma$ (i.e. using the vector pointing out of the gradient flow line at $q$). We give a sign $\pm 1$ to our flow line according to if this orientation agrees or disagrees with the orientation of $T_qX$ given by the unstable manifold of $p$; this is the sign in the oriented count of flow lines defining the Morse boundary operator.

We can change our Morse function by a sign, so that $p$ is the unique critical point of index zero and the $q_i$ are critical points of index one. From Morse theory as we've described it, there is thus a cell structure for $X$ with a single zero cell given by $p$ and one cells given by the gradient flow lines. In particular, each index one critical point $q$ has two flow lines from $q$ to $p$ that together determine a loop in $M$, which is a one cell with both ends glued to $p$. Hereafter, we change our function back by a sign, so again $p$ is the global maximum and the points $q_i$ have index $n-1$, with the new knowledge that these flow lines from $p$ to $q$ determine a 1-skeleton for a CW structure of $X$.

Note that by choice of a metric, $TX$ has structure group $\mathrm{O}(n)$ and orienting $TX$ is equivalent to a reduction of structure group to $\mathrm{SO}(n)$. Such reduction is the same as a section of the quotient bundle $Q$ with fibre $\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}_2$. Clearly $\mathbb{Z}_2$ has $\pi_0$ group $\mathbb{Z}_2$ and all its higher homotopy groups are trivial. We will conclude from obstruction theory, but one can probably prove directly, that there is an obstruction to extending a section of $Q$ from the 0-skeleton of $X$ (which can always be constructed trivially) to the 1-skeleton given by an element of $H^1(X; \mathbb{Z}_2)$ (this is the first Stiefel-Whitney class $w_1(TX)$). And once we have a section of $Q$ on the one-skeleton of $X$, there is no further obstruction to extending it to all of $X$.

We conclude that $X$ will be orientable if and only if we can define an orientation of $TX$ over the gradient flow lines from $p$ to the index $n-1$ critical points $q_i$. If we remove the points $q_i$, we are left with a "spider shaped" graph which is contractible and hence we can define an orientation of $TX$ over it. Thus $X$ is orientable if and only if the orientation extends to each index one critical point $q$. This will be the case if and only if the orientations as defined on the two flow lines converging to $q$ meet compatibly at $q$.

Now pick an orientation on the one-skeleton of gradient flow lines punctured at the index $n-1$ critical points. At one such point $q$, use the two gradient flow lines converging to $q$ to define two candidate orientations for $T_qX$. We can represent these orientations by an ordered basis for $T_qX$ whose first $n-1$ vectors span and induce a given orientation on the tangent space to the unstable manifold of $q$ and whose last vector spans the tangent to the gradient flow line, pointing in whichever direction necessary to agree with the overall orientation of the punctured one-skeleton. Since the first $n-1$ vectors in each oriented basis agree, we see that the orientation extends to $q$ if and only if the last vector in the pair of oriented bases point in the same direction. Because, by Hartman–Grobman, the two gradient flow lines meeting at $q$ lie on opposite sides of the tangent space to the unstable manifold of $q$, the orientations extend if and only if the orientation on exactly one of two gradient lines includes the direction agreeing with the direction of gradient flow. Referring back to our description of signs in Morse homology, the orientation extends to $q$ if and only if exactly one of the two gradient flow lines carries a positive sign in the boundary operator of the Morse complex. That is, the orientation extends to $q$ if and only if $\langle \partial p, q \rangle = 0$ (and not $\pm 2$). Repeating this analysis for every index $n-1$ critical point, we see that the orientation extends to the one skeleton of $X$, and hence to all of $X$ by our discussion above, if and only $\partial p = 0$.

So if $X$ is orientable, $p$ is a non-zero generator for $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$ While if $X$ is not orientable, $\partial p \neq 0$ and $H^n(M; \mathbb{Z}) = 0$. Nothing we said should use in a fundamental way that the manifold be closed, so we can with little work extend to the case of compact manifolds with boundary. Phew. ☐

Exercise 3.37. Show by excision that for any $x$ in the interior of $X$ a smooth manifold of dimension $n$, $H_i(X, X \setminus \{x\}) \cong H_i(S^n)$. This is called the *local homology* of $X$ at $x$. Furthermore, show that the map,

$$i_* : H_n(X, \partial X) \to H_n(X, X \setminus \{x\}),$$

induced by inclusion of the pair is an in isomorphism. Show that if $X$ is oriented, the local homology $H_n(X, X \setminus x)$ has a canonical generator induced by the orientation. ▲

If $X$ is oriented with a given orientation, then there is a canonical generator of $H_n(X, \partial X)$, described in the language of the above proof as follows. We consider a signed sum of the $n$-simplices of $X$ which assigns a coefficient $\pm 1$ to a simplex based off if the smooth embedding $\Delta^n \to X$ is orientation preserving or reversing. Equivalently, the isomorphism described in the above exercise should send the generator of $H_n(X, \partial X)$ to the canonical generator of the top local homology.

**Definition 3.38: Fundamental Class**

If $X$ is a compact oriented $n$-manifold, the canonical generator of $H^n(X, \partial X)$ is called the *fundamental class* and denoted $[X, \partial X]$. If $X$ is closed, we write $[X] \in H_n(X)$ for the fundamental class. If $X$ is non-oriented we use the same name and notation to refer to the unique non-zero element of $H_n(X, \partial X; \mathbb{Z}_2)$.

**Definition 3.39: Degree**

The *degree* of a continuous map $f : M \to N$ between compact $n$-dimensional oriented manifolds is the integer by which $f_* : H_n(M, \partial M) \to H_n(N, \partial N)$ multiplies elements by under the isomorphism of top homology with $\mathbb{Z}$. Equivalently, the degree $d$ satisfies $f_*([M, \partial M]) = d[N, \partial N]$.

We can always homotope our map to be smooth, for which we have the following equivalent definitions of degree.

(1) By Sard's theorem, $f$ has an open dense subset of regular values. The degree of $f$ is an oriented count of the preimage of any regular value.

(2) If $\omega$ is a volume form on $N$ normalized to have volume one, then the degree is given by,

$$d = \int_M f^*\omega.$$

To see these two are equivalent, note that the critical values are closed and measure zero, so do not affect the integral. On the regular value set, we can define a partition of unity and apply the change of variables theorem to deduce the equivalence. To see (1) is the same as the topological version, pick a regular value $y$ and note that the map,

$$H_n(M) \to H_n(M, M \setminus f^{-1}(y)) \to H_n(N, N \setminus y)$$

should be multiplication by the degree. But by excision, $H_n(M, M \setminus f^{-1}(y))$ is a direct sum of the local homology at each point of $f^{-1}(y)$, and so the above map factors as a multiplication by the sum of local degrees of the map near each point of $f^{-1}(y)$. For our smooth map, each local degree is just plus or minus one dependent on orientation from which we recover definition (1).

*Poincaré Duality.* We are now ready for the most fundamental result about the topology of manifolds.

**Theorem 3.40: Poincaré Isomorphism Theorem**

If $X$ is a smooth compact oriented $n$-manifold, one has for each $i$,

$$H_i(X, \partial X) \cong H^{n-i}(X) \quad \text{and} \quad H_i(X) \cong H^{n-i}(X, \partial X).$$

In particular, in the case of a closed oriented manifold, one has,

$$H_i(X) \cong H^{n-i}(X).$$

If $X$ is not oriented, the same isomorphisms hold over $\mathbb{Z}_2$. Moreover, these isomorphisms are induced by capping with the fundamental class $[X, \partial X]$ or its dual in cohomology.

Remark 3.41. The isomorphism in the case of a closed manifold is called *Poincaré duality*, while its generalization to manifolds with boundary is called *Lefschetz duality*. For non-compact manifolds, there is a more general statement provided we switch one of the groups in each isomorphism to (co)homology with compact support. This can be defined as $\operatorname{colim} H^*(M, M \setminus K)$ taken over compact sets $K$ directed by inclusion. Equivalently in de Rham theory, it is the cohomology of the complex of differential forms with compact support. $\triangle$

Proof. We follow a somewhat unorthodox argument through Morse theory. Consider a Morse function $f$; we can perturb $f$ so that all of its critical points have distinct critical values. Near some critical value $c$ denote $M_- = f^{-1}((-\infty, c - \varepsilon))$ and $M_+ = f^{-1}((-\infty, c + \varepsilon))$ for $\varepsilon$ sufficiently small.

Note that if there are no critical points on the interval $[a, b]$, then gradient flow determines a diffeomorphism between $f^{-1}((-\infty, a))$ and $f^{-1}((-\infty, b))$. In particular, the topology of $f^{-1}((-\infty, a))$ only changes when we pass through the level set of a critical point. Thus, we may inductively prove Poincaré duality by assuming it holds for $M_-$ and showing it then holds for $M_+$ for an arbitrary critical value $c$.

Recall, the Morse lemma says that there are local coordinates $(x_1, \ldots, x_n)$ near a critical point so that our Morse function has the form.

$$f(x) = f(c) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2,$$

where $i = \operatorname{ind}(c)$. Thus, $M_+$ is obtained from $M_-$ by gluing on a handle $H = D^i \times D^{n-i}$. The boundary of $H$ is the union of two pieces $\partial_1 H = \partial D^i \times D^{n-i}$ and $\partial_2 H = D^i \times \partial D^{n-i}$. We see that the handle $H$ is glued to $M_-$ along $\partial_1 H$.

Now assume inductively that Poincaré duality holds for $M_-$. The base case is Poincaré duality for a disk, which is clear. Consider the triple $(M_+, \partial M_+ \cup H, \partial M_+)$ and the pair $(M_+, M_-)$ We may consider the homology sequence of the triple and the cohomology sequence of the pair, which are connected by the cap product.

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M_+) \arrow[r] \arrow[ddd, "\frown[M_+,\partial M_+]"] & H^k(M_-) \arrow[r] \arrow[dd, "\cong"', "\frown[M_-,\partial M_-]"] & H^k(M_+, M_-) \arrow[r] \arrow[d, equal, "\text{ex.}"] & H^{k+1}(M_+) \arrow[r] \arrow[ddd, "\frown[M_+,\partial M_+]"] & \cdots \\
 & & & H^k(H, \partial_1 H) \arrow[d, "\cong"', "\frown[H,\partial H]"] & & \\
 & & H_{n-k}(M_-, \partial M_-) \arrow[d, equal, "\text{ex.}"'] & H_{n-k}(H, \partial_2 H) \arrow[d, equal, "\text{ex.}"] & & \\
\cdots \arrow[r] & H_{n-k}(M_+, \partial M_+) \arrow[r] & H_{n-k}(M_+, \partial M_+ \cup H) \arrow[r] & H_{n-k}(\partial M_+ \cup H, \partial M_+) \arrow[r] & H_{n-k-1}(M_+, \partial M_+) \arrow[r] & \cdots
\end{tikzcd}$$

Here, ex. denotes excision isomorphisms. We know the cap products in the middle two columns are isomorphisms, since we know Poincaré duality holds for the disk $H$ and we assume it holds for $M_-$. Thus, we have a morphism of long exact sequences, which is an isomorphism on two of every three columns. The five lemma implies the other columns are also isomorphisms, and so the map,

$$\frown [M_+, \partial M_+] : H^k(M_+) \to H_{n-k}(M_+, \partial M_+),$$

is an isomorphism. By induction, this extends to hold for $(M, \partial M)$. Dualizing our sequences and using the universal coefficient theorem gives the second isomorphism. The mod 2 case follows from the same argument using the mod 2 fundamental class. □

Proof. (Take Two) We can provide a simpler Morse theoretic proof of the isomorphism (without the fact it is induced by capping with $[M]$).

Take $(f, g)$ to be a Morse–Smale pair on $M$ . We can study its negative gradient flow to obtain the usual Morse complex and then the Morse homology. Now study the Morse homology of the pair $(-f, g)$. Note that the critical points of $f$ of index $i$ are in correspondence with the critical points of $-f$ of index $n-i$:

$$CM_i(M; f, g) \cong CM_{n-i}(M; -f, g). \tag{3.1}$$

Under this isomorphism, we can write the Morse chain complex of $-f$ as a cochain complex,

$$\cdots \to CM_{i-1}(M; f, g) \xrightarrow{\delta} CM_i(M; f, g) \xrightarrow{\delta} CM_{i+1}(M; f, g) \to \cdots . \tag{3.2}$$

Given critical points $p, q$ of index $i$ and $i+1$ with respect to $f$ (so $n-i$ and $n-i-1$ with respect to $-f$), the coefficient $\langle \delta p, q\rangle$ is an oriented count of negative gradient flow lines of $-f$ from $p$ to $q$. But a negative gradient flow line of $-f$ is the same as a positive gradient flow line of $f$. And so changing the direction, $\langle \delta p, q\rangle$ is an oriented count of negative gradient flow lines of $f$ from $q$ to $p$. As a warning, these orientations may be different from the ones in the Morse complex of $f$; we can only guarantee they agree in the case where $M$ is orientable and thus admits coherent orientations for all its unstable manifolds (think back to our fundamental class result). We conclude, modulo two, $\langle \delta p, q\rangle = \langle \partial q, p\rangle$ where $\partial : CM_{i+1}(M; f, g) \to CM_i(M; f, g)$ is the usual Morse boundary, and the equality holds over $\mathbb{Z}$ if $M$ is oriented.

Hence under 3.1, $\delta$ is the adjoint of $\partial$ and the complex 3.2 is dual to the Morse chain complex of $f$. Since the homology of $CM(M; f, g)$ computes the singular homology of $M$, our dual complex 3.2 computes the singular cohomology of $M$.

But on the other hand, $(-f, g)$ is a Morse–Smale pair and we know its Morse complex has to compute the singular homology of $M$. Since the Morse complex of $-f$ is the same as 3.2 after re-indexing, we conclude,

$$H_i(M; \mathbb{Z}_2) \cong H^{n-i}(M; \mathbb{Z}_2),$$

and for $M$ oriented this extends to integers coefficients.

For the case where $M$ has boundary, if we start with a Morse function $f$ pointing inward on the boundary it will compute $H_*(M; \mathbb{Z}_2)$. From the above analysis, the Morse complex of $-f$ will compute $H^{n-*}(M; \mathbb{Z}_2)$. On the other hand $-f$ is a Morse function pointing outward on the boundary and so its Morse complex will compute $H_*(M, \partial M; \mathbb{Z}_2)$. Conversely, if $\widetilde{f}$ points outward on the boundary, the Morse complex of $-\widetilde{f}$ simultaneously computes $H^{n-*}(M, \partial M; \mathbb{Z}_2)$ and $H_*(M; \mathbb{Z}_2)$. These results again extend over $\mathbb{Z}$ if $M$ is oriented. □

Poincaré duality together with the universal coefficient theorem give relatively strict restrictions on the potential homology of a smooth manifold. In particular we have for a closed oriented $n$-manifold $X$,

$$H_i^{\text{free}}(X) \cong H_{n-i}^{\text{free}}(X) \quad \text{and} \quad H_i^{\text{tor}}(X) \cong H_{n-i-1}^{\text{tor}}(X).$$

As an example of the applications of this, consider the following exercise.

Exercise 3.42. Suppose $M$ is a smooth closed 3-manifold with $H_1(M; \mathbb{Z}) = 0$. Apply the UCT and the (mod 2) Poincaré isomorphism to conclude that $H_3(M) = \mathbb{Z}$ and so by Theorem 3.36, $M$ is oriented (this also follows from obstruction theory). Then apply Poincaré duality and the UCT to show $M$ is a *homology sphere*, that is $H_*(M) = H_*(S^3)$. ▲

Remark 3.43. Let $M$ be a closed $n$-manifold which is not oriented or maybe not even orientable. A version of Poincaré duality still holds provided we use local coefficients. Define the local coefficient system $\mathscr{O}_x = H_n(M, M \setminus \{x\})$. The condition that $M$ is orientable is equivalent to finding a section of this system, i.e. a trivialization of it. Any closed manifold has a local coefficient fundamental class $[M] \in H_n(M; \mathscr{O})$ defined so that its image under $M \to (M, M \setminus \{x\})$ is the unit in $\mathscr{O}_x$. Capping with this fundamental class will then give an isomorphism,

$$H^k(M; \mathscr{G}) \cong H_{n-k}(M; \mathscr{G} \otimes \mathscr{O}),$$

for any local coefficient system $\mathscr{G}$. This can be proved just by inserting local coefficients into our proof above. △

*Intersection Theory.* We are suddenly going to be quite fast and loose with our arguments, favouring some basic geometric intuition. This is well in the spirit of intersection theory, which personally I feel is one of the crown jewels of algebraic topology.

First we briefly discuss a concept that will appear in greater detail in later chapters when we study vector bundles. Suppose $V$ is a rank $r$ oriented vector bundle over a closed base $n$-manifold. Let $DV$ be the corresponding unit disk bundle of $V$ with respect to some metric (or just some tubular neighbourhood of the zero-section of $V$) and $SV = \partial DV$ the unit sphere bundle.

**Definition 3.44: Thom Class**

Proposition 3.45. *There is a unique element $t_V \in H^r(DV, SV)$ so that for any fibre $D_xV$ of the bundle, the map*

$$H^r(DV, SV) \to H^r(D_xV, S_xV) \cong H^r(D^r, S^r),$$

*sends $t_V$ to the fundamental cohomology class $[D^r, S^r]$ (with respect to the orientation of the bundle).*

This element $t_V$ is called the *Thom class* of the bundle $V$. One can equivalently view it as living in $H^r(TV)$, where $TV$ is the *Thom space* of $V$ defined as $V$'s one point compactification.

Thom classes will allow us to answer the following basic question. Suppose $Z$ is an $n-m$ dimensional oriented submanifold of a closed oriented $n-$manifold $M$. We can define a fundamental submanifold class $[Z] \in H_{n-m}(M)$ as the image of the usual fundamental class $[Z] \in H_{n-m}(Z)$ under the map induced by inclusion. Our natural question is: what is the Poincaré dual of $[Z]$?

We are looking for $\alpha \in H^m(M)$ so that $[M] \frown \alpha = [Z]$. First, let $U$ be a tubular neighbourhood around $Z$; by a classical theorem, $U$ is diffeomorphic to a neighbourhood of the zero section of the normal bundle $\nu$ of the embedding $Z \to M$. We can instead include $Z \to U$ and consider $[Z] \in H^n(U)$. By the refinement lemma, we can thus restrict to look for $\alpha \in H^m(M, M \setminus U)$, which by excision is the same as $\alpha \in H^m(U, \partial U)$.

Again by refinement and some locality arguments for manifolds, it suffices to work locally. Pick a small neighbourhood of $Z$ homeomorphic to a disk $D^{n-m}$ over which the normal bundle $\nu$ trivializes. Taking a product with the fibre, we get a neighbourhood of $U$ of the form $D^{n-m} \times D^m$. On this neighbourhood, by excision $[M]$ becomes $[D^{n-m} \times D^m, \partial(D^{n-m} \times D^m)] = [D^{n-m}, \partial D^{n-m}] \times [D^m, \partial D^m]$ and $[Z]$ becomes $[D^m, \partial D^m]$. If we let $\widetilde{[X]}$ denote the fundamental cohomology class of $X$. It is a simple lemma that in $D^n = D^m \times D^{n-m}$,

$$[D^{n-m}, \partial D^{n-m}] \times [D^m, \partial D^m] \frown \widetilde{[D^{n-m}, \partial D^{n-m}]} = [D^n, \partial D^n].$$

We deduce that the restriction of $\alpha \in H^m(U, \partial U)$ should coincide with the fundamental cohomology class of the fibre $[D^{n-m}, S^{n-m}]$. By Proposition 3.45, this is exactly the Thom class of the normal bundle $t_\nu \in H^{n-m}(U, \partial U)$. By excision and pullback by inclusion, this becomes a class in $H^{n-m}(M)$, which we denote $t_Z$.

Proposition 3.46. *Given an oriented $(n-m)$ dimensional submanifold $Z$ of an oriented $n$-manifold $M$, the Thom class $t_Z \in H^{n-m}(M)$ of the normal* *bundle of $Z$, realized as a tubular neighbourhood of $Z \subset M$ satisfies,*

$$[M] \frown t_Z = [Z],$$

*and hence $t_Z$ is Poincaré dual to the fundamental class of $Z$. The same holds with boundary, or mod 2 in the unoriented case.*

From this we deduce the following very important theorem.

**Theorem 3.47**

Let $Z$ and $W$ be closed oriented submanifolds of a closed oriented manifold $M$. Perturb $Z$ and $W$ so that they intersect transversally. By convention, we orient $Z \cap W$ using the orientation of $Z$ and the co-orientation of $W$. Then,

$$t_Z \smile t_W = t_{Z \cap W},$$

or equivalently,

$$PD([Z]) \smile PD([W]) = PD([Z \cap W]).$$

The same holds modulo 2 in the unoriented case.

Proof. Let $U$ and $V$ be tubular neighbourhoods of $Z$ and $W$ respectively. Suppose $Z$ has codimension $r$ and $W$ has codimension $s$. We have $t_Z \in H^r(M, M \setminus U)$ and $t_W \in H^s(M, M \setminus V)$. Thus,

$$t_Z \smile t_W \in H^{r+s}(M, M \setminus (U \cap V)) \cong H^{r+s}(U \cap V, \partial(U \cap V)),$$

where we use excision. As before, by refinement and locality arguments, it suffices to analyze this cup product locally. Let $D^{n-r-s}$ be a small open disk in $Z \cap W$, over which all the normal bundles trivialize. The restrictions of the normal bundles $U, V$ over this disk have the form,

$$D^{n-r-s} \times D^r \quad \text{and} \quad D^{n-r-s} \times D^s$$

respectively. By transversality, the normal bundle to the embedding $Z \cap W \to M$ restricted to this disk has the form $D^{n-r-s} \times D^r \times D^s$.

By a similar computation to above for fundamental cohomology classes of disks inside $D^r \times D^s$,

$$[\widetilde{D^r, \partial D^r}] \smile [\widetilde{D^s, \partial D^s}] = [\widetilde{D^r \times D^s, \partial(D^r \times D^s)}].$$

Noting that the Thom classes of $Z$, $W$, and $Z \cap W$ restricted to this neighbourhood correspond to the fundamental classes of the fibres of the respective embeddings by Proposition 3.45, we conclude $t_Z \smile t_W = t_{Z \cap W}$. $\square$

This result is powerful for computing ring structures.

EXAMPLE 3.48. The cellular complex of $\mathbb{C}P^n$ has a single generator in even gradings and is zero in odd gradings. Thus the differential is trivial and we have,

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n, \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

By Poincaré duality, the cohomology is the same. Note from the cellular complex that a hyperplane $\mathbb{C}P^k$ for $k < n$ is a closed submanifold of dimension $2k$ which represents the generator of $H_{2k}(\mathbb{C}P^n)$. If we take two generic hyperplanes of codimensions $2n-2k$ and $2n-2j$, their intersection will be a hyperplane of codimension $4n-2k-2j$. We conclude from the above theorem that,

$$PD([\mathbb{C}P^k]) \cap PD([\mathbb{C}P^j]) = PD([\mathbb{C}P^{j+k-n}]).$$

We easily deduce that the cohomology ring is,

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1}), \quad |x| = 2.$$

Here $x$ is Poincaré dual to the fundamental class of a hyperplane $[\mathbb{C}P^{n-1}]$.

The same computation holds in the unoriented case for real projective spaces,

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1}), \quad |x| = 1.$$

Here $x$ is Poincaré dual to the fundamental class of a hyperplane $[\mathbb{R}P^{n-1}]$. ▲

*Some Geometric Applications.* The simplest application of homology is as a homotopy invariant to distinguish spaces which are not homeomorphic. But there is much more we can do with it. We outline a few of the additional invariants and results that can be proved from basic homology theory.

**Definition 3.49**

Let $C_*(X)$ be a chain complex with finitely many free and finite dimensional chain groups. The *Euler characteristic* of $C_*(X)$ is given as,

$$\chi(C_*(X)) := \sum_i (-1)^i \dim C_*(X).$$

It is a simple exercise that this equals,

$$\chi(C_*(X)) := \sum_i (-1)^i \dim H_i^{\text{free}}(X).$$

If the latter but not the former is well defined, we still call it the Euler characteristic. For the case where our chain complex is $\mathcal{C}_*(X)$ the cellular complex of a space with finite dimensional homology, we abbreviate to $\chi(X)$, which we call the Euler characteristic of $X$.

EXERCISE 3.50. Prove the following properties of the Euler characteristic.

(1) $\chi(M \times N) = \chi(M) \cdot \chi(N)$.

(2) If $\widetilde{M}$ is a $k$-sheeted cover of $M$, $\chi(\widetilde{M}) = k\chi(M)$.

(3) If $M, N$ are nice subsets of $X$ (so that Mayer–Vietoris applies), then,
$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N).$$

(4) A genus $g$ surface $\Sigma_g$ has $\chi(\Sigma_g) = 2 - 2g$. Deduce Euler's formula for a convex polyhedron: $V - E + F = 2$. What is the corresponding formula for a polyhedron homeomorphic to a three-holed torus? What about for an $n$-dimensional convex polytope?

We give one important result about the Euler characteristic.

**Definition 3.51**

Let $V$ be a vector field on a smooth $n$-manifold $M$. If $x$ is an isolated zero of $V$, then in a small neighbourhood of $x$, the normalized vector field $V(x)/\|V(x)\|$ (with respect to some metric) restricted to a small sphere around $x$ defines a map $S^{n-1} \to S^{n-1}$. This map's degree is invariant under continuous deformations, and so is independent of the metric (since the set of metrics is contractible) or the choice of sphere around $x$.
The *index of* $V$ at $x$ is,
$$\mathrm{ind}_x(V) : \deg\left(V/\|V\| : S^{n-1} \to S^{n-1}\right).$$

**Theorem 3.52: Poincaré–Hopf Theorem**

Let $V$ be a vector field on a smooth closed manifold $M$ with isolated zeroes. Then,
$$\sum_{x \in \mathrm{crit}(V)} \mathrm{ind}_x(V) = \chi(M).$$
If $M$ is compact with boundary, the same holds provided $V$ points outward along the boundary.

PROOF. We use Morse homology. One notes that the sum of the indices is independent of which vector field we use. There are a few ways to see this, using homotopy invariance, embedding $M$ into Euclidean space and studying the degree of the Gauss map, or realizing this sum as the pairing $\langle e(TM), [M]\rangle$. Whatever way, we may take $V$ to be a gradient vector field of a function $f$.

By locality, we may assume locally at a critical point $c$ of $f$ and a zero of $V$, $f$ looks like $f(c) - x_1^2 + \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2$. We may also take our metric to be standard with respect to these coordinates, so that the gradient vector field looks like,

$$V = -2\sum_{j=1}^{i} x_j \frac{\partial}{\partial x_j} + 2\sum_{k=i+1}^{n} x_k \frac{\partial}{\partial x_k}.$$

This is a linear vector field and hence it has degree plus or minus one depending on if it preserves orientation or not. It preserves orientation precisely when $i$ is even. Hence, using the definition of the Morse complex,

$$\sum_{x\in \mathrm{crit}(V)} \mathrm{ind}_x(V) = \sum_{x\in \mathrm{crit}(f)} (-1)^{\mathrm{ind}(x)} = \sum_{i=0}^{n} (-1)^i \dim(CM_i(M)) = \chi(CM_*(M)).$$

But note Morse and cellular homology agree, so $\chi(CM_*(M)) = \chi(M)$.

The boundary case follows similarly from the Morse complex for manifolds with boundary. $\square$

Corollary 3.53 (Hairy Ball Theorem). *Every smooth vector field on an odd dimensional sphere has a zero. In particular, the odd dimensional spheres are non-parallelizable.*

**Definition 3.54**

Let $X$ be a space with finitely generated homology and $f : X \to X$ continuous. The *Lefschetz number* of $f$ is defined as,

$$\Lambda(f) = \sum_i (-1)^i \operatorname{tr}\big(f_* : H_n^{\text{free}}(X) \to H_n^{\text{free}}(X)\big).$$

Note this is invariant under homotopies of $f$. And if $f$ is homotopic to the identity, $\Lambda(f) = \chi(X)$.

We have the following generalization of Brouwer's fixed point theorem.

**Theorem 3.55: Lefschetz Fixed Point Theorem**

If $X$ is a finite simplicial complex and $f : X \to X$ is continuous with $\Lambda(f) \neq 0$, then $f$ has a fixed point. In particular if $X$ has the rational homotopy type of a point, then every $f : X \to X$ has a fixed point.

Proof. Assume $f$ has no fixed points. By compactness, we know that $d(x, f(x))$ obtains a positive minimum $\varepsilon$. By subdivision, we may assume that every simplex in $X$ has diameter smaller than $\varepsilon/2$ and so $f$ maps every point out of its simplex. Analogous to the cellular approximation theorem, we prove by the Lebesgue number lemma that after subdivision of the codomain, we may approximate a continuous map $f$ of simplices by a simplicial map $g$ so that for a simplex $\sigma$, $f(\sigma)$ lies in the simplices containing $g(\sigma)$.

Apply this simplicial approximation to approximate $f$ by a simplicial map $g$. Note that for any simplex $\sigma$ and some $x \in \sigma$, all of $g(\sigma)$ lies within $\varepsilon/2$ from $f(x)$ and all $\sigma$ lies within $\varepsilon/2$ from $x$. Since $d(x, f(x)) \geq \varepsilon$, we conclude $g(\sigma) \cap \sigma = \varnothing$.

The induced map $g_*$ on the simplicial complex is described by the matrix of coefficients in our simplicial map. But since $g(\sigma) \cap \sigma = \varnothing$, the diagonal coefficients are zero. Hence the trace of $g_*$ on the cell complex is zero. As with the Euler characteristic, the alternating sum of traces of $g_*$ is invariant under passing to homology and hence $\Lambda(f) = \Lambda(g) = 0$. □

There is the following theorem which generalizes both Lefschetz fixed point and Poincaré–Hopf. We state it for interest, without proof.

**Theorem 3.56: Lefschetz–Hopf**

Let $X$ be a finite simplicial complex and $f : X \to X$ have finitely many fixed points. Then,

$$\Lambda_f = \sum_{x \in \mathrm{Fix}(f)} \mathrm{ind}_x(f),$$

where the index of $f$ at a fixed point is defined as for the zero of a vector field.

We prove one more very classical result using the full structure of the cohomology ring.

**Theorem 3.57: Borsuk–Ulam**

Any continuous map $f : S^n \to \mathbb{R}^n$ agrees on some pair of antipodes. That is there is $x \in S^n$ so that $f(-x) = f(x)$.

Proof. The case $n = 1$ is an easy consequence of the intermediate value theorem.

Now suppose for contradiction there is an $f : S^n \to \mathbb{R}^n$ that does not agree on any antipodes. Setting $g(x) = f(x) - f(-x)$, we obtain an odd function on

$S^n$ with no zeroes. Thus $h(x) = g(x)/|g(x)|$ is a well defined odd continuous function $S^n \to S^{n-1}$. We claim such a map cannot exist.

In the case $n = 2$, we have an odd continuous function $S^2 \to S^1$. Restricting to the equator, gives an odd map $t : S^1 \to S^1$. Note that $t(\pi)$ is antipodal to $t(0)$ and so $t$ must traverse an angle of $\pi + 2\pi k$ on $[0, \pi]$ for some $k \in \mathbb{Z}$. Since $t$ is odd, it traverses the same path on $[\pi, 2\pi]$ as on $[0, \pi]$ and so $t$ traverses an angle of $(1 + 2k)2\pi$ after a full revolution. Hence the map $t$ has an odd winding number. But on the other hand, $t$ extends to a map on the upper hemisphere of $S^2$ and hence is null homotopic. This is a contradiction with the fact $t$ represents an odd element in $\pi_1(S^1) \cong \mathbb{Z}$.

For $n > 2$, we can use the fact $h$ is odd to quotient by antipodal identification to get a map $h' : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Note that $H_1(\mathbb{R}P^n; \mathbb{Z}_2)$ is generated by $[\mathbb{R}P^1]$, which we can represent by a standard embedding $S^1 \to \mathbb{R}P^n$. Lifting to $S^n$, this map is a path half way along an equator. Composing with the map $h$, since it's odd, we get a path in $S^{n-1}$ connecting antipodes. Hence, under quotienting, it becomes a non-contractible loop in $\mathbb{R}P^{n-1}$ which thus represents $[\mathbb{R}P^1]$ the generator of $H_1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$.

We conclude that $h'$ induces an isomorphism on $H_1$. And thus by dualizing its pullback induces an isomorphism on $H^1$. Recall the structure of the cohomology rings,

$$H^*(\mathbb{R}P^n) = \mathbb{Z}[a]/(a^{n+1}) \quad \text{and} \quad H^*(\mathbb{R}P^{n-1}) = \mathbb{Z}[b]/(b^n).$$

We have just shown $(h')^*b = a$. By naturality, since $b^n = 0$,

$$0 = (h')^*b^n = ((h')^*b)^n = a^n.$$

This is obviously a contradiction. □

**3.5. Cohomology of Some Common Spaces.** For reference, we list the homology and cohomology rings for some important spaces. These can be computed either using cellular homology, or spectral sequences. We'll revisit some of the spectral sequence computations when we discuss them later.

Some Cohomology Groups

**Spheres:**

$$H^*(S^n) = \mathbb{Z}[\alpha]/\alpha^2 \quad |\alpha| = n, \alpha = [S^n].$$

**Projective Spaces:** As we found,

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[a]/(a^{n+1}) \quad |a| = 2, a = PD[\mathbb{C}P^{n-1}]$$

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[b]/(b^{n+1}) \quad |b| = 1, b = PD[\mathbb{R}P^{n-1}].$$

We find over $\mathbb{Z}$,

$$H^i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & i = 0 \text{ and } n \text{ if } n \text{ is odd} \\ \mathbb{Z}_2 & 0 < i < n \text{ even} \\ 0 & \text{else} \end{cases}$$

**Grassmannians:** Over $\mathbb{C}$, the differentials in the cell complex for the Schubert cell structure are trivial. So $H^{2i}(\mathbb{C}G(n,k))$ is the free abelian group generated by the $i$ square Young diagrams confined in the $k\times(n-k)$ rectangle and the odd cohomology groups are zero.

Over $\mathbb{R}$, we can deduce the homology from the following description of attaching maps. Suppose $\Delta, \Delta'$ are two Young diagrams of size $i$ and $i-1$ respectively which we identify with Schubert cells in $G(n,k)$. If $\Delta' \not\subset \Delta$, then the attaching map has degree zero. Otherwise, if $\Delta$ is $\Delta'$ union the square $(s,t)$, then the attaching map has degree,

$$\langle \partial\Delta, \Delta'\rangle = \begin{cases} 0 & : s+t \equiv 0 \pmod 2 \\ \pm 2 & : s+t \equiv 1 \pmod 2. \end{cases}$$

Whether the coefficient is plus or minus two can I think usually/always be fixed by the condition that $\partial^2 = 0$. In particular, the $\mathbb{Z}_2$ homology is just freely generated by Young diagrams as in the complex case.

**Surfaces:** By Mayer–Vietoris, we can deduce the effect on homology of attaching a handle. We can geometrically understand the cup product. We deduce for an orientable surface of genus $g$ $\Sigma_g$,

$$H^*(\Sigma_g) = \mathbb{Z}[a_1, b_1, \dots, a_g, b_g, c]/(a_i a_j, b_i b_j, a_i b_j - \delta_{ij} c),$$

where $|a_i| = |b_i| = 1, |c| = 2$. The homology of the non-orientable surface $S_k$, the connect sum of $k$ copies of $\mathbb{R}P^2$, is

$$H^0(S_k) = \mathbb{Z}, \quad H^1(S_k) = \mathbb{Z}^{2k} \oplus \mathbb{Z}_2, \quad \text{and} \quad H^2(S_k) = 0.$$

**Lie Groups:** From spectral sequences we will deduce,

$$H^*(\mathrm{SU}(n)) = \Lambda^*_{\mathbb{Z}}[x_3, x_5, \ldots, x_{2n-1}]$$
$$H^*(\mathrm{U}(n)) = \Lambda^*_{\mathbb{Z}}[x_1, x_3, \ldots, x_{2n-1}]$$
$$H^*(\mathrm{Sp}(n)) = \Lambda^*_{\mathbb{Z}}[x_3, x_7, \ldots, x_{4n-1}].$$

The cohomology of $\mathrm{SO}(n)$ is harder. Here are some results,

$$H^{*,\text{free}}(\mathrm{SO}(2k+1)) = \Lambda^*_{\mathbb{Z}}[x_3, x_7, \ldots, x_{4k-1}]$$
$$H^{*,\text{free}}(\mathrm{SO}(2k)) = \Lambda^*_{\mathbb{Z}}[a_1, a_7, \ldots, a_{4k-5}, a_{2k-1}]$$
$$H^*(\mathrm{SO}(n); \mathbb{Z}_2) = \bigotimes_{0<i<n \text{ odd}} \mathbb{Z}_2[b_i]/(b_i^{p_i}),$$

where $|b_i| = i$ and $p_i = \min\{2^k : |b_i|^{2^k} \geq n\}$.

## 4. Homotopy Theory

We now move on to the other main set of algebro-topological invariants of spaces: their homotopy groups. While conceptually simpler, the homotopy groups will not admit the same kind of algorithmic formulation as homology; even the homotopy groups of spheres are not fully known! Nevertheless, the ideas of homotopy theory will vastly expand our understanding of topology and the tools we have for the classification of spaces.

### 4.1. Homotopy Groups.

**Definition 4.1**

Given a pointed space $(X, x_0)$, its $n$th *homotopy group* as a set is given by pointed homotopy classes of maps from the $n$-sphere, called *spheroids*:

$$\pi_n(X, x_0) := [(S^n, *), (X, x_0)].$$

If $X$ is path connected, this is independent of basepoint, and so we usually ignore $x_0$ in our notation. The group structure on $\pi_n$ can be described as follows. Given two pointed maps $f, g : S^n \to X$, we may define $f * g$ by taking $S^n$, contracting the equator to a point, and mapping the two hemispheres by $f$ and $g$ respectively.

The group $\pi_n$ has two equivalent descriptions. We can instead view $\pi_n(X)$ as maps $[(S^n, S^n \setminus B), (X, x_0)]$, where $B$ is a ball inside $S^n$. Then the group operation is given by cutting out two balls from $S^n$ and identifying each with one of the composed elements. We may also view $\pi_n(X)$ as $[(I^n, \partial I^n), (X, x_0)]$.

Then the group operation is given by gluing two cubes $I^n$ together and mapping on each by one of the composed elements. Our three perspectives are shown below in Figure 2.

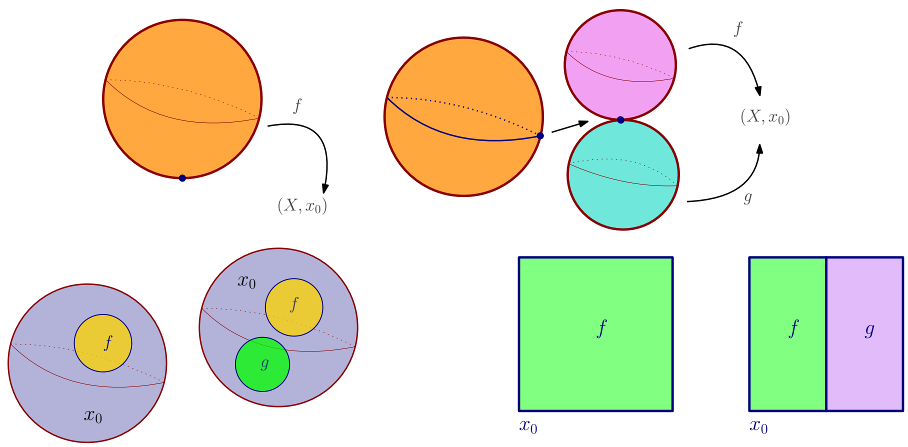



FIGURE 2. Three Models of $\pi_n(X, x_0)$

PROPOSITION 4.2. *For $n \geq 2$, the homotopy group $\pi_n(X)$ is abelian.*

PROOF. If we use the second picture, it is obvious. There is no preferred ordering of the balls on which $f$ and $g$ are defined inside $f * g$. The argument using our cube picture is important and worth walking through. An illustration of the proof is shown below in Figure 3. □

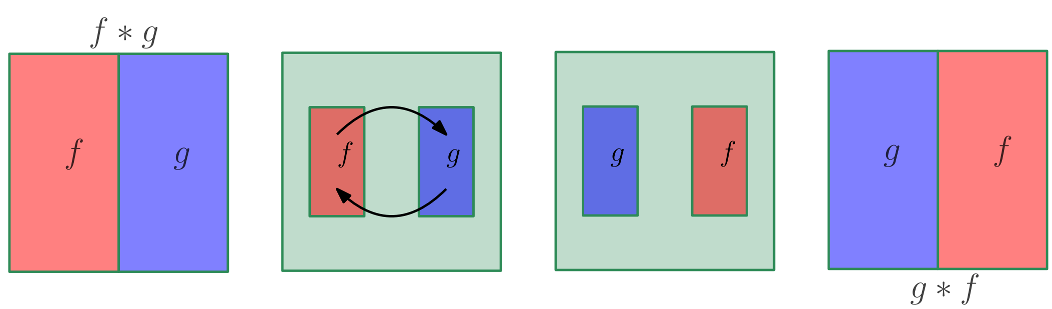



FIGURE 3. Proof that $\pi_n$ is abelian.

EXERCISE 4.3. Prove $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. ▲

Given a continuous map $f : X \to Y$, we can precompose it with a spheroid in $X$ to obtain one in $Y$. Thus $f$ induces a map $f_* : \pi_n(X) \to \pi_n(Y)$. It is immediate that this is a group homomorphism, natural with respect to composition, and only depends on the homotopy class of $f$. In particular we obtain the following.

**Theorem 4.4**

Homotopy equivalent spaces have the same homotopy groups.

Because the definition of the homotopy groups is so simple, it does allow for some easy computations.

PROPOSITION 4.5. *For $n < m$, $\pi_n(S^m) = 0$.*

PROOF. Suppose we have a map $f : S^n \to S^m$. By the cellular approximation theorem, we may homotope $f$ to be cellular with respect to the usual cell structures on the spheres. Hence $S^n$ must land in the $n$-skeleton of $S^m$, which is just a point since $n < m$. So $f$ is null-homotopic and $\pi_n(S^m)$ is trivial. □

PROPOSITION 4.6. *If $p : \widetilde{X} \to X$ is a covering map , then $\pi_n(\widetilde{X}) = \pi_n(X)$ for $n > 1$.*

PROOF. We have a map $p_* : \pi_n(\widetilde{X}) \to \pi_n(X)$. Because $S^n$ is simply connected for $n > 1$, the map lifting lemma implies that any map $f : S^n \to X$ lifts to a map $\widetilde{f} : S^n \to \widetilde{X}$ with chosen basepoint, hence $p_*$ is surjective. And moreover, if $p_* f$ is null homotopic, the map lifting lemma implies we may lift the homotopy $S^n \times I \to X$ to $\widetilde{X}$. The uniqueness part of the map lifting lemma tells us that this homotopy must connect $f$ to the constant map and so $f$ is null homotopic as well. Hence $p_*$ is injective and thus an isomorphism. □

COROLLARY 4.7. *Any space with contractible universal cover has trivial higher homotopy groups. In particular, this applies to $S^1$ and all surfaces of genus at least one.*

**Definition 4.8**

Given a pointed pair $(X, A, x_0)$, we may define its $n$th *relative homotopy group* as,

$$\pi_n(X, A) := [(I^n, \partial I^n, \partial I^n \setminus \{0\} \times I^{n-1}), (X, A, x_0)],$$

with the same product as in the absolute case. Note absolute homotopy groups $\pi_n(X)$ are relative homotopy groups for the pair $(X, x_0)$.

We have a couple ways to visualize this as in Figure 4.

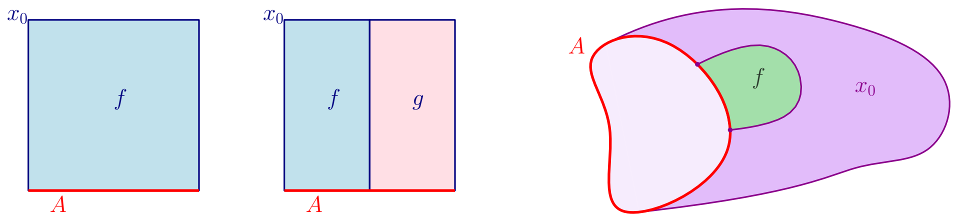



FIGURE 4. Two models for $\pi_n(X, x_0)$

EXERCISE 4.9. Prove that $\pi_n(X, A)$ is abelian for $n > 2$ and show it need not be for $n = 2$. Show the map $\pi_2(X) \to \pi_2(X, A)$ induced by inclusion of pairs lands in the centre of $\pi_2(X, A)$. ▲

REMARK 4.10. We can make sense of $\pi_0(X)$ as homotopy classes of pointed maps $S^0 \to X$ and hence homotopy classes of maps $* \to X$. That is, $\pi_0(X)$ is the *set* (not group) of path components of $X$. We may similarly define $\pi_1(X, A)$ as a set, although its meaning is less transparent. These sets are still compatible with the framework of homotopy groups, and they will play a role in the exact sequence we define in the next subsection.

Note that one can also define $\pi_{-1}(X)$, not as a set but as a Boolean variable, recording if $X$ is empty or not. △

### 4.2. Long Exact Sequences in Homotopy.

*Long Exact Sequence of a Pair.* We have a homotopical analogue of the exact sequence of a pair in homology.

**Theorem 4.11: Homotopy Long Exact Sequence of a Pair**

Let $(X, A)$ be a topological pair. There is a long exact sequence of homotopy groups,

$$\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \to \cdots .$$

Here $i_*$ and $j_*$ are induced by inclusion, and $\partial$ is the map induced by restricting relative spheroids $I^n \to X$ to the face $\{0\} \times I^{n-1} \to A$.

PROOF. Let's check exactness at $\pi_n(X)$. If $f \in \pi_n(A)$, then $i_*(f)$ lies only within $A$. But then the relative spheroid $j_* i_*(f)$ can be pushed down along the first axis to become constant. If $g \in \pi_n(X)$ has $j_*(g) = 0$, then there is a map $I^n \times I \to X$ which is constant on all the faces except $\{0\} \times I^{n-1} \times I$, where it lies in $A$. We can consider a one parameter family of angled cubes beginning from $I^n \times \{0\}$ and ending with $\{0\} \times I^n$ which defines a homotopy from $g$ to a spheroid contained in $A$. So $g$ comes from the image of $i_*$. Hence $\operatorname{im} i_* = \ker j_*$.

Let's check exactness at $\pi_n(X, A)$. If $f \in \pi_n(X)$, then $j_*(f)$ is a relative spheroid which is constantly $x_0$ on its base face. Hence $\partial j_*(f)$ is constant. If $g \in \pi_n(X, A)$ has $\partial g = 0$, then the base face of $g$ can be homotoped to be constant. By Borsuk's theorem, this homotopy can extend to all of $I^n$, and hence $g$ is homotopic to a spheroid which is constantly $x_0$ on its base, and thus lies in $\pi_n(X)$. So $\operatorname{im} j_* = \ker \partial$.

Lastly, we check exactness at $\pi_n(A)$. If $f \in \pi_{n+1}(X, A)$, then $f : I \times I^n \to X$ defines a homotopy between the base $\partial f : \{0\} \times I^n \to X$ and the constant spheroid $\{1\} \times I^n \to X$. Hence $\partial f$ is trivial under the inclusion map $i_*$. If $g \in \pi_n(A)$ has $i_*(g) = 0$, then there is a homotopy of $I \times I^n \to X$ connecting $g$ to a constant map. This homotopy is precisely the data of a relative spheroid $I^{n+1} \to X$. Hence $\operatorname{im} \partial = \ker i_*$. $\square$

Remark 4.12. The tail end of this sequence looks like,

$$\cdots \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X) \to 0.$$

After the first map, these are just maps of pointed sets, not groups. But this still makes sense as an exact sequence of pointed sets. So for example a path component $A$ (an element of $\pi_0(A)$) is in the basepoint component of $X$ ($0 \in \pi_0(X)$) precisely when there is a path in $X$ from the basepoint to an element of that path component of $A$ (i.e. it's in the image of $\pi_1(X, A)$). $\triangle$

Exercise 4.13. Prove a homotopy long exact sequence for triples. $\blacktriangle$

*Long Exact Sequence of a Fibration.* The homotopy long exact sequence of a pair is largely of instrumental use to give us the long exact sequence of a fibration. This is the one point on which homotopy has homology beat computationally, and we will try to milk it for all it's worth.

Lemma 4.14. *If $p : (E, x_0) \to (B, b_0)$ is a Serre fibration and $\pi^{-1}(b_0) = F$ then,*

$$p_* : \pi_n(E, F, x_0) \xrightarrow{\cong} \pi_n(B, b_0)$$

*is an isomorphism for each $n$.*

Proof. We first prove injectivity. Suppose $f \in \pi_n(E, F)$ has $p_* f = 0$. Then there is a homotopy $(I^n, \partial I^n) \times I \to (B, b_0)$ from $p_* f$ to a constant spheroid. By definition of a Serre fibration, this homotopy lifts to relative to $f$ and a chosen homotopy within $F$ on $\{0\} \times I^{n-1}$ to a homotopy

$(I^n, \partial I^n, \partial I^n \setminus \{0\} \times I^{n-1}) \times I \to (E, F, b_0)$ of relative spheroids. By construction this homotopy is between $f$ and a map lying in $p^{-1}(b_0) = F$. Hence, $f$ is null-homotopic in $\pi_n(E, F)$.

Now we prove surjectivity. Suppose $g \in \pi_n(B, b_0)$. Again by defining property of a Serre fibration, $g$ can be lifted relatively to a map $(I^n, \partial I^n, \partial I^n \setminus \{0\} \times I^{n-1}) \to (E, F, b_0)$ which gives an element of $\pi_n(E, F)$. Since this is a lift, it maps to $g$ under $p_*$. □

Our central result now falls out for free.

**Theorem 4.15: Homotopy Long Exact Sequence of a Fibration**

Let $p : E \to B$ be a Serre fibration with fibre $F$ over the base-point. Then there is a long exact sequence,

$$\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots ,$$

where $i_*$ is induced by inclusion and $p_*$ by projection.

Proof. Consider the homotopy exact sequence of the pair $(E, F)$. Using Lemma 4.14, we may replace the term $\pi_n(E, F)$ with $\pi_n(B)$. Based on the proof of the lemma we see the map $p_*$ is as described.

One should also check the end of the sequence:

$$\cdots \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0$$

is exact at $\pi_0(E)$, since the term $\pi_0(B)$ is replacing something ill-defined in the long exact sequence of a pair. But this is clear, since a path component of $E$ projects to the path component of the basepoint in $B$ precisely when it contains a path component of $F$. □

Remark 4.16. What is the map $\partial$ in the long exact sequence? We may view a spheroid $S^n \to B$ in $\pi_n(B)$ as a homotopy of spheroid $S^{n-1} \times I \to B$ from the constant spheroid to itself by sweeping out cross sections of the sphere. We may lift this homotopy to $E$ relative to one endpoint to obtain a homotopy of spheroids $S^{n-1} \times I \to E$ for which $S^{n-1} \times \{1\}$ maps to $F$ and hence defines an element of $\pi_{n-1}(F)$. This is essentially just repeating the proof of the above lemma.

Example 4.17. This long exact sequence is our main computational tool to find homotopy groups. Right away we can see a few important applications of this result.

(a) Consider the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. One way to see this is identifying $S^2 = \mathbb{C}P^1$ and then this fibration is just the defining quotient map of $\mathbb{C}P^1$ as points of $S^3$ modulo rescaling by complex units. We have, remembering $\pi_2(S^3) = 0$ by cell approximation,

$$\cdots \to \cancelto{0}{\pi_2(S^3)} \to \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \to \cancelto{0}{\pi_1(S^3)} \to \cdots .$$

Hence $\pi_2(S^2) = \mathbb{Z}$. We proved earlier that the higher homotopy groups of $S^1$ vanish and so for $n > 2$ we find,

$$\cdots \to \cancelto{0}{\pi_n(S^1)} \to \pi_n(S^3) \xrightarrow{\cong} \pi_n(S^2) \to \cancelto{0}{\pi_{n-1}(S^1)} \to \cdots .$$

Hence,

$$\boxed{\pi_n(S^3) = \pi_n(S^2) \quad \text{for} \quad n \geq 3.}$$

We will show shortly that $\pi_n(S^n) = \mathbb{Z}$ for all $n$, with homotopy classes of maps $S^n \to S^n$ classified by their degree. Hence in particular $\pi_3(S^2) = \mathbb{Z}$. The fibration long exact sequence show the unit of this group is exactly the Hopf fibration, and other elements are given by precomposing the Hopf fibration with higher degree maps of $S^3$.

(b) One has a generalized Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$. And the same long exact sequence argument shows $\pi_k(S^{2n+1}) = \pi_k(\mathbb{C}P^n)$ for all $k \geq 3$ and $\pi_2(\mathbb{C}P^n) = \mathbb{Z}$.

Taking a direct limit gives the fibration $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$. Because $S^\infty$ is contractible, one obtains $\pi_n(S^1) \cong \pi_{n+1}(\mathbb{C}P^\infty)$ for all $n$. Hence $\pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$ and all its other homotopy groups vanish.

From quaternions and octonions one has analogous Hopf bundles $S^3 \hookrightarrow S^7 \to S^4$ and $S^7 \hookrightarrow S^{15} \to S^8$ which give some non-trivial relations of homotopy groups.

(c) Given any pointed space $(X, x_0)$, it has a "path space fibration," $\Omega X \hookrightarrow EX \to X$ given by sending a path in $X$ beginning at a fixed endpoint $x_0$ to its other endpoint. We see that the fibre over $x_0$ is given by $\Omega X$. Note $EX$ is contractible by performing a homotopy retracting paths to their first endpoint $x_0$. Hence the long exact sequence of this fibration gives for all $n$,

$$\pi_n(X) \cong \pi_{n-1}(\Omega X).$$

In the case $n = 1$, this is just the obvious statement that homotopy classes of loops are the same as path components of the loop space. The full statement is actually just a special case of the *loop-suspension adjunction*:

$$[\Sigma X, Y] = [X, \Omega Y],$$

which holds for any pair of spaces $X, Y$. To see this is true, note that a pointed map $\Sigma X \to Y$ associates to each $x \in X$, a one-parameter family of points in $Y$ which begins and ends with the basepoint, i.e. a loop, and so we can interpret it as a map $X \to \Omega Y$. One then just has to easily check this association is compatible with homotopies. Applying this adjunction to $S^n$ and our original space $X$ and noting $\Sigma S^n = S^{n+1}$ gives the equality of homotopy groups.

This is in turn a special case of the adjunction,

$$[X \wedge A, Y] = [X, [A, Y]].$$

EXERCISE 4.18. Check this adjunctive isomorphism of homotopy classes is compatible with the group structure of $\pi_n$. ▲

(d) The determinant map gives a fibration $\mathrm{SU}(2) \xrightarrow{\det} \mathrm{U}(2) \to S^1$. Since $\mathrm{SU}(2) \cong S^3$, we deduce from the fibration long exact sequence that $\pi_3(\mathrm{U}(2)) \cong \pi_3(S^3) = \mathbb{Z}$, where we are again using the yet unproved fact $\pi_n(S^n) \cong \mathbb{Z}$.

The map given by taking an $n \times n$ unitary matrix and mapping to its first column defines a fibration $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \to S^{2n-1}$. As long as $n \geq 3$, $\pi_3(S^{2n-1}) = 0$. Hence we inductively find from the long exact sequence of the fibration that for all $n \geq 2$,

$$\pi_3(\mathrm{U}(n)) \cong \pi_3(\mathrm{U}(n-1)) \cong \cdots \cong \pi_3(\mathrm{U}(2)) = \mathbb{Z}.$$

(e) There are many more fibrations of Lie groups and homogeneous spaces that can be exploited to find homotopy groups, some we will need to apply later. Here are a few of note:

$$\begin{aligned} \mathrm{O}(n-1) &\hookrightarrow \mathrm{O}(n) &&\to S^{n-1} \\ \mathrm{U}(n-1) &\hookrightarrow \mathrm{U}(n) &&\to S^{2n-1} \\ \mathrm{Sp}(n-1) &\hookrightarrow \mathrm{Sp}(n) &&\to S^{4n-1} \\ V(n-k, m-k) &\hookrightarrow V(n,m) &&\to V(n,k) \\ \mathrm{O}(n) &\hookrightarrow V(n,k) &&\to \mathrm{Gr}(n,k). \end{aligned}$$

▲

**4.3. The Freudenthal Suspension Theorem and Stable Homotopy Theory.** Suppose $f : X \to Y$ is a continuous map. There is a natural map $\Sigma f : \Sigma X \to \Sigma Y$ induced by quotienting from the trivial product map $f \times \mathrm{id} : X \times I \to Y \times I$. Clearly we may suspend homotopies as well, so that if $f, g$ are homotopic, then $\Sigma f, \Sigma g$ are homotopic.

In the particular case of maps $S^n \to X$, we may suspend to a map $S^{n+1} \to \Sigma X$. Thus we obtain a map,

$$\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X).$$

It is simple enough to check $\Sigma(f+g)$ is homotopic to $\Sigma(f)+\Sigma(g)$ so that this is a group homomorphism.

**Definition 4.19**

We say a space $X$ is *$n$-connected* if $\pi_k(X) = 0$ for $0 \leq k \leq n$.

**Theorem 4.20: Freudenthal Suspension Theorem**

If $X$ is $n$-connected, the suspension homomorphism,

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k \leq 2n$ and surjective for $k = 2n+1$. In particular,

$$\Sigma : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphisms for $n > k+1$ and surjective for $n = k+1$.

We will not that prove this theorem since it is a bit of an involved argument. One method is by smoothly approximating. Another is through homotopy excision. We will prove the special case for the sphere where $k = 0$, since it is necessary for what we do later.

**Theorem 4.21**

For every $n \geq 0$, $\pi_n(S^n) = \mathbb{Z}$. These homotopy classes are labelled by the degree of maps $S^n \to S^n$.

Proof. Since homotopic maps have the same degree, we have a map,

$$\pi_n(S^n) \xrightarrow{\deg} \mathbb{Z}.$$

Note given two maps $f, g : S^n \to S^n$, their product $f+g$ as an element of $\pi_n$ is given by $S^n \to S^n \vee S^n \to S^n$ where we contract the equator and then map on the pair of spheres by $f$ and $g$.

We recall $H_n(S^n_a \vee S^n_b) = \mathbb{Z}^2$ is generated by $[S^n_a]$ and $[S^n_b]$, the fundamental classes under inclusion. It is clear that the equatorial contraction map $S^n \to S^n \vee S^n$ induces a map on homology that will send $[S^n]$ to $[S^n_a] + [S^n_b]$. Almost by definition,

$$f_*[S^n_a] = \deg(f)[S^n] \quad \text{and} \quad g_*[S^n_b] = \deg(g)[S^n].$$

Hence,

$$(f+g)_*[S^n] = f_*[S^n_a] + g_*[S^n_b] = (\deg(f) + \deg(g))[S^n].$$

Thus the degrees are additive under the group operation of $\pi_n$, and so the above degree map we wrote down is a group homomorphism.

We know 1 is in the image of deg, since the identity map has degree one, and so the homomorphism is surjective.

EXERCISE 4.22. Consider $S^n \subset \mathbb{C} \times \mathbb{R}^{n-1}$ with coordinates $(z, \vec{x})$. Show for $k \geq 0$ that the maps,

$$f(z, \vec{x}) = (z^k, \vec{x}) \quad \text{and} \quad g(z, \vec{x}) = (\overline{z}^k, \vec{x})$$

have $\deg(f) = k$ and $\deg(g) = -k$. ▲

It only remains to show the homomorphism is injective. Suppose $f : S^n \to S^n$ has degree zero. We can homotope $f$ to be smooth, then a generic point $p$ with have preimages $q_1, \ldots, q_{2n}$ so that near $q_i$, $f$ is a local diffeomorphism which is orientation preserving at half of the $q_i$'s and orientation reversing on the others. Now pair up the $q_i$'s in pairs with opposite orientations and draw paths connecting the pairs. In a tubular neighbourhood of each path, we may homotope $f$ so it corresponds in this neighbourhood to the sum of 1 and $-1$ in the fundamental group of $S^n$. Hence we may homotope $f$ to be constant on this neighbourhood, and in particular not equal $p$. Doing this on each tubular neighbourhood, $p$ will not be in the image of a function homotopic to $f$. But then this map factors as $S^n \to S^n \setminus \{p\} \hookrightarrow S^n$. Since $S^n \setminus \{p\}$ is contractible, we conclude $f$ is null homotopic. □

If we take the Freudenthal Suspension Theorem on faith, it leads us to the following definition,

**Definition 4.23**

The *kth stable homotopy group* of a space $X$ is,

$$\pi_k^{\mathrm{st}}(X) := \lim_{n\to\infty} \pi_{n+k}(\Sigma^n X).$$

Later results will imply that the $(n+1)$st suspension of any space is $n$-connected. And hence this limit stabilizes after $n = k+2$,

$$\pi_{2k+2}(\Sigma^{k+2} X) = \pi_{2k+3}(\Sigma^{k+3} X) = \cdots = \pi_k^{\mathrm{st}}(X).$$

In particular, we have the *stable homotopy groups of the sphere*,

$$\pi_k^{\mathrm{st}} := \pi_{2k+2}(S^{k+2}).$$

It turns out that $\pi_k^{\mathrm{st}}(\cdot)$ is an extraordinary homology theory in the sense of Definition 3.20. The homology groups of a point are the stable homotopy groups of the sphere. Stable homotopy is in some sense the most fundamental extraordinary cohomology theory and largely initiated their study.

We have already computed the 0th stable homotopy group of the sphere $\pi_0^{\mathrm{st}} = \pi_2(S^2) = \mathbb{Z}$. The first stable homotopy group $\pi_1^{\mathrm{st}} = \pi_4(S^3) = \mathbb{Z}_2$ will be computed later using spectral sequences. The stable homotopy groups of a sphere are difficult to calculate, but some tools exist, for example the Adams spectral sequence and chromatic homotopy theory. The *unstable homotopy groups* $\pi_{n+k}(S^n)$ with $n \leq k+1$ are even harder. In Table 1, the first several homotopy groups of low dimensional spheres are shown; in gray are groups we know to be trivial, and the coloured diagonals are the stable groups. While there are patterns, the possible groups are varied and sometimes complicated.

| | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_{10}$ | $\pi_{11}$ | $\pi_{12}$ | $\pi_{13}$ | $\pi_{14}$ | $\pi_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S^1$ | $\mathbb{Z}$ | | | | | | | | | | | | | | |
| $S^2$ | | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{12} \times \mathbb{Z}_2$ | $\mathbb{Z}_{84} \times \mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ |
| $S^3$ | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{12} \times \mathbb{Z}_2$ | $\mathbb{Z}_{84} \times \mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ |
| $S^4$ | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z} \times \mathbb{Z}_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_{24} \times \mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$ | $\mathbb{Z}_{84} \times \mathbb{Z}_2^5$ |
| $S^5$ | | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{30}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_{72} \times \mathbb{Z}_2$ |
| $S^6$ | | | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{60}$ | $\mathbb{Z}_{24} \times \mathbb{Z}_2$ | $\mathbb{Z}_2^3$ |
| $S^7$ | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{120}$ | $\mathbb{Z}_2^3$ |
| $S^8$ | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z} \times \mathbb{Z}_{120}$ |
| $S^9$ | | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ |

TABLE 1. Homotopy Groups of Spheres

In Table 2, is a list of the first few stable homotopy groups. Memorizing these is probably high on the list of bar tricks to impress a mathematician.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_k^{\mathrm{st}}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{240}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\mathbb{Z}_6$ | $\mathbb{Z}_{504}$ | 0 | $\mathbb{Z}_3$ | $\mathbb{Z}_4$ |

TABLE 2. Stable Homotopy Groups

One important result which one could maybe guess from the above tables is the following.

PROPOSITION 4.24 (Serre's Finiteness Theorem). *The homotopy groups of the spheres $\pi_{n+k}(S^n)$ for $k \geq 0$ are finite abelian groups except for:*

*(1) $k = 0$ in which case $\pi_n(S^n) = \mathbb{Z}$.*

*(2) $k = 2m$ and $n = 2m - 1$ in which case,*

$$\pi_{4m-1}(S^{2m}) = \mathbb{Z} \oplus F_m,$$

*for $F_m$ finite abelian.*

*In particular, the stable homotopy groups $\pi_k^{\mathrm{st}}$ are finite for $k \geq 1$.*

We will prove this result later using spectral sequences and describe a free generator of $\pi_{4m-1}(S^{2m})$.

This is the beginning of *stable homotopy theory*, in which spaces are *stabilized* by repeated suspension and homotopy groups are studied in this direct limit. In the modern perspective on algebraic topology, spaces are replaced by *spectra*, which encapsulate this direct limit under stabilization. Spectra turn out to be extremely important because it turns out extraordinary cohomology theories are all representable functors and their representing objects are precisely spectra; this is the Brown representability theorem. We will discuss some other generalized cohomology theories and their associated spectra later in these notes.

### 4.4. Homotopy and CW Complexes.

*Whitehead Theorem.* At this stage, a reasonable question to ask is: how much of the homotopy type of a space is captured by its homotopy groups? A partial answer is that for CW complexes the homotopy groups are almost enough.

Recall we said that spaces $X$ and $Y$ weak homotopy equivalent if there is a natural bijection $[Z, X] \to [Z, Y]$ for any CW complex $Z$. A special case of this, when the bijection is induced by a map, is described in the following theorem.

**Theorem 4.25**

Let $f : X \to Y$ be a continuous map. The following are equivalent.
(i) $f_* : [Z, X] \to [Z, Y]$ is a bijection for any CW complex $Z$.
(ii) $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n$.

(iii) For any CW pair $(Z, W)$ with maps $h, g$ satisfying the following commutative diagram,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ W \arrow[u, "h"] \arrow[r, hook] & Z, \arrow[u, "g"'] \end{tikzcd}$$

there exists a map $\widetilde{h} : Z \to X$ extending $h$ so that $f \circ \widetilde{h} \sim g$.

In this case we call $f$ a *weak homotopy equivalence.*

Proof. $(i) \implies (ii)$: Taking $Z = S^n$ gives the result.

$(iii) \implies (i)$: Take pair $(Z, \varnothing)$. This shows $f_*$ is surjective. Take pair $(Z \times I, Z \times \{0\} \sqcup Z \times \{1\})$. Given $\varphi, \psi : Z \to X$ homotopic in $Y$, we thus obtain a homotopy in $X$. So, $f_*$ is injective.

$(ii) \implies (iii)$: Let $C_f$ be the mapping cylinder of $f$. This is homotopy equivalent to $Y$ and so from the long exact sequence of the pair $(C_f, X)$ and the fact $f_*$ is an isomorphisms on homotopy groups, we conclude $\pi_n(C_f, X) = 0$ for all $n$.

Suppose we have a CW pair $(Z, W)$ and maps $h : W \to X$ and $g : Z \to Y$ so that $g|_W = f \circ h$. Because $(Z, W)$ is built out of cells and because $\pi_n(C_f, X) = 0$, the map $i \circ g : Z \to C_f$ on cells of $Z \setminus W$ can be homotoped to a map $\widetilde{h}$ lying in $X \subset C_f$. By construction, composing $\widetilde{h}$ with $f$ gives a map homotopic to $g$. $\square$

This has the following crucial corollary.

**Theorem 4.26: Whitehead Theorem**

Let $f : X \to Y$ be a weak homotopy equivalence between CW complexes. Then $f$ is a homotopy equivalence.

Proof. This proof is the same as Theorem 1.3. By Theorem 4.25, $f$ induces a bijection $[Z, X] \to [Z, Y]$. Applying this to $X$ and $Y$, since they are CW complexes, shows $f_* : [X, X] \to [X, Y]$ and $f_* : [Y, X] \to [Y, Y]$ are bijections. Let $g = (f_*)^{-1}(\mathrm{id}_Y)$. This implies $f_*(g) = f \circ g \sim \mathrm{id}_Y$. On the other hand, $f_*(g \circ f) = (f_* g) \circ f \sim \mathrm{id}_Y \circ f \sim f_*(\mathrm{id}_X)$. Since $f_*$ is a bijection, $g \circ f \sim \mathrm{id}_X$. So $f$ and $g$ determine a homotopy equivalence. $\square$

Note that it is not enough that $X$ and $Y$ have the same homotopy groups, we need a map realizing this isomorphism in order to conclude the spaces are homotopy equivalent. The following exercise demonstrates this.

EXERCISE 4.27. Show that $S^2$ and $S^3 \times \mathbb{C}P^\infty$ have the same homotopy groups but are not weak homotopy equivalent. Show the same for $\mathbb{R}P^3 \times S^2$ and $S^3 \times \mathbb{R}P^2$. ▲

We can do even worse than this. The spaces $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ have the same homotopy and homology groups, but are not homotopy equivalent (which can be seen by studying the cohomology ring). Worse still, there are two $S^3$ bundles on $S^2$. They have the same homotopy groups and cohomology ring but are not homotopy equivalent (which can be seen by showing the non-trivial bundle is not spin and applying the Wu formula).

*Attaching Cells.* We need several preliminary results dealing with how attaching cells to CW complexes affects homotopy. These will be applied in the next few sections.

PROPOSITION 4.28. *If $X$ is an $n$-connected CW complex, it is homotopy equivalent to a CW complex with one $0$-cell and no cells of dimension $1$ through $n$.*

PROOF. We work by induction. In the base case, if $n$ is 0-connected, it is path connected. Thus for any pair of 0-cells, there is a path connecting them, which by cellular approximation can be taken to be a one cell joining the 0-cells. We may identify this 1-cell to a point and the resulting cell structure will be homotopy equivalent with our two 0-cells identified. Doing this inductively for all pairs of 0-cells and using the weak topology gives a homotopy equivalent complex with only one 0-cell.

Now suppose that $X$ is $n$-connected with one 0-cell and no cells of dimension 1 through $n-1$. Thus the $n$-skeleton of $X$ is a wedge sum of $n$-spheres. Because $X$ is $n$-connected, each such $n$-sphere is contractible in $X$. By cellular approximation, this homotopy $S^n \times I \to X$ to a point defines a disk $D^{n+1}$ lying in the $n+1$ skeleton of $X$. We can form a CW complex $\widetilde{X}$ by adding a disk $D^{n+2}$ bounded by two copies of $D^{n+1}$ for each $n$-sphere in the $n$-skeleton of $X$. By construction, $D^{n+2}$ is contractible and so $\widetilde{X}$ is homotopy equivalent to $X$. If $A$ is the wedge-product of these $(n+2)$-disks, we have that $X \sim \widetilde{X}/A$ and $\widetilde{X}/A$ has no $n$-cells. □

As we probably should have mentioned before, but only use now, there is an obvious $\pi_1(X)$ action on the higher homotopy groups of $X$ given by composing a spheroid with some loop.

**Theorem 4.29: Cell-Attaching Lemma**

Let $X$ be a CW complex and $Y$ a complex obtained from $X$ by attaching a disk $D^{n+1}$ with $n \geq 2$ via $\varphi : \partial D^{n+1} \to X_n$. Then the inclusion $i : X \to Y$ induces an isomorphism on homotopy groups $\pi_q$ for $q < n$. For $q = n$, it induces a surjection with kernel generated by the $\pi_1(X)$ orbit of the homotopy class of the attaching map $[\varphi]$.

Proof. (Sketch) Given a spheroid $S^q \to Y$, if $q \leq n$, cellular approximation implies it can be homotoped to lie in $X$. Hence the map $\pi_q(X) \to \pi_q(Y)$ is surjective. If a spheroid $S^q \to X$ is contractible under inclusion into $Y$ and $q < n$, then cellular approximation implies the contraction $D^q \to Y$ can be homotoped to lie in $X$. Hence the spheroid $S^q \to X$ was contractible to begin with and $\pi_q(X) \to \pi_q(Y)$ is injective.

It only remains to check the kernel condition for $q = n$. Suppose $f : S^n \to X$ is non-trivial and in the kernel of $i_*$ so that it extends to a contraction $\widetilde{f} : D^{n+1} \to Y$. We may homotope $\widetilde{f}$ to be smooth. Since $f$ is not null-homotopic, $\widetilde{f}$ must land at some point in $D^{n+1} \subset Y$. Looking at a preimage of a neighbourhood of regular value of $\widetilde{f}$ in $D^{n+1}$ gives a collection of neighbourhoods in $D^{n+1}$. We can homotope $\widetilde{f}$ so the domain is a bouquet of spheres each containing one of the preimage neighbourhoods. By retracting this neighbourhood onto the entire sphere, and simultaneously onto all of $D^{n+1} \subset Y$, $\widetilde{f}$ becomes a connected sum of maps homotopic to $\varphi$ up to composition with a loop on the boundary. □

Corollary 4.30. *Let $X$ and $Y$ be CW complexes.*

*(i)* $\pi_n(X) = \pi_n(X_{n+1})$.

*(ii) If $X$ and $Y$ are $p$ and $q$ connected respectively for $p, q \geq 1$, then,*

$$\pi_n(X \vee Y) = \pi_n(X) \oplus \pi_n(Y) \quad \textit{for} \quad n \leq p + q.$$

*(iii)* $\pi_n(\bigvee_m S^n) = \mathbb{Z}^m$ *for $n \geq 2$ and $m \in \mathbb{N} \cup \infty$, with generators corresponding to the inclusion map of spheres in the bouquet.*

Proof. (i) This is immediate from the cell-attaching lemma.

(ii) By Proposition 4.28, we may suppose $X \vee Y$ has one 0-cell at the joint vertex and $X, Y$ have no cells of dimensions 1 through $p$ and 1 through $q$ respectively. Note that $X \times Y \setminus X \vee Y$ consists only of cells of dimension at least $p + q + 2$. Hence the result follows from cell-attaching.

(iii) This is immediate from (ii). $\square$

*CW Approximation.* We have seen that homotopy captures almost all the information about homotopy type of CW complexes. We now have a partial converse in that homotopy groups and weak homotopy equivalence captures only as much about a space as can be encoded in a CW complex. The upshot of this is that it means often one can conclude a result about homotopy or homology holds for all spaces just by checking for CW complexes.

**Theorem 4.31: CW Approximation Theorem**

For any space $X$ there is a CW complex $X^{CW}$ with a weak homotopy equivalence $f : X^{CW} \to X$ called a *CW approximation* to $X$. The CW approximation of $X$ is unique up to (homotopically unique) homotopy equivalence. Moreover, two space are weak homotopy equivalent if and only if they have the same CW approximation.

PROOF. We construct $X^{CW}$ inductively as a direct limit,

$$X_0^{CW} \subset X_1^{CW} \subset X_2^{CW} \subset \cdots \subset X^{CW}.$$

Let $X_0^{CW}$ be a point and $f_0 : X_0^{CW} \to X$ the map to the basepoint. Suppose we have $f_{n-1} : X_{n-1}^{CW} \to X$ which induces an isomorphism on homotopy group $\pi_q$ for $q \leq n-1$.

Let $Y_n = X_{n-1}^{CW} \bigvee_\alpha S_\alpha^n$, with $\alpha$ indexing a generating set for $\pi_n(X)$. We may define a map $g_n : Y_n \to X$ which acts as $f_{n-1}$ on $X_{n-1}^{CW}$ and as a spheroids $\varphi_\alpha$ representing the indexed generators of $\pi_n(X)$ on each $S_\alpha^n$. Then, by the cell attaching lemma, $\pi_q(X)$ and $\pi_q(Y_n)$ are the same for $q \leq n-1$ with isomorphisms induced by $g_n$. Also the map,

$$(g_n)_* : \pi_n(Y_n) \to \pi_n(X)$$

is surjective since its image contains a generating set $\{\varphi_\alpha\}$ for $\pi_n(X)$. Now let $\{\psi_\beta : \partial D^{n+1} \to Y_n\}$ be a generating set for the kernel of $(g_n)_*$ so that $g_n \circ \psi_\beta$ extends to a map $\widetilde{\psi_\beta} : D^{n+1} \to X$ for each $\beta$. Set $X_n^{CW} = Y_n \cup_\beta D_\beta^{n+1}$ glued by $\psi_\beta$. Define a map $f_n : X_n^{CW} \to X$ which is $g_n$ on $Y_n$ and $\widetilde{\psi_\beta}$ on $D_\beta^{n+1}$. By the cell-attaching lemma, $f_n$ induces isomorphisms on $\pi_q$ for $q \leq n-1$, and actually also for $q = n$, since the generators of $\ker((g_n)_*)$ are trivial in $\pi_n(X_n^{CW})$.

To check uniqueness, suppose $f_1 : X_1^{CW} \to X$ and $f_2 : X_2^{CW} \to X$ are two cellular approximations. These define weak homotopy equivalences, which by transitivity, tells us $X_1^{CW}$ and $X_2^{CW}$ are weak homotopy equivalent. It is an easy extension of Whitehead's theorem that weak homotopy equivalent CW complexes are homotopy equivalent, as desired.

Lastly, suppose $X$ and $Y$ have the same CW approximation $Z$. Then $X$ and $Z$ are weak homotopy equivalent and $Y$ and $Z$ are weak homotopy equivalent. By transitivity of weak homotopy equivalence, so are $X$ and $Y$. Conversely, suppose $X$ and $Y$ are weak homotopy equivalent. And consider CW approximations $Z_1 \to X$ and $Z_2 \to Y$. These define weak homotopy equivalences and so again by transitivity, $Z_1$ and $Z_2$ are weak homotopy equivalent, and since they are CW, they are homotopy equivalent. $\square$

REMARK 4.32. From this theorem we can infer another definition of a CW approximation in terms of a universal property.

The CW approximation $X^{CW} \to X$ of a space $X$ is the unique up to homotopy equivalence space so that any map $f : Z \to X$ for $Z$ a CW complex uniquely factors in **hTop** through $X^{CW}$. That is, we obtain the following diagram commuting up to homotopy:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists!"'] \arrow[dr, "f"] & \\
X^{CW} \arrow[r, "\cong_{WHE}"'] & X.
\end{tikzcd}$$

In more categorical language, we can say that CW approximation defines a functor $\mathbf{hTop} \to \mathbf{hCW}$ which is the right adjoint to the inclusion functor $\mathbf{hCW} \hookrightarrow \mathbf{hTop}$. $\triangle$

*Homotopy and Homology: The Hurewicz Theorem.* We will now study the first non-trivial homotopy group of a CW complex and compare it to homology. We need the following proposition first.

PROPOSITION 4.33. *A weak homotopy equivalence $f : X \to Y$ induces an isomorphism of homology groups.*

PROOF. Let $\Sigma$ be an $n$ dimensional CW complex representing a given $n$ dimensional singular cycle in $Y$ under a map $\sigma : \Sigma \to Y$. By definition of weak homotopy equivalence, there is a unique homotopy class of map $\widetilde{\sigma} : \Sigma \to X$ so that $\sigma = f \circ \widetilde{\sigma}$. This map defines an $n$-cycle in $X$ mapping to $\sigma$ under $f_*$, and so $f_*$ is surjective.

Now suppose $\sigma : \Sigma \to X$ is an $n$-cycle so that $f_*(\sigma) = \partial\tau$ for $\tau$ a singular $n$-chain. We can thus extend $\Sigma$ to an $n+1$ complex $T$ so that $T \to Y$ extends $f \circ \sigma$ and represents $\tau$. By Proposition 4.25 in its relative incarnation, we can lift $\Sigma \to X$ to $T \to X$ and so $\sigma$ is a boundary in $X$. $\square$

**Definition 4.34**

Given a space $X$ and a class of spheroid $\varphi : S^n \to X$, we may define an element
$$h([\varphi]) = \varphi_*([S^n]) \in H_n(X).$$
This only depends on the homotopy class of $\varphi$, and it's easy to see that adding spheroids in $\pi_n$ will sum their image under $h$ in $H_n$. Hence we obtain a group homomorphism,
$$h : \pi_n(X) \to H_n(X),$$
called the *Hurewicz homomorphism*.

**Theorem 4.35: Hurewicz Theorem**

For any space $X$, the Hurewicz homomorphism $h : \pi_1(X) \to H_1(X)$ is surjective, with kernel the commutator subgroup of $\pi_1(X)$. Moreover, if $X$ is simply connected and $n \geq 2$ then the following are equivalent:

(i) $\pi_k(X) = 0$ for $k = 1, \ldots, n-1$,

(ii) $H_k(X) = 0$ for $k = 1, \ldots, n-1$.

In this case, $h : \pi_n(X) \to H_n(X)$ is an isomorphism.

Proof. Without loss of generality, $X$ is a CW complex. If not, we can CW approximate and use that homology and homotopy are invariants of weak homotopy equivalence.

Suppose $X$ is $(n-1)$-connected for $n > 1$. By Proposition 4.28, we may assume $X$ has a single 0-cell and no other cells of dimension less than $n$. In this case, the chain groups $C_k(X)$ are zero for $k < n$. So, $H_n(X)$ is the free abelian group generated by the $n$-cells of $X$ quotiented by the image of the boundary of $(n+1)$-cells. Meanwhile by the cell attaching lemma, $\pi_n(X)$ is the free abelian group generated by the $n$-cells of $X$ quotiented by the $\pi_1(X)$ orbit of the attaching spheroids of $(n+1)$-cells. Since $\pi_1(X) = 0$, this is just the free abelian group generated by attaching maps. But expressing the attaching map of an $(n+1)$-cell as a sum of copies of the inclusion of each $n$-cell is easily seen to be the same as the boundary map on that $(n+1)$-cell in homology. Hence our descriptions of $\pi_n(X)$ and $H_n(X)$ are the same and so these groups are isomorphic. Since the Hurewicz homomorphism sends the generators of $\pi_n(X)$ to the generators of $H_n(X)$ it induces this isomorphism.

In the case where $n = 1$, everything holds verbatim except that $\pi_1(X)$ is generated instead of abelian generated by 1-cells. Hence to recover $H_1(X)$, one has to abelianize, after which we obtain an isomorphism. The Hurewicz homomorphism must factor through the abelianization of $\pi_1(X)$ by universal properties, after which it sends generators to generators and thus again must give an isomorphism.

Finally, to show (ii) implies (i), suppose $X$ is simply connected $H_k(X) = 0$ for $k = 1, \ldots, n-1$. If $\pi_k(X) \neq 0$ for any of these groups $k > 1$, what we have just established shows $H_k(X) = \pi_k(X) \neq 0$. Since we know $H_k(X) = 0$, we conclude $\pi_k(X) = 0$ for $k = 1, \ldots, n-1$. $\square$

This result allows us to prove a homological version of the Whitehead theorem (assuming we know the Serre spectral sequence).

**Theorem 4.36: Homological Whitehead Theorem**

Suppose $f : X \to Y$ is a continuous map between simply connected spaces. If $f_* : H_*(X) \to H_*(Y)$ is an isomorphism, then $f$ is a weak homotopy equivalence. More generally, the following are equivalent:

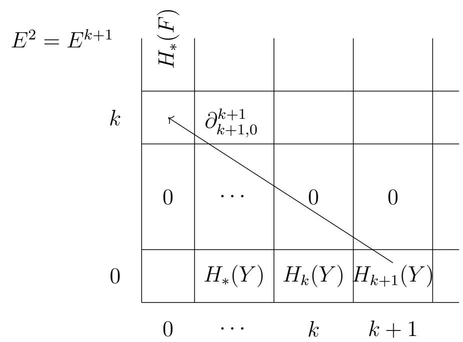
(i) $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and surjective for $k = n$.

(ii) $f_* : H_k(X) \to H_k(Y)$ is an isomorphism for $k < n$ and surjective for $k = n$.

Proof. We can homotope $f : X \to Y$ to a fibration with homotopy fibre $F$. From the long exact sequence of a fibration we conclude that condition (i) in the theorem is equivalent to $\pi_{k-1}(F) = 0$ for $k \leq n$.

Because $\pi_1(Y) = 0$, our fibration is homologically simple and so we can consider the associated Serre spectral sequence. Suppose $f_* : H_k(X) \to H_k(Y)$ is an isomorphism for $k < n$ and surjective for $k = n$. We work inductively to show $H_{k-1}(F) = 0$ for $2 \leq k \leq n$.

Suppose $H_0(F) = \mathbb{Z}$ and $H_1(F) = \cdots = H_{k-1}(F) = 0$. There are no non-trivial differentials on the spectral sequence until the page $k+1$:




This is the last time the illustrated squares participate in a non-trivial differential. We conclude $E^\infty_{k,0} = H_k(Y)$, $E^\infty_{k+1,0} = \ker \partial^{k+1}_{k+1,0}$ and $E^\infty_{0,k} = \operatorname{coker} \partial^{k+1}_{k+1,0}$.

We have that the $f_* : H_k(X) = E^\infty_{p+q=k} \to E^\infty_{k,0} = H_k(Y)$ is a surjection with kernel $E^\infty_{0,k}$. We also know that $E^\infty_{k+1,0} = \operatorname{im} f_* : H_{k+1}(X) \to H_{k+1}(Y)$ . In particular, we have an exact sequence,

$$H_{k+1}(X) \xrightarrow{f_*} H_{k+1}(Y) \xrightarrow{\partial^{k+1}_{k+1,0}} H_k(F) \xrightarrow{i_*} H_k(X) \xrightarrow{f_*} H_k(Y) \to 0.$$

It follows that if $H_0(F) = \mathbb{Z}$ and $H_1(F) = \cdots = H_{n-1}(F) = 0$, then $f_* : H_k(X) \to H_{k+1}(Y)$ is an isomorphism for $k < n$ and surjective for $k = n$. Conversely, if $f_* : H_k(X) \to H_{k+1}(Y)$ is an isomorphism for $k < n$ and surjective for $k = n$, we can inductively apply versions of the above exact sequence for increasing $k$ to show $H_k(F) = 0$ for $0 < k < n$. Hence the condition (ii) in the theorem is equivalent to $H_{k-1}(F) = 0$ for $1 < k \leq n$.

Thus we have reduced our theorem to showing $\pi_{k-1}(F) = 0$ for $k \leq n$ if and only if $H_{k-1}(F) = 0$ for $1 < k \leq n$. From the long exact sequence of a fibration,

$$\pi_2(Y) \to \pi_1(F) \to \cancelto{0}{\pi_1(X)} \to \cancelto{0}{\pi_1(Y)},$$

we conclude $\pi_1(F)$ is abelian and hence equal to $H_1(F)$. The rest of the equivalence is then just the Hurewicz theorem. □

Remark 4.37. A more standard proof goes the other way. We homotope $f : X \to Y$ to a cofibration and then prove a relative version of Hurewicz.

Also note that we did not quite use the full strength of the assumption that $X, Y$ are simply connected. If we only assume they are *homotopically simple* so that their fundamental groups are abelian and act trivially on the higher homotopy groups, then the result still holds. △

**4.5. Obstruction Theory.** One very beautiful synergy of homotopy and homology is to ask about whether a pair $(X, A)$ admits extensions of maps

$A \to F$ to maps $X \to F$. Or more generally, we can ask about extending sections over $A$ of a bundle with fibre $F$ to sections over $X$.

*Obstruction Classes.* We will deal with a fibration $F \hookrightarrow E \to B$ with $B$ a CW complex and try to extend sections over some subcomplex of $B$ to additional cells. We will need some assumptions on the fibration to make this work. We assume $F$ is *homotopically trivial* so that the action of $\pi_1(F)$ on its homotopy groups is trivial (in particular $\pi_1(F)$ should be abelian). We also assume the fibration is *homotopically simple* so that $E$ trivializes over any loop in $B$ (this holds for example if $\pi_1(B) = 0$ or if $E$ is a trivial fibration).

Suppose we have a section $s : B_n \to E$ defined over the $n$ skeleton of $B$ which we wish to extend to an $\widetilde{s}$ defined on an $n+1$ cell $D^{n+1}$. Since Feldbau's lemma implies our bundle trivializes over a disk, we are dealing with the following diagram:

$$\begin{tikzcd} F \times \partial D^{n+1} \arrow[r, hook] \arrow[d] & F \times D^{n+1} \arrow[d] \\ \partial D^{n+1} \arrow[r, hook] \arrow[u, bend right, "s"'] & D^{n+1}. \arrow[u, bend right, dashed, "\widetilde{s}?"'] \end{tikzcd}$$

**Definition 4.38**

The map $s$ projected to $F$ defines an $n$-spheroid $c_s(D^{n+1}) \in \pi_n(F)$. Doing this for each $(n+1)$-cell of $B$ defines an element of the cochain complex $c_s \in \mathcal{C}^{n+1}(B; \pi_n(F))$ called the *obstruction cochain* of $s$.

Given two sections $s, s'$ on $B_n$ that agree on $B_{n-1}$, they together define for each $n$-cell $D^n$ an $n$-spheroid $d_{s,s'}(D^n) \in \pi_n(F)$. Doing this for each $n$-cell of $B$ defines an element $d_{s,s'} \in \mathcal{C}^n(B; \pi_n(F))$ called the *difference cochain* of $s, s'$.

Remark 4.39. Note that we used homtopic simplicity of $F$ to ignore basepoints and we used homotopical triviality of the fibration to identitfy $\pi_n(F)$ over different fibres. △

Given a fibration $E \to B$ and a map $\varphi : \widetilde{B} \to B$, we can pullback the bundle to given $\varphi^* E \to \widetilde{B}$. A section $s$ of $E$ has a pullback section $\varphi^* s$ of $\varphi^* E$. The obstruction cochains we have defined are natural in the sense that,

$$c_{\varphi^* s} = \varphi^* c_s.$$

To see this, note that for any $n+1$ cell $\widetilde{D}$ of $\widetilde{B}$

$$\varphi^* c_s(\widetilde{D}) = c_s(\varphi_* \widetilde{D}) = c_s(\sum_\beta \deg_\beta D_\beta),$$

where $\deg_\beta$ is the degree of the map $\widetilde{B}/\widetilde{B} \setminus \widetilde{D} \xrightarrow{\varphi} B/B \setminus D_\beta$ for each $n$-cell $d_\beta$ of $B$. Approximating $\varphi$ smoothly and looking at the preimage of a regular value, $\varphi^* s$ on $\widetilde{D}$ is homotopic to a map on the preimages of the cells $D_\beta$ with appropriate coefficients given by the degrees, as above.

**Theorem 4.40:** [**A**, §16.3]

Suppose we have sections $s, s'$ defined on the $n$-skeleton of the base of a fibration $E \to B$. The following hold.

(i) The obstruction cochains are cocycles: $\delta c_s = 0$.

(ii) Obstruction cochains agreeing on $B_{n-1}$ are cohomologous via the difference cochain: $c_s - c_{s'} = \delta d_{s,s'}$.

(iii) The *obstruction class* $[c_s] \in H^{n+1}(B; \pi_n(F))$ is zero if and only if $s$ extends to $B^{n+1}$, possibly after modifying $s$ on $B^n \setminus B^{n-1}$.

Proof. (i) Consider a $(n+2)$-cell $D^{n+2}$ of $B$. We wish to show $\delta c_s(D^{n+2}) = c_s(\partial D^{n+2}) = 0$. Denote by $\Gamma \subset B$ the subcomplex $\partial D^{n+2}$. Consider the following diagram, where $Z$ denotes the space of cellular cycle,

$$\begin{aligned} \mathcal{C}_{n+1}(\Gamma; \pi_n(F)) \xrightarrow{\partial} Z_n(\Gamma; \pi_n(F)) &= Z_n(\Gamma_n; \pi_n(F)) \\ &= H_n(\Gamma_n; \pi_n(F)) \xleftarrow{h} \pi_n(\Gamma_n) \xrightarrow{s_*} \pi_n(F). \end{aligned}$$

Here $h$ is the Hurewicz theorem and $s_*$ is induced by $s : D^{n+2} \to F \times D^{n+2} \to F$. For $n > 1$, $\Gamma_n$ is $n-1$ connected and $h$ is an isomorphism by Hurewicz, and so $\psi = s_* \circ h^{-1}$ is well defined. If $n = 1$, $s_*$ factors through the abelianization of $\pi_n(\Gamma_n)$, and so we can still make sense of $\psi : Z_n(\Gamma; \pi_n(F)) \to \pi_n(F)$.

Since $\mathcal{C}_{n-1}(\Gamma; \pi_n(F))$ is free abelian, we may express $Z_n(\Gamma; \pi_n(F))$ as a direct summand of $\mathcal{C}_n(\Gamma; \pi_n(Y))$. In particular $\psi$ can be extended to a map,

$$\widetilde{\psi} : \mathcal{C}_n(B; \pi_n(F)) \to \pi_n(F).$$

We have $c_s(D^{n+1}) = s_* \circ h^{-1}(\partial D^{n+1}) = \widetilde{\psi}(\partial D^{n+1})$. Hence,

$$c_s(\partial D^{n+2}) = \widetilde{\psi}(\partial \partial D^{n+2}) = 0,$$

as desired.

(ii) We want to show that for any given $(n+1)$-cell $D^{n+1} \subset B$,

$$c_s(D^{n+1}) - c_{s'}(D^{n+1}) = d_{s,s'}(\partial D^{n+1}).$$

But this is almost obvious. The difference of the homotopy classes of maps $s, s' : \partial D^{n+1} \to F$ is the homotopy class of the the glued map $s \cup -s' : \partial D^{n+1} \to F$.

(iii) We know that $c_s = 0$ if and only if $s$ extends to $B_{n+1}$ essentially by definition. Then changing $s$ to $s'$ by altering it on $B_n$ in any way gives, $c_{s'} = c_s + \delta d_{s',s} = \delta d_{s',s}$. Hence $[c_{s'}] = 0$ and so $[c_s] = [c_{s'}] = 0$.

Conversely, if $c_s = \delta f$ for $f \in H^n(B; \pi_n(F))$ so that $[c_s] = 0$, we can define $s'$ by altering $s$ on each $n$-cell $D^n$ to be a spheroid representing $s(D^n) + f(D^n)$. Then,

$$c_{s'} = c_s - \delta d_{s,s'} = c_s - \delta f = 0.$$

So $s$ can be altered on $B_n$ to $s'$ which extends to $B_{n+1}$. □

We will not elaborate on the details, but one can easily extend this to a relative version. Suppose $E \to B$ is a fibration and $(B, A)$ a CW pair. If we want to extend a section $s$ from $B_n \cup A$ to $B_{n+1} \cup A$, we can define a relative obstruction cochain $c_s \in \mathcal{C}^{n+1}(B, A; \pi_n(F))$ which turns out to be a cocycle and for two sections agreeing on $B_{n-1} \cup A$, a difference cochain $d_{s,s'} \in \mathcal{C}^n(B, A; \pi_n(F))$. Then $[c_s] = 0$ if and only if $s$ extends from $B_n \cup A$ to $B_{n+1} \cup A$ after altering it on $B_n \setminus B_{n-1} \cup A$.

Remark 4.41. One can remove the assumption of homotopic triviality using local coefficients. Given a homotopically simple fibration $\pi : E \to B$, we define for each $k \geq 0$ the local coefficient system $\{\pi_k(E_x)\}$ on $B$. This local system is trivial for each $k$ and identified with $\pi_k(F)$ exactly when the fibration is homotopically trivial. Given a section $s$ of $E$ over the $k$-skeleton of $B$, we can define its obstruction class,

$$c_s \in H^{k+1}(B; \{\pi_k(E_x)\}).$$

All our results are exactly analogous to the homotopically simple case. △

Example 4.42. The relative obstruction case has the following important example for constructing homotopies. Suppose we have sections $s, s'$ defined on a CW complex $B$ and a homotopy between them on $B_{n-1}$, i.e. a map $B_{n-1} \times I \to E$ agreeing with $s$ and $s'$ at the endpoints of $I$. We wish to extend this to a homotopy on $B_n$.

The obstruction class $c_{s \cup s'}$ lives in $\mathcal{C}^{n+1}(B \times I, B \times \{0\} \cup \{1\}; \pi_n(F))$ which is the same as $\mathcal{C}^n(B; \pi_n(F))$. One can check explicitly that $c_{s \cup s'} = d_{s,s'}$. We have,

$$\delta c_{s \cup s'} = \delta d_{s,s'} = c_s - c_{s'} = 0,$$

since $s$ and $s'$ extend to $B$. We have that $[c_{s \cup s'}] = 0$ in $H^n(B; \pi_n(F))$ if and only if this homotopy extends to $B_n \times I$ without alteration on $B_{n-1} \times I$. ▲

EXAMPLE 4.43. There is another generalization that we may be useful. Suppose $p : E \to B$ is a fibration and there is a map $f : X \to B$. We can ask about the existence of a lift $g : X \to E$. Suppose such a lift $g$ exists on the $n$-skeleton of $X$ and we wish to extend it to the $(n+1)$-skeleton. For a given $(n+1)$-cell $D^{n+1}$, its boundary defines an attaching map $S^n \xrightarrow{\alpha} X \xrightarrow{g} E$. Composing with $p$, we get a map $S^n \to B$ that we know extends to the disk $D^{n+1}$ via $f$. Hence we have a homotopy $F_t : S^n \times I \to B$ between $f \circ \alpha$ and a constant. Since we are dealing with a fibration, we may lift $F_t$ to a map $G_t : S^n \times I \to E$ so that $G_0 = g \circ \alpha$ and $p \circ G_1$ is constant. Hence we may homotope $g \circ \alpha : S^n \to E$ to a map $S^n \to F$ lying in a single fibre; to extend this to $D^{n+1} \to F$ requires the spheroid be contractible. Under our standard topological assumptions, the obstruction to lifting $g$ to a map $X_{n+1} \to F$ thus defines an obstruction cochain $c_g \in H^{n+1}(X; \pi_n(F))$. Similarly we define difference cochains and our same results will hold. The case of sections is just this construction applied when the map $f$ we begin with on the base is $\mathrm{id}_B : B \to B$. ▲

*Another Proof of Homological Whitehead.* We can use obstruction theory to reprove the homological case of the Whitehead theorem without the need for spectral sequences.

**Theorem 4.44: Homological Whitehead Theorem**

Let $X$ and $Y$ be homotopically simple CW complexes and $f : X \to Y$ a continuous map inducing isomorphisms in homology. Then $f$ is a homotopy equivalence.

PROOF. We may homotope $f$ to be a cofibration $X \hookrightarrow Y$. From the homology long exact sequence of a pair, because $f_*$ is an isomorphism, $H_*(Y, X) = 0$. We conclude from the universal coefficient theorem that,

$$H^*(Y, X; \pi_n(X)) = 0 \quad \text{for all} \quad n.$$

We would like to extend the identity map $X \to X$ to a map $g : Y \to X$ completing the following diagram which commutes up to homotopy,

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[dl, dashed, "g"] \\ X. & \end{tikzcd}$$

Because $X$ is homotopically simple we can do obstruction theory, The relative obstruction to extending $g$ to the $i$ skeleton of $Y$ lives in $H^i(Y, X; \pi_{i-1}(X)) = 0$. Hence there is no obstruction to finding such a map $g : Y \to X$ so that $g \circ f \cong \mathrm{id}_X$.

Now note since id and $f$ both induce isomorphisms on homology groups, the above diagram implies that $g_* : H_*(Y) \to H_*(X)$ is also an isomorphism.

But now, since $Y$ is homotopically simple, we can apply the same obstruction theory argument to $g$ as we did to $f$ and find a map $h : X \to Y$ so that $h \circ g \cong \mathrm{id}_Y$.

But then,

$$f \circ g \cong h \circ g \circ f \circ g \cong h \circ g \cong \mathrm{id}_Y.$$

Hence $f$ is a homotopy equivalence between $X$ and $Y$. □

*First Obstructions.* If our fibre $F$ is highly connective, there may be no obstruction to extending a section to some higher skeleta of the base $B$. The first time we do run into an obstruction will carry some intrinsic data about the fibration independent of the choice of section. This concept of "first obstructions" will give important geometric realizations of characteristic classes which we study in depth in a later chapter.

Consider a homotopically trivial fibration $p : E \to B$ with CW base $B$ and fibre $F$. Suppose that,

$$\pi_0(F) = \pi_1(F) = \cdots = \pi_{n-1}(F) = 0,$$

so that $\pi_n(F)$ is the first non-trivial homotopy group of the fibre. From obstruction theory, we know that we can always obtain sections of $E$ over the $n$-skeleton $B_n$ of the base.

**Theorem 4.45**

Given any two sections $s, s' : B_n \to E$, we have,

$$[c_s] = [c_{s'}] \in H^{n+1}(B; \pi_n(F)).$$

Proof. Note that if $s|_{B_{n-1}}$ and $s''|_{B_{n-1}}$ agree then $c_s - c_{s''} = \delta d_{s,s''}$ and so $c_s$ and $c_{s''}$ are cohomologous. Thus it suffice to show $s'$ is homotopic to a section $s''$ agreeing with $s$ on $B_{n-1}$. Then we will have $[c_{s'}] = [c_{s''}] = [c_s]$.

We can prove this inductively. Say $s|_{B_{k-1}} = s'|_{B_{k-1}}$ for some $0 \leq k < n$. We would like to homotope $s'$ to agree with $s$ on $B_k$. The obstruction to such a homotopy is precisely,

$$d_{s|_{B_k}, s'|_{B_k}} \in \mathcal{C}^k(B; \pi_k(F)) = 0.$$

Hence they are homotopic on $B_k$. We can extend this homotopy to a map on all of $B \times I$ by a simple extension of Brosuk's theorem. Hence $s'$ is homotopic to a section agreeing with $s$ on $B_k$. Repeating this argument, we obtain our result. □

**Definition 4.46**

In this setting, we define the *first obstruction class* of the fibration $E$ to be the cohomology class,

$$C(E) := c_s \in H^{n+1}(B; \pi_n(F)),$$

for any section $s$ of $E$ defined on the $n$-skeleton of $B$ (this is independent of $s$ by the above theorem).

Suppose we have a cellular map $f : B' \to B$ and a fibre bundle $E$ on $B$. We showed above that obstruction classes are natural with respect to pullback. Hence,

$$C(f^*E) = f^*C(E). \tag{4.1}$$

By cellular approximation this extends to non-cellular maps. In particular, $C(E)$ does not depend on the CW structure we put on $B$. If $B$ is not CW, we may take a CW approximation $\varphi : B^{CW} \to B$, and pullback our bundle $E$ and then define $C(E)$ as $(\varphi^*)^{-1}C(\varphi^*E)$.

An association of a cohomology class in $H^*(B)$ to any bundle $E \to B$ that satisfies the naturality property 4.1 is called a *characteristic class.*

Thus, the first obstruction class defines a characteristic class of our fibre bundle sometimes called the *primary characteristic class.*

We will have much more to say about characteristic classes in the case of vector bundles later, but as a taste we will see one example now.

**Definition 4.47**

Let $\xi \to B$ be an oriented rank $n$ vector bundle and $S\xi$ be its associated sphere bundle, defined by picking some metric on $\xi$ and restricting to vectors of norm one in each fibre. The fibres $F = S^{n-1}$ of $S\xi$ are $(n-2)$-connected and $\pi_{n-1}(F) = \mathbb{Z}$. The fact the bundle is oriented means $S\xi$ is homotopically trivial. Thus we obtain a first obstruction,

$$e(\xi) := C(S\xi) \in H^n(B),$$

called the *Euler class* of $\xi$.

Here is some motivation for the name of the Euler class.

Proposition 4.48. *Let $M$ be a closed oriented $n$-manifold. Then,*

$$\langle e(TM), [M] \rangle = \chi(M).$$

*Equivalently, $e(TM)$ is Poincaré dual to $\chi(M) \in H_0(M)$.*

Proof. Pick a cell structure on $M$ by use of Morse theory. Pick a generic section $V$ of $TM$, i.e. a vector field, so that the zeroes of $V$ are discrete and non-degenerate. We can assume the zeroes all lie in the interior of $n$-cells of $M$. Hence $V$ defines a section of the sphere bundle $STM$ over the $n-1$ skeleton of $M$. Our proof of the fundamental class theorem shows $[M]$ can be written as the sum of all $n$ cells of $M$, oriented compatibly with the orientation of $M$. Given a single $n$-cell $D^n$, the value of $\langle e(TM), [D^n] \rangle$ is the value of the obstruction class $c_V(D^n)$. The spheroid $V : \partial D^n \to STM|_{D^n} \cong S^{n-1} \times D^n$ extends over $D^n$ minus some small balls around the zeroes of $V$ and so the homotopy class of this spheroid (i.e. the degree of this map $V : \partial D^n \to S^{n-1} \times D^n$) is the sum of the homotopy classes around each zero. Near each of these zeroes, $V$ determines a map $S^n \to S^n$ whose degree is by definition the index of $V$ at the zero. Hence, $\langle e(TM), [D^n] \rangle$ is the sum of the indices of all interior zeroes. Summing over all cells,

$$\langle e(TM), [M] \rangle = \sum_{V(x)=0} \mathrm{ind}_x(V) = \chi(M),$$

where the last equality is just Poincaré–Hopf. □

We can give a more general geometric description of the Euler class.

Proposition 4.49. *Let $\xi$ be a rank $r$ oriented vector bundle on $B$ and let $t_\xi \in H^r(\xi, \xi \setminus B) \cong H^r(D_\xi, \partial D_\xi)$ be its Thom class. The pullback of $t_\xi$ under the inclusion of the zero section $i : B \hookrightarrow \xi$ is $i^*(t_\xi) = e(\xi) \in H^r(B)$.*

*Moreover, suppose $B$ is an oriented $n$-manifold and $s$ is a section of $\xi$ that transversely intersects the zero section. Let $Z$ be the zero locus of $s$, which is a submanifold of $B$ of dimension $n-r$. Then $e(\xi)$ is Poincaré dual to $[Z] \in H_{n-r}(B)$.*

Proof. Pick some non-vanishing section $s$ of $\xi$ over the $(k-1)$-skeleton of $B$. The Euler class measure the first obstruction to extending the section to the $k$-skeleton of $B$. Extend $s$ to a section on $B_k$ transversely intersecting the zero section on each cell. The Euler class evaluated on each $k$-cell $D^k$ is an oriented count of the zeroes of $s$ on $D^k$ (see the proof of the previous proposition).

Because $s$ is homotopic to the zero section on $D^k$, we can compute our restriction of $t_\xi$ by pulling back to $s(D^k)$. Since $s$ is non-zero on the $(k-1)$-skeleton, we can pullback $t_\xi$ to define an element of $H^k(B_k, B_{k-1})$. The section $s$ considered in $\xi$ modulo $\xi \setminus B$ is homotopic to an oriented sum of the fibres of $\xi$ over which $s$ vanishes. The Thom class on the other hand represents the fundamental class of each fibre. So restricting the Thom class to the section and then considering this as an element in the homology of the base gives the cohomology class counting the oriented zeroes of $s$ on each $k$-cell $D^k$. But this is exactly what the Euler class does. Hence $t_\xi|_B = e(\xi)$.

For the second statement, recall from intersection theory that $t_\xi$ is Poincaré dual to $[B]$, the fundamental class of the base in the total space of $\xi$. Pulling back $t_\xi$ to the base gives on the one hand $e(\xi)$ by above, while on the other hand it is Poincaré dual to the class of the self intersection of $[B]$ in $\xi$. The self intersection of $[B]$ can be found by perturbing $B$ to a section of $\xi$ that has transverse intersection with the zero section and looking at the class of this intersection. This is exactly what we claimed. □

Example 4.50. The above proposition for a tangent bundle is a special case of this result. There is another important case of the normal bundle $\nu_N$ of an embedding $N \hookrightarrow M$ of smooth oriented manifolds. The Euler class $e(\nu_N)$ of this embedding will be Poincaré dual to the self-intersection of $N$, $[N] \cdot [N]$ inside $N$. In the case where $\dim M = 2 \dim N$, the Euler class is identified with an integer which is the *self-intersection number* of $N \subset M$. As the proposition shows, this is computed by an oriented count of intersections of $N$ with some generic isotoped copy of itself. The submanifold $N \subset M$ can be displaced from itself only if its self-intersection number is zero. ▲

There is one final easy but important corollary of what we have said.

Corollary 4.51. *The Euler class is* unstable. *If $\xi$ is an oriented vector bundle on $B$ and $\underline{\mathbb{R}}$ is the trivial line bundle on $B$, then $e(\xi \oplus \underline{\mathbb{R}}) = 0$.*

Proof. The section $s(x) = (0, 1)$ of $\xi \oplus \underline{\mathbb{R}}$ is non-vanishing on $B$ and so there is no obstruction to extending a non-vanishing section of the bundle to all the skeleta of $B$. □

Remark 4.52. Even if $\xi \to B$ is not oriented, we can use the construction of obstruction classes with local coefficients to define an Euler class,

$$e(\xi) \in H^{n+1}(B; \{H_n(E_x)\}).$$

Here we use that $H_n(E_x) \cong \pi_n(E_x) \cong \mathbb{Z}$ for each $x \in B$.

### 4.6. Some Important Constructions.

*Eilenberg–MacLane Spaces.* Another very reasonable question to ask is whether there exist spaces realizing any sets of homotopy groups. This can easily be seen to be true.

EXERCISE 4.53. Let $G_1, G_2, \ldots$ be a collection of groups, all abelian except possibly $G_1$. Let $\rho_i$ for $i = 2, 3, \ldots$ be a collection of group actions of $G_1$ on $G_i$. Construct a connected CW complex whose homotopy groups are $\pi_i(X) = G_i$ and whose action of $\pi_1(X)$ on $\pi_i(X)$ is $\rho_i$. ▲

The general spaces constructed in the above exercise are usually not so interesting. But, they turn out to be very important in the case when only one of the homotopy groups is non-zero.

**Definition 4.54**

Let $n$ be a positive integer and $G$ a group, abelian if $n > 1$. A space is called an *Eilenberg–MacLane space* and denoted $K(G, n)$ if it satisfies,

$$\pi_i(K(G,n)) = \begin{cases} G & i = n \\ 0 & \text{otherwise} \end{cases}.$$

While in general the homotopy groups are not enough to determine the weak homotopy type of a space, they are sufficient in the case of Eilenberg–MacLane spaces.

**Theorem 4.55: Eilenberg–MacLane Spaces**

There exists a unique Eilenberg–MacLane space $K(G, n)$ up to weak homotopy equivalence for any $G$ and $n$. In particular, all CW Eilenberg–MacLane spaces $K(G, n)$ are homotopy equivalent.

PROOF. Let's construct a CW complex which is an Eilenberg–MacLane space $K(G, n)$. Let $\{g_\alpha\}$ be a generating set for $G$. Consider $X_n$ a bouquet of $n$-spheres indexed by the generators $g_\alpha$. We have that $\pi_i(X_n) = 0$ for $i < n$ and $\pi_n(X_n)$ is the free abelian group (or just free group if $n = 1$) generated by the $g_\alpha$. We can thus find a subgroup $H$ of $\pi_n(X_n)$ so that $\pi_n(X_n)/H \cong G$. Let $\{h_\beta\}$ be a generating set for $H$. We may attach $n+1$ cells $D_\beta$ to $X_n$ using $h_\beta : S^n \to X_n$ as the attaching maps, to obtain a $n+1$ dimensional CW complex $X_{n+1}$. Note by the cell attaching lemma that $\pi_i(X_{n+1}) = \pi_i(X_n) = 0$ for $i < n$. Moreover, $\pi_n(X_{n+1}) = \pi_n(X_n)/H = G$.

Now, we finish inductively. Suppose we have an $m+1$ dimensional CW complex $X_{m+1}$ for $m \geq n$ so that,

$$\pi_i(X_{m+1}) = \begin{cases} G & i = n \\ 0 & i \leq m, i \neq n. \end{cases}$$

Set $\pi_{m+1}(X_{m+1}) = G_{m+1}$. Suppose this group is generated by spheroids $f_\gamma : S^{m+1} \to X_{m+1}$. We may attach a collection of $m+2$ cells $D_\gamma$ to $X_{m+1}$ using $f_\gamma$ as the attaching maps, to obtain a $m+2$ dimensional CW complex $X_{m+2}$. By the cell attaching lemma, $\pi_i(X_{m+2}) = \pi_i(X_{m+1})$ for $i \leq m$. Moreover, by the cell attaching lemma, $\pi_{m+1}(X_{m+1}) = 0$.

Hence taking the direct limit of these CW complexes with the weak topology gives a space $X = \lim_{n\to\infty} X_n$ whose homotopy groups all vanish except for $\pi_n(X) = G$. That is $X$ is an Eilenberg–MacLane space.

Now suppose $Y$ is another CW version of $K(G,n)$. From our construction of $X$, we know $\pi_n(X) \cong G$ is generated by the inclusions of spheres $g_\alpha : S^n_\alpha \hookrightarrow \bigvee_\beta S^n_\beta = X_n \subset X$. Let $s_\alpha : S^n \to Y$ a collection of spheroids representing the same generators under an isomorphism $\pi_n(Y) \cong G$. Now define a map on the $n$-skeleta $F : X_n \to Y$ which on each sphere in the bouquet $S^n_\alpha \subset X_n$ restricts to the map $s_\alpha$.

Note from obstruction theory, the series of obstructions to extending $F$ to a map $\widetilde{F} : X \to Y$ live in $H^{i+1}(X, \pi_i(X))$ for $i \geq n$. Note that every $n+1$ cell we attach to $X_n$ bounds a spheroid $h_\alpha$ which is null homotopic after composition with $F$ (since $F_*$ sends generators of $\pi_n(X_n)$ to the corresponding generators of $G$, and $h_\alpha$ was chosen as a relation on the generators of $G$). Hence the first obstruction $c_F \in H^{n+1}(X, \pi_n(X))$ vanishes and we can extend $F$ to the $n+1$ skeleton. For any $i > n$, the obstruction class must vanish since $X$ is a $K(G,n)$ and so $F$ extends to a map $\widetilde{F} : X \to Y$ with $\widetilde{F}|_{X_n} = F$.

It is clear that under isomorphism with $G$, $\widetilde{F}_* : \pi_n(X) \to \pi_n(Y)$ acts as the identity on generators and hence is an isomorphism. For all other $i$, $\widetilde{F} : \pi_i(X) \to \pi_i(Y)$ is necessarily the trivial isomorphism.

Hence $\widetilde{F}$ is a weak homotopy equivalence, and by the Whitehead theorem is a homotopy equivalence. Given any space $Z$ which is a $K(G,n)$, it has a CW approximation $Z^{CW} \to Z$. Note $Z^{CW}$ is a CW $K(G,n)$ which must be homotopy equivalent to $X$ by above. Hence every Eilenberg–MacLane space is weak homotopy equivalent to $X$. $\square$

Example 4.56. While our proof of existence was constructive, it was far from explicit, as in general we do not know what the $(m+1)$st homotopy group of our $(m+1)$-skeleton that we need to kill will be.

For example, suppose we want to build a CW model for $K(\mathbb{Z}, n)$. We know we may take its $n$-skeleton to be $S^n$. But already, we do not in general know what $\pi_{n+1}(S^n)$ is so we do not know what $(n+2)$-cells to attach to ensure the $(n+1)$st homotopy group vanishes.

We can make a few Eilenberg–MacLane space constructions explicit, as we discuss now.

(i) If we wish to find an Eilenberg–MacLane space $K(G, 1)$, one method is to find a space with fundamental group $G$ and a contractible universal cover. Conversely, if $G$ is discrete, we can find a contractible space with a free $G$-action and take the quotient.

   For example, quotienting $\mathbb{R}^n$ by a lattice gives, $K(\mathbb{Z}^n, 1) = T^n$, the $n$-torus. In particular, $K(\mathbb{Z}, 1) = S^1$.

(ii) Recall the sphere $S^\infty$ is contractible. The quotient by antipodal identification gives $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$.

(iii) More generally, $S^\infty \subset \mathbb{C}^\infty$ has a $\mathbb{Z}_m$ action by,

$$q \cdot (z_1, z_2, \ldots) = (z_1 e^{2\pi i q/m}, z_2 e^{2\pi i q/m}, \ldots).$$

   The quotient $L_m^\infty = S^\infty/\mathbb{Z}_m$ is called an *infinite lens space* and is a model for $K(\mathbb{Z}_m, 1)$.

(iv) Any surface $\Sigma$ which is not $S^2$ or $\mathbb{R}P^2$ has $\mathbb{R}^2$ as its universal cover and hence is a model for $K(\pi_1(\Sigma), 1)$.

(v) Recall from the infinite Hopf fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$ that $\pi_i(\mathbb{C}P^\infty) = \pi_{i-1}(S^1)$. We conclude that $\mathbb{C}P^\infty$ is a model for $K(\mathbb{Z}, 2)$. This is essentially the only space $K(G, n)$ for $n > 1$ that we can make totally explicit.

(vi) Let $\mathrm{Sym}^n(X)$ denote the $n$th symmetric product of $X$, i.e. the quotient $X^n/S_n$ where $S_n$ is the symmetric group acting by permutations of factors. Note if $X$ has a basepoint $x_0$, there is an inclusion map $\mathrm{Sym}^n(X) \hookrightarrow \mathrm{Sym}^{n+1}(X)$ sending $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, x_0)$. Let $\mathrm{Sym}^\infty(X)$ be the direct limit as $n \to \infty$ of this construction. One can construct a fibration to show for any path-connected Hausdorff space $X$ that $\pi_n(\mathrm{Sym}^\infty X) = \pi_{n+1}(\mathrm{Sym}^\infty \Sigma X)$. In particular we deduce, $K(\mathbb{Z}, n) = \mathrm{Sym}^\infty(S^n)$.

(vii) More generally, one has the Dold–Thom theorem, which says that for a connected CW complex $X$,

$$\pi_n(\mathrm{Sym}^\infty(X)) \cong \widetilde{H}_n(X).$$

One concludes that we can obtain any Eilenberg–McLane space $K(G,n)$ by finding a space $X$ whose reduced homology is concentrated in degree $n$ and equals $G$ there. Such a space is called a *Moore space* $M(G,n)$. These are not unique like Eilenberg-MacLane spaces, for example there are homology spheres which have the same homology as a sphere but a non-trival perfect fundamental group. A CW Moore space $M(\mathbb{Z}_k, n)$ can be constructed easily. Simply take an $n$-sphere and glue on an $n+1$ dimensional disk by a degree $k$ map.

(viii) Let $\mathbf{Top}(n)$ and $\mathbf{PL}(n)$ be respectively the topological groups of homeomorphisms and piecewise linear homeomorphisms of $\mathbb{R}^n$ fixing the origin. Let $\mathbf{Top}$ and $\mathbf{PL}$ be the direct limit of these groups under inclusions into one dimension higher by the identity on the last factor. It is a result of Kirby and Siebenmann that $\mathbf{Top}/\mathbf{PL}$ is a model for $K(\mathbb{Z}_2, 3)$. This has the following consequence in differential topology.

Associated to a manifold $M$ with an $\mathcal{M}$-structure, there is a classifying map for its tangent bundle $M \to B\mathcal{M}$, where $B\mathcal{M}$ is the classifying space of the group $\mathcal{M}$ (we will talk about these later). So, to make a topological manifold $M$ piecewise linear (and hence smooth, since those categories are equivalent) is to ask for a lift of the classifying map $M \to B\mathbf{Top}(n)$ to $M \to B\mathbf{PL}(n)$.

As for any classifying spaces, there is a fibration,

$$\mathbf{Top}(n)/\mathbf{PL}(n) \hookrightarrow B\mathbf{PL}(n) \to B\mathbf{Top}(n).$$

By obstruction theory (our example 4.43), the obstructions to lifting live in the groups $H^{i+1}(M; \pi_i \mathbf{Top}(n)/\mathbf{PL}(n))$. It is known for $n \geq 5$ that the map $\mathbf{Top}(n)/\mathbf{PL}(n) \to \mathbf{Top}/\mathbf{PL}$ is $(n+1)$-connected and hence the only obstruction to lifting is a characteristic class $ks(M) \in H^4(M; \mathbb{Z}_2)$. This obstruction is called the *Kierby–Siebenmann class* and by what we've claimed topological manifolds of dimension at least 5 are smoothable precisely when it vanishes. ▲

Exercise 4.57. Here are some properties of Eilenberg–MacLane spaces.

(1) Prove that $K(\pi_1, n) \times K(\pi_2, n) \cong K(\pi_1 \oplus \pi_2, n)$. Construct as explicitly as possible $K(F, n)$ for any finitely generated abelian group.

(2) Prove that $K(\pi, n-1) \cong \Omega K(\pi, n)$ and conclude $\Omega \mathbb{C}P^\infty$ is weak homotopy equivalent to $S^1$.

(3) Prove that a group homomorphism $G \to H$ induces a map of Eilenberg–MacLane spaces $K(G,n) \to K(H,n)$. Prove a short exact sequence of abelian groups $0 \to A_1 \to A_2 \to A_3 \to 0$ induces a long exact sequence of Eilenberg–MacLane spaces,

$$\cdots \to K(A_1,n) \to K(A_2,n) \to K(A_3,n) \to K(A_1,n+1) \to \cdots .$$

What does the beginning of this sequence look like for the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$? ▲

*An Application to Homology.* One very important application of Eilenberg–MacLane spaces is to give a homotopical formulation of homology groups. This can be useful in proving some properties of homology. But its main benefit is reducing some homotopical questions, for example the classification of principal and vector bundles which we will intensively study later, to homological considerations.

For some space $X$ and some Eilenberg–MacLane space $K(G,n)$, consider the homotopy classes of pointed maps $[X, K(G,n)]$. Note this has a group structure described as follows. By the exercise above, $K(G,n) \cong \Omega K(G,n+1)$. There is a map,

$$\Omega K(G,n+1) \times \Omega K(G,n+1) \to \Omega K(G,n+1),$$

given by composition of loops. Applying the $[X,-]$ functor and the above isomorphism, this induces a map,

$$[X, K(G,n)] \times [X, K(G,n)] \to [X, K(G,n)].$$

This gives a group operation. The identity is the constant map and the inverse is given by taking a map in $[X, K(G,n)] \cong [X, \Omega K(G,n+1)]$ and composing with the map reversing directions of loops.

There is also a graded ring structure on $\bigoplus_{n\in\mathbb{N}}[X, K(\mathbb{Z},n)]$ described as follows. Consider the smash product $K(\mathbb{Z},n) \wedge K(\mathbb{Z},m)$. Using our CW model of Eilenber–MacLane spaces, we deduce this space is $(n+m-1)$ connected and its $\pi_{n+m}$ group is $\mathbb{Z}$. Hence we may glue on cells of dimension $n+m+2$ and higher to obtain a cofibration $K(\mathbb{Z},n)\wedge K(\mathbb{Z},m) \hookrightarrow K(\mathbb{Z},n+m)$. Given maps $f : X \to K(\mathbb{Z},n)$ and $g : X \to K(\mathbb{Z},m)$, we obtain a map $f \wedge g$ from $X \times X$ into the smash product which we may precompose with diagonal $\Delta : X \to X \times X$ and post-compose with inclusion to get a map $X \to K(\mathbb{Z},n+m)$. Thus we obtain a map,

$$[X, K(\mathbb{Z},n)] \times [X, K(\mathbb{Z},m)] \to [X, K(\mathbb{Z},m+n)].$$

Exercise 4.58. Show this map distributes over our addition map. Show this defines a ring multiplication on $\bigoplus_{n\in\mathbb{N}}[X, K(\mathbb{Z}, n)]$. Extend this to a map,

$$[X, K(G, n)] \times [X, K(H, m)] \to [X, K(G \otimes_{\mathbb{Z}} H, n + m)],$$

for $\mathbb{Z}$-modules $G, H$. In particular for $G = H = G \otimes H$ a field. ▲

As the final ingredient we need for the upcoming theorem, we note there is a canonical element of $H^n(K(G, n); G)$. Namely, note by the Hurewicz theorem and the universal coefficient theorem that $H^n(K(G, n); G) \cong \mathrm{Hom}(G, G)$. Consider the element $F_G \in H^n(K(G, n); G)$ which under this isomorphism represents the identity map in $\mathrm{Hom}(G, G)$.

This element has a couple equivalent definitions. Recall that we constructed $K(G, n)$ as a CW complex whose $n$-skeleton was a bouquet of $n$-spheres $S^n_\alpha$ indexed by generators $g_\alpha$ of $G$. We can define a cellular cochain $F_G \in \mathrm{Hom}(\mathcal{C}_n(K(G, n)), G)$ so that $F_G(S^n_\alpha) = g_\alpha$ and then extend linearly. For any $n + 1$ cell $D^{n+1}$,

$$\delta F_G(D^{n+1}) = F_G(\partial D^{n+1}).$$

But the boundary of $(n+1)$-cells we attach correspond to relations in $G$, and so $F_G$ must evaluate to zero on them. Hence $F_G$ defines a cohomology class and will agree with $F_G$ above.

For an obstruction theory perspective, we can view $F_G$ as the difference cochain $d_{\mathrm{id},*} \in \mathcal{C}^n(K(G, n); G)$ between the identity and the constant maps $K(G, n) \to K(G, n)$ on the $n$-skeleton. We know that both id and $*$ extend to all of $K(G, n)$ and so $\delta d_{\mathrm{id},*} = c_{\mathrm{id}} - c_* = 0 - 0$. Hence $d_{\mathrm{id},*} \in H^n(K(G, n); G)$ is a cohomology class. Studying this explicitly on the $n$-skeleton, we see it again coincides with $F_G$.

**Theorem 4.59: Homotopical Formulation of Cohomology**

For any CW complex $X$ and any abelian group $G$ there is a bijection,

$$[X, K(G, n)] \to H^n(X; G) \quad \text{given by} \quad f \mapsto f^*(F_G),$$

which is natural with respect to $X$. Moreover, this is a group isomorphism using the loop composition map on the left and addition on the right. Taking the direct sum over all $n$ and $G = \mathbb{Z}$ or a field $F$, this is a graded ring isomorphism using the smash product on the left and the cup product on the right.

Proof. Consider a cellular cocycle representative of a cochain $c \in H^n(X; G)$. Consider the constant map $f_0 : X_{n-1} \to * \subset K(G, n)$. We can extend this to $f_n : X_n \to K(G,n)$ as follows. Because $f_0$ is constant, we may factor $f_n$ as $X_n \to X_n/X_{n-1} \to K(G,n)$. $X_n/X_{n-1}$ is a bouquet of spheres $S^n_\alpha$ corresponding to the $n$-cells $D^n_\alpha$ of $X$. We define $f_n$ on $S^n_\alpha$ to represent the spheroid $c(D^n_\alpha) \in G \cong \pi_n(K(G,n))$. We have by construction and naturality,

$$c = d_{f_n,*} = f_n^*(d_{\mathrm{id},*}) = f_n^* F_G.$$

So $c$ is in the image of our map at least on $X_n$. Also, since $c$ is cocycle,

$$0 = \delta c = \delta d_{f_n,*} = c_{f_n} - c_* = c_{f_n}.$$

This implies that $f_n$ extends to a map $f_{n+1} : X_{n+1} \to K(G,n)$. But then $\pi_k(K(G,n)) = 0$ for $k > n$ and so the map extends to all higher skeleta and we obtain $f : X \to K(G,n)$ with $f|_{X_n} = f_n$. Hence $f^*F_G = c$. Thus, our map is surjective.

Now suppose $f, g : X \to K(G,n)$ satisfy $[f^*F_G] = [g^*F_G]$. By cellular approximation we may assume $f, g$ are constant on the $(n-1)$-skeleton. On $X_n$,

$$d_{f,g} = d_{f,*} - d_{g,*} = f^*F_G - g^*F_G \in \mathrm{im}(\delta).$$

Hence $\delta d_{f,g} = 0$. By example 4.42, we can find a homotopy between $f$ and $g$ on $X_n$. Since the higher homotopy groups of $K(G,n)$ vanish, obstruction theory implies this homotopy extends to $X$. Hence $f, g$ are in the same homotopy class and our map is injective.

The naturality of the map with respect to $X$ is clear, since if we have a map $g : Y \to X$ and $f \in [X, K(G,n)]$ then $(f \circ g)^*F_G = g^*(f^*F_G)$.

To show the map is a group homomorphism, note that the inclusion $\Omega K \xhookrightarrow{\mathrm{id}\times *} \Omega K \times \Omega K \xrightarrow{\circ} \Omega K$ is homotopic to the identity, as is including by $* \times \mathrm{id}$ instead. Applying this to $K(G,n) = \Omega K(G,n+1)$, our group operation precomposed with inclusion from either factor is homotopic to the identity.

Let $i_1, i_2$ denote the two factor inclusions and $\mu$ our multiplication. We thus have that $i_1^*\mu^*F_G = i_2^*\mu^*F_G = F_G$. By the universal coefficient theorem and the Künneth formula,

$$H^n(K(G,n) \times K(G,n); G) = \mathrm{Hom}(G \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes G, G).$$

We thus have, $\mu^*F_G = F_G \oplus F_G$. But then for any $f_1, f_2 : X \to K$,

$$(f_1 \times f_2)^*\mu^*F_G = (f_1 \times f_2)^*(F_G \oplus F_G) = f_1^*(F_G) + f_2^*(F_G).$$

Now finally we show it is a ring homomorphism, beginning with some notation. Let $p_1, p_2$ denote the projections from $K(G,n) \times K(G,m)$ to the two factors. Let $q$ be the composition of quotient and inclusion $K(G,n) \times K(G,m) \to K(G,n) \wedge K(G,m) \to K(G,n+m)$. Let $F^i_G$ denote our canonical element in $H^i(K(G,i);G)$. It is simple enough to check explicitly in terms of cellular descriptions using Künneth and otherwise that,

$$p_1^* F_G^n \smile p_2^* F_G^m = q^* F_G^{n+m}.$$

We can rephrase this as follows. Applying the bijection we have just proved to all the terms in the cup product gives a map,

$$[X, K(G,n)] \times [X, K(G,n)] \to [X, K(G, n+m)].$$

If we plug in $X = K(G,n) \times K(G,m)$, then we just concluded that the image of $(p_1, p_2)$ under this map is the map $q$. Now to verify that this cup product map always agrees with the one coming from smash products is essentially just a matter of naturality.

Given $f \in [X, K(G,n)]$ and $g \in [X, K(G,m)]$, the smash product we have described maps this pair to $q \circ (f \times g) \circ \Delta \in [X, K(G, n+m)]$. Hence under our isomorphism, this product is sent to the cohomology class,

$$\Delta^*(f \times g)^* q^* F_G^{n+m} = \Delta^*(f \times g)^*(p_1^* F_G^n \smile p_2^* F_G^m).$$

Note that $p_i \circ f \times g = f \circ \pi_i$ where $\pi_i$ are the projections $X \times X \to X$,

$$= \Delta^*(\pi_1^* f^* F_G^n \smile \pi_2^* g^* F_G^m).$$

Recalling the definition of the cup product in terms of the diagonal embedding into the product,

$$= f^* F_G^n \smile g^* F_G^m$$

Thus the cup product applied after the isomorphism is the same as the smash product applied before the isomorphism. Hence, we have a ring map. □

We have the following corollary.

Corollary 4.60. *Reduced homology of spaces is stable under suspension:*

$$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X).$$

Proof. By CW approximating, we may take $X$ to be a CW complex so our theorem applies. One has group isomorphisms for $i \geq 1$,

$$H^i(X) \cong [X, K(\mathbb{Z}, i)] \cong [X, \Omega K(\mathbb{Z}, i+1)] \cong [\Sigma X, K(\mathbb{Z}, i)] \cong H^{i+1}(\Sigma X),$$

where we used loop-suspension adjunction in the middle. By the universal coefficient theorem, we get an isomorphism of homology as well. For $i = 0, -1$, this is easy to prove directly.

The isomorphism can be more explicitly realized by taking a singular simplex $\sigma : \Delta^n \to X$ and suspending to $\Sigma\sigma : \Sigma\Delta^n \to \Sigma X$. We may identify $\Sigma\Delta^n$ with two copies of $\Delta^{n+1}$ glued along a pair of faces so that $\Sigma\sigma$ defines a singular $n+1$ chain.

One could also prove the isomorphism with Mayer–Vietoris. $\square$

The fact we mentioned much earlier that the Freudenthal suspension theorem implies stability of the homotopy groups of repeated suspensions can now be proved.

Corollary 4.61. *The $(n+1)$st suspension of a space $X$ is $n$-connected. Hence, $\pi_k^{\mathrm{st}}(X) = \pi_{2k+2}(\Sigma^{k+2}X)$.*

Proof. We have $\widetilde{H}_i(\Sigma^{n+1}X) = \widetilde{H}_{i-n-1}(X)$ is zero as long as $i < n+1$. By Hurewicz, we conclude $\pi_i(\Sigma^{n+1}X) = 0$ for $i \leq n$. The second statement follows directly from Freudenthal suspension. $\square$

Remark 4.62. We should have probably discussed this earlier, but better late than never. We define a homotopical generalization of topological groups (and a dual object which doesn't exist in **Top**!).

**Definition 4.63**

An *H-space* (H for Hopf) is a space $X$ equipped with a multiplication $\mu : X \times X \to X$ so that the following diagram commutes up to homotopy,

$$\begin{tikzcd} X \vee X \arrow[d, hook] \arrow[dr, "\mathrm{id}\vee\mathrm{id}"] & \\ X \times X \arrow[r, "\mu"] & X. \end{tikzcd}$$

A *co-H-space* is a space $Y$ equipped with a comultiplication $c : Y \to Y \vee Y$ so that the following diagram commutes up to homotopy,

$$\begin{tikzcd} & Y \vee Y \arrow[d, hook] \\ Y \arrow[ur, "c"] \arrow[r, "\Delta"'] & Y \times Y. \end{tikzcd}$$

Every loop space $\Omega X$ is an H-space via loop composition. Every topological group (or even monoid) is by group multiplication (this is actually a special case of loop spaces since $G \cong \Omega BG$ as we will prove later).

Every suspension $\Sigma Y$ is a co-H-space by collapsing the equator. More generally $Y \wedge A$ is a co-H-space for any co-H-space $A$.

Note that if either $X$ is a co-H-space or $Y$ is an $H$-space, then the set $[X, Y]$ obtains a multiplication (and if we're lucky a group structure). In the case $X = S^n = \Sigma S^{n-1}$ or $Y = K(G, n) = \Omega K(G, n+1)$, these are just the familiar group operations on the homotopy and homology groups. If both $X$ is a co-H-space and $Y$ is an $H$-space, we will obtain two multiplications, but it turns out these must coincide and moreover are commutative and associative by the Eckmann–Hilton argument. So for example the isomorphisms,

$$\pi_n(K(G,n)) \cong [S^n, K(G,n)] \cong H^n(S^n; G)$$

are all compatible with group structures.

Similarly if $X$ is an H-space or $Y$ is a co-H-space, then $[X,Y]$ obtains a co-multiplication structure. For example, the homotopy groups $H^*(G;K)$ of a topological group $G$ have a co-multiplication. It turns out this is compatible with the multiplication structure of the cup product, so that $H^*(G)$ is a *Hopf algebra*. Dualizing, the homology $H_*(X)$ of an H-space $X$ has a multiplication; this multiplicative structure is called the *Pontryagin ring* of $X$.

As one last note motivating the definition of $H$-spaces, the smash product can be seen as a topological version of the tensor product. And the definition of an $H$-space really says we have something like an operation on the smash product $X \wedge X$ up to homotopy. △

Combining this description of homology with the cellular approximation theorem and our knowledge about attaching cells may allow explicit descriptions of homotopy classes $[X,Y]$ for certain pairs $X, Y$. This will be fruitful when we wish to study principal $G$-bundles, which are known to be classified by $[X, BG]$. Here is a preliminary example of this.

Corollary 4.64 (Hopf Theorem). *Let $X$ be an $n$ dimensional connected CW complex. Then,*

$$[X, S^n] \cong H^n(X; \mathbb{Z}).$$

Proof. By cellular approximation, any map $X \to K(\mathbb{Z}, n)$ can be assumed to lie in the $n$-skeleton of $K(\mathbb{Z}, n)$ and any homotopy $X \times I \to K(\mathbb{Z}, n+1)$ can be assumed to lie in the $(n+1)$-skeleton. So restricting the codomain to the $(n+1)$-skeleton gives the same set of maps. From our Eilenberg–MacLane space construction we see that a CW $K(\mathbb{Z}, n)$ can be constructed by starting from $S^n$ and gluing cells of dimension at least $n+2$, so that $K(\mathbb{Z}, n)$ has a cell structure with $(n+1)$-skeleton $S^n$. Hence,

$$H^n(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)] \cong [X, \mathrm{sk}_{n+2} K(\mathbb{Z}, n)] \cong [X, S^n].$$

□

*Cohomology Operations.* The results we just proved lend themselves to a compact description of the functorial operations one can perform on cohomology.

**Definition 4.65: Cohomology Operation**

A *cohomology operation* of type $(q, \pi, G)$ and degree $n$, for integers $n, q$ and abelian groups $\pi, G$ is a natural transformation of the functors,

$$\Phi : H^q(-;\pi) \to H^{q+n}(-;G).$$

Equivalently, $\Phi$ assigns to each space $X$ a map $\Phi_X : H^n(X;\pi) \to H^q(X;G)$ which is natural with respect to $X$.
A *stable cohomology operation* $\Phi$ of degree $n$ is a sequence of cohomology operations $\{\Phi^q\}$ of type $(q, \pi, G)$ ranging over $q \in \mathbb{N}$ so that for every space $X$ the diagram,

$$\begin{array}{ccc} H^q(X;\pi) & \xrightarrow{\Phi^q} & H^{q+n}(X;G) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ H^{q+1}(\Sigma X;\pi) & \xrightarrow{\Phi^{q+1}} & H^{q+n+1}(\Sigma X;G) \end{array}$$

is commutative.

Recall a functor $F : \mathbf{C} \to \mathbf{Set}$ is *representable* if there is an element $X \in \mathbf{C}$ so that $F(Y) = \mathrm{Hom}(X, Y)$ for all $Y \in \mathbf{C}$. Similarly for contravariant functors. The possible cohomology operations are characterized by the following foundational result of category theory.

Lemma 4.66 (Yoneda). *Let $F, F'$ be functors from a category $\mathbf{C}$ to $\mathbf{Set}$. Suppose $F$ is representable: $F(-) = \mathrm{Hom}(-, Z)$. Then there is a canonical bijection between natural tranformations $\Phi : F \to F'$ and elements $\varphi \in F'(Z)$.*

Proof. Consider $\mathrm{id}_Z \in F(Z) = \mathrm{Hom}(Z, Z)$. Given $\Phi$, let $\varphi = \Phi(\mathrm{id}_Z)$. Conversely, given $\varphi$, define $\Phi_X : F(X) \to F'(X)$ by $\Phi_X(f) = f^*(\varphi)$ for any $f \in \mathrm{Hom}(X, Z)$, where $f^* = F'(f) : F'(Z) \to F'(X)$. One can check this is natural and these are inverse constructions. $\square$

Corollary 4.67. *Given representable functors $\mathrm{Hom}(-, Z)$ and $\mathrm{Hom}(-, Z')$, there is a canonical bijection between natural transformations $\Phi : \mathrm{Hom}(-, Z) \to \mathrm{Hom}(-, Z')$ and elements $\varphi \in \mathrm{Hom}(Z, Z')$.*

**Theorem 4.68**

Cohomology operations of type $(q, \pi, G)$ and degree $n$ are in canonical bijection with elements of $H^{q+n}(K(\pi, q); G)$.

Proof. Such a cohomology operation is a natural transformation $H^q(-; \pi) \to H^{q+n}(-; G)$. These are both representable by Theorem 4.59. So this is the same as a natural transformation,

$$[-, K(\pi, q)] \to [-, K(G, q+n)].$$

By the above corollary, these are in bijection with $[K(\pi, q), K(G, q+n)]$. Again by Theorem 4.59, this is isomorphic to $H^{q+n}(K(\pi, q); G)$. $\square$

Corollary 4.69. *All non-zero cohomology operations have degree* $n \geq 0$.

Proof. By Hurewicz, the first non-trivial cohomology group of $K(\pi, q)$ is in degree $q$. By our theorem, we need to have $n \geq 0$ to have any non-zero cohomology operations. $\square$

This gives us a good reason to want to understand the homology of Eilenberg-MacLane spaces. We will study some of this with spectral sequences. For example, it will be an exercise later to prove that for $\pi$ a finitely generated abelian group of rank $r$,

$$H^*(K(\pi, n); \mathbb{Q}) = \begin{cases} \Lambda_{\mathbb{Q}}(x_1, \ldots, x_r), \ |x_i| = n & : n \text{ odd} \\ \mathbb{Q}[x_1, \ldots, x_r], \ |x_i| = n & : n \text{ even.} \end{cases}$$

From this one can deduce for example if $q$ is odd then the only cohomology operations $H^q(-, \mathbb{Z}) \to H^{q+n}(-, \mathbb{Q})$ have $n = 0$ and come induced from the inclusion $\mathbb{Z} \to \mathbb{Q}$ multiplied by a rational constant.

As another example, one can show any degree one cohomology operation of type $(n, C, A)$ is a Bockstein homomorphism (i.e. the connecting homomorphism in the induced long exact sequence) from a short exact sequence $0 \to A \to B \to C \to 0$. Over a field, there are cohomology operations given by taking cup product powers of a cohomology class : $x \mapsto x \smile \cdots \smile x$.

Over $\mathbb{Z}_2$ there are many novel cohomology operations.

**Definition 4.70: Steenrod Squares**

For $n \geq 0$ there are stable cohomology operations

$$Sq^n : H^q(X; \mathbb{Z}_2) \to H^{q+n}(X; \mathbb{Z}_2),$$

called *Steenrod squares*. These are uniquely determined by the following properties.

(i) $Sq^0$ is the identity.

(ii) $Sq^n(x) = x^2$ when $x$ has degree $n$ and $Sq^n(x) = 0$ if $x$ has degree less than $n$.

(iii) The Steenrod squares applied to a product satisfy the Cartan formula:

$$Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y).$$

There is a multiplication on all $\mathbb{Z}_2$ cohomology operations under composition forming the *Steenrod algebra*. This Steenrod squares are not free in this algebra. The multiplicative structure of the Steenrod squares is determined by the *Adem relations*:

$$Sq^iSq^j = \sum_{0\leq k\leq \lfloor i/2\rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k}Sq^k.$$

It turns out however that the Steenrod squares multiplicatively generate the Steenrod algebra, so that all $\mathbb{Z}_2$ cohomology operations can be described as linear combinations of iterated Steenrod squares. Using the Adem relations, we can obtain a subset of Steenrod squares that freely generate the Steenrod algebra. Let $F_n \in H^n(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ be the fundamental/tautological class defined in the previous section.

**Theorem 4.71: (Serre)**

$H^*(K(\mathbb{Z}_2, q); \mathbb{Z}_2)$ is a polynomial algebra generated by certain iterates of Steenrod operations applied to $F_q$. More precisely, if $Sq^I = Sq^{i_1}\cdots Sq^{i_j}$ for $I = (i_1, \dots, i_j)$ a multi-index, the generators are $Sq^I F_q$, where the multi-index $I$ satisfies $i_r \geq 2i_{r+1}$ and $i_1 < i_2 + \ldots + i_j + q$.

We will later give a construction of the Steenrod squares from spectral sequences. Although we will not prove their properties, we will give some intuition for this result of Serre.

Remark 4.72. Steenrod also constructed corresponding stable operations on $\mathbb{Z}_p$ homology for primes $p > 2$ called the *reduced p-th power operations*

$$P^n : H^q(X; \mathbb{Z}_p) \to H^{q+2n(p-1)}(X; \mathbb{Z}_p).$$

These have the same characterizing properties as $Sq^n$ except that (ii) is replaced with the fact $P^n(x) = x^p$ for $|x| = 2n$ and 0 for $|x| < 2n$. They also satisfy similar but more complicated Adem relations. It was proved by Cartan that $H^*(K(\mathbb{Z}_p, q); \mathbb{Z}_p)$ is multiplicatively generated by the reduced power operations along with the Bockstein homomorphism $\beta_p$ coming from the short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$, and the cohomology is in fact a polynomial algebra generated by some collection of compositions of the $P^n$ possibly composed with $\beta_p$ (for $p = 2$, $\beta_2 = Sq^1$). △

*Capping and Killing Spaces.* Recall in our construction of $K(G, n)$ we glued on cells of dimensions $n+2$ and higher to kill all higher homotopy groups. None of this was special to the Eilenberg–MacLane space and we can do the same to kill the higher homotopy groups of any CW complex $X$.

PROPOSITION 4.73. *For any CW complex $X$ and any $n \in \mathbb{N}$, there is a CW complex $X^n$ and a cofibration $f_n : X \to X^n$ so that $(f_n)_* : \pi_i(X) \to \pi_i(X^n)$ is an isomorphism for $i \le n$ and so that $\pi_j(X^n) = 0$ for $j > n$.*

**Definition 4.74**

The space $X^n$ given in the above proposition is called the $n$th *capping space* of $X$. By construction,

$$\pi_i(X^n) = \begin{cases} \pi_i(X) & i \le n \\ 0 & i > n. \end{cases}$$

EXERCISE 4.75. Show that the capping space $X^n$ as described in the above proposition is homotopically unique. Moreover for any pair of capping space $X_1^n, X_2^n$, the cofibration $X \to X_2^n$ factors up to homotopy through the cofibration $X \to X_1^n$ and a homotopy equivalence $X_1^n \to X_2^n$. ▲

Let $X$ be $(n-1)$-connected and consider our capping space $f_n : X \to X^n$. By definition, $X^n$ is a cellular $K(\pi_n(X), n)$. We may homotopy $f_n$ into a fibration with homotopy fibre $X|_{n+1}$. One can iteratively apply the construction to kill off the $n+2$ homotopy group and so on, and obtain a space $X|_m$ for any $m \ge n$.

**Definition 4.76**

The space $X|_m$ above is called the $m$th *killing space* of $X$. By the homotopy long exact sequence of a fibration, it must satisfy,

$$\pi_j(X|_m) = \begin{cases} \pi_j(X) & j \ge m \\ 0 & j \le m. \end{cases}$$

Because $X|_{n+1}$ is the fibre of $f_n$, we obtain a map $X|_{n+1} \to X$. We can homotope this to a fibration and the fibre will be $K(\pi_n(X), n-1)$. Similarly there is a map $X|_{m+1} \to X|_m$ for each $m$ and composing them gives a map $X|_m \to X$.

Example 4.77. Consider $S^3|_4 \to S^3$, a fibration with fibre $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. By the Hurewicz theorem, $\pi_4(S^3) = \pi_4(S^3|_4) = H^4(S^3|_4)$. Once we have the Serre spectral sequence, we will be able to use the homology of $S^3$ and $\mathbb{C}P^\infty$ to deduce the homology of $S^3|_4$ and hence compute $\pi_3(S^4)$, the second stable homotopy group. ▲

Given a capping space $X \to X^n$ and $m > n$, we can consider the capping space $X^n \to (X^n)^m$. Looking at the homotopy groups, we conclude by exercise 4.75 that $(X^n)^m$ is homotopy equivalent to $X^m$. Hence we conclude that we can assemble all the capping spaces into a sequence of maps. Further, we can homotope the maps to be fibrations for which the fibres will be Eilenberg–MacLane spaces. We obtain a diagram like the following.

$$\begin{tikzcd}
 & \vdots \arrow[d, "K(\pi_{n+1}(X), n+1)"] \\
 & X^n \arrow[d, "K(\pi_n(X), n)"] \\
X \arrow[ur] \arrow[dr] \arrow[ddr] & \vdots \arrow[d, "K(\pi_3(X), 3)"] \\
 & X^2 \arrow[d, "K(\pi_2(X), 2)"] \\
 & X^1 = K(\pi_1(X), 1)
\end{tikzcd}$$

This is called a *Postnikov system* or *Postnikov tower* for $X$. It can be seen as an attempt to build $X$ from Eilenberg–MacLance spaces through a series of fibrations. Indeed $X$ looks something like a limit of this tower, which we can make precise with the aid of the following lemma.

Lemma 4.78. *Given a sequence of fibrations $\cdots \to X_2 \to X_1$, the natural map*

$$\pi_i(\varprojlim X_n) \to \varprojlim \pi_i(X_n),$$

*is surjective. It is injective if the maps $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ are surjective for large $n$.*

PROOF. An element of $\varprojlim \pi_i(X_n)$ is given by a collection of spheroids $f_n : S^i \to X_n$ compatible in the sense that the fibration $p_n : X_n \to X_1$ carries $f_n$ to $f_{n-1}$ up to homotopy. By definition of a fibration, we may homotope $f_n$ so that $p_n \circ f_n = f_{n-1}$. We may do this inductively for all $n$, which gives the surjectivity of our map.

Now we prove injectivity. We may ignore a finite set of terms from our inverse limit, and so we may assume all the maps $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ are surjective. Suppose we have a spheroid $f : S^i \to \varprojlim X_n$ so that each map $f_n : S^i \to X_n$ is null-homotopic via some $F_n : D^{i+1} \to X_n$.

We have $p_n \circ F_n = F_{n-1}$ when restricted to $S^i$, hence we may glue them together to a map $g_{n-1} : S^{i+1} \to X_{n-1}$. Since $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ is surjective, we may pick $F_n$ so that $g_{n-1}$ is null homotopic and thus $p_n \circ F_n \cong F_{n-1} \mathrm{rel} S^i$. By the relative homotopy lifting property, we can ensure $p_n \circ F_n = F_{n-1}$. Doing this inductively for all $n$ to obtain a null homotopy of $f$. $\square$

COROLLARY 4.79. *The inverse limit of the Postnikov tower for $X$ is weak homotopy equivalent to $X$. Hence $X$ is the CW approximation for $\varprojlim X^n$.*

PROOF. The lemma above implies $\varprojlim X^n$ has the correct homotopy groups and moreover the map $X \to \varprojlim X^n$ induces an isomorphism on all these homotopy groups (since it does for $n$ large enough). $\square$

The homotopy fibres of the maps $X \to X^n$ are the killing spaces $X|_n$. We can form a sequence out of them, and homotope all the maps to be fibrations,

$$* \to \cdots \to X|_{n+1} \xrightarrow{K(\pi_{n+1}(X),n)} X|_n \to \cdots \to X|_2 \xrightarrow{K(\pi_2(X),1)} X|_1 \to X.$$

This is called the *Whitehead tower* of $X$. It is dual to the Postnikov tower and gives $X$ as a colimit of a tower of spaces again built from fibrations with Eilenberg–MacLane spaces as the fibres. Note that $X|_1$ has the same homotopy groups as the universal cover of $X$ so we can also see this tower as a generalization of the universal covering construction where we find $n$-connected spaces $X|_n$ fibered over $X$.

We will not prove it, but it turns out that if (and only if) $X$ is homotopically simple, then each of the fibrations $X^{n+1} \to X^n$ in the Postnikov tower can be extended to fibre sequences,

$$K(\pi_{n+1}(G), n+1) \to X^{n+1} \to X^n \to K(\pi_{n+1}(G), n+2).$$

In this case, $X^{n+1}$ is the pullback under $X^n \to K(\pi_{n+1}(G), n+2)$ of the pathspace fibration of $K(\pi_{n+1}(G), n+2)$ and so the homotopy type of $X^{n+1}$ is determined by the homotopy class of the map $X^n \to K(\pi_{n+1}(G), n+2)$.

This defines an element $k^n \in [X^n, K(\pi_{n+1}(G), n+2] \cong H^{n+2}(X^n; \pi_{n+1}(X))$ called the *nth Postnikov k invariant*. Because $k^n$ tells us the homotopy type of $X^{n+1}$ and the homotopy class of the fibration $X^{n+1} \to X^n$, we can recover the weak homotopy type of $X$ from its homotopy groups along with some obstruction-type classes which tell us how to fibre the spaces $K(\pi_n(X), n)$ over each other to re-obtain $X$. For example, if all the $k$ invariants vanish, then these fibrations are trivial and $X$ is the product of the Eilenberg–MacLane spaces corresponding to each of its homotopy groups.

## 5. Spectral Sequences

Spectral sequences are powerful tools to compute the homology of some graded (or more generally filtered) complex (or even more generally differential group) in homological algebra. There are several important spectral sequences in algebraic topology: the Serre and Eilenberg–Moorse spectral sequences for fibrations, the Adams spectral sequence for stable homotopy, and the Atiyah–Hirzebruch spectral sequence for K-theory and generalized homology. We will focus on the Serre spectral sequence, which is the easiest and of the greatest importance. The initial notational framework of spectral sequences is very daunting, but most of it can be ignored, and once understood they become an extremely useful tool to have in attempting computations. We will mostly omit proofs of the basic homological results as they are unenlightening and of little importance in practice to computations.

As some elementary motivation for the construction, consider a complex $C_*$ of abelian groups with a subcomplex $D_*$. Then we obtain a short exact sequence,

$$0 \to D_* \to C_* \to C_*/D_* \to 0,$$

which induces a long exact sequence in homology,

$$\cdots \to H_n(D_*) \to H_n(C_*) \to H_n(C_*/D_*) \xrightarrow{\partial_n} H_{n-1}(D_*) \to \cdots .$$

Suppose we know the homologies $H_*(D_*)$ and $H_*(C_*/D_*)$. We can consider the homology of the two term chain complex,

$$H_*(C_*/D_*) \xrightarrow{\partial_*} H_*(D_*),$$

which will be two groups $G_1H_*$ and $G_0H_*$. Then, decomposing the long exact above, we obtain short exact sequences,

$$0 \to G_0H_* \to H_*(C_*) \to G_1H_* \to 0.$$

This determines $H_*(C_*)$ up to an extension problem.

The goal of spectral sequences will be to generalize this from a subcomplex to a sequence of nested subcomplexes, i.e. a filtration. Computing the homology of the associated graded groups through a sequence of approximations will yield the homology of our original complex, up to a series of extension problems.

**5.1. Spectral Sequences in Homological Algebra.** We introduce spectral sequences in full generality. This discussion can be freely skimmed.

**Definition 5.1**

A *differential abelian group* $(F, \mathrm{d})$ is an abelian group $F$ equipped with a differential $\partial : F \to F$ so that $\mathrm{d}^2 = 0$. A *filtration* on $F$ is a nested sequence of subgroups $0 \subset F_0 \subset F_1 \subset \cdots$ which union to all of $F$ and so that $\mathrm{d} : F_p \to F_p$ preserves the filtration. In this case, $F$ has an *associated graded group*,

$$\mathrm{Gr}F := \bigoplus_p F_p/F_{p-1}, \quad \mathrm{Gr}_n F := F_n/F_{n-1}.$$

In the cases we are interested in $F = \bigoplus_n C_n$ will be a chain complex. A *graded filtered differential group* $F$ is a filtered differential abelian group so that $F = \bigoplus_n C_n$ and so that the differential restricts to a map $\mathrm{d} : C_n \to C_{n-1}$. Further, we ask the filtration and grading are compatible in the sense that,

$$F_p = \bigoplus_n F_p \cap C_n.$$

For ease of notation, we abbreviate $F_p \cap C_n$ by $F_p C_n$.

*Spectral Sequence of a Filtered Differential Group.* Associated to a filtered differential group is its *spectral sequence*, which is a sequence of graded differential groups $(E^r, \mathrm{d}_r)$ for $r = 0, 1, \ldots, \infty$. In terms of the grading, $E^r = \bigoplus_p E^r_p$ and the differential restricts to a map $\mathrm{d}^r_p : E^r_p \to E^r_{p-r}$.

The first term in the sequence is $E^0_p = \mathrm{Gr}_p F$ with differential $\mathrm{d}^0$ given by the overall differential d on $F$ descended to the quotient. We recursively define $(E^{r+1}, \mathrm{d}_{r+1})$ to be the homology of the previous term in the sequence: $E^{r+1} = H_p(E^r_p, \mathrm{d}^r)$. The differential $\mathrm{d}^{r+1}$ is inherited from the original d by whatever "remnant" is left in the quotient group we have defined.

More explicitly, we have,

$$E^r_p = \frac{F_p \cap \mathrm{d}^{-1} F_{p-r}}{F_{p-1} \cap \mathrm{d}^{-1} F_{p-r} + F_p \cap \mathrm{d} F_{p+r-1}}.$$

And the differential,

$$\frac{F_p \cap \mathrm{d}^{-1}F_{p-r}}{F_{p-1}\cap \mathrm{d}^{-1}F_{p-r} + F_p \cap \mathrm{d}F_{p+r-1}} \xrightarrow{\mathrm{d}^r} \frac{F_{p-r}\cap \mathrm{d}^{-1}F_{p-2r}}{F_{p-r-1}\cap \mathrm{d}^{-1}F_{p-2r} + F_{p-r}\cap \mathrm{d}F_{p-1}} = E^r_{p-r},$$

is just the differential d applied to this quotient in the obvious way. It is elementary but tedious to check this is all well defined and matches the description above that $E^{r+1}$ is the homology of $E^r$.

What happens as $r \to \infty$? Using the expression for $E^r$ above, we see heuristically,

$$E^\infty_p = \frac{F_p \cap \mathrm{d}^{-1}(0)}{F_{p-1}\cap \mathrm{d}^{-1}(0) + F_p \cap \mathrm{d}F} = \mathrm{Gr}_p(H(F, \mathrm{d})).$$

One often speaks of the spectral sequence *converging* to the homology of $F$ and one writes,

$$E^1_p = H(\mathrm{Gr}_p(F)) \implies \mathrm{Gr}_p H(F).$$

Sometimes one omits the Gr on the right side and says the spectral sequence converges to $H(F)$, even if this is technically not true.

We say that the spectral sequence *degenerates* or *collapses* at index $k$ if $\mathrm{d}^i = 0$ for $i \geq k$, and hence $E^k = E^{k+1} = \cdots$. In this case we will obtain $E^\infty$ after finitely many steps. If the filtration is finite, then the spectral sequence always collapses after finitely many pages. In general, the sequence may never collapse, although the $E^\infty$ term can always be realized as a direct limit since the filtered differential $\mathrm{d}^r_p$ will always collapse for fixed $p$ and large enough $r$. We will not be so interested in this subtlety as in the cases we apply the spectral sequence will almost always degenerate.

*Spectral Sequence of a Graded Filtered Differential Group.* In the cases we are interested in, our differential abelian group will be a chain complex, and hence will be graded by degree. So suppose that $F = \bigoplus_q C_q$ is a graded filtered differential group. Because the filtration and grading are compatible, the associated spectral sequence $E^r$ of $F$ now inherits the $q$ grading from the $C_q$ and hence will be bigraded,

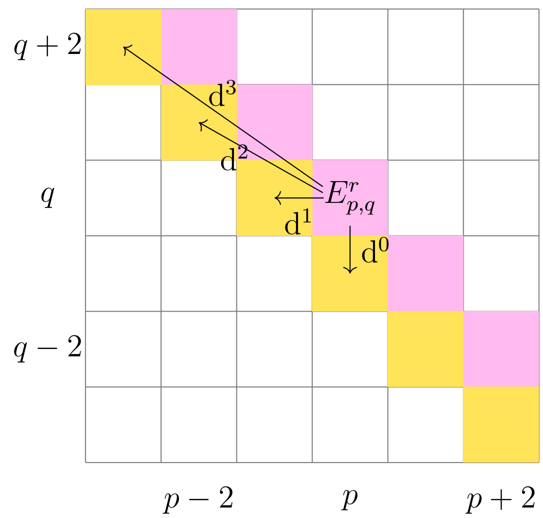
$$E^r_{p,q} := \frac{F_pC_{p+q} \cap \mathrm{d}^{-1}(F_{p-r}C_{p+q-1})}{[F_{p-1}C_{p+q}\cap \mathrm{d}^{-1}(F_{p-r}C_{p+q-1})] + [F_pC_{p+q}\cap \mathrm{d}(F_{p+r-1}C_{p+q+1})]}.$$

The differential is also compatible with this bigrading and restricts to a map,

$$\mathrm{d}^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

It is again elementary but tedious to check this all works.

This is a lot of indices to keep track of in our spectral sequence. Fortunately, there is a graphical way to efficiently organize all this information. We refer to the total group $\bigoplus_{p,q} E^r_{p,q}$ as the $r$th *page* of the spectral sequence. Each page of the spectral sequence is represented by a grid of the terms $E^r_{p,q}$, with $p$ increasing along the horizontal axis and $q$ increasing along the vertical axis. The terms along a downwards diagonal will all have the same *total degree* $n = p + q$. See the following picture showing the differentials on the first few pages. Diagonals of fixed total degree are shown in pink and yellow.




On each page the differentials all point in the same direction. We obtain the next page by taking the homology of all the diagonal complexes on the page. To help remember how the differentials go, you can remember the second differential $\mathrm{d}^2$ (usually the first one we care about) is a knight's move and later differentials are extended knight's moves, where all differentials need to decrease the total degree by one.

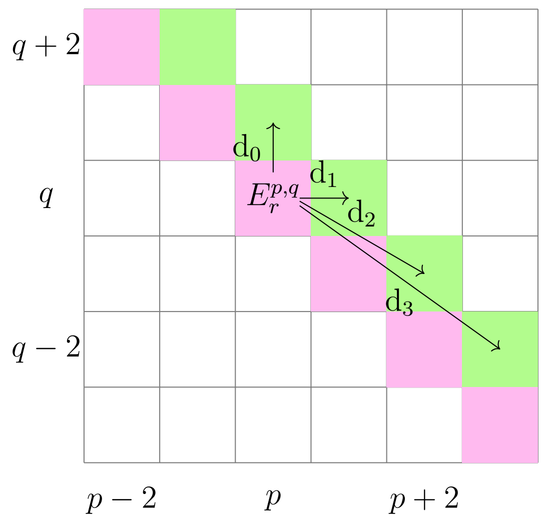
What happens as $r \to \infty$ in this case? We have,

$$E^\infty_{p,q} = \frac{F_p C_{p+1} \cap \mathrm{d}^{-1}(0)}{F_{p-1} C_{p+q} \cap \mathrm{d}^{-1}(0) + F_p C_{p+q} \cap \mathrm{d}F} = \frac{F_p H_n}{F_{p-1} H_n} = \mathrm{Gr}_p H_{p+q},$$

where $n = p+q$ is the total degree and $H_n$ is the $n$th homology group of $F$ with respect to the grading of $F$. Again, in the cases we care about, the sequence usually degenerates after finitely many pages and we converge to $E^\infty$.

The relation between the actual homology of $F$ and these associated graded pieces can be a little subtle. But if on the $E^\infty$ page the diagonal of total degree $n$ is empty, then we must have $H_n(F) = 0$. And if that diagonal has only one non-zero entry, that entry must equal $H_n(F)$.

*Spectral Sequences in Cohomology.* Of course we can do a very similar construction in cohomology with a differential that increases the graded degree. A trick to make the construction especially easy is to notice that by flipping the indexing of the grading, the cohomological differential becomes a homological differential. So, we may flip the indexing, apply the techniques above, and the flip back to get our cohomological spectral sequence. The result will be diagrams that look the same, except with the differentials pointing in the opposite direction. Because we are dealing with cohomology, the grading indices should now be superscripts. Correspondingly, we move our page indices to subscripts. See the picture below.




*Naturality.* Suppose we have a homomorphism of filtered differential groups $C \to C'$ which is compatible with the filtrations. If our groups are graded, we ask the homomorphism be compatible with that as well.

Then it is clear we will induce morphisms of the corresponding spectral sequences $E \to E'$ on each page. This extends to the page at infinity, where the morphism $\mathrm{Gr}H(C) \to \mathrm{Gr}H(C')$ should be the usual one induced on homology (and then to the associated graded quotient).

**5.2. Basic Examples of Spectral Sequences.** We will begin with a few simple spectral sequences to get a sense for the subject. One can also refer back to our usage of spectral sequences in proving the universal coefficient theorem and Künneth formula.

*The Hochschild–Serre Spectral Sequence.* This is a spectral sequence for computing group cohomology of a group $G$ by studying a normal subgroup $H$

and the quotient $G/H$, which may be considerably simpler. We will look at the particular case of Lie algebra cohomology.

Let $\mathfrak{g}$ be a complex Lie algebra and $M$ a $\mathfrak{g}$-module, which will be our coefficient space. We define a cochain complex with $C^n(\mathfrak{g};M) = \mathrm{Hom}_{\mathbb{C}}(\Lambda^n\mathfrak{g}, M)$ and differential $\mathrm{d} : C^n(\mathfrak{g};M) \to C^{n+1}(\mathfrak{g};M)$ given by,

$$\begin{aligned}\mathrm{d}c(g_1 \wedge \cdots \wedge g_{n+1}) = \sum_{s<t}&(-1)^{s+t-1}c([g_s, g_t] \wedge g_1 \wedge \cdots \widehat{g_s} \cdots \widehat{g_t} \cdots \wedge g_{n+1}) \\ &- \sum_u (-1)^u g_u c(g_1 \wedge \cdots \widehat{g_u} \cdots \wedge g_{n+1}),\end{aligned}$$

where $\widehat{\cdot}$ denotes an omitted term.

The *Lie algebra cohomology* $H^n(\mathfrak{g};M)$ of $\mathfrak{g}$ is the cohomology of the complex $C^n(\mathfrak{g};M)$.

To apply spectral sequence techniques, consider a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Define a filtration on our cochain spaces by,

$$F^pC^{p+q}(\mathfrak{g};M) = \left\{ c \in C^{p+q}(\mathfrak{g};M) : c(g_1 \wedge \cdots \wedge g_{p+q}) = 0 \quad \text{if} \quad g_1, \ldots, g_{q+1} \in \mathfrak{h} \right\}.$$

It is simple enough to see that this filtration is compatible with the differential and grading. Hence we obtain a spectral sequence converging to $\mathrm{Gr}H^n(\mathfrak{g};M)$ called the *Hochschild–Serre spectral sequence.*

On the $E_0$ page we have,

$$E_0^{p,q} = \frac{F^pC^{p+q}}{F^{p-1}C^{p+q}} = \mathrm{Gr}_p(C^{p+q}) = C^q(\mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M)),$$

where the verification of the last equality is left to the reader. The effect of the differential $\mathrm{d}_0$ is to increase the degree of $q$ by one, and so it is no surprise that, $E_1^{p,q} = H^q(\mathfrak{h}; \mathrm{Hom}_{\mathbb{C}}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M))$.

In the case where $\mathfrak{h}$ is an ideal, we can rewrite this as, $E_1^{p,q} = C^p(\mathfrak{g}/\mathfrak{h}; H^q(\mathfrak{h};M))$. Hence taking homology,

$$E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{h}; H^q(\mathfrak{h};M)) \implies H^{p+q}(\mathfrak{g};M) = E_\infty^{p+q}.$$

So the cohomology of $\mathfrak{g}$ can be deduced from the homology of $\mathfrak{h}$ and a quotient $\mathfrak{g}/\mathfrak{h}$ using a spectral sequence. For proofs of these claims, see the original paper [**HS**].

*Spectral Sequence of a Filtered Topological Space.* Suppose we have a topological space $X$ filtered by subspaces:

$$\varnothing \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots = X.$$

If the filtration is infinite, $X$ should have the weak topology with respect to the filtration (so $X$ is the colimit of inclusion maps).

The space of singular chains on $X$ has a filtration with $p$th part $F_pC_n := C_n(X_p)$, consisting of singular chains living in $X_p$. This makes the singular chains on $X$ into a graded filtered group and we can study the corresponding spectral sequence.

Note that $E^0_{p,q} = \mathrm{Gr}_p(C_{p+q}(X)) = C_{p+q}(X_p, X_{p-1})$. The differential $\mathrm{d}_0$ is just inherited from the usual differential map on homology and so,

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}).$$

The differential on the first page,

$$\mathrm{d}^1 : E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \to E^1_{p-1,q} = H_{p+q-1}(X_{p-1}, X_{p-2}),$$

is the connecting homomorphism in the long exact sequence of the triple $(X_p, X_{p-1}, X_{p-2})$. This spectral sequence will converge to the associated graded group of $H(X)$.

Let us briefly consider the special case of a pair, where our filtration of $X$ is $\varnothing \subset A \subset X$. Then the first three pages in our spectral sequence have the following form.

$$\begin{tikzcd}
2 & C_2(A) \arrow[d] & C_3(X,A) \arrow[d] \\
1 & C_1(A) \arrow[d] & C_2(X,A) \arrow[d] \\
0 & C_0(A) & C_1(X,A) \arrow[d] \\
-1 & 0 & C_0(X,A) \\
E^0 & 0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
2 & H_2(A) & H_3(X,A) \arrow[l] \\
1 & H_1(A) & H_2(X,A) \arrow[l] \\
0 & H_0(A) & H_1(X,A) \arrow[l] \\
-1 & 0 & H_0(X,A) \\
E^1 & 0 & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
2 & \operatorname{coker}\partial_* & \ker\partial_* \\
1 & \operatorname{coker}\partial_* & \ker\partial_* \\
0 & \operatorname{coker}\partial_* & \ker\partial_* \\
-1 & 0 & H_0(X,A) \\
E^2 & 0 & 1
\end{tikzcd}$$

After this, the sequence degenerates. We conclude that $H_n(X)/\operatorname{coker}(\partial_*) \cong$

$\ker \partial_*$. This is just equivalent to the long exact sequence of a pair,

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial_*} H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots .$$

*Spectral Sequence of Cellular Complex.* Let $X$ be a CW complex. As before, consider a filtration of $X$ but now specifically given by the successive skeleta of $X$:

$$\varnothing \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots = X.$$

As before, we get a spectral sequence converging to the homology of $X$. We see that,

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) = \widetilde{H}_{p+q}(X_p/X_{p-1}) = \begin{cases} \mathcal{C}_p(X) & q = 0 \\ 0 & q \neq 0. \end{cases}.$$

Hence the $E^1$ page of our spectral sequence looks like the following.

$$\begin{tikzcd}
1 & 0 & 0 & 0 & 0 \\
0 & \mathcal{C}_0(X) & \mathcal{C}_1(X) \arrow[l] & \mathcal{C}_2(X) \arrow[l] & \mathcal{C}_3(X) \arrow[l] \\
E^1 & 0 & 1 & 2 & 3
\end{tikzcd}$$

Referring back to our discussion of cellular homology, one sees the map here is precisely the boundary map in the cellular chain complex (alternatively we could use this spectral sequence to define the complex). Thus the $E^2$ page looks like the following.

$$\begin{tikzcd}
1 & 0 & 0 & 0 & 0 \\
0 & H_0^{\text{cell}}(X) & H_1^{\text{cell}}(X) & H_2^{\text{cell}}(X) \arrow[llu] & H_3^{\text{cell}}(X) \arrow[llu] \\
E^2 & 0 & 1 & 2 & 3
\end{tikzcd}$$

The spectral sequence degenerates here. There is only one entry in each diagonal and hence,

$$H_i(X) = H_i(\mathcal{C}_*(X), \partial_*),$$

which gives another proof that singular and cellular homology agree.

**5.3. The Serre Spectral Sequence.** We now move on to the main subject of this chapter. The Serre spectral sequence allows us to relate the homology of a total space of a fibration with the homology of the base and the fibre.

*The Homological Sequence.* Suppose $\pi : E \to B$ is a Serre fibration with base $B$ a CW complex and fibre $F$. We will need to assume our fibration is *homologically simple*. By this we mean that the homotopy equivalence from Theorem 1.10 between fibres of $\pi$ is independent of the path used to define it. More succinctly, we might say we require $\pi_1(B)$ to act trivially on $H_*(F)$. In practice it is easiest to verify this by assuming $B$ is simply connected.

There is a filtration of $B$ by its skeleta: $\varnothing \subset B_0 \subset B_1 \subset \cdots = B$. This in turn determines a filtration of $E$ by $E_p = \pi^{-1}(B_p)$. From this we get a filtration of the singular chain complex of $E$:

$$0 \subset C_*(E_0) \subset C_*(E_1) \subset \cdots \subset C_*(E),$$

by singular chains landing in $E_p$. This makes $C_*(E)$ a graded filtered differential group, and hence it has a spectral sequence. The zeroth page of the sequence is $E^0_{p,q} = C_{p+q}(E_p, E_{p-1})$ with standard differential,

$$\partial : C_{p+q}(E_p, E_{p-1}) \to C_{p+q-1}(E_p, E_{p-1}).$$

Hence $E^1_{p,q} = H_{p+q}(E_p, E_{p-1})$.

Now, for each $p$-cell of $B_p$, pick a small ball $D^p_\alpha$. The inclusion,

$$B^{p-1} \hookrightarrow B^p \setminus \bigsqcup_\alpha D^p_\alpha$$

is a homotopy equivalence. Hence,

$$\pi^{-1}(B^{p-1}) = E_{p-1} \hookrightarrow \pi^{-1}\left(B^p \setminus \bigsqcup_\alpha D^p_\alpha\right)$$

is a weak homotopy equivalence from the long exact sequence of a fibration and the five lemma. Hence,

$$H_{p+q}(E_p, E_{p-1}) = H_{p+q}\left(E_p, E_p \setminus \pi^{-1}(\bigsqcup_\alpha D^p_\alpha)\right).$$

By excision,

$$\begin{aligned} &= H_{p+q}\left(\pi^{-1}(\bigsqcup_\alpha D^p_\alpha), \pi^{-1}(\bigsqcup_\alpha \partial D^p_\alpha)\right) \\ &= \bigoplus_\alpha H_{p+q}\big(\pi^{-1}(D^p_\alpha), \pi^{-1}(\partial D^p_\alpha)\big). \end{aligned}$$

By Feldbau's lemma, the fibration trivializes on each disk $D^p_\alpha$. Using our homological simplicity to speak unambiguously about the homology of fibres on each path component and identifying fibres on different path components by an arbitrary isomorphism,

$$= \bigoplus_\alpha H_{p+q}(F \times D^p_\alpha, F \times \partial D^p_\alpha).$$

By the Künneth formula,

$$\begin{aligned} &= \bigoplus_\alpha H_q(F) \\ &= \mathcal{C}_p(B; H_q(F)). \end{aligned}$$

The boundary map on the $E^1$ page is,

$$E^1_{p,q} = \mathcal{C}_p(B; H_q(F)) \xrightarrow{\partial} \mathcal{C}_{p-1}(B; H_q(F)) = E^1_{p-1,q}.$$

Comparing the explicit construction of this boundary map to the cellular boundary map reveals they are the same. Hence,

$$E^2_{p,q} = H_p(B; H_q(F)).$$

This spectral sequence should converge to the associated graded $\mathrm{Gr}_p H_{p+q}(E)$ of the homology of the total space $E$. If the spaces $B$ or $F$ are finite dimensional, then the spectral sequence will degenerate by the page $\min\{\dim B, \dim F\} + 1$. If the spaces are infinite, the sequence may not degenerate at a finite page. But at a square $E^r_{p,q}$, all differentials involving this square are zero for $r > \max\{p, q+1\}$, and so on any fixed diagonal $\bigoplus_{p+q=n} E^r_{p,q}$ the sequence is constant once $r = n+2$. Hence there is no ambiguity in what it means to converge to the homology of $E$. We thus have a proof of the following central result.

**Theorem 5.2: Serre Spectral Sequence**

Let $\pi : E \to B$ be a homologically simple Serre fibration with a connected CW base and a fibre $F$. Then there is a spectral sequence,

$$E^2_{p,q} = H_p(B; H_q(F)) \implies H_{p+q}(E).$$

That is, it has $E^2$ page given by the homology of the base with homology of the fibre coefficients and converges to (the associated graded of) the homology of $E$.

As we noted before, the relationship between $H_n(E)$ and $\mathrm{Gr}_p(H_{p+q}(E))$ may be difficult to work out, since entries $E^\infty_{p,q}$ of the $E^\infty$ page are quotients $F_p H_{p+q}(E)/F_{p-1} H_{p+q}(E)$ of intermediate subgroups of $H_{p+q}(E)$ (i.e. there is still a sequence of extension problems involved in determining $H_n(E)$). But the fact that our spectral sequence is contained in the first quadrant of each page does give us some important information. We know that the entries $E^\infty_{0,n}$ in the zeroth column of the infinity page are equal to $F_0H_n(E)$ and in particular are a subgroup of $H_n(E)$. Correspondingly, the entries $E^\infty_{n,0}$ in the zeroth row of the infinity page are equal to $H_n(E)/F_{n-1}H_n(E)$ and in particular are quotient groups of $H_n(E)$. Thus we get canonical injections and surjections for each $n$,

$$E^\infty_{0,n} \hookrightarrow H_n(E) \twoheadrightarrow E^\infty_{n,0}. \tag{5.1}$$

Here is an important immediate corollary of our spectral sequence.

COROLLARY 5.3. *If $E \to B$ is a homologically simple fibration with fibre $F$ so that $B$ is a finite CW complex and each homology group of $F$ is finite rank then,*

$$\chi(E) = \chi(B)\chi(F).$$

PROOF. We compute from the Künneth formula,

$$\sum_{p,q}(-1)^{p+q}\dim E^1_{p,q} = \sum_{p,q}(-1)^{p+q}\dim \mathcal{C}_p(B)\dim H_q(F) = \chi(B)\chi(F).$$

But passing to homology does not affect the Euler characteristic. Hence going from the $E^1$ page to the $E^\infty$ page will not affect the answer,

$$\chi(B)\chi(F) = \sum_{p,q}(-1)^{p+q}\dim E^\infty_{p,q} = \sum_n (-1)^n \dim H_n(E) = \chi(E).$$

□

REMARK 5.4. If our fibration is not homologically simple, there is still a way to interpret the second page of our spectral sequence. Instead of considering the homology of $B$ with coefficients in $H_*(F)$, we should use a *local coefficient system* where the homology has coefficients in the locally constant sheaf $H_q(\pi^{-1}(x))$ on $B$. The details can be found in [**FF**, §22.2].

If $p : E \to B$ is a map, not necessarily a fibration, we can still get a sheaf of abelian groups on $B$ as coefficients. Namely, $\check{H}^q(p^{-1}(U);\underline{\mathbb{Z}})$ for $U$ an open set of $B$, where $\check{H}^*$ denotes sheaf or Čech cohomology and $\underline{\mathbb{Z}}$ is the usual locally constant sheaf (this is a right derived functor of the direct image $p_*$ of $\underline{\mathbb{Z}}$). Denote this sheaf by $\check{H}^q(F)$. Then there is a cohomological spectral sequence,

$$E_2^{p,q} = \check{H}^p(B;\check{H}^q(F)) \implies \check{H}^{p+q}(E;\underline{\mathbb{Z}}).$$

This is called the *Leray–Serre spectral sequence* and it directly generalizes the Serre spectral sequence as well as the local coefficient case. It is an important tool in algebraic geometry and more advanced topics in algebraic topology like equivariant cohomology. △

*Cohomology.* As we saw for the homology of any spectral sequence of a filtered complex, we can define analogously a spectral sequence for cohomology. The diagrams will look the same will differential arrows pointing in the opposite direction. We conclude by the same proof as above that we have a cohomological Serre spectral sequence,

$$E_2^{p,q} = H^p(B; H^q(F)) \implies H^{p+q}(E).$$

As with (5.1), we obtain important injection and surjection maps on the $E^\infty$ page but now reversed due to the change in direction of the filtration,

$$E_\infty^{n,0} \hookrightarrow H^n(E) \twoheadrightarrow E_\infty^{0,n}.$$

*Morphisms.* Suppose that one has a morphism $f : E \to E'$ of fibrations, meaning there is a commutative diagram of the following form.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "g"] & B' \end{tikzcd} \tag{5.2}$$

Further suppose the map on the base $g : B \to B'$ is cellular. In this case we get homomorphism between the chain complexes of $E$ and $E'$ compatible with the filtration by skeleta of $B, B'$. Thus we get a morphism of spectral sequences $f_* : E^r_{p,q} \to E'^r_{p,q}$. This has the following properties.

(1) $f_* : E^1_{p,q} \to E'^1_{p,q}$ is the usual induced map on relative homology $H_{p+1}(E_p, E_{p-1}) \to H_{p+1}(E'_p, E'_{p-1})$.

(2) The map $f_*$ on the $(r+1)$st page is the induced map on homology from the map $f_*$ on the $r$th page.

(3) The map $f_*$ on the $E^\infty$ page is the usual induced map $f_* : H_n(E) \to H_n(E')$ on the associated graded groups.

We also of course get a contravariant induced map on spectral sequences in cohomology, $f^* : E'^{p,q}_r \to E^{p,q}_r$. The corresponding properties hold.

Actually, even if the map $f : E \to E'$ gives a map $g : B \to B'$ on the base which is not cellular, we can still construct an induced map $f_*$ on spectral sequences from the second page onwards. The first step is to cellulary approximate $g$, but the full argument is more involved for a Serre fibration. We provide a full proof since the result is important, although it may freely be skipped on first reading.

Proposition 5.5. *Let $f : E \to E'$ be a morphism of homologically simple fibrations (i.e. inducing a diagram like (5.2)) with CW bases. Then starting* *from the $E^2$ page, there is a well defined and canonical induced morphism of Serre spectral sequences $f_*$.*

Proof. We define a homotopy $G : B \times [0,1] \to G'$ between $G_0 = g$ and a cellular map $G_1 = g'$. We obtain a pullback fibration $\mathcal{E} = G^* E'$ via the following diagram,

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\widetilde{G}"] \arrow[d, "F'"] & E' \arrow[d, "F'"] \\ B \times [0,1] \arrow[r, "G"] & B'. \end{tikzcd}$$

We denote by $\mathcal{E}_t$ the inverse image of $B \times \{t\}$ under the above fibration. By restriction, we have the following diagram,

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, "\widetilde{g}'"] \arrow[d, "F'"] & E' \arrow[d, "F'"] \\ B \times \{1\} \arrow[r, "g'"] & B'. \end{tikzcd} \tag{5.3}$$

By the universal property of pullback fibrations, we get a map $f' : E \to \mathcal{E}_0$ completing the following diagram,

$$\begin{tikzcd} E \arrow[r, dotted, "f'"] \arrow[rr, bend left, "f"] \arrow[d, "F"] & \mathcal{E}_0 \arrow[r, "\widetilde{g}"] \arrow[d, "F'"] & E' \arrow[d, "F'"] \\ B \arrow[r, "\mathrm{id}"] & B \times \{0\} \arrow[r, "g"] & B'. \end{tikzcd} \tag{5.4}$$

We have the following two diagrams coming from inclusions,

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, hook, "i_0"] \arrow[d, "F'"] & \mathcal{E} \arrow[d, "F'"] \\ B \times \{0\} \arrow[r, hook] & B \times [0,1] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_1 \arrow[r, hook, "i_1"] \arrow[d, "F'"] & \mathcal{E} \arrow[d, "F'"] \\ B \times \{1\} \arrow[r, hook] & B \times [0,1]. \end{tikzcd} \tag{5.5}$$

Note the morphisms of fibrations $\widetilde{g}', f', i_0$ and $i_1$ depicted in (5.3)–(5.5) have cellular bases. Hence they define morphisms of spectral sequences which we now analyze starting from the $E^2$ page. We know that on the $E^2$ page, the induced morphism is the one coming from the induced map on homology of the base and fibre. So for diagram (5.3), we have,

$$(\widetilde{g}')^2_* : H_*(B; H_*(F')) \to H_*(B'; H_*(F')),$$

induced by $g'_* : H_*(B) \to H_*(B')$ and the identity on the fibres. For diagram (5.4), we have

$$(f')^2_* : H_*(B; H_*(F)) \to H_*(B; H_*(F')),$$

induced by $h_* H_*(F) \to H_*(F')$ and the identity on the base. For the diagrams in (5.5), we see that the map on the fibres is the identity and the maps on the base are homotopy equivalences. Hence $(i_0)_*$ and $(i_1)_*$ give isomorphisms of the $E^2$ pages of the spectral sequences for the three fibrations $\mathcal{E} \to B \times [0,1]$, $\mathcal{E}_0 \to B$, and $\mathcal{E}_1 \to B$.

Composing these morphisms gives the following,

$$E^2_{p,q}(E) = H_p(B; H_q(F)) \xrightarrow{(f')^2_*} E^2_{p,q}(\mathcal{E}_0) \xrightarrow{\cong} E^2_{p,q}(\mathcal{E}_1) \xrightarrow{(\widetilde{g}')^2_*} H_p(B'; H_q(F')) = E^2_{p,q}(E').$$

It is clear from our definition of these morphisms above, that the total morphism is given by $h_* : H_q(F) \to H_q(F')$ and $g'_* : H_p(B) \to H_p(B')$. Since $g$ and $g'$ are homotopic, they induce the same maps in homology. Thus the map of $E^2$ pages, $E^2_{p,q}(E) \to E^2_{p,q}(E')$, is precisely the one we would expect for the map $f : E \to E'$ of fibrations.

To conclude, we just need to show this morphism of spectral sequences extends to all higher pages. If a morphism of spectral sequences is an isomorphism on a certain page it is an isomorphism on all later pages including $\infty$ (since the induced map on homology will always be an isomorphism). Thus (5.5) gives isomorphisms of spectral sequences for $\mathcal{E}_0, \mathcal{E}$ and $\mathcal{E}_1$ on all pages 2 and beyond. Composing with the other morphisms, we obtain for all $p, q$ and all $k > 2$,

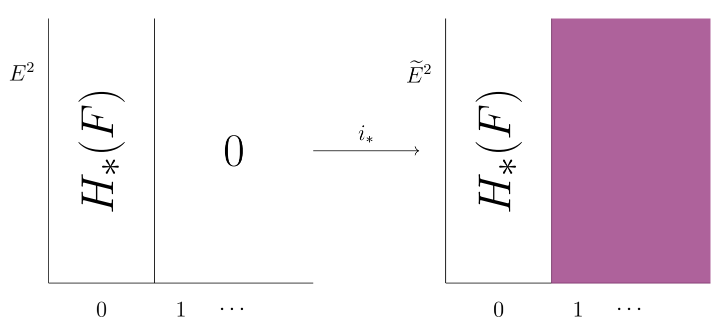
$$E^k_{p,q}(E) \xrightarrow{(f')^k_*} E^k_{p,q}(\mathcal{E}_0) \xrightarrow{\cong} E^k_{p,q}(\mathcal{E}_1) \xrightarrow{(\widetilde{g}')^k_*} E^k_{p,q}(E').$$

Thus we see that $g$ gives a morphism of Serre spectral sequences on all pages 2 and greater. □

From this we conclude that from the $E^2$ page onwards the Serre spectral sequence is independent of the cellular structure on the base. Moreover, this means if we have a homologically simple fibration $E \to B$, even if $B$ is not a CW complex, we may consider a CW approximation $B^{CW} \to B$. We can define the Serre–spectral sequence for $B^{CW}$ with respect to the pullback fibration. The long exact sequence of a fibration and the five lemma implies this gives a weak homotopy equivalence of total spaces. Thus we may extend the Serre spectral sequence to our non-CW fibration and there is no ambiguity on the $E^2$ page and beyond due to the above result on morphisms.

Remark 5.6. The original construction of Serre does not require a CW structure. In that approach, we work with homology defined in terms of cubical singular chains $\sigma : I^{p+q} \to E$. We may filter $C_{p+q}(E)$ so that $F_pC_{p+q}(E)$ is generated by cubical singular chains $\sigma : I^{p+q} \to E$ for which $\pi \circ \sigma : I^{p+q} \to B$ is independent of the last $q$ coordinates. The resulting spectral sequence of this filtration replicates the Serre spectral sequence as we have constructed it. A full proof (as well as a very good general presentation of spectral sequences) is given here. △

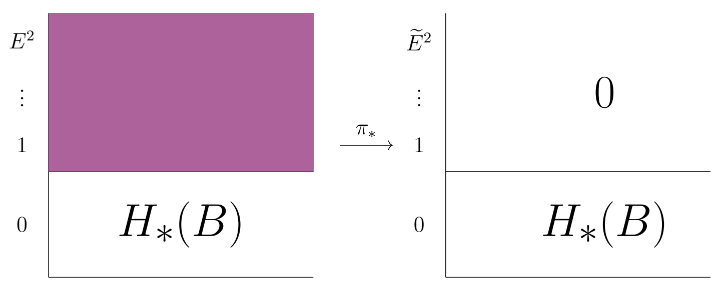
*Edge Morphisms.* There are two simple but important examples of morphisms of spectral sequences associated to any fibration $\pi : E \to B$ that we wish to study, both coming from the edges of the spectral sequence. For the first one, let $F_{x_0} \to x_0$ be the fibration $\pi : E \to B$ restricted to a fibre over a single point $x_0 \in B$. Then we have a morphism of fibrations coming from the embedding $i : F_{x_0} \hookrightarrow E$. The corresponding of morphism of spectral sequences on the $E^2$ page is represented by the embedding of the zeroth column as shown below, where the magenta region is unknown.



On the left side, the sequence degenerates: $E^2 = E^\infty$. On the right, the sequence may continue and the zeroth column $\widetilde{E}^\infty_{0,*}$ will end up as some quotient of $H_*(F)$ (since all differentials out of $E_{0,*}$ are trival). On $E^\infty$, the morphism should just be the map on homology induced by embedding $i_* : H_*(F) \to H_*(E)$. Hence, using (5.1), this map factors as follows:

$$\begin{tikzcd} H_n(F) \arrow[r, "i_*"] \arrow[d, two heads] & H_n(E) \\ \widetilde{E}^\infty_{0,n}. \arrow[ur, hook] & \end{tikzcd}$$

The second morphism of fibrations we want to consider is the projection $\pi$ from $\pi : E \to B$ to the identity fibration $B \to B$. On the $E^2$ page, the corresponding morphism of spectral sequences is the projection killing all the rows above the zeroth row, as shown below.




On the right side, the sequence degenerates: $\widetilde{E}^2 = \widetilde{E}^\infty$. On the left, the sequence may continue and the zeroth row $E^\infty_{*,0}$ will be some subgroup of $H_*(B)$ (since all differentials into $E_{*,0}$ are trival). One $E^\infty$, the morphism must correspond to the map on homology induced by projection $\pi_* : H_*(E) \to H_*(B)$. Hence, again using (5.1), this map factors as follows:

$$\begin{tikzcd} H_n(E) \arrow[r, "\pi_*"] \arrow[d, two heads] & H_n(B) \\ E^\infty_{n,0}. \arrow[ur, hook] & \end{tikzcd}$$

We get analogous diagrams applying these morphisms to the cohomological spectral sequence:

$$\begin{tikzcd} H^n(B) \arrow[r, "\pi^*"] \arrow[d, two heads] & H^n(E) \\ E_\infty^{n,0} \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} H^n(E) \arrow[r, "i^*"] \arrow[d, two heads] & H^n(F) \\ E_\infty^{0,n}. \arrow[ur, hook] & \end{tikzcd}$$

*Transgression.* Given a first quadrant spectral sequence (homological or cohomological), there are a series of maps that connect the zeroth row and column:

$$\mathrm{d}^r_{r0} : E^r_{r,0} \to E^r_{0,r-1} \quad \text{and} \quad \mathrm{d}^{0,r-1}_r : E^{0,r-1}_r \to E^{r,0}_r.$$

These maps are called *edge maps*. They are the last non-trivial map in the spectral sequence that these squares interact with. In light of (5.1), we can think of these as partially defined maps (i.e. maps from a subgroup to a quotient group),

$$\tau : E^2_{r,0} \to E^2_{0,r-1} \quad \text{and} \quad \tau : E^{0,r-1}_2 \to E^{r,0}_2.$$

These maps are called *transgressions* and elements of the domain of $\tau$ are called *transgressive*. Understanding the transgressions is important for understanding the differentials of a spectral sequence. They can be thought of as generalizations of the connecting homomorphisms in the snake lemma.

In the Serre spectral sequence case, the transgressions are partially defined maps,

$$H_r(B) \to H_{r-1}(F) \quad \text{and} \quad H^{r-1}(F) \to H^r(B).$$

Thankfully these have a nice geometric interpretation.

**Theorem 5.7: Transgressions in the Serre Spectral Sequence**

Let $\pi : E \to B$ be a homologically simple fibration. The transgression maps in the homological and cohomological Serre spectral sequence of $\pi$ coincide respectively with the following partially defined maps:

$$H_n(B) = H_n(B,*) \xrightarrow{(\pi_*)^{-1}} H_n(E,F) \xrightarrow{\partial_*} H_{n-1}(F)$$

$$H^{n-1}(F) \xrightarrow{\delta^*} H^n(E,F) \xrightarrow{(\pi^*)^{-1}} H^n(B,*) = H^n(B).$$

Here $\partial_*, \delta^*$ come from the long exact sequences of a pair, and $\pi_*, \pi^*$ may not be invertible, hence these are only partially defined.

Proof. We prove just the homological version, the other being analogous. By restricting to a path component of the base, we may assume $B$ has only a single 0-cell $*$. The group $E^n_{n,0}$ is

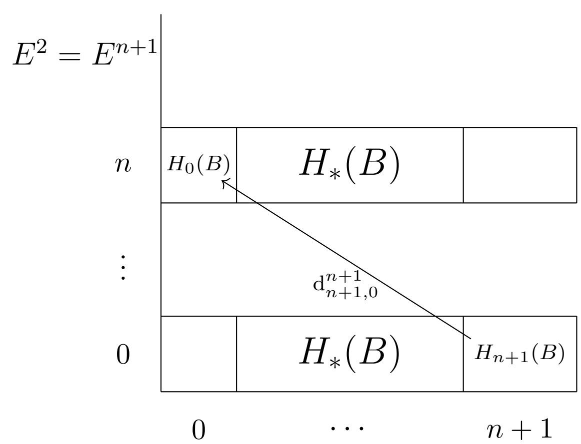
$$\frac{F_nC_n(E) \cap \mathrm{d}^{-1}F_0C_{n-1}(E)}{F_{n-1}C_n(E) \cap \mathrm{d}^{-1}F_0C_{n-1}(E) + F_nC_n(E) \cap \mathrm{d}F_{2n-1}C_{n+1}}.$$

Representatives of this group consist of chains $c \in C_n(\pi^{-1}(B_n))$ with boundaries in $C_{n-1}(\pi^{-1}(*)) = C_{m-1}(F)$. I.e. by relative cycles of the pair $(\pi^{-1}(B_n), F)$. We identify $E^n_{n,0}$ with a subgroup of $H_n(B)$ by sending $c$ to $\pi_*(c)$. The differential $\mathrm{d}^n_{n,0}$ acts by sending $c$ to $\partial c \in C_{n-1}(F)$. This completes the proof. □

*Two Exact Sequences.* We discuss now a couple of exact sequences that can be derived from the Serre spectral sequence. They both deal with fibrations involving spheres.

The first is the Gysin sequence for sphere bundles. Suppose $\pi : E \to B$ is a homologically simple fibration with spherical fibre $S^n$. If this is a smooth sphere bundle, the homological simplicity assumption is the same as asking that the bundle be orientable.

We know the homology of $S^n$ and so we deduce that the $E^2$ page of the spectral sequence is two copies of the homology of $B$ in rows 0 and $n$. The only non-trivial differential can occur on the page $E^{n+1}$ when we get a family of maps $\mathrm{d}^{n+1}_{r,0} : E^{n+1}_{r,0} \to E^{n+1}_{r-n-1,n}$ for each $r \geq n+1$. See the figure below.




We see these differentials give rise to maps $H_r(B) \to H_{r-n-1}(B)$ for each $r$.

The cohomology of this page is $E^{n+2} = E^\infty$ We conclude that each diagonal $E^\infty_{p+q=m}$ consists of at most two non-zero terms, namely $E^\infty_{r,0} = \ker(\mathrm{d}^{n+1}_{r,0})$ and $E^\infty_{r-n,n} = \mathrm{coker}(\mathrm{d}^{n+1}_{r+1,0})$. We know that because it lies in the last non-trivial row, $E^\infty_{r-n,n}$ is a subgroup of $H_r(E)$, while because it lies in the first non-trivial row, $E^\infty_{r,0}$ is a quotient of $H_r(E)$ (see (5.1)). So we have short exact sequences,

$$0 \to \mathrm{coker}(\mathrm{d}^{n+1}_{r+1,0}) \to H_r(E) \to \ker(\mathrm{d}^{n+1}_{r,0}) \to 0.$$

This is equivalent to the exact sequence,

$$H_{r+1}(B) \xrightarrow{\mathrm{d}^{n+1}_{r+1,0}} H_{r-n}(B) \to H_r(E) \to H_r(B) \xrightarrow{\mathrm{d}^{n+1}_{r,0}} H_{r-n-1}(B).$$

Because the differential squares to zero, these can be merged to a long exact sequence,

$$\cdots \xrightarrow{\pi_*} H_{r+1}(B) \xrightarrow{\mathrm{d}} H_{r-n}(B) \xrightarrow{\pi^!} H_r(E) \xrightarrow{\pi_*} H_r(B) \to \cdots .$$

This is called the *Gysin sequence*. The fact the map $H_r(E) \to H_r(B)$ coincides with $\pi_*$ follows directly from our earlier discussion of edge morphisms of spectral sequences. The map $\pi^!$ maps a homology class in $B$ to its inverse image under $\pi$. If we are dealing with a smooth bundle, then $\pi^!$ is the "wrong-way" map on homology given by the pullback on cohomology composed on both sides with the Poincaré isomorphism:

$$\pi^! : H_{r-n}(B) \xrightarrow{PD} H^{b-r+n}(B) \xrightarrow{\pi^*} H^{b-r+n}(E) \xrightarrow{PD} H_r(E),$$

where $\dim(B) = b$. In this context, this map $\pi^!$ is usually called the *Gysin homomorphism*. The map d, which came from the differential in the spectral sequence, can be interpreted as the cap product with the Euler class of our sphere bundle $e(E) \in H^{n+1}(B)$.

This last fact is a consequence of the following proposition.

Proposition 5.8. *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a homologically simple fibration for which $F$ is $(n-1)$-connected. Define a tautological cohomology class $c_F \in H^n(F; \pi_n(F))$ so that $c_F(h(\alpha)) = \alpha$ for any $\alpha \in \pi_n(F)$, where $h$ is the Hurewicz homomorphism. Then under the transgression $\tau$ in the Serre spectral sequence of $\pi$, $c_F$ is sent to the first obstruction of our fibration:*

$$\tau(c_F) = C(E) \in H^{n+1}(B; \pi_n(F)).$$

Proof. By Theorem 5.7, we want to show that $\delta^*(c_F) = \pi^*(C(E)) \in H^{n+1}(E, F; \pi_n(F))$.

Since $C(E)$ is the first obstruction to lifting a section of $E$, $\sigma : B \to E$ to the $n+1$ skeleton of $B$, we conclude $\pi^*(C(E))$ is the obstruction to lifting a fibrewise map $\widetilde{\sigma} : E_n \to E$ to the $(n+1)$-skeleton of $E$. Equivalently, this is the first obstruction to lifting the identity map $F \to F$ to a map $E \to F$ (refer to example 4.43).

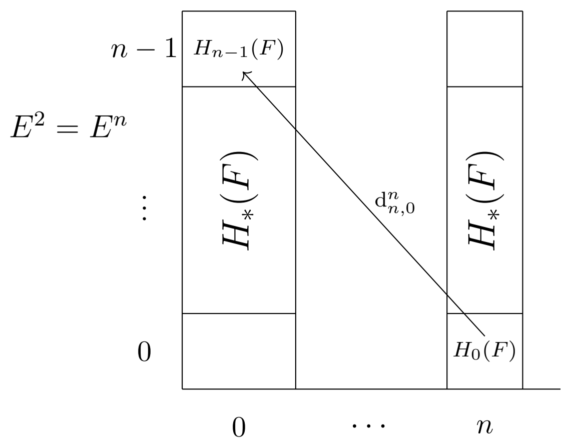
Now note $\delta^*(c_F)$ evaluated on some relative $(n+1)$-cell of $(E, F)$ is $c_F$ evaluated on its boundary, an $n$-cell of $F$. Since $c_F$ is tautological, this is just the homotopy class of the boundary. But this precisely describes $\delta^*(c_F)$ as the first obstruction to lifting the identity map $F \to F$ to a map $E \to F$ just like $\pi^*(C(E))$. $\square$

We have an analogous Gysin sequence in cohomology:

$$\cdots \xrightarrow{\mathrm{d}} H^r(B) \xrightarrow{\pi^*} H^r(E) \xrightarrow{\pi_!} H^{r-n}(B) \xrightarrow{\mathrm{d}} H^{r+1}(B) \to \cdots .$$

Now d is the cup product with $e(E)$ and in the smooth case $\pi_!$ is the wrong-way map in cohomology.

Now we consider the Wang sequence. Suppose $\pi : E \to S^n$ is a fibration over a sphere with fibre $F$. If $n = 1$, we should assume homological simplicity, otherwise it is automatic. The $E^2$ page of the corresponding specral sequence consists of two copies of the homology of $F$ on the zeroth and $n$ columns. The only non-trivial differential can occur on the page $E^n$ when we get a family of maps $\mathrm{d}^n_{n,r-n+1} : E^n_{n,r-n+1} \to E^n_{0,r}$ for each $r \geq n-1$. See the figure below.




We see the differential gives rise to maps $H_r(B) \to H_{r-n+1}(B)$ for each $r$. We have $E^{n+1} = E^\infty$ and as with the Gysin sequence we obtain short exact sequences,

$$0 \to \operatorname{coker}(\mathrm{d}^n_{n,r-n+}) \to H_r(E) \to \ker(\mathrm{d}^n_{n,r-n}) \to 0,$$

which patch together into a long exact sequence,

$$\cdots \xrightarrow{i_!} H_{r-n+1}(F) \xrightarrow{\mathrm{d}} H_r(F) \xrightarrow{i_*} H_r(E) \xrightarrow{i_!} H_{r-n}(F) \to \cdots .$$

This is called the *Wang sequence*. The map $i_*$ is induced from inclusion, as we know from our discussion on morphisms. The map $i_!$ is given by restricting a cycle to a fibre; in the smooth situation this is the wrong way map corresponding to $i$. The map d coming from the differential can be dscribed as follows. Consider $h : D^n \to D^n/\partial D^n = S^n$ and let $h^*E$ be the pullback of our fibration; being over $D^n$ this is necessarily trivial. Thus the map $h^*E \to E$ gives a map $D^n \times F \to E$ which restricts to a map $\widetilde{h} : S^{n-1} \times F \to F$. The map d is given by $\mathrm{d}(\alpha) = \widetilde{h}([S^{n-1} \times \alpha)$. A proof is left to the reader. We also have an analogous cohomological Wang sequence,

$$\cdots \xrightarrow{\mathrm{d}} H^{r-n}(F) \xrightarrow{i^!} H^r(E) \xrightarrow{i^*} H^r(F) \xrightarrow{\mathrm{d}} H^{r-n+1}(F) \to \cdots .$$

*Multiplication.* As for cohomology vs. homology, the cohomological Serre spectral sequence has a major advantage over the homological version in that it has the structure of a graded ring.

We can use the multiplication to understand the cup product structure of the total space of a fibration, or the cup product on the fibre and base. Additionally, the product structure sometimes tells us a priori what the differential maps of the spectral sequence need to be without having to work through their quite involved definition.

We begin in full generality. Suppose $F = F^0 \supset F^1 \supset \cdots$ is a filtered differential algebra equipped with a multiplication $F^p \times F^q \to F^{p+q}$ compatible with the differential in the sense that there is a Leibniz rule,

$$\delta(ab) = (\delta a)b + (-1)^{|a|}a(\delta b).$$

We are assuming here that $F$ is $\mathbb{Z}_2$ or $\mathbb{Z}$-graded. If not then just we can set $|a| = 0$ for all $a$.

This multiplication descends to a map $F \otimes F \to F$. Note that this tensor product has an induced filtration and differential given by,

$$(F \otimes F)^n = \sum_{p+q \leq n} F^p \otimes F^q \quad \text{and} \quad \delta(a \otimes b) = \delta a \otimes b + (-1)^{|a|} a \otimes \delta b.$$

This makes $F \otimes F \to F$ a map of filtered differential algebras and so we get a corresponding map of spectral sequences. This begins with the induced map on the associated graded groups $E_0 \otimes E_0 \to E_0$. By passing to the cohomology, we get a map,

$$H^*(E_0) \otimes H^*(E_0) \to H(E_0 \otimes E_0) \to H(E_0).$$

Here the first map is the embedding from the Künneth formula and the second is the induced map on $E_1$ pages from $F \otimes F \to F$. Thus, we obtain a map $E_1 \otimes E_1 \to E_1$. In this exact way, we pass to all later pages. If $F$ possesses a grading, then this whole structure is compatible with the bigrading of the spectral sequence, as we expand on in the following central result.

**Theorem 5.9: Multiplication in the Serre Spectral Sequence**

Let $\pi : E \to B$ be a homologically simple fibration with fibre $F$. Then the cohomological Serre spectral sequence of $\pi$ obtains a multiplication on each page beginning at $E_2$ possessing the following properties:

(i) The multiplication is compatible with the bigrading in the sense that if $\alpha \in E_r^{p,q}$ and $\beta \in E_r^{s,t}$, then $\alpha\beta \in E_r^{p+s,q+t}$.

(ii) The differentials satisfy a bigraded Leibniz rule: if $\alpha \in E_r^{pq}$ and $\beta \in E_r^{st}$ then,

$$\mathrm{d}_r^{p+s,q+t}(\alpha\beta) = (\mathrm{d}_r^{p,q}\alpha)\beta + (-1)^{p+q}\alpha(\mathrm{d}_r^{s,t}\beta).$$

(iii) The multiplication on the page $E_{r+1}$ is induced from the multiplication on $E_r$, so that if $\alpha, \beta \in E_{r+1}$ are represented by $\mathrm{d}_r$-cycles $a, b$, then $\alpha\beta$ is represented by $ab$.

(iv) On the $E_2$ page, the multiplication is just the multiplication in $H^*(B)$ with coefficients in the ring $H^*(F)$.

(v) The multiplication on $E_\infty$ is inherited from the multiplication on $H^*(E)$ in the following sense. If $a \in F^pH^m(E)$ and $b \in F^pH^n(E)$ represent $\alpha \in E_\infty^{p,m-p}$ and $\beta \in E_\infty^{q,n-q}$, then $ab \in F^{p+q}H^{m+n}(E)$ represents $\alpha\beta \in E_\infty^{p+q,m+n-p-q}$.

Proof. We need only construct the multiplication on $E_2$. The facts (i)–(iii) are just general consequences of our set up described above.

By CW approximation, we can restrict to the case where $B$ is cellular. Consider the diagonal embedding of fibrations factored through the pullback bundle.

$$\begin{tikzcd}
E \arrow[r] \arrow[rr, bend left, "\Delta_E"] \arrow[d, "F"] & \Delta_E^* E \times E \arrow[r] \arrow[d, "F\times F"] & E \times E \arrow[d, "F\times F"] \\
B \arrow[r, "\mathrm{id}"] & B \arrow[r, "\Delta_B"] & B \times B
\end{tikzcd}$$

By Proposition 5.5, we may CW approximate $\Delta_B$ by $\Delta_B^{CW}$ and obtain a morphism of Serre spectral sequences beginning at the page $E_2$. Note the cup product is usually defined as the pullback under $\Delta_B$ of the cross product of cohomology classes. We may equally well use $\Delta_B^{CW}$ since they are homotopic and hence induce the same maps on cohomology.

From the proof of Proposition 5.5, the induced map on $E_2$ factors as in the commutative diagram above into maps,

$$H^*(B \times B; H^*(F \times F)) \to H^*(B; H^*(F \times F)) \tag{5.6}$$

$$H^*(B; H^*(F \times F)) \to H^*(B; H^*(F)). \tag{5.7}$$

We also have a morphism given by the Künneth formula, which on the $E_2$ page looks like,

$$H^*(B; H^*(F)) \otimes H^*(B; H^*(F)) \to H^*(B \times B; H^*(F \times F)) \tag{5.8}$$

The composition of (5.6) with (5.8) is the cup product on $H^*(B)$ and the cross product on $H^*(F)$. Composing with (5.7), the total construction is the cup product on $H^*(B)$ and $H^*(F)$. This is our desired map of spectral sequences.

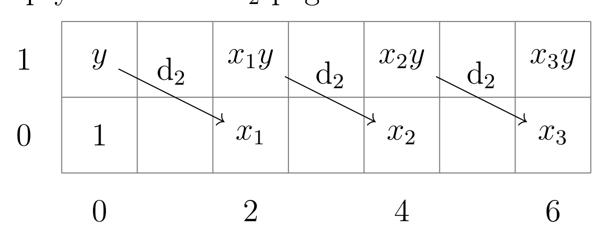
From the usual properties of morphisms of spectral sequences, the map on $E_\infty$ is the associated graded of the cup product on $H^*(E)$, proving (v). $\square$

Remark 5.10. The induced multiplication on $E_\infty = \mathrm{Gr}(H^*(E))$ may be much less rich than that on $H^*(E)$. For example, even if the multiplication on $E_\infty$ is trivial, we only know that $F^pH^*(E) \cdot F^qH^*(E) \subset F^{p+q+1}H^*(E)$. $\triangle$

**5.4. Computations with the Serre Spectral Sequence.** We will now take a look at several applications of computing homology (and homotopy) from the Serre spectral sequence. The sequence is useful for computing the stable homotopy groups of spheres and the homology of Eilenberg-MacLane spaces and we will briefly touch on these subjects. Our main interest will be in finding the homology rings of some Lie groups and homogeneous spaces; the Serre spectral sequence is very useful for this purpose since Lie groups and homogeneous spaces come equipped with many geometrically motivated fibrations, some of which we touched on earlier. We will also give a proof of the Thom isomorphism theorem.

*Some Miscellaneous Examples.* We begin with a few examples to become acquainted with explicit calculations using spectral sequences.

To ease into things, let us re-compute the ring structure of $H^*(\mathbb{C}P^\infty)$ which we previously found from intersection theory. We will use the cohomological Serre spectral sequence associated to the infinite complex Hopf fibration $S^1 \hookrightarrow * = S^\infty \to \mathbb{C}P^\infty$, which is homologically simple. We know the cohomology ring of $S^1$ is $H^*(S^1) = \mathbb{Z}[y]/(y^2)$ where $|y| = 1$. From cellular homology we know that $H^i(\mathbb{C}P^\infty)$ is zero in odd degrees and $\mathbb{Z}$ in even degrees. Let $x_i$ denote a generator of $H^{2i}(\mathbb{C}P^\infty)$. For ease of computation, it is customary when drawing a page of a spectral sequence to write the generator of each square in place of the group it generates. If a square is the trivial group, we usually leave it empty. Thus our $E_2$ page will look like the following.



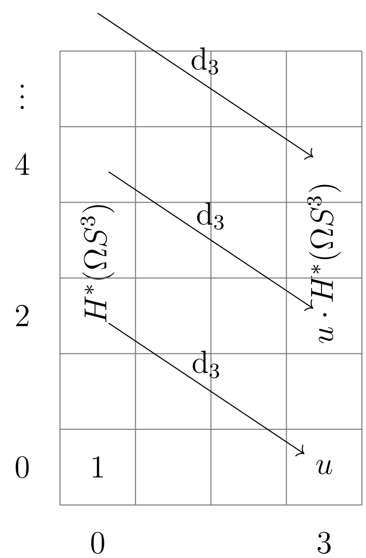

This is the last page with non-trivial differentials and so $E_3 = E_\infty$. Since $S^\infty$ is contractible, the $E_\infty$ page should be empty apart from $E_\infty^{0,0}$. Hence all the diagonal maps $\mathrm{d}_2 : E_2^{2k,1} \to E_2^{2k+2,0}$ must be isomorphisms. Without loss of generality, we can inductively modify the generators $x_i$ up to a sign so that $\mathrm{d}_2(y) = x$ and $\mathrm{d}_2(x_i y) = x_{i+1}$ for $i \geq 1$. Since the differential leaving $x_i$ is trivial, the graded derivation structure implies that,

$$\mathrm{d}_2(x_i y) = \cancelto{0}{\mathrm{d}_2(x_i)} y + x_i \mathrm{d}_2 y = x_i x_1.$$

We conclude, $x_2 = x_1^2$, $x_3 = x_2 x_1 = x_1^3$, and so on. So that $x_i = x_1^i$. We conclude all the powers of $x_1$ are non-zero and that the $2n$th cohomology group of $\mathbb{C}P^\infty$ is generated by $x_1^n$. Thus we rediscover

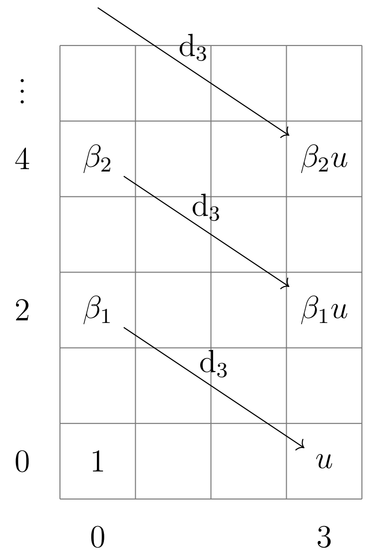
$$H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x], \quad \text{where} \quad |x| = 2.$$

Let us compute the cohomology of the loop space $\Omega S^3$ of $S^3$. Note there is a path space fibration $\Omega S^3 \hookrightarrow * = ES^3 \to S^3$. This is homologically simple and so we can use the Serre spectral sequence. We know the cohomology ring of $S^3$ is $H^*(S^3) = \mathbb{Z}[u]/(u^2)$, where $|u| = 3$. The $E_2 = E_3$ page of the cohomological spectral sequence of our fibration looks like the following.




This is the last page on which non-trivial differentials occur. Since the spectral sequence must converge to the trivial homology of $ES^3$, all the maps $\mathrm{d}_3 : E_3^{0,n} \to E_3^{3,n-2}$ for $n \geq 1$ must be isomorphisms. We conclude inductively that $H^n(\Omega S^3) = \mathbb{Z}$ if $n$ is even and $0$ if $n$ is odd. This could have been deduced from (really it is the same computation as) the Wang sequence.

We can now look at the $E_3$ page again to deduce the multiplicative structure Let $\beta_n$ denote a generator for $H^{2n}(\Omega S^3)$. The $E_3$ page looks as follows.




Inductively, we may choose $\beta_i$ so that $\mathrm{d}_2(\beta_1) = u$ and $\mathrm{d}_3(\beta_i) = \beta_{i-1}u$. for $i > 1$. We would like to determine the relationship between the products of the generators. Suppose that $\beta_k \cdot \beta_\ell = \lambda_{k,\ell}\beta_{k+\ell}$ for some coefficient $\lambda_{k,\ell} \in \mathbb{Z}$. We determine from the derivation structure,

$$\mathrm{d}_3(\beta_k\beta_\ell) = \mathrm{d}_3(\beta_k)\beta_\ell + \beta_k\mathrm{d}_3(\beta_\ell) = \beta_{k-1}\beta_\ell u + \beta_k\beta_{l-1}u = (\lambda_{k-1,\ell} + \lambda_{k,\ell-1})\beta_{k+\ell-1}u.$$

On the other hand,

$$\mathrm{d}_3(\beta_k\beta_\ell) = \mathrm{d}_3(\lambda_{k,\ell}\beta_{k+\ell}) = \lambda_{k,\ell}\beta_{k+\ell-1}u.$$

We conclude that the coefficients satisfy the coupled integer equations,

$$\lambda_{k,\ell} = \lambda_{k-1,\ell} + \lambda_{k,\ell-1} \quad \text{for} \quad k, \ell \in \mathbb{N}.$$

But note these are exactly the relations for $\lambda_{k,\ell}$ the $k$th entry of the $(k+\ell)$th row of Pascal's triangle. So our coefficients are binomial coefficients:

$$\beta_k\beta_\ell = \binom{k+\ell}{k}\beta_{k+\ell}.$$

We conclude the ring structure is,

$$H^*(\Omega S^3) = \mathbb{Z}[x^k/k! : k = 1, 2, \ldots] \quad \text{where} \quad |x| = 2.$$

EXERCISE 5.11. Show that if $n$ is odd, then $H^*(\Omega S^n)$ has the same ring structure but where $|x| = n-1$. Find the ring structure of $H^*(\Omega S^n)$ for $n$ even. ▲

*Homotopy Groups of Spheres.* As a first application to stable homotopy theory, we can finally compute the first stable homotopy group of the sphere, namely $\pi_4(S^3)$. Consider the killing space $S^3|_4$. We have by Hurewicz,

$$\pi_4(S^3) = \pi_4(S^3|_4) = H_4(S^3|_4).$$

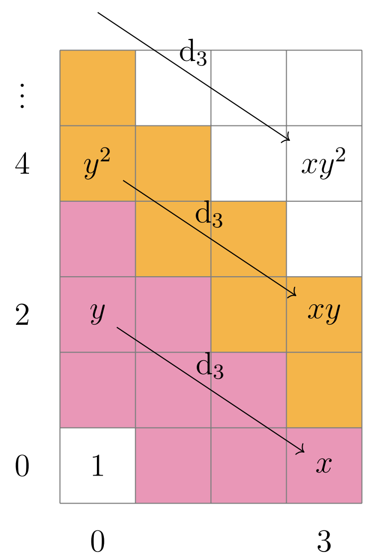
Recall from our construction of killing spaces, $S^3|_4$ is the total space of a Serre fibration,

$$K(\mathbb{Z}, 2) \hookrightarrow S^3|_4 \to S^3.$$

This fibration is homologically simple and so we can study the associated Serre spectral sequence. We know the cohomology of $S^3$ and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ as rings:

$$H^*(S^3) = \mathbb{Z}[x]/(x^2) \quad \text{and} \quad H^*(K(\mathbb{Z}, 2)). = \mathbb{Z}[y],$$

where $|x| = 3$ and $|y| = 2$. The $E_2 = E_3$ page of the spectral sequence looks like the following.



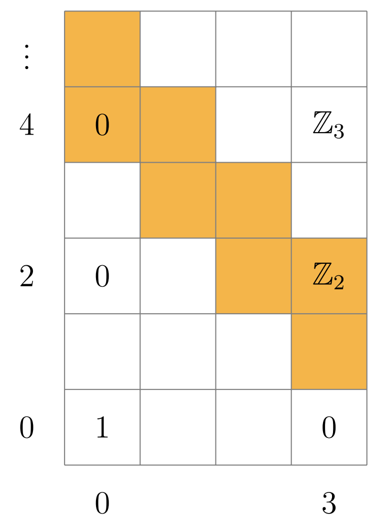

This will be the last page on which non-trivial differentials occur so that $E_4 = E_\infty$. We know by the Hurewicz theorem that $H^i(S^3|_4) = \pi_i(S^3|_4) = 0$ for $i = 1, 2$. Hence all contents of the pink squares must be killed by the $E_\infty$ page. By the universal coefficient theorem, $H_4(S^3|_4)$ is the free part of $H^4(S^3|_4)$ direct sum the torsion part of $H^5(S^3|_4)$, so to compute our homotopy group we need to understand the yellow squares of the spectral sequence on the $E_\infty$ page.

In order that the contents of the pink squares are killed, we need the transgressive map $\mathrm{d}_3^{0,2}$ to be an isomorphism. Without loss of generality, changing the sign of $x$ if necessary, we can set $\mathrm{d}_3 y = x$. Applying this to the next map

$d_3^{0,4}$ and using the graded derivation structure,

$$d_3(y^2) = 2y\, d_3 y = 2xy.$$

We conclude the $E_4 = E_\infty$ page looks like,




Since each diagonal has at most one non-trivial entry we have,

$$H^4(S^3|_4) = 0 \quad \text{and} \quad H^5(S^3|_4) = \mathbb{Z}_2.$$

Thus, $H_4(S^3|_4) = Z_2$. In fact we easily compute all higher homotopy groups:

$$H_n(S^3|_4) = \begin{cases} \mathbb{Z}_m & n = 2m \\ 0 & \text{else.} \end{cases}$$

We conclude $\pi_4(S^3) = H_4(S^3|_4) = \mathbb{Z}_2$. And so the first stable homotopy group of the sphere is $\mathbb{Z}_2$. By the Freudenthal suspension theorem, we know this homotopy class of map $S^4 \to S^3$ is given by suspending the Hopf fibration $S^3 \to S^2$. The preceding computation demonstrates that if we precompose the Hopf fibration with a degree two map $S^3 \to S^3$ and then suspend (or suspend the Hopf fibration and precompose with a degree two map $S^4 \to S^4$) the resulting map is null-homotopic.

This looks like a promising method for computing homotopy groups of spheres, and indeed it is. But further computations will not be as easy because understanding the cohomology of the capping spaces becomes harder as its relation to the sphere in the associated Whitehead tower becomes more complicated.

We now give another application to homotopy theory by proving Serre's finiteness theorem (Proposition 4.24). We begin with a lemma.

LEMMA 5.12. *Let $X$ be a simply connected space with finitely generated homology which is a rational odd homology sphere. I.e. $H^*(X;\mathbb{Q}) = H^*(S^m;\mathbb{Q})$ for $m$ odd. Then $\pi_q(X)$ is finite for $q \neq m$ and has rank one if $q = m$.*

PROOF. We have $H^m(X;\mathbb{Q}) = \mathbb{Q}$ and so there is an element of infinite order $\alpha \in H^m(X;\mathbb{Z})$. Let $F_m \in H^m(K(\mathbb{Z},m);\mathbb{Z})$ be the tautological class. By obstruction theory, we can construct $f : X \to K(\mathbb{Z},m)$ so that $f^*(F_m) = \alpha$.

EXERCISE 5.13. Inductively apply the Serre spectral sequence to the fibrations $K(\mathbb{Z}, n-1) \hookrightarrow * \to K(\mathbb{Z},n)$ to conclude the rational homology of these Eilenberg–MacLane spaces is,

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \Lambda_{\mathbb{Q}}[x] & n \text{ odd} \\ \mathbb{Q}[x] & n \text{ even} \end{cases} \quad \text{where} \quad |x| = n.$$

Apply the spectral sequence of the more general fibration $K(G,n-1) \hookrightarrow * \to K(G,n)$, to show the homology $H^q(K(G,n)$ is finitely generated for any $q$ and any $G$ finitely generated abelian. ▲

Having proved this exercise, we conclude $f^*$ is an isomorphism on rational homology.

We may homotope $f$ to be a fibration with homotopy fibre $F$ and consider the corresponding Serre spectral sequence. Recall from our discussion of edge morphisms that the induced pullback projection map $f^* : H^n(K(\mathbb{Z},m);\mathbb{Q}) \to H^*(X,\mathbb{Q})$ has to factor through the quotient map $H^n(K(\mathbb{Z},m;\mathbb{Q})) \to E_\infty^{*,0}$ onto the first row of the infinity page. Because $f^*$ is an isomorphism, this quotient must be an isomorphism and so the zeroth row of the spectral sequence can never participate in any differentials.

On the other hand, because $H^n(K(\mathbb{Z},m);\mathbb{Q}) \cong H^*(X,\mathbb{Q})$, the only contents of the $E_\infty$ page of the spectral sequence can be in the first row, since additional non-zero squares would complicate the homology of the total space. Suppose $H^m(F;\mathbb{Q})$ is the non-trivial rational homology group of $F$ with minimal $m > 0$. Then as we just noted, $E_2^{0,m} = H^m(F;\mathbb{Q})$ must be killed on some page of the sequence. But the only row below $m$ containing non-zero squares is the zeroth row. So $E_2^{0,m}$ can only be killed by a transgression. But this would force a non-trivial map to hit the zeroth row, which we said could not happen. We obtain a contradiction and so the homology $H^m(F;\mathbb{Q}) = 0$ for all $m > 0$.

Hence by the universal coefficient theorem, all the homology groups of $F$ are torsion. Because the integral homology of $X$ and $K(\mathbb{Z},m)$ are finitely generated (by the above exercise) the same must be true of $F$ (using the integral

Serre spectral sequence). Hence $F$ has finite homology groups. Now inductively studying the fibrations,

$$K(H_n(F|_n); n-1) \hookrightarrow F|_{n+1} \to F|_n,$$

we conclude from the corresponding spectral sequences that the killing spaces $F|_n$ all have finite integral homology groups. Hence $\pi_n(F) = H_n(F|_n)$ is finite for each $n$.

Now we can look at the homotopy long exact sequence of the fibration $F \hookrightarrow X \to K(\mathbb{Z}, m)$. We conclude the maps $f_* : \pi_n(X) \to \pi_n(K(\mathbb{Z}, m))$ all have finite kernel and cokernel. Hence using the homology of $K(\mathbb{Z}, m)$ when $m$ is odd, we obtain the result. $\square$

PROOF. (of Serre's finiteness theorem) We wish to show $\pi_q(S^n)$ is finite except for $q = n$ or in the case $\pi_{4m-1}(S^{2m})$ where the rank is one. If $n$ is odd, then we are done directly by the above lemma.

Let $n = 2m$ be even and consider the fibration $K(\mathbb{Z}, n-1) \hookrightarrow S^n|_{n+1} \to S^n$. The rational cohomological spectral sequence will look like the following with the first (and last) non-trivial differential on the $n$th page.

$$\begin{tikzcd}
n-1 & \mathbb{Q} \arrow[rrrddd, "\mathrm{d}_n"] & & & \mathbb{Q} \\
 & & & & \\
\vdots & & & & \\
0 & \mathbb{Q} & & & \mathbb{Q} \\
 & 0 & & \cdots & n
\end{tikzcd}$$

Because $H^{n-1}(S^n|_{n+1}) = H^n(S^n|_{n+1}) = 0$, the illustrated map $\mathrm{d}_n^{0,n-1}$ has to be an isomorphism. Hence, $H^*(S^n|_{n+1}; \mathbb{Q}) \cong H^*(S^{2n-1}; \mathbb{Q})$.

We conclude from the previous lemma that for $q > 2m$, $\pi_q(S^{2m}|_{2m+1}) = \pi_q(S^{2m})$ is finite unless $q = 4m-1$, in which case it has rank one. $\square$

In fact we can give a nice construction of a free generator in $\pi_{4m-1}(S^{2m})$. There is a graded Lie algebra structure on the direct sum of the homotopy groups of a space called the *Whitehead bracket*,

$$[\cdot, \cdot] : \pi_k(X) \times \pi_\ell(X) \to \pi_{k+\ell-1}(X).$$

This defines a graded Lie algebra in the sense it is bilinear, supersymmetric, and satisfies a super/graded Jacobi identity.

To construct this bracket, note $S^k \times S^\ell$ is formed from $S^k \vee S^\ell$ by attaching a $(k+\ell)$-cell. Hence there is a cell attaching map $\psi : S^{k+\ell-1} \to S^k \vee S^\ell$. Then the Whitehead bracket is defined by,

$$[f,g] = \psi^*(f \vee g).$$

PROPOSITION 5.14. *Let* $\mathrm{id}_{2n} \in \pi_{2n}(S^{2n})$ *be the homotopy class of the identity map. Then* $[\mathrm{id}_{2n}, \mathrm{id}_{2n}] \in \pi_{4n-1}(S^{2n})$ *has infinite order.*

PROOF. (Sketch) Given $f : S^{4n-1} \to S^{2n}$, form its cone $C_f = S^{2n} \cup_f D^{4n}$ which up to homotopy equivalence depends only on the class $[f] \in \pi_{4n-1}(S^{2n})$. The cohomology ring of $C_f$ is easily seen to be,

$$H^*(C_f) = \mathbb{Z}[\alpha,\beta]/(\alpha\beta, \beta^2, \alpha^2 - h(f)\beta), \quad \text{where} \quad |\alpha| = 2n, |\beta| = 4n.$$

Here $h(f)$ is some integer depending on $f$, called the *Hopf invariant*.

It can be shown that $h : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$ is a group homomorphism. To see this, one studies the space $Y_{f,g} = S^{2n} \cup_{f\vee g} (D^{4n} \vee D^{4n})$ and the effect on cohomology of the natural maps $C_f, C_g, C_{f+g} \to Y_{f,g}$.

One can also show $h([\mathrm{id}_{2n}, \mathrm{id}_{2n}]) = 2$. For this, one can form $C_{[\mathrm{id}_{2n},\mathrm{id}_{2n}]}$ as the quotient of $S^{2n} \times S^{2n}$ by identifying points $(x, x_0)$ with $(x_0, x)$, where $x_0$ is the basepoint. An explicit study of the cohomology of this space gives the Hopf invariant.

We conclude that non-zero multiples of $[\mathrm{id}_{2n}, \mathrm{id}_{2n}]$ in $\pi_{4n-1}(S^{2n})$ all have non-zero Hopf invariants and hence can never be null-homotopic. $\square$

Lastly for this section, let us compute the second stable homotopy group of the sphere. Consider the identification $\Sigma S^2 \to S^3$. By the loop-suspension adjunction, we obtain a map $f : S^2 \to \Omega S^3$. We may homotope $f$ to a fibration with homotopy fibre $F$. From the homotopy long exact sequence of this fibration, $F$ is 2-connected, and so by Hurewicz $\pi_3(F) = H_3(F)$. We computed earlier the ring structure of $H^*(\Omega S^3)$. We deduce from this that $H_4(\Omega S^3) = \mathbb{Z}$. Considering the spectral sequence of our fibration, since $H_3(S^2) = H_4(S^2) = 0$, the trangressive map on the $E^4$ page, $H_4(\Omega S^3) \to H_3(F)$ must be an isomorphism (it is the first and last non-trivial map to hit the squares $E_{4,0}$ and $E_{0,3}$). Hence, $\mathbb{Z} = H_3(F) = \pi_2(F)$.

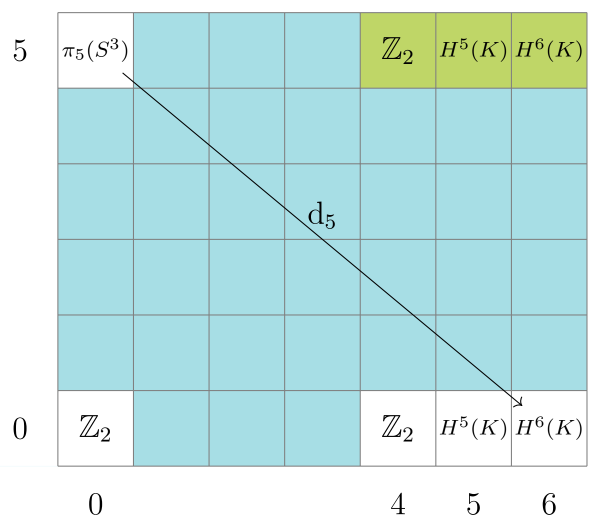
We consider the following segment of the homotopy long exact sequence of our fibration.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_4(S^2) \arrow[r, "f_*"] \arrow[d, equal] & \pi_4(\Omega S^3) \arrow[r] \arrow[d, equal] & \pi_3(F) \arrow[r] \arrow[d, equal] & \pi_3(S^2) \arrow[r, "f_*"] \arrow[d, equal] & \pi_3(\Omega S^3) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathbb{Z}_2 \arrow[r] & \pi_5(S^3) \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}_2 \arrow[r] & \cdots
\end{tikzcd}$$

We have used $\pi_4(S^2) = \pi_4(S^3) = \mathbb{Z}_2$ from the long exact sequence of the Hopf fibration and the fact $\pi_n(\Omega S^3) = \pi_{n+1}(S^3)$. The map $\mathbb{Z} \to \mathbb{Z}_2$ is either the quotient map or trivial. Hence the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by $\pm 2$ or $\pm 1$, in particular it is injective. Thus the map $\pi_5(S^3) \to \mathbb{Z}$ is trivial and so $\mathbb{Z}_2 \to \pi_5(S^3)$ is a surjection. We now want to show it is non-trivial and hence equals $\mathbb{Z}_2$.

Now consider the fibration $K(\mathbb{Z}, 2) \to S^3|_4 \to S^3$. We computed $\pi_4(S^3|_4) = \mathbb{Z}_2$. So construct a map $g : S^3|_4 \to K(\mathbb{Z}_2, 4)$ inducing an isomorphism on $\pi_4$ (this is clearly doable by obstruction theory). We can homotope $g$ to a fibration with homotopy fibre $Y$. From this fibration, we conclude that $Y$ is 4-connected and so by Hurewicz and the fibration exact sequence we have $H^5(Y) = \pi_5(S^3)$. Since $\pi_5(S^3)$ is 0 or $\mathbb{Z}_2$, we may reduce mod 2: $H^5(Y; \mathbb{Z}_2) = \pi_5(S^3)$.

Now consider the cohomological spectral sequence of the fibration $Y \hookrightarrow S^3|_4 \to K(\mathbb{Z}_2, 4)$ over $\mathbb{Z}_2$. On the $E_2$ page the sequence looks like the following.




Here the blue squares are all zero and we write $H^n(K)$ to denote $H^n(K(\mathbb{Z}_2, 4); \mathbb{Z}_2)$. The green squares are not relevant to our computation. Clearly no non-trivial differentials affect the squares we are interested in up until the $E_5$ page and none will affect them again afterwards. On the $E_5$ page the only important differential is the pictured transgression,

$$d_5^{0,5} : H^5(Y; \mathbb{Z}_2) = \pi_5(S^3) \to H^6(K(\mathbb{Z}_2, 4); \mathbb{Z}_2).$$

By Serre's theorem on cohomology operations, we know that $H^5(K(\mathbb{Z}_2, 4); \mathbb{Z}_2)$ and $H^6(K(\mathbb{Z}_2, 4); \mathbb{Z}_2)$ are generated by $Sq^1 F_4$ and $Sq^2 F_4$ respectively, where $F_4 \in H^4(K(\mathbb{Z}_2, 4); \mathbb{Z}_2)$ is the tautological class. (We will shortly give some indication of how to prove Serre's theorem for those concerned by this gap in our argument). Hence we know the entries of our spectral sequence $E_5^{0,5} = E_5^{0,6} = \mathbb{Z}_2$. Now recall in finding the first stable homotopy group of the sphere we determined the homology of $S^3|_4$, in particular $H^5(S^3|_4; \mathbb{Z}_2) = \mathbb{Z}_2$ and $H^6(S^3|_4) = 0$. Since the above transgression $\mathrm{d}_5^{0,5}$ is the last map impacting the relevant region of our spectral sequence, we must have in order to converge to the correct cohomology of $S^3|_4$ that $E_6^{0,5} = E_6^{6,0} = 0$. Hence, the map $\mathrm{d}_5^{0,5}$ is an isomorphism. In particular,

$$\pi_5(S^3) = H^6(K(\mathbb{Z}_2, 4); \mathbb{Z}_2) = \mathbb{Z}_2.$$

We are almost done. Consider the quaternionic Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$. From this it is clear that $\pi_n(S^4) = \pi_{n-1}(S^3)$ for $n \leq 6$. Hence, $\pi_6(S^4) = \pi_5(S^3) = \mathbb{Z}_2$. This is the second stable homotopy group. Note the standard Hopf fibration also tells us $\pi_5(S^2) = \mathbb{Z}_2$. At this stage we know the following chart of homotopy groups depicted in Table 3. The coloured squares are the stable groups. Referring back to the Table 1, we see that $\pi_6(S^2) = \pi_6(S^3) = \mathbb{Z}_{12}$, although we haven't computed that.

| | $S^1$ | $S^2$ | $S^3$ | $S^4$ |
|---|---|---|---|---|
| $\pi_1$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| $\pi_2$ | 0 | $\mathbb{Z}$ | 0 | 0 |
| $\pi_3$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 |
| $\pi_4$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $\pi_5$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $\pi_6$ | 0 | ? | ? | $\mathbb{Z}_2$ |

TABLE 3. The Computed Homotopy Groups of Spheres

It is possible to go a lot further in building out this chart than we have done, but it requires progressively more work. See Chapters 3–5 of [**FF**] for a comprehensive discussion of the theory and a computation of the first thirteen stable homotopy groups. The low dimensional unstable groups can also be computed using similar methods involving Steenrod squares; see [**HatcherSS**, Thm 5.4.1] for a computation of $\pi_6(S^2)$.

*Constructing Steenrod Squares.* We now use the homology of Eilenberg–MacLane spaces to give a quick construction of Steenrod squares. Recall the Steenrod squares were stable (mod 2) cohomology operations,

$$Sq^i : H^n(X; \mathbb{Z}_2) \to H^{i+n}(X; \mathbb{Z}_2).$$

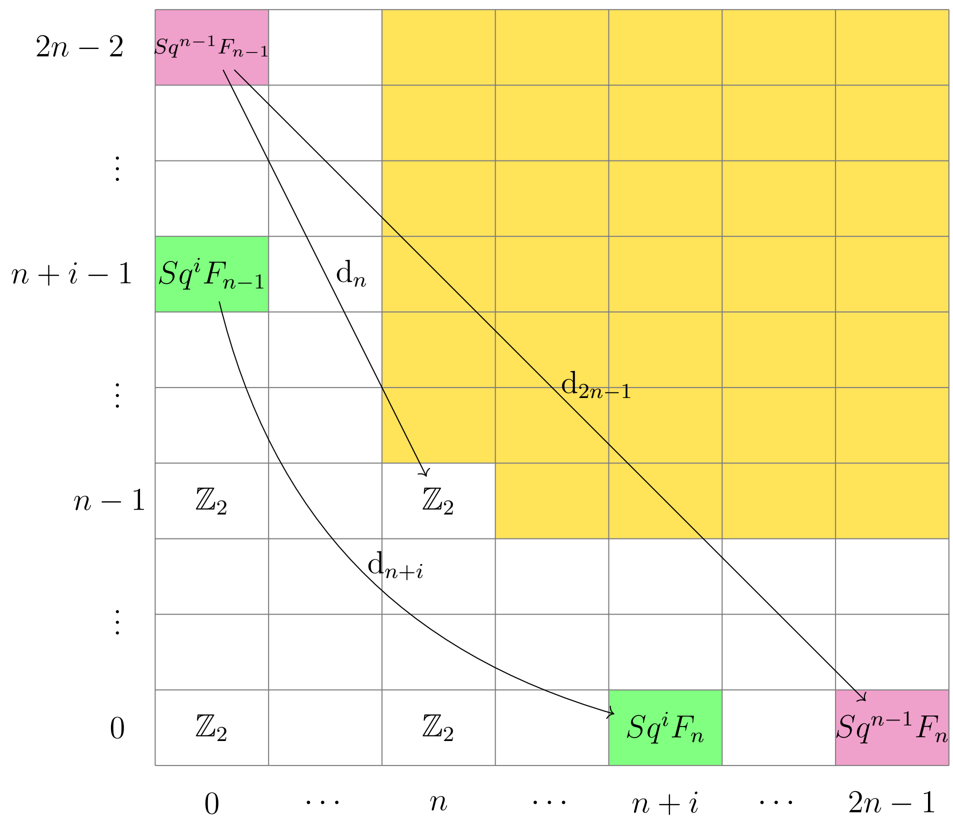
By the Yoneda lemma, these are determined by knowing

$$Sq^i(F_n) \in H^{i+n}(K(\mathbb{Z}_2, n); \mathbb{Z}_2),$$

where $F_n \in H^n(K(\mathbb{Z}_2; n); \mathbb{Z}_2)$ is the canonical cohomology class.

The axiomatic properties of Steenrod squares force $Sq^0 F_1 = F_1, Sq^1 F_1 = F_1^2$ and $Sq^i F_1 = 0$ for $i > 1$. We now work inductively, supposing we know $Sq^i F_{n-1}$ for all $i$ and determining $Sq^i F_n$.

Consider the fibration $K(\mathbb{Z}_2, n-1) \hookrightarrow * \to K(\mathbb{Z}_2, n)$ and the corresponding cohomological spectral sequence over $\mathbb{Z}_2$. Suppose $i < n-1$. We then have the following schematic picture of several pages of our sequence.




The yellow squares contain potentially non-zero entries while all the clear empty squares are zero from $E_2$ onwards. Since $i < n-1$, the only non-trivial differential impacting the squares $E^{0,n+i-1}$ and $E^{0,n+i}$ is the transgression,

$$\mathrm{d}_{n+i}^{0,n+i-1} : H^{n+i-1}(K(\mathbb{Z}_2, n-1); \mathbb{Z}_2) \to H^{n+i}(K(\mathbb{Z}_2, n); \mathbb{Z}_2).$$

Because this spectral sequence has to converge to the homology of a point, this transgression must be an isomorphism. Hence we can define,

$$Sq^i F_n := \mathrm{d}_{n+i} Sq^i F_{n-1}.$$

Now consider the case of $Sq^n$. The square $E^{0,2n-2}$ is involved in two potentially non-trivial differentials: the map $\mathrm{d}_n^{0,2n-2}$ and the map $\mathrm{d}_{2n-1}^{0,2n-2}$, both shown above. Note since the top Steenrod square is the actual square,

$$\mathrm{d}_n(Sq^{n-1}F_{n-1}) = \mathrm{d}_n(F_{n-1}^2) = 2F_{n-1}\mathrm{d}_n(F_{n-1}) = 0 \pmod 2.$$

Hence $Sq^{n-1}F_{n-1}$ persists to the $E_{2n-1}$ page and is in the domain of the transgression,

$$\mathrm{d}_{2n-1}^{0,2n-2} : \ker(\mathrm{d}_n^{0,2n-2}) \to H^{2n-1}(K(\mathbb{Z}_2, n); \mathbb{Z}_2).$$

Since this has to again be an isomorphism, we may define,

$$Sq^{n-1}F_n := \mathrm{d}_{2n-1}Sq^{n-1}F_{n-1}.$$

We know axiomatically that we must have $Sq^nF_n = F_n^2$ and $Sq^iF_n = 0$ for $i > n$. Thus we have inductively defined the Steenrod squares for all $i, n$.

The stability of these operations can be expressed as saying that the sequence of elements $Sq^rF_n$ should be obtained by applying the suspension maps,

$$H^{r+n-1}(K(\mathbb{Z}, n-1); \mathbb{Z}) \xrightarrow{\Sigma} H^{r+n}(\Sigma K(\mathbb{Z}, n-1); \mathbb{Z}) \xrightarrow{(i_n^*)^{-1}} H^{r+n}(K(\mathbb{Z}, n); \mathbb{Z}).$$

But one can show these maps agree with the transgression maps in our above spectral sequence. Hence we find the Steenrod squares are stable. Conversely, we see that to be stable, the Steenrod squares need to commute with transgressions. This proves the uniqueness of our axiomatic definition of the squares since we inductively defined them just in terms of transgressions and the other axioms. The proof of Cartan's formula proceeds from some geometric manipulations with the Eilenberg–MacLane spaces. See [**FF**, §29–30] for the details of these proofs as well as a proof of the Adem relations.

Lastly, let us sketch an argument for Serre's theorem on the fact the Steenrod squares multiplicatively generate all $\mathbb{Z}_2$ cohomology operations. The key ingredient is a theorem of Borel.

PROPOSITION 5.15 (Borel's Theorem). *Suppose $F \hookrightarrow E \to B$ is a fibration with simply connected base and weakly $\mathbb{Z}_2$ contractible total space. Further, suppose $H^*(F; \mathbb{Z}_2)$ is multiplicatively generated by elements $a_i$ which are transgressive in the Serre spectral sequence and so that the monomials $a_{i_1}a_{i_2}\cdots a_{i_k}$ with $i_1 < i_2 < \cdots < i_k$ form an additive basis for $H^*(F; \mathbb{Z}_2)$. Then $H^*(B; \mathbb{Z}_2)$ is the polynomial algebra generated by the images $b_i = \tau(a_i)$ under transgressions.*

PROOF. (Sketch) We can construct an abstract multiplicative spectral sequence whose $\widetilde{E}_2$ page is $H^*(F; \mathbb{Z}_2) \otimes \mathbb{Z}_2[b_1, b_2, \ldots]$ and compare to the Serre spectral sequence $E_2$ of the fibration. We let the differentials act trivially on each $b_i$ and act trivially on the $a_i$'s until the appropriate page $|a_i| + 1$ where

$\mathrm{d}a_i = b_i$. It is routine to check this is all well defined and gives a multiplicative spectral sequence. Each generator of the $\widetilde{E}_2$ page either eventually has non-trivial image under a differential or lies in the image of a differential. Hence the $\widetilde{E}_\infty$ page of this sequence is trivial.

There is a map between our two spectral sequence by sending the corresponding elements $a_i, b_i$ to each other. Now suppose this spectral sequence $\widetilde{E}_2$ differs from the Serre spectral sequence of $F \hookrightarrow E \to F$ on the $E_2$ page, so that the multiplicative map between the first rows is not an isomorphism. Then the first row of $E_2$ either contains an element $c$ (of minimal dimension) which is not generated by the image of the transgressions or contains a non-trivial relation $f(b_i) = 0$ (of minimal dimension) between the image of the transgressions. But comparing with the spectral sequence we constructed, there is nothing on $E_2$ to kill $c$ and so $E_\infty$ can't be trivial, or there is nothing on $\widetilde{E}_2$ to kill $f(b_i)$ and so $\widetilde{E}_\infty$ can't be trivial. $\square$

PROOF. (of Serre's theorem) We work inductively on $n$ in $K(\mathbb{Z}_2, n)$. The algebra $H^*(K(\mathbb{Z}_2, 1); \mathbb{Z}_2) = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[e_1]$ is the polynomial algebra on $Sq^0$ as desired.

Suppose we have obtained the desired description of $H^*(K(\mathbb{Z}_2, n-1); \mathbb{Z}_2)$. We may pick a suitable indexed set of Steenrod squares $Sq^I F_{n-1}$ and their powers which multiplicatively generate and whose monomials with increasing indices additively generate $H^*(K(\mathbb{Z}_2, n-1); \mathbb{Z}_2)$ (this can be done for any polynomial algebra, for example $\mathbb{Z}[x]$ is generated by $x^{2^i}$ in the sense we desire). We can then apply Borel's theorem to the fibration $K(\mathbb{Z}_2, n-1) \to * \to K(\mathbb{Z}_2, n)$. We conclude that $H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ is also a polynomial algebra generated by images of transgressions of the generators. By construction of the Steenrod squares, the image of the transgressions will be Steenrod squares $Sq^I F_n$ and their powers. Fiddling around with the indexing for a while, one can show the set of generators matches the description given in the statement of Serre's theorem. $\square$

In addition to helping compute homotopy groups of spheres, Steenrod squares also have several other applications throughout algebraic topology. They can be used as a homotopy invariant to distinguish spaces with the same cohomology/homotopy, like lens spaces or the following example.

EXERCISE 5.16. Prove with Steenrod squares that $\Sigma\mathbb{C}P^2$ and $\Sigma(S^2 \vee S^4)$ are not homotopy equivalent. ▲

They are also express an important property of Stiefel–Whitney classes, which we present in the next chapter.

*Thom Isomorphism Theorem.* We now prove an extremely important result for understanding the topology of vector bundles that we will need for our chapter on characteristic classes.

**Theorem 5.17: Thom Isomorphism Theorem**

Let $\pi : E \to B$ be a rank $n$ oriented vector bundle. The cup and cap product with the Thom class $t_E \in H^n(E, E \setminus B; \mathbb{Z})$ determine isomorphisms,

$$\smile t_E : H^i(B; \mathbb{Z}) \xrightarrow{\cong} H^{i+n}(E, E \setminus B; \mathbb{Z})$$

$$\frown t_E : H_{n+i}(E, E \setminus B; \mathbb{Z}) \xrightarrow{\cong} H_i(B; \mathbb{Z}).$$

In the unoriented case, the same holds over $\mathbb{Z}_2$.

Proof. Pick a metric on our vector bundle so that we can deal with the unit disk and sphere bundles $DE$ and $SE$ which are deformation retracts of $E$ and $E \setminus B$ respectively.

It is a simple extension of the Serre spectral sequence to work for relative fibrations. So consider the fibration of pairs,

$$(D^n, S^{n-1}) \hookrightarrow (DE, SE) \to B.$$

We will ignore coefficients to treat $\mathbb{Z}$ and $\mathbb{Z}_2$ simultaneously. The important point is that in the oriented case, the coefficients coming from the homology of the fibres are untwisted and we can freely work over $\mathbb{Z}$. The $E_2$ page of the relative cohomological spectral sequence is given, using the universal coefficient theorem, by

$$E_2^{p,q} = H^p(B; H^q(D^n, S^{n-1})) \cong H^p(B) \otimes H^q(D^n, S^{n-1}) = \begin{cases} 0 & q \neq n \\ H^p(B) & q = n. \end{cases}$$

This isomorphism is given by tensoring with the fundamental class of the fibre $[D^n, S^{n-1}]$. Since there is only one non-trivial row of this spectral sequence, there are no non-trivial differentials and $E_2^{p,q} = E_\infty^{p,q}$. This spectral sequence must converge to the homology of the pair $(DE, SE)$. Hence,

$$H^p(B) \cong H^{p+n}(DE, SE).$$

Let $t_E \in H^n(DE, SE) \cong H^0(B)$ be the multiplicative unit under the isomorphism. Note that by our discussion of edge morphisms, pulling back $t_E$ by inclusion of a fibre can be seen by looking at the $E_2$ page to give the fundamental class of the fibre. Hence $t_E$ agrees with our earlier definition of the Thom class.

Note because the spectral sequence was degenerate and respects multiplication, that since the isomorphism sends $1 \in H^0(B)$ to $t_E$, it must send $\alpha \in H^0(B)$ to $t_E \smile \alpha$. So our isomorphism is as described.

For the second isomorphism, the universal coefficient theorem implies that $H_{i+n}(E, E \setminus B) \cong H_i(B)$. By duality, this isomorphism is given by $x \mapsto z$ where $\beta(z) = (\beta \smile t_E)(x)$ for all $\beta \in H^i(B)$. But,

$$(\beta \smile t_E)(x) = \beta \frown (t_E \frown x).$$

Hence $z = t_E \frown x$ and the isomorphism on homology comes from capping with the Thom class. $\square$

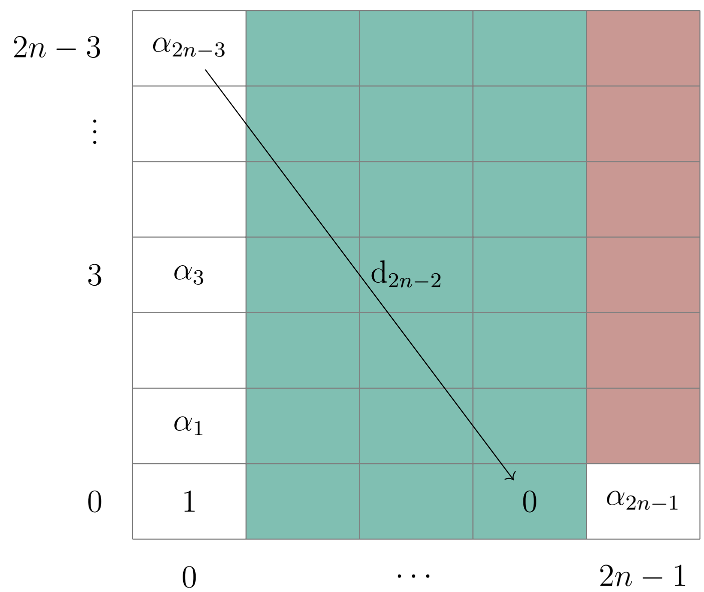
*Cohomology of Lie Groups.* As an essential application of spectral sequences, we find the cohomology rings of the classical compact Lie groups. Some of these results were stated at the end of the chapter on homology.

PROPOSITION 5.18. *The cohomology ring of* $\mathrm{U}(n)$ *is,*

$$H^*(\mathrm{U}(n)) = \Lambda_{\mathbb{Z}}[\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}] \quad \text{where} \quad |\alpha_i| = i.$$

PROOF. We work by induction, the case $H^*(\mathrm{U}(1)) = H^*(S^1) = \Lambda_{\mathbb{Z}}[\alpha_1]$ is known to us. Suppose the cohomology ring of $\mathrm{U}(n-1)$ is as claimed. Consider the fibration $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \to S^{2n-1}$ induced by sending a unitary matrix $A$ to the vector in $\mathbb{R}^{2n}$ represented by the first column of $A$. The last $n-1$ columns represent a unitary basis of the orthogonal complement of the first column (hence an element of $\mathrm{U}(n-1)$).

Let $\alpha_{2n-1}$ denote a generator of $H^{2n-1}(S^{2n-1})$. The $E_2$ page of our Serre spectral sequence looks like the following, where we only write down the multiplicative generators.




The green region consists entirely of zeroes. Note all the differentials leaving the elements $\alpha_i$ are trivial. Since these multiplicatively generate the whole page and the differential satisfies a Leibniz rule, we conclude all differentials are zero and thus $E_2 = E_\infty$. We conclude that additively $H^*(\mathrm{U}(n))$ has the claimed description. It remains to show the multiplicative structure is correct (since the fact $E_\infty$ is associated graded to $H^*(\mathrm{U}(n))$ may distort the multiplication). We do know that $\alpha_1, \ldots, \alpha_{2n-1}$ have representatives in $H^*(\mathrm{U}(n))$ (which we'll refer to by the same name) that multiplicatively generate the homology of $\mathrm{U}(n)$. These elements must all anti-commute in the cohomology ring by the super-commutativity of the cup product. Because there is no 2-torsion, we also know $\alpha_i^2 = 0$. There can be no other relations between the $\alpha_i$, otherwise the rank of $H^*(\mathrm{U}(n))$ would be less than the rank of $\Lambda_{\mathbb{Z}_2}[\alpha_1, \ldots, \alpha_{2n-1}]$ contradicting the additive isomorphism. Thus the ring structure of $H^*(\mathrm{U}(n))$ is as claimed. $\square$

Proposition 5.19. *The cohomology ring of* $\mathrm{SU}(n)$ *is,*

$$H^*(\mathrm{SU}(n)) = \Lambda_{\mathbb{Z}}[\alpha_3, \alpha_5, \ldots, \alpha_{2n-1}] \quad \textit{where} \quad |\alpha_i| = i.$$

Proof. This is the same argument as the $\mathrm{U}(n)$ case. We know $H^*(\mathrm{SU}(2)) = H^*(S^3) = \Lambda_{\mathbb{Z}}[\alpha_3]$ as claimed. We can then work by induction using the fibration $\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \to S^{2n-1}$. The spectral sequence will look like the picture above just without the $\alpha_1$ square. The differentials are again all trivial and hence $E_2 = E_\infty$ is additively the correct group. The same reasoning shows a multiplicative isomorphism holds as well. $\square$

Proposition 5.20. *The cohomology ring of the complex Stiefel manifold* $\mathbb{C}V(n,k)$ *is,*

$$H^*(\mathbb{C}V(n,k)) = \Lambda_{\mathbb{Z}}[\beta_{2(n-k)+1}, \beta_{2(n-k)+3}, \ldots, \beta_{2n-1}], \quad \textit{where} \quad |\beta_i| = i.$$

Proof. Again the argument is the same. We work inductively on $n$ and $k$ simultaneously. In the base case, $\mathbb{C}V(n,1) = S^{2n-1}$ and we have the correct description.

We can then use the Serre spectral sequence of the fibration $\mathbb{C}V(n-1,k-1) \hookrightarrow \mathbb{C}V(n,k) \to \mathbb{C}V(n,1) = S^{2n-1}$. All the differentials will be trivial giving an additive isomorphism and by the same argument a multiplicative isomorphism. □

Proposition 5.21. *The cohomology ring of* $\mathrm{Sp}(n)$ *is,*

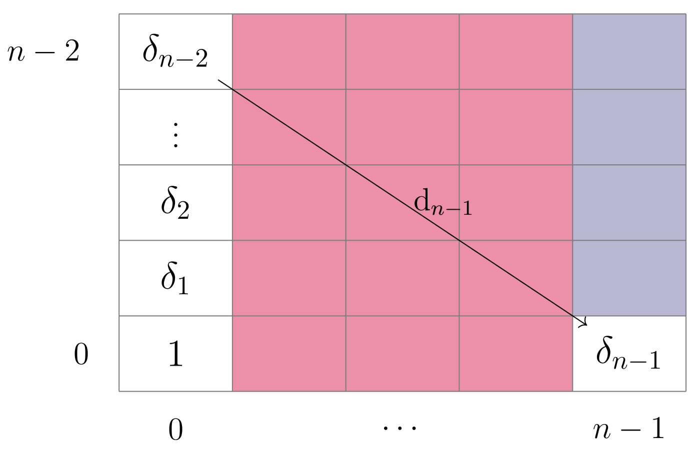
$$H^*(\mathrm{Sp}(n)) = \Lambda_{\mathbb{Z}}[\gamma_3, \gamma_7, \ldots, \gamma_{4n-1}] \quad \textit{where} \quad |\gamma_i| = i.$$

Proof. Proceed as above, inductively, applying the Serre spectral sequence to the fibration $\mathrm{Sp}(n-1) \hookrightarrow \mathrm{Sp}(n) \to S^{4n-1}$. □

Proposition 5.22. *The mod 2 cohomology of* $\mathrm{SO}(n)$ *is* ***additively*** *given by,*

$$H^*(\mathrm{SO}(n); \mathbb{Z}_2) = \Lambda_{\mathbb{Z}_2}[\delta_1, \delta_2, \ldots, \delta_{n-1}] \quad \textit{where} \quad |\alpha_i| = i.$$

Proof. We work inductively. The case $n = 2$ is clear. Suppose we know the cohomology of $\mathrm{SO}(n-1)$. Consider the fibration $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \to S^{n-1}$. Our spectral sequence looks like the following, where we let $\delta_{n-1}$ denote the generator of $H^{n-1}(S^{n-1}; \mathbb{Z}_2)$.




Here the pink squares are all zero. The differentials acting on $\delta_i$ are all necessarily zero, except maybe on $\delta_{n-2}$. We need to check the transgression $\mathrm{d}_{n-1}^{0,n-1}$

sends $\delta_{n-2}$ to zero. Consider the following morphism of fibrations given on total spaces by sending a matrix to its first two columns,

$$\begin{tikzcd}
S^{n-2} \arrow[r, hook] & V(n,2) \arrow[r] & S^{n-1} \\
\mathrm{SO}(n-1) \arrow[r, hook] \arrow[u] & \mathrm{SO}(n) \arrow[r] \arrow[u] & S^{n-1}. \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

This induces a corresponding morphism in the other direction of cohomological Serre spectral sequences. On the square $E_2^{0,n-2}$, this must send the fundamental class of $S^{n-2}$ to $\delta_{n-2}$ (since we inductively know the fibration $\mathrm{SO}(n-1) \to S^{n-2}$ induces an injection in cohomology). So it suffices to show $\mathrm{d}_{n-1}^{0,n-2}$ is zero (mod 2) in the spectral sequence of $V(n,2) \to S^{n-1}$. We know from the Gysin sequence that this differential takes the fundamental class of $S^{n-2}$ to the Euler class of the sphere bundle $V(n,2) \to S^{n-1}$. Note this bundle $V(n,2) \to S^{n-1}$ is actually the unit tangent bundle of $S^{n-1}$ (consider the tangent bundle as a sub-bundle of $T\mathbb{R}^n$ under $S^{n-1} \hookrightarrow \mathbb{R}^n$ using the standard metric). Hence,

$$\mathrm{d}_{n-1}^{0,n-2}[S^{n-2}] = \chi(S^{n-1})[S^{n-1}].$$

But spheres all have even Euler characteristic, and so the differential is zero mod 2. □

**5.5. The Atiyah–Hirzebruch Spectral Sequence.** We briefly[1] describe a generalization of the Serre spectral sequence to generalized homology, amely the Atiyah–Hirzebruch spectral sequence.

*K-theory.* To apply the Atiyah–Hirzebruch spectral sequence, we should know a little more about generalized cohomology. Recall a generalized cohomology theory is a contravariant functor from spaces to abelian groups satisfying all the Eilenberg–Steenrod axioms except possibly the dimension axiom. We know that singular homology and stable (co)homotopy are two such theories. We introduce a couple more, beginning with K-theory.

Let $X$ be a space and consider the commutative monoid of equivalence classes of finite dimensional vector bundles on $X$ with addition given by direct sum. There is a classic construction to turn a commutative monoid into an abelian group, called its *Grothendieck group*. What we do is simply consider formal differences $a - b$ of elements of the monoid with the obvious addition and the equivalence $c - c = 0$. In our case, we want to study *virtual vector bundles* $(V_+, V_-)$ on $X$ with the equivalence $(V, V) \sim (0, 0)$. The Grothendieck group of vector bundles on $X$ is called the *K-theory* of $X$. If we are considering

[1] Whoops.

complex vector bundles, this is denoted $K^0(X)$. For real vector bundles, we write $KO^0(X)$ and for quaternionic vector bundles we write $KSp^0(X)$.

It turns out, for suitably nice spaces, that for every vector bundle $V$, there is a bundle $W$ so that $V \oplus W$ is trivial. This implies that all the elements of $K^0(X)$ have the form $(V, \underline{\mathbb{C}}^n)$, where $\underline{\mathbb{C}}^n$ is a trivial bundle, and this form is unique up to stabilization by direct summing both with another trivial bundle (there are analogous descriptions for real and quaternionic cases).

As we will see in the next chapter, there are classifying spaces $BO(n)$, $BU(n)$, and $BSp(n)$ so that $n$-dimensional real, complex, or quaternionic vector bundles on a finite CW complex $X$ are in bijection with homotopy classes of maps from $X$ to the corresponding classifying space. These spaces can be described as the Grassmannian of $n$-dimensional subspaces of infinite dimensional real, complex, or quaternionic space.

Taking a colimit of the inclusions $\mathrm{U}(1) \hookrightarrow \mathrm{U}(2) \hookrightarrow \cdots$ gives a space $\mathrm{U}$ with a classifying space $BU$. Homotopy classes of maps $[X, BU]$ classify stable isomorphism classes of complex vector bundles on $X$. Similarly to define $BO$ and $BSp$. Our $K$-theory group records pairs $(V, n)$ where $V$ is an stable isomorphism class of bundle on $X$ and $n$ is an integer, the dimension of our virtual vector bundle. Thus we have the following description of our $K$-theory groups for suitably nice spaces $X$,

$$\begin{aligned} K^0(X) &= [X_+, BU \times \mathbb{Z}], \\ KO^0(X) &= [X_+, BO \times \mathbb{Z}], \\ KSp^0(X) &= [X_+, BSp \times \mathbb{Z}]. \end{aligned}$$

Here $X_+$ denotes $X$ with an adjoined basepoint, which we need to do in keeping with the convention that $[-,-]$ denotes based homotopy classes.

It turns out these define the zeroth groups of generalized cohomology theories. Using the suspension axiom for cohomology and the loop-suspension adjunction, we are forced into the following definition for the higher $K$ groups of $X$.

**Definition 5.23: $K$-theory**

The *complex $K$-theory* of a space $X$ is the graded group,

$$K^{-i}(X) := [X_+, \Omega^i(BU \times \mathbb{Z})].$$

Similarly, the *real $K$-theory* of $X$ is,

$$KO^{-i}(X) := [X_+, \Omega^i(BO \times \mathbb{Z})],$$

and the *quaternionic $K$-theory* is,

$$KSp^{-i}(X) := [X_+, \Omega^i(BSp \times \mathbb{Z})].$$

This only defines the negative graded part of $K$-theory but we will shortly see from Bott periodicity that this extends to all $i \in \mathbb{Z}$.

These have a group structure coming from the fact loop spaces are $H$-spaces (we'll see in a second $BU, BO, BSp$ are themselves loop spaces too). In fact in the case of $K$ and $KO$, these generalized cohomology theories have a graded multiplication, like the cup product in usual cohomology, given by taking the tensor product of vector bundles. Formally, this means some $K$-theories are ring spectra.

Because $K$-theory is a generalized cohomology theory, the $K$-theory of a point should be non-trival. We see that, $K^{-i}(\mathrm{pt})$ is the group of homotopy classes of maps from a point to $\Omega^i(BU \times \mathbb{Z})$, or equivalently, $\pi_i(BU \times \mathbb{Z})$. Analogously for real and quaternionic theories. We see immediately that $K^0(\mathrm{pt}) = \mathbb{Z}$; this is just the fact that virtual vector spaces (i.e. virtual bundles over a point) are classified by virtual dimension. The same holds in the other cases.

PROPOSITION 5.24. *If $G$ is a topological group, then $G$ is weak homotopy equivalent to $\Omega BG$.*

PROOF. As we will explain in the next chapter, the *classifying space $BG$* is defined in general as the base of a fibration $EG \to BG$ with fibre $G$ and a contractible total space $EG$ called the *universal bundle*. Consider the path space fibration $\Omega BG \hookrightarrow EBG \to BG$. Since $EBG$ is contractible, we may also think of this as the universal bundle of $\Omega BG$. Let $\mathcal{E}$ be the pullback by $EBG \to BG$ of the universal bundle; by universal properties it is also the pullback by $EG \to BG$ of the universal bundle of $\Omega BG$. We have the following commutative diagram.

$$\begin{tikzcd}
 & G \arrow[d, hook] & G \arrow[d, hook] \\
\Omega BG \arrow[r, hook] & \mathcal{E} \arrow[r] \arrow[d] & EG \sim \mathrm{pt} \arrow[d] \\
\Omega BG \arrow[r, hook] & EBG \sim \mathrm{pt} \arrow[r] & BG.
\end{tikzcd}$$

By the long exact sequence of a fibration applied to the horizontal and vertical pullbacks, we see that $\mathcal{E}$ is weak homotopy equivalent to $G$ and to $\Omega BG$. By transitivity, $G \cong_{WHE} \Omega BG$. $\square$

Since $\Omega BG \times \mathbb{Z} = \Omega BG$, we conclude that for $i \geq 1$ the $K$-theories are given by homotopy classes of maps into iterated loop spaces of $U, O$, and $Sp$. In particular for $i \geq 1$,

$$K^{-i}(\mathrm{pt}) = \pi_{i-1}(U), \quad KO^{-i}(\mathrm{pt}) = \pi_{i-1}(O), \quad \text{and} \quad KSp^{-i}(\mathrm{pt}) = \pi_{i-1}(Sp).$$

One of the magical results of algebraic topology is that these groups are periodic. Moreover, iterated loopings of the groups $U, O, Sp$ return back to the group themselves. This implies the $K$-theory of all spaces is periodic. We will not prove this wonderful result; the classical methods are by Morse theory although there are many other approaches.

**Theorem 5.25: Bott Periodicity**

Up to weak homotopy equivalence,

$$\Omega^2 U \cong U, \quad \Omega^8 O \cong O, \quad \text{and} \quad \Omega^8 Sp \cong Sp.$$

Explicitly, there are periodic sequences of loop spaces and their corresponding periodic $\pi$ groups, as depicted in Tables 4 and 5. In particular,the complex $K$-theory of any space is 2-periodic, while the real and quaternionic $K$-theories are 8-periodic. Further, quaternionic $K$-theory is the same as real $K$-theory up to a shift in degree by 4.

| $i \pmod 2$ | 0 | 1 |
|---|---|---|
| $\Omega^i U$ | $U$ | $BU \times \mathbb{Z}$ |
| $K^{i+1}(\mathrm{pt}) = \pi_i(U)$ | 0 | $\mathbb{Z}$ |

TABLE 4. Bott Periodicity for $U$

| $i \pmod 8$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\Omega^i O$ | $O$ | $O/U$ | $U/Sp$ | $BSp \times \mathbb{Z}$ | $Sp$ | $Sp/U$ | $U/O$ | $BO \times \mathbb{Z}$ |
| $KO^{-i+1}(\mathrm{pt}) = \pi_i(O)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |

TABLE 5. Bott Periodicity for $O$ and $Sp$

As rings, these have the following descriptions [S],

$$\begin{aligned} K^*(\mathrm{pt}) &= \mathbb{Z}[\nu, \nu^{-1}], \\ KO^*(\mathrm{pt}) &= \mathbb{Z}[\eta, \alpha, \beta, \beta^{-1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta) \\ KSp^*(\mathrm{pt}) &= KO^*(\mathrm{pt}) \cdot \theta, \end{aligned}$$

where $|\nu| = -2$, $|\eta| = -1$, $|\alpha| = -4$, $|\beta| = -8$ and $|\theta| = -4$. Here, $\nu \in \pi_2(BU)$ represents the tautological complex line bundle on $\mathbb{C}P^1$, $\eta \in \pi_1(BO)$ represents the tautological line bundle on $\mathbb{R}P^1$, $\theta \in \pi_4(BSp)$ is the tautological quaternionic line bundle on $\mathbb{H}P^1$ with $\alpha \in \pi_4(BO)$ the underlying real bundle, and $\beta \in \pi_8(BO)$ is the tautological octonionic line bundle on $\mathbb{O}P^1 = S^8$.

Example 5.26. Let us find the complex $K$-theory of a sphere. We have,

$$K^0(S^n) = [S^n_+, BU\times\mathbb{Z}] = \mathbb{Z}\oplus\pi_n(BU) = \mathbb{Z}\oplus\pi_{n-1}(U) = \begin{cases} \mathbb{Z} & n \cong 1 \pmod 2 \\ \mathbb{Z}^2 & n \cong 0 \pmod 2. \end{cases}$$

And,

$$K^1(S^n) = [S^n_+, U] = \mathbb{Z} \oplus \pi_n(U) = \begin{cases} \mathbb{Z}^2 & n \cong 1 \pmod 2 \\ \mathbb{Z} & n \cong 0 \pmod 2. \end{cases}$$

In the case of $n = 1$, we will see in the next chapter that all complex line bundles on $S^1$ are trivial and hence $K^0(S^1) = \mathbb{Z}$ just tracks the dimension of a trivial bundle.

In the case $n = 2$, the complex line bundles on $S^2$ are tensor products of copies of the tautological/Hopf line bundle $\xi$ and its dual. Since $c_1(\xi^{\otimes n}) = nc_1(\xi)$ are all different for different $n$, the integers index a family of stably distinct bundle. Since $K^0(S^2) = \mathbb{Z}^2$, we conclude the stable classes of bundles on $S^2$ are given by $n\underline{\mathbb{C}} \oplus \xi^{\otimes m}$ for $m, n \in \mathbb{Z}$. It turns out that $(\xi \otimes \xi) \oplus \underline{\mathbb{C}} \cong \xi \oplus \xi$. Hence the $K$-theory of $S^2$ may be described as a ring,

$$K^0(S^2) \cong \mathbb{Z}[\xi]/(\xi - 1)^2.$$

▲

*Bordism Theory.* Now we move on to a brief description of bordism. We begin with some important intuition. In homotopy theory, we understand a space $X$ in terms of its holes; we consider all the ways to map spheres $S^n$ into $X$ up to identifying maps $S^n \to X$ connected by a map of cylinders $S^n \times I \to X$. This may seem awfully restrictive in what types of holes we can capture. Why don't we consider maps of arbitrary $n$-manifolds $M$ into $X$? Because we now are dealing with many distinct spaces mapping into $X$ we should define identify maps not just through cylinders but through arbitrary $(n+1)$-manifolds connecting maps of $n$-manifolds. This is essentially the content of bordism theory (although we are often interested in adding extra structure to these manifolds). As an extra note, one could see homology theory as taking this one step further: we consider maps not just from manifolds into $X$ but from arbitrary "closed" CW complexes (i.e. cycles) into $X$ up to boundaries (this analogy is not quite right because there is the extra structure of a chain complex in the homology case).

Consider a space $X$ and "singular closed $n$-manifolds" in $X$, i.e. continuous maps $f : M^n \to X$ from a closed smooth $n$-manifold $M$ to $X$. Two such singular $n$-manifolds $f : M \to X$, $g : N \to X$ are called *bordant* (or *cobordant*) if there is a compact $(n+1)$-manifold $W$ and a continuous map $F : W \to X$ so that $\partial W = M \sqcup N$ and $F|_{\partial W} = f \sqcup g$. In this case, we call $W$ a *bordism* (or *cobordism*) between $f$ and $g$. If $f : M \to X$ has a bordism to the empty map $\varnothing \to X$ (i.e. there is $g : W \to X$ with $M = \partial W$ and $g|_{\partial W} = f$) we say $f$ is *null-bordant*.

**Definition 5.27: Bordism Groups**

Given a space $X$, its *nth bordism group* is the set,

$$\Omega_n^O(X) := \{\text{singular } n\text{-manifolds in } X\}/\{\text{null-bordant elements}\},$$

equipped with the group operation of disjoint union. That is, we identify bordant singular $n$-manifolds.

If we require that our singular $n$-manifolds be oriented and that our bordisms are oriented compatibly (so that $\partial W = M \sqcup -N$), then we obtain the *nth oriented bordism group*,

$$\Omega_n^{SO}(X) := \{\text{oriented singular } n\text{-manifolds}\}/\{\text{oriented bordism}\}.$$

A *stable almost complex structure* on a smooth manifold $M$ is a complex vector bundle structure on the normal bundle $\nu$ of some embedding $M \hookrightarrow \mathbb{R}^N$. We can study singular $n$-manifolds up to bordisms $\partial W = M \sqcup N$ with stable almost complex structures inducing[a] the given stable almost complex structures on $M$ and on $-N$:

$$\Omega_n^U(X) := \{\text{stable a.c. singular } n\text{-manifolds}\}/\{\text{stable a.c. bordism}\}.$$

There are several more versions of this construction, $\Omega_n^G(X)$ given by asking the stable normal bundles to have a $G$-structure. For example, we could take $G = \mathrm{Spin}, \mathrm{SU}, \mathrm{Sp}$, etc.. More exotically, we could consider $\Omega_n^{\mathrm{fr}}(X)$, cobordisms of manifolds with stable normal framings (i.e. stable trivializations of the normal bundle). Also there are $\Omega_n^{\mathrm{PL}}(X), \Omega_n^{\mathrm{Top}}(X)$, where we extend from cobordisms of smooth to topological or piecewise linear manifolds.

[a] Given a stable almost complex structure on a bordism $W$ coming from an embedding $i : W \to \mathbb{R}^n$ with a normal bundle $\nu_W$ with a complex structure, it will restrict to a bundle $\nu_M \oplus \mathbb{R}$ on a boundary component $M$ with a complex structure. Composing $i|_M$ with the standard embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$ gives an embedding of $M$ with normal bundle $\nu_M \oplus \mathbb{R}$, which we know has a complex structure and so gives a stable almost complex structure on $M$.

These all define generalized homology theories; the corresponding cohomology theories are called the *cobordism groups*. Since this is a generalized theory, the homology of a point is non-trivial. The graded groups $\Omega_*^G := \Omega_*^G(\mathrm{pt})$ are called the *cobordism rings*. These are the groups of all closed smooth $n$-manifolds up to abstract cobordism (with the appropriate stable $G$-structure). They have a graded ring structure coming from taking the Cartesian product of spaces.

When $X$ is a manifold, these bordism groups also possess a multiplication, which can be described as follows. Suppose $f : M \to X$ and $g : N \to X$ are two singular manifolds of dimension $n$ and $m$ and $X$ is a closed manifold of dimension $k$. We can consider the product, $f \times g : M \times N \to X \times X$. Let $X_\Delta \subset X \times X$ be the diagonal. We can homotope $f \times g$ so that $P = (f \times g)^{-1}(X_\Delta)$ is a smooth submanifold of $M \times N$ of dimension $n + m - k$. Thus we get a singular manifold $(f \times g)|_P : P \to X$ which up to cobordism is independent of our choice of homotopy and hence defines a multiplication $\Omega_n^G(X) \times \Omega_m^G(X) \to \Omega_{n+m-k}^G(X)$.

Note that every element of $\Omega_*^O$ is 2-torsion. This is because a cylinder $M \times I$ defines a null bordism of $M \sqcup M$ for any closed manifold $M$. The same is not true in $\Omega_*^{SO}$ or $\Omega_*^U$ because $M \times I$ defines the wrong orientation on one of the boundary components, instead this shows us that $-M$ is the additive inverse to $M$ in the bordism group.

Let us compute these groups $\Omega_n^G$ for $G = O, SO$ in low dimensions $n = 0, 1, 2$. Zero dimensional manifolds are points and bordisms between 0-manifolds are line segments. An orientation on a point is given by a choice of sign. We conclude $\Omega_0^O = \mathbb{Z}_2$ (by counting points mod 2) and $\Omega_0^{SO} = \mathbb{Z}$ (by counting signed points).

One dimensional closed manifolds are all circles. These are always null-bordant via the disk. Hence $\Omega_0^O = \Omega_0^{SO} = 0$.

In the two dimensional case, oriented two manifolds are surfaces $\Sigma_g$, which are all null-bordant via handlebodies. Hence $\Omega_2^{SO} = 0$.

To find $\Omega_2^O = 0$ we thus need only study the non-orientable surfaces, which are connected sums of projective planes. The projective plane $\mathbb{R}P^2$ is not null-bordant. This can be seen from the fact $\chi(\mathbb{R}P^2) = 1$ and the following proposition.

Proposition 5.28. *Any closed null-bordant manifold has even Euler characteristic.*

Proof. If $M$ is odd dimensional, by Poincaré duality it has even Euler characteristic. If $M = \partial N$ is even dimensional, then $X = N \sqcup_M N$ has Euler characteristic $\chi(X) = 2\chi(N) - \chi(M)$. But $X$ is closed and odd dimensional, hence has even Euler characteristic, which implies $M$ does too. $\square$

The Klein bottle $K = \mathbb{R}P^2 \# \mathbb{R}P^2$ is null-bordant. To see this, note the Klein bottle can be formed by gluing the ends of a cylinder with opposite orientations. A null-bordism is just given by filling in the cylinder: see figure 5 where the pink Klein bottle is filled in by the green interior.

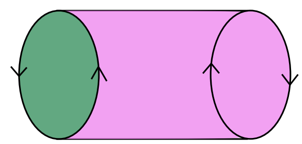


FIGURE 5. Null Bordism of Klein Bottle

Given compact manifolds with boundary $M, N$, we can define a boundary connect sum $M \#_\partial N$ by cutting out small hemispheres in boundary charts of $M$ and $N$, with the flat faces of the hemispheres sitting in the boundaries of $M$ and $N$, and gluing together along a cylinder with boundary gluing the boundaries together. It is clear that $\partial(M \#_\partial N) = \partial M \# \partial N$. In this way, we see that we can boundary connect sum Klein bottles to show that connect sums of Klein bottles are null-bordant. This shows that connect sums of even numbers of copies of $\mathbb{R}P^2$ are null-bordant, and it remains to deal with the odd number cases. For that, we have the following proposition which generalizes the results we just stated on Klein bottles.

PROPOSITION 5.29. *The group operation of disjoint union in the cobordism ring is equivalent to the operation of connect sum. That is, given closed manifolds $M$ and $N$, the connect sum $M \# N$ is cobordant to $M \sqcup N$.*

PROOF. Take a look at the picture proof in figure 6. $\square$

From this we conclude that $\#^{2n+1}\mathbb{R}P^2 = (\#^n T^2) \# \mathbb{R}P^2 = \Sigma_n \# \mathbb{R}P^2$ is bordant to $\mathbb{R}P^2$, since $\Sigma_n$ is null-bordant. Thus, every surface is either null-bordant or bordant to $\mathbb{R}P^2$ and we conclude $\Omega_2^O = \mathbb{Z}_2$.

Analyzing the case $n = 3$ is possible using handle decompositions and one finds $\Omega_3^{SO} = 0$. For $n \geq 4$ the classification of manifolds is too intractable to

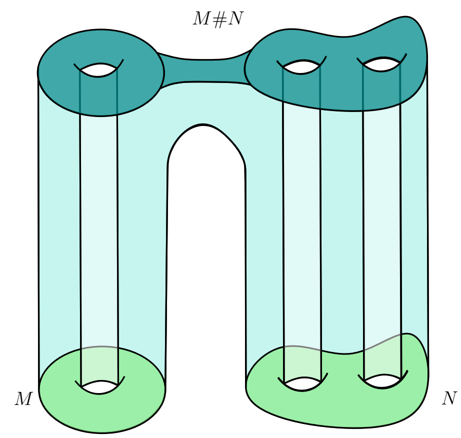



FIGURE 6. A bordism between $M \sqcup N$ and $M \# N$

directly compute bordism groups. It turns out that,

$$
\begin{aligned}
\Omega_4^{SO} &= \mathbb{Z} \text{ generated by } \mathbb{C}P^2, \\
\Omega_5^{SO} &= \mathbb{Z}_2 \text{ generated by } SU(3)/SO(3), \\
\Omega_6^{SO} &= 0, \\
\Omega_7^{SO} &= 0, \\
\Omega_8^{SO} &= \mathbb{Z}^2 \text{ generated by } \mathbb{C}P^4 \text{ and } \mathbb{C}P^2 \times \mathbb{C}P^2.
\end{aligned}
$$

All higher oriented bordism groups turn out to be non-zero. In spite of the geometric complexity, the methods of characteristic classes, stable homotopy, and spectra allow for computations of the bordism rings, as first undertaken by Thom [**Thom**].

**Theorem 5.30: (Dold, Milnor, Novikov, Quillen, Thom, Wall)**

The unoriented bordism ring is,

$$\Omega_*^O \cong \mathbb{Z}_2[x_i : 1 \leq i \neq 2^k - 1] \cong \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \dots],$$

where $|x_i| = i$. For even $i$ we may take $x_i$ to be the cobordism class of $\mathbb{R}P^i$. For odd $i$ we may take $x_i$ to be the cobordism class of $P(2^r-1, s2^r)$ where $i = 2^r(2s+1)$ and $P(m,n)$ is the *Dold manifold* defined as the quotient of $S^m \times \mathbb{C}P^n$ under a $\mathbb{Z}_2$ action given by swapping antipodes on $S^m$ and complex conjugation on $\mathbb{C}P^n$.

Rationally, the oriented and complex bordism rings are,

$$\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[y_2, y_4, y_6 \cdots]$$
$$\Omega_*^{U} \otimes \mathbb{Q} = \mathbb{Q}[y_1, y_2, y_3, \cdots],$$

where $|y_i| = 2i$ and we may take $y_i$ to be the cobordism class of $\mathbb{C}P^i$ in both cases. These complex projective spaces do not generate integrally, although it turns out that,

$$\Omega_*^U = \mathbb{Z}[a_1, a_2, a_3, \ldots] \quad \text{where} \quad |a_i| = 2i,$$

and modulo 2-torsion, $\Omega_*^{SO}$ is integrally generated by $b_i$ of degree $4i$.

Remark 5.31. There is a known family of manifolds called the *Milnor hypersurfaces* $H_{ij}$ for $i \le j$ which multiplicatively generate $\Omega_*^U$ over $\mathbb{Z}$. They are defined as subvarieties of $\mathbb{C}P^i \times \mathbb{C}P^j$ given by,

$$H_{ij} = \{([z_0 : \cdots : z_i], [w_0 : \cdots : w_j]) \in \mathbb{C}P^i \times \mathbb{C}P^j |\ z_0w_0 + \cdots + z_iw_i = 0\}$$

These are $\mathbb{C}P^{j-1}$-bundles over $\mathbb{C}P^i$. However, there are algebraic relations in the bordism ring between these surfaces so that they are not a free set of generators; there is not a known nice general description for the $a_i$ which form a multiplicative basis.

*The Atiyah–Hirzebruch Spectral Sequence.* Now we can study the Atiyah–Hirzebruch spectral sequence for the computation of generalized homology. We will prove its construction and give a few simple applications.

Recall back to our construction of the Serre spectral sequence that given a homologically simple Serre fibration $p : E \to B$ over a CW base, there was a filtration of the chain complex of $E$ by singular chains contained in the preimages of successive skeleta of $B$, $E_p = p^{-1}(B_p)$. This gave the zeroth page of a spectral sequence: $E_0^{p,q} = C_{p+q}(E_p, E_{p-1})$. The first page of this spectral sequence then had entries $E_1^{p,q} = H_{p+q}(E_p, E_{p-1})$, which we were able to show agreed with $\mathcal{C}_p(B; H_q(f))$.

Now suppose $h_*$ is a generalized homology theory. The first page of a spectral sequence still makes sense,

$$E^1_{p,q} = h_{p+q}(E_p, E_{p-1}).$$

We follow the argument we used for the Serre case, but diverge to avoid the use of Künneth (which does not hold for generalized homology). By excision and Feldbau's lemma,

$$h_{p+q}(E_p, E_{p-1}) = \bigoplus h_{p+q}(F \times D^p_\alpha, F \times \partial D^p_\alpha),$$

where $F$ is the fibre, and $D^p_\alpha$ are small balls contained in the $p$-cell $e_\alpha$ of $B$. The definition of reduced homology and the relation between relative homology and reduced homology of a cofibre still make sense for a generalized theory,

$$= \bigoplus_\alpha \widetilde{h}_{p+q}(F \times D^p / F \times \partial D^p).$$

With some thought, one sees this quotient as repeated suspensions,

$$= \bigoplus_\alpha \widetilde{h}_{p+q}(\Sigma^p F).$$

It follows from excision that the suspension isomorphism extends to generalized theories,

$$\begin{aligned} &= \bigoplus_\alpha h_q(F) \\ &= \mathcal{C}_p(B; h_q(F)). \end{aligned}$$

As before, the differential will be the usual cellular differential and hence we obtain,

$$E^2_{p,q} = H_p(B; h_q(F)).$$

Going to the $E^\infty$ page, we should obtain the (associated graded) generalized homology of the total space $E$. The question of convergence is a bit subtle, as $h_*(F)$ and $H_*(B)$ may both be non-trivial in infinitely many degrees. But if $B$ is a finite CW then we are definitely okay. Of course an analogous idea holds in cohomology, and the same proof as for Serre gives a multiplication whenever $h^*$ is multiplicative (i.e. represented by a ring spectrum).

**Theorem 5.32: Atiyah–Hirzebuch Spectral Sequence**

Let $p : E \to B$ be a homologically simple Serre fibration with a finite CW base and a fibre $F$. Let $h_*$ be an extraordinary homology theory. Then there is a spectral sequence,

$$E^2_{p,q} = H_p(B; h_q(F)) \implies h_{p+q}(E).$$

And analogously in cohomology,

$$E_2^{p,q} = H^p(B; h^q(F)) \implies h^{p+q}(E).$$

In particular, considering the fibration $* \hookrightarrow X \xrightarrow{\text{id}} X$, one has a spectral sequence,

$$E^2_{p,q} = H^p(X; h^q(*)) \implies h^{p+q}(X).$$

If $h^*$ is a multiplicative cohomology theory, then the cohomological spectral sequences possess a multiplication with the usual properties.

Remark 5.33. The general spectral sequence is sometimes called the Atiyah–Hirzebruch–Serre spectral sequence, with the name Atiyah–Hirzebruch spectral sequence being reserved for the special case of a trivial fibration. The case of a trivial fibration is particularly useful because if we know the cohomology of a space and the stable homotopy groups $h^*(*)$ of a generalized cohomology theory (like in the case of $K$-theory) then we can directly compute the generalized cohomology using our spectral sequence. △

Let us explore a couple properties of this sequence and apply it to a few examples, beginning with $K$-theory.

In the case of complex $K$ theory of a space $X$, remembering $K^0(*) = \mathbb{Z}$ and $K^1(*) = 0$, the Atiyah–Hirzebruch sequence yields the following $E_2$ page.

| | | | | | |
|---|---|---|---|---|---|
| 6 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | | | | | |
| 4 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | | | | | |
| 2 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | | | | | |
| 0 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | | | | | |
| $-2$ | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | 0 | | | | 4 |

Clearly the differentials on this page are trivial, and so the first possible action happens on the $E_3$ page.

One can also construct the Atiyah–Hirzebruch spectral sequence using ideas related to Postnikov towers and it turns out the first differential comes from a Postnikov $k$-invariant. In particular, the first differential in an Atiyah–Hirzebruch spectral sequence must be a stable cohomology operation.

Remark 5.34. One interesting corollary of this is the following. It is not hard to show that every stable cohomology operation over $\mathbb{Q}$ is trivial (i.e. the cohomology $H^*(K(\mathbb{Q}, n); \mathbb{Q})$ is concentrated in degree $n$). Hence over $\mathbb{Q}$ there can be no non-trivial first differentials. Hence we have,

$$h^q(X) \otimes \mathbb{Q} = \bigoplus_{r+s=q} H^r(X; \mathbb{Q}) \otimes h^s,$$

for any homology theory $h$.

For the case of complex $K$-theory, we are looking for an operation $H^*(X;\mathbb{Z}) \to H^{*+3}(X;\mathbb{Z})$. It turns out there is only one such non-trivial operation, and one can contruct an example where this differential is non trivial. Hence we have the following.

Proposition 5.35. *The first non-trivial differential in the complex $K$-theory Atiyah–Hirzebruch spectral sequence is,*

$$\mathrm{d}_3 = \beta \circ Sq^2 \circ r : H^*(X;\mathbb{Z}) \to H^{*+3}(X;\mathbb{Z}),$$

*where $\beta : H^*(X;\mathbb{Z}_2) \to H^{*+1}(X;\mathbb{Z})$ is the Bockstein operation coming from $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2$ and $r : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}_2)$ is reduction mod 2.*

In the case of real $K$-theory, the $E_2$ page looks as follows.

| | | | | | |
|---|---|---|---|---|---|
| 8 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | $H^0(X;\mathbb{Z}_2)$ | $H^1(X;\mathbb{Z}_2)$ | $H^2(X;\mathbb{Z}_2)$ | $H^3(X;\mathbb{Z}_2)$ | $H^4(X;\mathbb{Z}_2)$ |
| 6 | $H^0(X;\mathbb{Z}_2)$ | $H^1(X;\mathbb{Z}_2)$ | $H^2(X;\mathbb{Z}_2)$ | $H^3(X;\mathbb{Z}_2)$ | $H^4(X;\mathbb{Z}_2)$ |
| | | | | | |
| 4 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | | | | | |
| 2 | | | | | |
| | | | | | |
| 0 | $H^0(X)$ | $H^1(X)$ | $H^2(X)$ | $H^3(X)$ | $H^4(X)$ |
| | $H^0(X;\mathbb{Z}_2)$ | $H^1(X;\mathbb{Z}_2)$ | $H^2(X;\mathbb{Z}_2)$ | $H^3(X;\mathbb{Z}_2)$ | $H^4(X;\mathbb{Z}_2)$ |
| $-2$ | $H^0(X;\mathbb{Z}_2)$ | $H^1(X;\mathbb{Z}_2)$ | $H^2(X;\mathbb{Z}_2)$ | $H^3(X;\mathbb{Z}_2)$ | $H^4(X;\mathbb{Z}_2)$ |
| | 0 | | | | 4 |

In this case, there can be differentials on the $E_2$ page leaving from the $8k$th and $(8k-1)$th rows and they are given by,

$$\begin{aligned} \mathrm{d}_2^{*,8k} &= Sq^2 \circ r : E_2^{*,8k} &&= H^*(X;\mathbb{Z}) \to H^{*+2}(X;\mathbb{Z}_2) = E_2^{*+2,8k-1} \\ \mathrm{d}_2^{*,8k-1} &= Sq^2 : E_2^{*,8k-1} &&= H^*(X;\mathbb{Z}_2) \to H^{*+2}(X;\mathbb{Z}_2) = E_2^{*+2,8k-2}. \end{aligned}$$

On $E_3$, the first non-trivial differential leaving the $(8k+6)$th rows is,

$$\mathrm{d}_3^{*,8k+6} = \beta \circ Sq^2 : E_3^{*,8k+6} = H^*(X;\mathbb{Z}_2) \to H^{*+3}(X;\mathbb{Z}) = E_3^{*+3,8k+4}.$$

And finally on $E_5$ the differential leaving the $(8k+4)$th rows is,

$$\mathrm{d}_5^{*,8k+4} = \beta \circ Sq^4 \circ r : E_5^{*,8k+4} = H^*(X;\mathbb{Z}) \to H^{*+5}(X;\mathbb{Z}) = E_5^{*+5,8k}.$$

Example 5.36. Note since every non-trivial differential of Atiyah–Hirzebruch for complex $K$-theory is on an odd page, the differentials always move an odd horizontal distance. Hence if the cohomology is concentrated in odd or even degrees, the sequence has to degenerate on the $E_2$ page.

For example, we know the cohomology of $\mathbb{C}P^n$ and the complex Grassmannians is contained in purely in even grades. For the case of $\mathbb{C}P^n$, we have,

$$E_2^{p,q} = E_\infty^{p,q} = \begin{cases} \mathbb{Z} & p, q \text{ even and } 0 \leq p \leq 2n, \\ 0 & \text{otherwise.} \end{cases}$$

We have the associated graded to $K^i(\mathbb{C}P^n)$ is $\bigoplus_p E_\infty^{p,i-p}$ which is $\mathbb{Z}^{n+1}$ if $i$ is even and zero otherwise. Recall that a short exact sequence of abelian groups with final group $\mathbb{Z}$ always splits (i.e. $\mathbb{Z}$ is projective). Hence there are no extension problems and the associated graded is isomorphic to the actual group:

$$K^0(\mathbb{C}P^n) = \mathbb{Z}^{n+1} \quad \text{and} \quad K^1(\mathbb{C}P^n) = 0.$$

Actually since nothing happened in this sequence, and everything was free, this isomorphism is multiplicative,

$$K^*(\mathbb{C}P^n) = \mathbb{Z}[\nu, \nu^{-1}, x]/(x^{n+1}), \quad \text{where} \quad |\nu| = |x| = 1.$$

The element $\nu$ is the complex Hopf line bundle generating $K^*(\mathrm{pt})$ and $x$ is the usual generator of $H^2(\mathbb{C}P^n)$. So we see that $K^0(\mathbb{C}P^n)$ has an explicit basis given by $x^i\nu^{-i}$ for $0 \leq i \leq n$. We already computed the $K$ theory of $S^2 = \mathbb{C}P^1$ in example 5.26. By the naturality of the embedding $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$, we conclude that $x\nu^{-1}$ can be represented by $(1-\xi)$, where $\xi$ is the tautological line bundle on $\mathbb{C}P^n$. Hence,

$$K^0(\mathbb{C}P^n) = \mathbb{Z}[\xi]/(1-\xi)^{n+1}.$$

Since the first non-trivial differential in our sequence was on the $E_3$ page, another setting in which everything immediately degenerates is if the homology is concentrated in degrees $0, 1$, and $2$. For example consider the genus $g$ orientable surface $\Sigma_g$. We see that,

$$E_2^{p,q} = E_\infty^{p,q} = \begin{cases} \mathbb{Z} & q \text{ even and } p = 0, 2, \\ \mathbb{Z}^{2g} & q \text{ even, } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The associated graded to $K^i(\Sigma_{2g})$ is $\mathbb{Z}^{2g+2}$ when $i$ is even and zero otherwise. Again everything is free and so this gives the $K$-theory:

$$K^0(\Sigma_g) = \mathbb{Z}^{2g+2} \quad \text{and} \quad K^1(\Sigma_g) = 0.$$

Other examples can be done similarly.

EXERCISE 5.37. Compute the complex $K$-theory of $\mathbb{R}P^n$ and of the complex Grassmannians $\mathbb{C}G(n, k)$. ▲

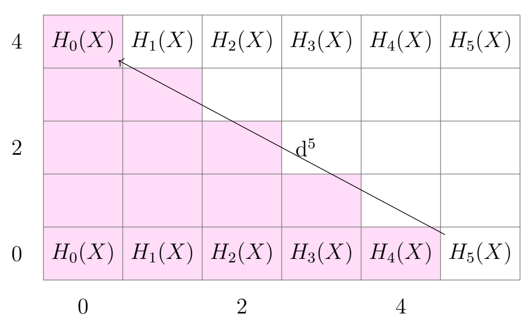
In the case of complex bordism $\Omega_*^U$, the groups for a point are restricted to even dimensions like with complex $K$-theory. We conclude that for spaces with only non-trivial even homology groups the Atiyah–Hirzebruch spectral sequence degenerates on the $E^2$ page. Hence we find for example, that additively,

$$\Omega_*^U(\mathbb{C}P^n) = \Omega_*^U(\mathrm{pt})[x]/(x^{n+1}), \quad \text{where} \quad |x| = 1.$$

If we switch from bordism to cobordism, we can make this isomorphism multiplicative.

Remark 5.38. This is really expressing that $KU$ and $\Omega_U$ are *complex oriented cohomology theories*. These are multiplicative generalized cohomology theories $E^*$ so that the map $i^* : E^2(\mathbb{C}P^\infty) \to E^2(S^2)$ induced by inclusion is surjective, along with a choice of $t \in \widetilde{E}^2(\mathbb{C}P^\infty)$ so that $i^*(t) \in \widetilde{E}^2(S^2) = E^0(*)$ is the unit element (this is something like a first Chern class). Such generalized cohomology theories are well studied and form the basis for chromatic homotopy theory. △

For oriented bordism, we know the groups in low dimensions are $\Omega_i^{SO}(*) = 0$ for $i = 1, 2, 3$ and $\mathbb{Z}$ for $i = 0, 4$. So for a space $X$, the beginning of the $E^2$ page of the homological Atiyah–Hirzebruch spectral sequence looks like the following.




Recall from our discussion of edge morphisms in the Serre spectral sequence (and it extends easily to Atiyah–Hirzebruch), that the map $\Omega_n(\mathrm{pt}) \to \Omega_n(X)$ should factor through $E^\infty_{n,0}$ by taking the quotient from $E^2$ to $E^\infty$ and then including $E^\infty_{n,0}$ into $\bigoplus_{p+q=n} E^\infty_{p,q}$. But we know that $\Omega_n(\mathrm{pt}) \to \Omega_n(X)$ must be an injection, because there is a right inverse induced from the constant map $X \to \mathrm{pt}$. Hence the quotient map $E^2_{n,0} \to E^\infty_{n,0}$ must be the identity, and so the maps $\mathrm{d}^r : E^r_{r,n-r+1} \to E^r_{0,n}$ in the spectral sequence have to be trivial. In particular, the transgressive map $\mathrm{d}^5$ shown above must be trivial. Hence the contents of the pink squares never participate in any non-trivial differentials and persist to $E^\infty$.

This tells us the zeroth through third oriented bordism groups of $X$. Assume $X$ is connected so that $H_0(X) = \mathbb{Z}$. To determine $\Omega_4^{SO}(X)$, we need to solve the extension problem,

$$0 \to \mathbb{Z} \to \Omega_4^{SO}(X) \to H_4(X) \to 0.$$

As just discussed, the first map can be identified via the edge morphism with $\Omega_4^{SO}(\mathrm{pt}) \to \Omega_4^{SO}(X)$. But we know this map has a right inverse, and hence by the splitting lemma the above short exact sequence splits. Actually, the inverse $\sigma : \Omega_4^{SO}(X) \to \mathbb{Z}$ can be taken to be the signature map, which sends $f : M \to X$ to the signature $\sigma(M)$ (the signature of the quadratic intersection form $H_2(M) \times H_2(M) \to \mathbb{Z}$); Thom showed this is a cobordism invariant.

Hence we conclude,

$$\Omega_i^{SO}(X) = H_i(X) \quad \text{for} \quad i = 0, 1, 2, 3 \quad \text{and} \quad \Omega_4^{SO}(X) = \mathbb{Z} \oplus H_4(X).$$

We could try to extend this analysis further, since we know some higher bordism groups of a point, but we would have to deal with non-trivial differentials.

This has beautiful applications to the problem of geometric realization, also known as *Steenrod's problem*, which asks if homology classes of a general space can be represented by the fundamental class of singular manifolds. These results are due to Thom [**Thom**].

One can show that the edge morphism $\Omega_n^{SO}(X) \to H_n(X)$ is the map sending a singular manifold $f : M \to X$ to the pushforward homology class $f_*([M]) \in H_n(X)$. Since cobordant singular manifolds define homologous cycles, this is well defined. Since this edge map is a surjection for $n \leq 4$, we conclude that any homology class in $H_n(X)$ is represented by a manifold. For $n > 4$, there may be non-trivial differentials leaving the square $E_{n,0}^*$ and so we cannot say if the morphism $\Omega_n^{SO}(X) \to H_n(X)$ will continue to be surjective. In fact it will be surjective as long as $n \leq 6$. For $n \geq 7$ surjectivity fails, for example Thom discovered a non-geometrically realizable class in $L^7(3) \times L^7(3)$, where $L^7(3)$ is the lens space $S^7/\mathbb{Z}_3$.

However notice, as we remarked earlier, if we work over $\mathbb{Q}$, the Atiyah–Hirzeburch spectral sequence has to immediately degenerate. Hence our rational edge morphism is always surjective, which has the following amazing upshot.

Proposition 5.39 (Thom). *Given a space $X$, and an integral homology class $\alpha \in H_n(X)$. There is an integer $k$ so that $k\alpha$ is represented by $f_*([M])$ for some oriented singular manifold $f : M \to X$ of dimension $n$.*

In fact this can be strengthened slightly. It happens that the spectral sequence immediately degenerates modulo odd torsion (for the same reason as unoriented bordism below). The proof can be found in [**CF**, §14] or [**R**, IV §7]. This implies that the coefficient $n$ in the preceding proposition may always be taken to be odd.

If we instead consider unoriented bordism, it can be shown that the unoriented cobordism spectrum is just a direct sum of Eilenberg–MacLane spectra. Since the Atiyah–Hirzerbuch spectral sequence for ordinary homology always immediately degenerates, the same must be true for unoriented cobordism. Since $\mathbb{Z}_2$ is a vector space, there can be no extension problems and so one has,

$$\Omega_n^O(X) = \bigoplus_{p+q=n} H_p(X; \Omega_q^O(\mathrm{pt})).$$

This gives a complete description of oriented bordism groups. It also means the edge morphism we just considered is always surjective for $\Omega_n^O$. This of course has the following corollary.

Proposition 5.40 (Thom). *Given a space $X$, and a mod 2 homology class $\alpha \in H_n(X; \mathbb{Z}_2)$, $\alpha$ is represented by $f_*([M])$ for some singular manifold $f : M \to X$ of dimension $n$.*

One final possible application for the Atiyah–Hirzebruch spectral sequence is to stable homotopy. Suppose there were a space $X$ whose homology groups and stable homotopy groups we knew or were not too hard to determine. Then we could apply Atiyah–Hirzebruch:

$$H_p(X; \pi_q^{\mathrm{st}}) \implies \pi_{p+q}^{\mathrm{st}}(X)$$

to possibly determine the stable homotopy groups $\pi_q^{\mathrm{st}}$ of the sphere.

This is actually a promising approach. Kochman is able to compute the first 64 stable homotopy groups of the sphere using this idea and some computer assistance [**K**]. The space $X$ he uses is actually not a space but a spectrum, namely the Brown–Peterson spectrum $BP$. We will later see that complex cobordism $\Omega_U^*$ is represented by the complex Thom spectra $MU$. It happens that $MU$ when localized at a prime $p$ becomes homotopy equivalent to a wedge sum of suspensions of spectra; these wedge summands are precisely the BP. It can be computed that the homotopy of $BP$ are $\mathbb{Z}_p$ polynomial algebras and the homology can be computed explicitly as a module over the Steenrod algebra. This makes these spaces perfect for the above application.

## 6. Characteristic Classes

### 6.1. Vector Bundles and Classifying Spaces.

**Definition 6.1: Principal $G$-Bundles and Vector Bundles**

A locally trivial fibration (we will call this a fibre bundle) $p : E \to B$ is a *principal $G$-bundle* for a topological group $G$ if there is a free right $G$ action on $E$ and $B = E/G$. That is, $G$ acts freely and transitively on the fibres of $p$.

A *rank $n$ vector bundle* is a fibre bundle $p : V \to B$ with fibre $\mathbb{R}^n$ so that the bundle has a covering of local trivializations so that overlapping trivializations differ by continuously varying linear isomorphisms of $\mathbb{R}^n$. Similarly for complex or quaternionic vector bundles but with $\mathbb{C}^n$ or $\mathbb{H}^n$ instead of $\mathbb{R}^n$.

A principal $G$-bundle can be expressed by local data as follows. Suppose $B$ is covered by open sets $U_\alpha$ over which the bundle trivializes as $p : U_\alpha \times G \to U_\alpha$ (we will see taking the $U_\alpha$ contractible suffices). Given two overlapping trivializations $U_\alpha, U_\beta$, there should be a transition map $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ so that if $(x, g)$ and $(x, h)$ represent $\widetilde{x} \in E$ in the two trivializations, then $\varphi_{\alpha\beta}(x) = g^{-1}h$. For these transition functions to define a prinicipal $G$-bundle, it is necessary and sufficient that they satisfy the following cocyle conditions,

$$\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1} \quad \text{and} \quad \varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = \mathrm{id}.$$

Vector bundles can be similarly defined in terms of transition functions $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^n)$.

Remark 6.2. Note that the fibres of $G$-bundles are $G$ torsors, they are homeomorphic to $G$ with an action of $G$ acting like left multiplication but there is no preferred identity element. One way to think of this condition is like a non-linear version of an affine vector space. Complementarily, the fibres of a vector space have a preferred zero element, but there is no way to define an addition action on the vector space. One way to think of this is like an abstract vector space with no reference to a standard basis or an addition.

If we give either bundle its missing structure: i.e. give the fibres of the principal $G$-bundle an identity element or the fibres of a vector bundle an addition action by $\mathbb{R}^n$, then they become trivial. $\triangle$

**Definition 6.3**

A morphism between principal $G$-bundles $P, Q$ over $B$ is a $G$-equivariant map $P \to Q$. A morphism between vector bundles $E, E'$ over $B$ is a map commuting with the projection to the base which is linear on the fibres in a local trivialization.

Proposition 6.4. *Principal $G$-bundles have the following properties.*

*(i) A morphism of principal $G$-bundles is an isomorphism.*

*(ii) A principal $G$-bundle with a section is trivial.*

*(iii) Every principal $G$-bundle and every vector bundle over a contractible paracompact base is trivial.*

Proof. (i) Consider a morphism $\sigma : P \to Q$. If $P$ and $Q$ are trivially equal to $B \times G$, then the morphism has the form $\sigma(b, g) = (b, gf(b))$ for a map $f : B \to G$. But then right multiplication by $f(b)^{-1}$ gives an inverse to $\sigma$. If $P, Q$ non-trivial, we know they are locally trivial and hence $\sigma$ is still an isomorphism since it is locally.

(ii) Given a section $s : B \to P$ of a principal $G$-bundle $P \to B$, there is a morphism $\varphi : B \times G \to P$ given by $\varphi(b, g) = gs(b)$. By (i) this gives an isomorphism of $P$ to a trivial bundle.

(iii) By Feldbau's lemma, a principal $G$-bundle $P \to B$ with $B$ contractible is trivial as a fibration. We need to show this trivialization is $G$-equivariant. Subdivide $B$ into contractible pieces over which $P$ trivializes. On each such piece $P$ possesses a section. Inductively, we can extend our section to each new local trivialization since each additional trivializable neighbourhood is contractible (we need paracompactness for this induction). Thus we obtain a section on all of $B$ and hence $P$ is trivial. For the vector bundle case, look at the associated frame bundle (defined shortly) which must be trivial. $\square$

We now discuss a couple way to get new bundles from old ones .

**Definition 6.5: Pullback Bundles**

Let $p : E \to B$ be a principal $G$-bundle or a vector bundle (or any fibre bundle). And $f : B' \to B$ be a continuous map. The *pullback bundle* $f^*E$ is defined to make the following diagram a pullback square in the categorical sense,

$$\begin{array}{ccc} f^*E & \xrightarrow{\widetilde{f}} & E \\ \big\downarrow{\scriptstyle f^*p} & & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

In terms of universal properties, $f^*E$ is defined so that any bundle $F \to B''$ with a bundle map $F \to E$ that factors as $B'' \to B' \xrightarrow{f} B$ on the base, must factor through $\widetilde{F}$ on the total space. Explicitly, the pullback is given as $f^*E = \{(b', e)|p(e) = f(b')\} \subset B' \times E$. This bundle will be a principal $G$-bundle or a vector bundle when $p$ is.

Recall a $G$-space $F$ is a topological space with a continuous left $G$-action.

**Definition 6.6: Associated Bundle**

If $F$ is a $G$-space and $p : E \to B$ is a principal $G$-bundle, we can form the *associated bundle* $E \times_G F$ defined as the quotient of $E \times F$ by the $G$-action $g \cdot (e, f) = (xg^{-1}, gf)$. Categorically, this is defined to make the following diagram a pushout square,

$$\begin{array}{ccc} E \times_G F & \longleftarrow & E \times F \\ \big\uparrow & & \big\uparrow{\scriptstyle \rho_F} \\ E \times F & \xleftarrow[\rho_E]{} & E \times G \times F \end{array}$$

Here the maps $\rho_{E,F}$ denote the action of $G$ on $E, F$ with the identity on the other factor, and the two maps $E \times F \to E \times_G F$ are the quotient map. Note $E \times_G F$ will be a fibre bundle with fibre $F$. Furthermore it will be a bundle with structure group $G$, meaning the transition maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$ factor through the $G$-action $G \to \mathrm{Homeo}(G)$.

Conversely, suppose $X \to B$ is an $F$-bundle with structure group $G$. Then we obtain a principal $G$-bundle using the transition data $U_\alpha \cap U_\beta \to G$ to determine the local data of the $G$ bundle. If the $G$-action on $F$ is faithful, i.e. $G \to \mathrm{Homeo}(F)$ has no kernel, then this will be inverse to the associated bundle construction.

In our settings of interest $F$ will be a vector space with a faithful $G$-representation. Then the associated bundle $E \times_G F$ will be a vector bundle. Conversely, given a vector bundle $V$ with a faithful $G$-representation on the fibre vector space, we will get a principal $G$-bundle, which we call the *associated frame bundle*.

Because the associated constructions are inverse to each other, classifying vector bundles with a given faithful $G$-representation is the same as classifying principal $G$-bundles. We will see some examples shortly.

Proposition 6.7. *Sections of an associated bundle $E \times_G F$ are the same thing as $G$-equivariant maps $E \to F$.*

Proof. A good exercise. $\square$

Consider a bundle $E \to B$ with structure group $G$. Suppose $G \subset H$. It is always possible to extend our bundle to have structure group $H$ just by considering the transition functions as living in $H$. If $H \subset G$, then the question of *reduction of the structure group* to $H$ is more subtle.

More generally, suppose we have some homomorphism $\varphi : H \to G$. We can ask whether the bundle $E \to B$ with structure group $G$ reduces[2] to an $H$-structure so that the transition maps factor through $\varphi$. In terms of associated principal bundles, a principal $G$-bundle $P$ has a reduction to a principal $H$-bundle $Q$ if there is an (iso)morphism, $Q \times_H G \to P$, where we use $\varphi$ to think of $G$ as an $H$-space.

If $H \subset G$, then the reducibility of structure group of $P$ from $G$ to $H$ is the same as the fibre bundle $P/H$ being trivial. And a choice of reduction is a choice of trivialization, or equivalently a choice of section. This is because a section $\sigma : X \to P/H$ gives a pullback bundle $\sigma^*(P \to P/H)$ which is a reduction of $P$ to structure group $H$. Conversely, a choice $Q \times_H G \cong P$ gives,

$$P/H \cong P \times_G (G/H) \cong Q \times_H G \times_G (G/H) \cong Q \times_H (G/H) \cong B \times G/H.$$

Example 6.8. Let us see some examples of the relationship between vector bundles and their associated frame bundles.

(i) Consider a rank $n$ real vector bundle $E$. Using its standard representation, this has structure group $GL(\mathbb{R}^n)$. It has an associated frame bundle $F_E$ which is a principal $GL(\mathbb{R}^n)$-bundle. Explicitly, the fibre of $F_E$ over $x$ consists of linear isomorphisms of $E_x$. Hence classifying real vector bundles on $X$ is the same as classifying principal $GL(\mathbb{R}^n)$-bundles on $X$. The same analysis works for complex and quaternionic bundles with $GL(\mathbb{C}^n)$ and $GL(\mathbb{H}^n)$.

(ii) Consider $O(n)$ with its standard representation. A rank $n$ vector bundle with structure group $O(n)$ is the same thing as a vector bundle

[2] If $\varphi$ is a surjection rather than injection this is sometimes called a lift instead of reduction of structure group.

with a Riemannian metric, since the fact transition maps are isometries means we can extend a metric across trivializations. The associated frame bundle has fibres consisting of orthogonal transformations of the fibre.

Any $GL(\mathbb{R}^n)$ bundle has a reduction to an $O(n)$ bundle. We can see this in two ways. Geometrically, assuming $X$ is paracompact, a partition of unity argument show a vector bundle on $E$ always inherits a metric. Topologically, $O(n)$ is a deformation retract of $GL(\mathbb{R}^n)$ by Gram–Schmidt; polar decomposition for matrices implies $GL(\mathbb{R}^n)/O(n)$ is the space of positive definite matrices, hence a trivialization is a choice of continuously varying positive definite matrix (this is just a non-degenerate bilinear form). Since $GL(\mathbb{R}^n)/O(n)$ is contractible, a section of $P/O(n)$, i.e. a metric, always exists. This means classifying real vector bundles is the same as classifying principal $O(n)$-bundles.

Analogously, having structure group $U(n)$ or $Sp(n)$ means the bundle is complex with a Hermitian metric or quaternionic with a quaternionic Hermitian metric respectively. By the same argument as above, $GL(\mathbb{C}^n)$ bundles and $GL(\mathbb{H}^n)$ bundles always have a reduction to $U(n)$ and $Sp(n)$ bundles. Hence classifying complex (resp. quaternionic) vector bundles is the same as classifying principal $U(n)$-bundles (resp. $Sp(n)$-bundles).

(iii) Given a real vector bundle, we can ask if it has a reduction of structure group to $SL(\mathbb{R}^n)$. Or equivalently by above a reduction to $SO(n)$. This is the same as the bundle being oriented, and a choice of trivialization is an orientation. To see this note if $F$ is the associated $O(n)$ frame bundle, then $F/SO(N)$ is a $\mathbb{Z}_2$-bundle measuring the sign of the determinant of a choice of orthonormal basis for each fibre of the bundle. A section of this quotient bundle must be constant, i.e. we need a consistent orientation for the fibres of our space. So classifying oriented vector bundles is the same as classifying principal $SO(n)$-bundles. Note $U(n) \subset SO(2n)$, hence every complex vector bundles has a relaxation of its structure group to an underlying oriented real bundle.

The reduction $SU(n) \subset U(n)$ does not have as obvious a geometric interpretation. One way to express it is that a complex vector bundle $E$ reduces to structure group $SU(n)$ if and only if its complex determinant line bundle $\Lambda^{\mathrm{top}}_{\mathbb{C}} E$ is trivial, and a choice of reduction is a choice of trivialization. This is analogous to the fact a real orientation is a trivialization of the real determinant line bundle. Note $Sp(n) \subset SU(n)$ so every quaternionic vector bundle relaxes to a complex vector with a trivial determinant bundle, and even more generally to an oriented real bundle.

(iv) Consider an oriented real bundle $E$ with structure group $SO(n)$. Recall that for $n \geq 3$, $SO(n)$ had a simply connected double cover $Spin(n)$. We can ask about a lift of the structure group via the covering $Spin(n) \to SO(n)$. If such a lift of the associated frame bundle exists, we say $E$ is *spin* and a choice of lift is a *spin structure*. If the tangent bundle of an oriented manifold $M$ is spin, we say $M$ is spin. Suppose $P$ is a principal $Spin(n)$-bundle representing a spin structure on some vector bundle or manifold. Given some representation of $Spin(n)$ on a vector space $V$ (there are several such representations we might care about), the associated bundle $P \times_{Spin(n)} V$ is called a *spinor bundle*. These are of much interest in mathematical physics because sections of spinor bundles correspond to classical fields representing fermionic particles.

(v) The group $Spin(n)$ has a central extension to the group $Spin^{\mathbf{c}}(n) = Spin(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$, which can be thought of as a complex Spin group. There is an obvious map $Spin^{\mathbf{c}}(n) \to SO(n)$. If the associated frame bundle of an oriented vector bundle $E$ admits a lift with respect to this map, a choice of such lift is called a *spin*$^{\mathbf{c}}$ *structure* on $E$. These are mostly of interest because they allow for many of the same constructions as spin structures, and many more spaces admit spin$^{\mathbf{c}}$ structures on their tangent bundle than spin structures.

▲

Associated to any topological group $G$, there is an important space that will help us classify principal $G$-bundles and hence vector bundles with structure group $G$.

**Definition 6.9: Classifying Space**

Given a topological group $G$, the *classifying space* $BG$ of $G$ is the space uniquely defined up to weak homotopy equivalence, so that there is a weakly contractible space $EG$ and a principal $G$-bundle $EG \to BG$ called the *universal bundle* of $G$.

The fact that classifying spaces are unique will follow shortly from our central theorem. But we can prove they always exist in a fairly explicit way.

**Theorem 6.10: Milnor's Construction**

For any topological group $G$, there is a classifying space $BG$.

Proof. Take $EG$ to be the infinite join of copies of $G$ (i.e. the colimit of the join of $n$ copies of $G$),

$$EG = G * G * G * \cdots .$$

A more intuitive way of writing this is that $EG$ consists of infinite vectors $(t_1 g_1, t_2 g_2, \ldots)$ where $t_i \in [0,1]$, $g_i \in G$, only finitely many of the $t_i$ are non-zero, and the $t_i$ sum to one (note we are also identifying entries $0g_i = 0\widetilde{g_i}$). The group $G$ has a continuous free action on $EG$ by right multiplication on each entry of our infinite vector. Set $BG = EG/G$. By construction, this is a principal $G$-bundle (in fact it is trivial, since picking the representative $(t_1 e, t_2 g_2, \cdots)$ defines a section).

Note $EG$ is contractible. Given a spheroid $S^n \to EG$, it must land in some finite join $G * \cdots * G$. But then it must be contract inside the next join since $(A, 0)$ is always contractible inside $A * B$. $\square$

If $EG \to BG$ is a universal bundle and $H \subset G$ is a subgroup, then we can consider first the quotient of $EG$ by $H$ and then by $G/H$:

$$\begin{tikzcd}
EG \arrow[dd, "G"'] \arrow[dr, "H"] & \\
& EG/H \arrow[dl, "G/H"] \\
BG &
\end{tikzcd}$$

We conclude that $BH = EG/H$ and we obtain a fibration,

$$G/H \hookrightarrow BH \to BG.$$

It is also clear, using products of universal bundles that,

$$B(G_1 \times G_2) = BG_1 \times BG_2.$$

Example 6.11. We list several classifying spaces of interest.

(1) If $G = \mathbb{Z}_2$, then $S^\infty \to \mathbb{R}P^\infty$ is a principal $\mathbb{Z}_2$ bundle with contractible total space. So $B\mathbb{Z}_2 = \mathbb{R}P^\infty$.

(2) If $G = \mathbb{Z}$, then $\exp(2\pi i -) : \mathbb{R} \to S^1$ is a principal $\mathbb{Z}$-bundle. So $B\mathbb{Z} = S^1$.

(3) More generally, whenever $G$ is discrete $BG = K(G, 1)$.

(4) For $G = O(n)$, the universal bundle is $V(\infty, n) \to G(\infty, n)$. This bundle is given by sending an orthonormal $n$-frame in $\mathbb{R}^\infty$ to the subspace it spans. Note that $V(\infty, 1) = S^\infty$ is contractible, and the fibration,
$$V(\infty, n-1) \hookrightarrow V(\infty, n) \to S^\infty,$$
shows inductively by the homotopy long exact sequence that $V(\infty, n)$ is always weakly contractible (and contractible by Whitehead). Hence $BO(n) = G(\infty, n)$ and $EO(n) = V(\infty, n)$.

For analogous reasons, $BU(n) = \mathbb{C}G(\infty, n)$ and $BSp(n) = \mathbb{H}G(\infty, n)$, and $EU(n) = \mathbb{C}V(\infty, n)$ and $ESp(n) = \mathbb{H}V(\infty, n)$. Of particular interest is that $BU(1) = \mathbb{C}P^\infty$. Using the fact about classifying spaces of products, $BT^n = (\mathbb{C}P^\infty)^n$.

(5) For $G = SO(n)$, note this is a subgroup of $O(n)$. Hence,
$$BSO(n) = V(\infty, n)/SO(n) = G_+(\infty, n).$$
We also obtain a fibration $O(n)/SO(n) \to G_+(\infty, n) \to G(\infty, n)$ which is just the usual double cover. Analogously one has $BSU(n) = \mathbb{C}V(\infty, n)/SU(n)$.

Consider a discrete group $G$ and a nice enough space $X$, say a CW complex. A path connected principal $G$-bundle $P$ on $X$ is a connected covering space of $X$. The $G$-action on the fibres are the deck transformations of the covering and the fact the $G$-action is transitive on fibres, means that $P$ is a normal covering. Conversely, any normal covering of $X$ with deck transformation group $G$ is a connected principal $G$-bundle on $X$ and an isomorphism of normal coverings gives an isomorphism of bundles and vice versa. Since the covering is normal, isomorphism classes of coverings of $X$ with deck transformation group $G$ are in bijection with groups $H$ so that $\pi_1(X)/G \cong H$. This is the same as an exact sequence,
$$0 \to H \to \pi_1(X) \to G \to 0.$$
Up to isomorphism class of $H$, it is enough that we just know the surjective map $\pi_1(X) \to G$. We conclude connected principal $G$-bundles are classified by surjective homomorphisms $\pi_1(X) \to G$.

If we generalize to consider principal $G$-bundles $P$ that are not connected, then we know each path component is a regular covering of $X$ for a subgroup $G' \subset G$. Moreover, the set of path components will have the structure of a $G/G'$ torsor. Thus, we find principal $G$-bundles are classified by arbitrary homomorphisms $\varphi : \pi_1(X) \to G$; the map $\pi_1(X) \to \mathrm{im}(\varphi)$ determines a connected bundle and the inclusion $\mathrm{im}(\varphi) \to G$ describes the torsor structure of the set of path components.

To reiterate, we have found for $G$ discrete,

$$\{\text{Principal } G\text{-bundles on } X\}/\sim \quad \longleftrightarrow \quad \mathrm{Hom}(\pi_1(X), G).$$

We can make this characterization purely in terms of $X$ instead of its fundamental group. Using the representability of cohomology and the universal coefficient theorem (these will both work fine for $G$ non-abelian in rank one) along with Hurewicz,

$$[X, K(G,1)] = H^1(X; G) = \mathrm{Hom}(H_1(X), G) = \mathrm{Hom}(\pi_1(X), G).$$

Since $K(G,1) = BG$, we conclude,

$$\{\text{Principal } G\text{-bundles on } X\}/\sim \quad \xrightarrow{\cong} \quad [X, BG].$$

We can actually make this isomorphic explicit. We see that tracing through the definitions, a principal bundle $P$ on $X$ is determined from the map $f : X \to BG$ by pulling back the universal bundle: $P = f^*EG$.

This wonderful result, that principal $G$ bundles for $G$ discrete are represented by maps to $BG$, generalizes verbatim to arbitrary topological groups. The classifying spaces $BG$ are so named because they classify isomorphism classes of bundles. That is, $BG$ is a representing object for the functor,

$$\mathrm{PBund}_G : \mathbf{CWhTop}^{\mathrm{op}} \to \mathbf{Set},$$

taking a CW complex $X$ to the set of isomorphism classes of principal $G$-bundles on $X$.

**Theorem 6.12: The Classification of Principal Bundles**

Let $\mathrm{PBund}_G(X)$ denote the equivalence classes of principal $G$-bundles on a CW complex $X$. Then there is a correspondence,

$$\mathrm{PBund}_G(X) \cong [X, BG],$$

given by sending $f : X \to BG$ to the pullback bundle $f^*EG$.

LEMMA 6.13. *Suppose $p : E \to B$ is a principal $G$-bundle and $B'$ is a CW complex. If two maps $g, h : B' \to B$ are homotopic, then the pullback bundles $g^*E$ and $h^*E$ are equivalent.*

PROOF. Let $F : B' \times [0,1] \to B$ be a homotopy between $g$ and $h$. There is a pullback bundle $\mathcal{E} = F^*E$ defined over $B' \times [0,1]$. It suffices to show

$\mathcal{E}$ is equivalent to $h^*E \times [0,1]$, or equivalently to construct a morphism $\mathcal{E} \to h^*E \times [0,1]$. This is the same thing as finding a section of $\mathcal{E} \times_G (h^*E \times [0,1]) \to B' \times [0,1]$. On the zero skeleton of $B$, this bundle is defined over a $[0,1]$, hence it is trivial and must possess a section. Now inductively, since this bundle is in particular a Serre fibration, we can extend this section over increasing skeleta of $B'$ to obtain a section and hence a morphism $\mathcal{E} \to h^*E \times [0,1]$. $\square$

PROOF. (of Theorem 6.12) The above lemma shows the correspondence in the theorem statement is a well defined map from $[X, BG]$ to $\mathrm{PBund}_G(X)$. We wish to show it is a bijection.

We begin with surjectivity. Given a principal $G$-bundle $P \to B$, consider the associated bundle $P \times_G EG \to B$. The fibre of this bundle is $EG$, which is weakly contractible, and hence this bundle has a section (by obstruction theory). Such a section defines a $G$-equivariant map $P \to EG$. Since this is $G$-equivariant, we may pass to the quotient spaces to obtain a map $f : B \to BG$. By universal properties, $f^*EG = P$.

Now we show injectivity. Suppose $f_0, f_1 : B \to BG$ are two maps so that $f_0^*EG$ and $f_1^*EG$ are isomorphic. Let $P = (f_0^*EG) \times [0,1] \to B \times [0,1]$. There is an associated bundle $Q = P \times_G EG \to B \times [0,1]$. The $G$-equivariant bundle maps $f_0^*EG \to EG$ and $f_0^*EG \xrightarrow{\cong} f_1^*EG \to EG$ define two sections of $(f_0^*EG) \times_G EG \to B$ and hence a section of $Q$ over $B \times \{0\} \cup B \times \{1\}$. Since $Q$ has weakly contractible fibre, this section extends to one over all of $B \times [0,1]$ (by obstruction theory). This then gives a $G$-equivariant bundle map $P \to EG$. We may pass to the quotient to obtain a map $B \times [0,1] \to BG$ which is a homotopy between $f_0$ and $f_1$. $\square$

COROLLARY 6.14. *The space $BG$ is unique up to weak homotopy equivalence.*

PROOF. Suppose we have another model for a universal bundle $\widetilde{EG} \to \widetilde{BG}$. By Theorem 6.12, there is a map $f : \widetilde{BG} \to BG$ so that $f^*EG = \widetilde{EG}$. Let $F : \widetilde{EG} \to EG$ be the induced map on total spaces. We thus have a morphism of fibrations,

$$\begin{tikzcd}
G \arrow[r, hook] & EG \arrow[r] & BG \\
G \arrow[u, "\mathrm{id}"] \arrow[r, hook] & \widetilde{EG} \arrow[u, "F"] \arrow[r] & \widetilde{BG}. \arrow[u, "f"]
\end{tikzcd}$$

Note that $\mathrm{id} : G \to G$ clearly induces an isomorphism on homotopy groups. And $F$ does as well since $EG, \widetilde{EG}$ are contractible. But then by the five lemma applied to the homotopy long exact sequences of these fibration, $f$ must also induce isomorphisms of homotopy groups. Hence $f$ is a weak homotopy equivalence of $\widetilde{BG}$ and $BG$. □

One can also show on general abstract nonsense grounds that any space representing the functor $\mathrm{PBund}_G$ must be weak homotopy equivalent to $BG$, and hence is the base of a $G$-bundle with contractible total space.

EXAMPLE 6.15. This is a very powerful result and we can make preliminary classifications of some low dimensional bundles using it.

(1) Consider the set of rank one real vector bundles on $X$. By above, this is the same as classifying principal $O(1) = \mathbb{Z}_2$ bundles on $X$. Note $B\mathbb{Z}_2 = \mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$. So,

$$\mathrm{PBund}_{O(1)}(X) \cong [X, B\mathbb{Z}_2] \cong [X, \mathbb{R}P^\infty] \cong H^1(X; \mathbb{Z}_2).$$

So real line bundles are classified by elements of $H^1(X; \mathbb{Z}_2)$. We will see shortly that the cohomology class corresponding to a line bundle $E$ under this classification is called the first Stiefel-Whitney class $w_1(E)$.

In the case of $X = S^1$, $H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$. The two isomorphism classes of line bundles on $S^1$ are the trivial bundle and the Möbius strip (which can't be trivial since it is non-orientable). By naturality, we conclude that for any connected CW complex $X$ and a line bundle $E \to X$, the classifying cohomology class $w_1(E)$ evaluated on a 1-cell $e_1$ of $X$ gives 0 or $1 \in \mathbb{Z}_2$ if $E$ is the trivial bundle or the Möbius strip respectively when pulled back to $e_1 \sqcup e_0$.

(2) Now consider rank one complex vector bundles on $X$ or equivalently principal $U(1)$-bundles. Note $BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$. So,

$$\mathrm{PBund}_{U(1)}(X) \cong [X, BU(1)] \cong [X, \mathbb{C}P^\infty] \cong H^2(X; \mathbb{Z}).$$

Thus complex line bundles are classified by elements of $H^2(X; \mathbb{Z})$. The cohomology class corresponding to a complex line bundle $\mathcal{E}$ under this classification is called the first Chern class $c_1(\mathcal{E})$.

For $X = S^2$, $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. These bundles are classified by their degree: $S^2 = \mathbb{C}P^1$ has tautological line bundle $\mathcal{O}(-1)$, its dual $\mathcal{O}(1)$, and a $\mathbb{Z}$'s worth of bundles $\mathcal{O}(\pm k) = \mathcal{O}(\pm 1)^{\otimes k}$.

(3)

EXERCISE 6.16. Classify principal $\mathrm{SL}(2, \mathbb{C})$-bundles on $\mathbb{C}P^2$. ▲

Suppose one has a map of topological groups $f : G \to H$. Consider the associated bundle $EG \times_G H$, where we let $G$ act on $H$ via $f$. This is a principal $H$-bundle over $BG$ and so by Theorem 6.12, it must be induced by a unique homotopy class of map $Bf : BG \to BH$. This map can also be recovered as follows. Given a principal $G$-bundle $P$ on $X$, we obtain an associated principal $H$-bundle $P \times_G H$ on $X$. This gives a map $\mathrm{PBund}_G(X) \to \mathrm{PBund}_H(X)$ for any $X$, i.e. a natural transformation $\mathrm{PBund}_G \to \mathrm{PBund}_H$. By the Yoneda lemma, this corresponds to a map $BG \to BH$ which must agree with $Bf$. It is clear this map is functorial in the appropriate senses. Hence the "classifying operation" $B(\cdot)$ defines a functor from topological groups **TopGp** to **hTop**.

EXAMPLE 6.17. Consider the usual short exact sequence of a subgroup $H \subset G \to G/H$. Functorially, we get a fibration,

$$BH \to BG \to B(G/H).$$

This fits in with a previously obtained map to give a fibration sequence,

$$G/H \to BH \to BG \to B(G/H).$$

We have proved that $\Omega B(G/H) \cong G/H$ and so this fibre sequence is nothing but a portion of the Puppe sequence for $BH \to BG$. △

**6.2. The Cohomology of Classifying Spaces.** We saw in Example 6.15 that we could classify line bundles using cohomology. While this is too strong to hope for in general, we may reasonably conclude that homotopy classes of maps $[X, BG]$ and hence $G$-bundles can be distinguished through invariants with values in cohomology classes. This is precisely the idea of characteristic classes.

**Definition 6.18: Characteristic Classes**

A *characteristic class* $c$ of $G$-bundles over a coefficient group $F$ is an assignment of a cohomology class $c(E) \in H^*(X; F)$ to every principal $G$-bundle $E$ on every space $X$ which is natural with respect to pullbacks. That is for $f : X \to Y$ and a bundle $E$ on $Y$,

$$c(f^*E) = f^*c(E).$$

More succinctly, a characteristic class is a natural transformation,

$$c : \mathrm{PBund_G} \to H^*(-; F).$$

We will usually consider characteristic classes for vector bundles which we take to be the characteristic class of their associated frame bundle.

Note that a characteristic class is uniquely determined by its value on the universal bundle $EG$, since every bundle is the pullback of $EG$ and using naturality. And moreover, any cohomology class of $BG$ defines a characteristic class $c(E)$ by looking at its pullback under the defining map $X \to BG$ of a bundle $E \to X$. Hence (this is just the Yoneda lemma) characteristic classes of $G$-bundles are the same thing as cohomology classes of $BG$.

So, to understand characteristic classes on real/complex/oriented vector bundles, we will want to understand the cohomology of $BO(n)$, $BU(n)$ and $BSO(n)$. We will obtain the cohomology of these spaces, and hence a list of important characteristic classes, from spectral sequences.

Remark 6.19. The preceding definition could be replaced with asking for a natural transformation $\mathrm{PBund_G} \to E^*$, where $E$ is some generalized cohomology theory. This will give us characteristic classes in a generalized cohomology theory and by Yoneda they are determined by $E^*(BG)$. These are of interest in many situations, particularly for $K$-theory. Also well studied are characteristic classes in the equivariant cohomology of an equivariant bundle. △

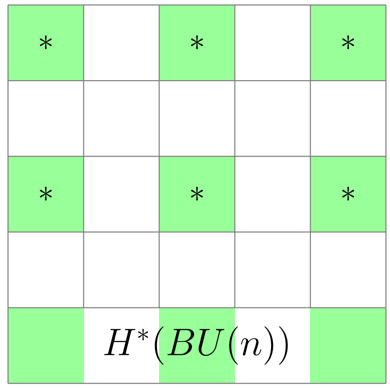
Also note that whenever we have a map $G \to H$, and hence a map $BG \to BH$, this gives a way to pullback characteristic classes of $H$-bundles to characteristic classes of $G$-bundle corresponding to a pullback of bundles. Hence whenever we have a geometric way to turn one bundle into another, we will get some relation between characteristic classes; we will use this idea several times.

Now we get to work computing the cohomology of our classifying spaces.

**Homology of $BU(n)$:** Note that the torus $T^n$ is a subgroup of $U(n)$ by considering unitary matrices which are diagonal. Thus we get a fibration,

$$U(n)/T^n \hookrightarrow BT^n \to BU(n).$$

Note that $U(n)/T^n$ is the manifold of complete flags in $\mathbb{C}^n$. This has a cell structure with only even dimensional cells by the standard Bruhat description. $BU(n)$ is the infinite dimensional complex Grassmannians and using the Schubert cell decomposition in the direct limit, again the cells are all even. Consider the associated cohomological spectral sequence on the $E_2$ page.




Note the only non-trivial entries are in the green squares, since the cellular chain groups and hence cohomology groups of $BU(n)$ and $U(n)/T^n$ are only non-trivial in even degrees. But then we see that all differentials are trivial and so $E_2 = E_\infty$ is the associated graded of the cohomology of $BT^n$. In particular, the natural edge morphism given by pullback under the fibration,

$$\pi^* : H^*(BU(n)) \to H^*(BT^n),$$

is an injection.

Note that $BT^n = (BS^1)^n = (\mathbb{C}P^\infty)^n$. Its cohomology is the free polynomial algebra generated by elements $x_1, \ldots, x_n$ each of degree two corresponding to each copy of $\mathbb{C}P^\infty$ in $(\mathbb{C}P^\infty)^n$. We claim that $\pi^*$ lands in the subalgebra of $\mathbb{Z}[x_1, \ldots, x_n]$ generated by symmetric polynomials. To see this let $\sigma \in S_n$ act on $(\mathbb{C}P^\infty)^n$ by permutation of factors. By naturality, the map $\pi : BT^n \to BU(n)$ must pull back $EU(n)$ to the $ET^n$. Let $\xi$ be the associated $\mathbb{C}^n$ bundle to $EU(n)$ and $\eta$ the complex line bundle associated to $S^\infty \to \mathbb{C}P^\infty$. We have a diagram of the above form,

$$\begin{tikzcd}
\bigoplus^n \eta = \pi^*\xi \arrow[r] \arrow[d] \arrow[dd, bend right] & \xi \arrow[d] \\
BT^n \arrow[r, "\pi"] & BU(n) \\
BT^n. \arrow[u, "\sigma"] \arrow[ur, "\pi"] &
\end{tikzcd}$$

This should commute up to homotopy, because the permutation $\sigma$ induces an isomorphism of the vector bundles $\pi^*\xi$ lying above (it just shuffles the copies of $\eta$). Thus given $\alpha \in H^*(BU(n))$, we have

$\pi^*\alpha = \sigma^*\pi^*\alpha$. Since clearly $\sigma^*(x_{\sigma(i)}) = x_i$, we conclude $\pi^*$ is an injection into the space of symmetric polynomials of the $x_i$. In fact it has to be an isomorphism onto this subalgebra. This just follows from cell counting. We know $\dim H^i(BU(n))$ is the number of Young Tableaux diagrams with $i$ squares in an "$\infty \times n$ grid". It is a basic combinatorially exercise to see this is also the dimension of degree $i$ symmetric polynomials in $n$ variables. Note the map $\pi^*$ is also a ring map. Thus we conclude as rings,

$$H^*(BU(n)) \cong \mathbb{Z}[\sigma_1(\vec{x}), \ldots, \sigma_n(\vec{x})] \quad \text{where} \quad |\sigma_i| = 2i.$$

Here $\sigma_i(\vec{x})$ is the $i$th elementary symmetric polynomial in $x_1, \ldots, x_n$. It is customary to write $c_i$ for $\sigma_i(\vec{x})$; this is called the *ith universal Chern class*. The corresponding characteristic class $c_i(E)$ for a complex vector bundle $E$ is called the *ith Chern class of E*. The elementary symmetric polynomials and thus the classes $c_i$ can be expressed in terms of the $x_i$ using the following simple formula,

$$(1 + x_1) \cdots (1 + x_n) = 1 + c_1 + \cdots + c_n := c \in H^*(BU(n)),$$

where $c_i$ is the homogeneous part of degree $2i$. This element $c$ in the cohomology ring (and its pullback via any classifying map) is called the *total Chern class*. By naturality, the inclusion $BU(n) \hookrightarrow BU(n+1)$ induces a pullback map which is the identity on the classes $c_i$ for $i \le n$ and sends $c_{n+1}$ to zero. In particular, the first Chern class $c_1$ can clearly be seen to agree with the characteristic class for complex line bundles we defined before under the same name.

**Homology of $BSp(n)$:** Exactly the same analysis goes through for the quaternionic case. Let $H^*(B(S^3)^n) = H^*((\mathbb{H}P^\infty)^n) = \mathbb{Z}[y_1, \ldots, y_n]$ where $|y_i| = 4$. We obtain an injection,

$$H^*(BSp(n)) \to H^*((\mathbb{H}P^\infty)^n),$$

which is an isomorphism onto symmetric polynomials of the $y_i$. So if we let,

$$(1 + y_1) \cdots (1 + y_n) = 1 + p_1 + \cdots + p_n,$$

where $p_i$ is the homogeneous part of degree $4i$, we find,

$$H^*(BSp(n)) \cong \mathbb{Z}[p_1, \ldots, p_n] \quad \text{where} \quad |p_i| = 4i.$$

The element $p_i$ is called the *ith universal Pontryagin class* and for a quaternionic vector bundle $E$ it corresponds to a characteristic class $p_i(E)$ called *ith Pontryagin class of E*.

**Homology of $BO(n)$ over $\mathbb{Z}_2$:** The integral homology of $O(n)$ is more complicated, but we can work modulo 2 and obtain a very similar analysis.

Let $H^*(B(\mathbb{Z}_2)^n;\mathbb{Z}_2) = H^*((\mathbb{R}P^\infty)^n) = \mathbb{Z}_2[t_1,\dots,t_i]$ where $|t_i| = 1$. We can consider the usual map induced by inclusion,

$$H^*(BO(n);\mathbb{Z}_2) \to H^*(B\mathbb{Z}_2^n;\mathbb{Z}_2).$$

We may consider the fibration $O(n)/\mathbb{Z}_2^n \to B\mathbb{Z}_2^n \to BO(n)$ and the associated spectral sequence (it is slightly subtle to show this fibration is homologically simple, but we will omit the details). Over $\mathbb{Z}_2$ we know the dimension of the homology of all these spaces (by cell counts) and we deduce that the spectral sequence must immediately degenerate (otherwise $H^*(B\mathbb{Z}_2^n;\mathbb{Z}_2)$ would not have enough generators). Hence the map above is an injection and by the same argument as above, using the commutative diagram and a cell count, it defines an isomorphism onto the subalgebra of symmetric polynomials.

So letting,

$$(1+t_1)\cdots(1+t_n) = 1 + w_1 + \cdots + w_n,$$

where $w_i$ is the homogeneous part of degree $i$, we find,

$$H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\dots,w_n] \quad \text{where} \quad |w_i| = i.$$

The element $w_i$ is called the *ith universal Stiefel–Whitney class* and for a real vector bundle $E$ it corresponds to a characteristic class $w_i(E)$ called *ith Stiefel–Whitney class of $E$*.

**Homology of $BSO(n)$ over $\mathbb{Q}$:** We need a certain lemma first. Recall a largest abelian subgroup of a compact connected Lie group $G$ is necessarily isomorphic to $T^n$ for some $n$ and is called the *maximal torus*. There is then an exact sequence,

$$1 \to T^n \to N_G(T^n) \to W \to 1,$$

defining a finite group $W$ known as the *Weyl group* of $G$.

Proposition 6.20. *For a compact connected Lie group $G$ with maximal torus $T$ and Weyl group $W$,*

$$H^*(BG;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W,$$

*where $H^*(\cdot)^W$ denotes the $W$-invariant subalgebra under $W$'s action induced by left multiplication on $T \subset G$.*

Proof. (Sketch) We proceed in a few steps.

(1) Let $p : \widetilde{X} \to X$ be a normal covering map with the structure of a connected principal $\pi$-bundle for $\pi$ a finite discrete group. Then, by the map lifting lemma, there is an isomorphism of singular chain groups $C_*(\widetilde{X};\mathbb{Q})^\pi \cong C_*(X;\mathbb{Q})$. Passing to homology,
$$H^*(X;\mathbb{Q}) = H^*(\widetilde{X};\mathbb{Q})^\pi.$$

(2) The quotient $G/T$ is a "generalized flag manifold," and there is always a generalized Bruhat partition allowing us to realize $G/T$ as a cell complex with $|W|$ cells of only even dimension. We have, by (1),
$$\begin{aligned} H^*(G/N_G(T);\mathbb{Q}) &\cong H^*(G/T;\mathbb{Q})^W \\ \implies \chi(G/N_G(T)) &= \chi(G/T)/|W| \\ &= 1. \end{aligned}$$
But $G/N_G(T)$ has only homology in even degrees (since $G/T$ does). Thus $G/N_G(T)$ must have the rational homology of a point.

(3) We have a pair of fibrations,
$$BT = EG/T \xrightarrow{W} EG/N_G(T) = BN_G(T) \xrightarrow{G/N_G(T)} BG.$$
The second map has rationally contractible fibre, so the Serre spectral sequence implies $H^*(BG;\mathbb{Q}) = H^*(BN_G(T);\mathbb{Q})$. Applying (1) to the first map, gives $H^*(BN_G(T);\mathbb{Q}) = H^*(BT;\mathbb{Q})^W$.

□

We can apply this proposition to our special orthogonal groups. We need to split into cases based on parity. Let's begin with the odd case of $BSO(2n+1)$. Let $R_\theta$ be the $2\times 2$ rotation matrix by angle $\theta$. Then, the maximal torus of $SO(2n+1)$ is,
$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & & \\ & \ddots & & \\ & & R_{\theta_n} & \\ & & & 1 \end{pmatrix} : \theta_1, \dots \theta_n \in S^1 \right\}.$$
The Weyl group consists of transformations permuting the $2\times 2$ blocks and possibly changing some by a sign. This is the symmetry group of an $n$-dimensional cube $W = S_n \rtimes \mathbb{Z}_2^n$. By our above proposition, we know that,
$$H^*(BSO(2n+1);\mathbb{Q}) = H^*(BT;\mathbb{Q})^W = \mathbb{Q}[x_1,\dots,x_n]^W,$$

for $|x_i| = 2i$. Note that $W$ acts on $\mathbb{Z}[x_1, \ldots, x_n]$ by permuting the $x_i$ and changing them by a sign. In order that a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ be invariant under a change of any $x_i$ by a sign it must be a polynomial in the squares $x_1^2, \ldots, x_n^2$. Hence, $H^*(BSO(2n+1); \mathbb{Q})$ is the algebra of symmetric rational polynomials in $x_1^2, \ldots, x_n^2$. Let $p_i$ denote the $i$th symmetric polynomial function of the $x_i$, so that,

$$(1 + x_1^2) \cdots (1 + x_n^2) = 1 + p_1 + \cdots + p_n,$$

where $p_i$ is the homogeneous part of degree $4i$. Hence,

$$H^*(BSO(2n+1); \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_n] \quad \text{where} \quad |p_i| = 4i.$$

Now we deal with the even case $BSO(2n)$. The maximal torus is then,

$$T = \left\{ \begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_n} \end{pmatrix} : \theta_1, \ldots \theta_n \in S^1 \right\}.$$

The Weyl group $W$ consists of transformations permuting $2 \times 2$ blocks with sign changes as before, with the new restriction that the total number of sign changes must be even. Let $P$ be the convex hull of $2^{n-1}$ vertices of an $n$-dimensional cubes which pairwise do not share an edge. $W$ is the symmetry group of $P$, isomorphic to $S_n \rtimes \mathbb{Z}_2^{n-1}$. Again,

$$H^*(BSO(2n); \mathbb{Q}) = H^*(BT; \mathbb{Q})^W = \mathbb{Q}[x_1, \ldots, x_n]^W,$$

for $|x_i| = 2i$. All the symmetric polynomials in the $x_i^2$ will be in the cohomology as before. Additionally, since any element of $W$ acts by an even number of sign changes on the $x_i$, the element $x_1 \cdots x_n$ is also in the cohomology. By a dimension count, this generator along with the symmetric polynomials gives the entire cohomology. Let $p_i$ denote the $i$th elementary polynomial in $x_i$'s as in the odd case. Let $e = x_1 \cdots x_n$. We conclude,

$$H^*(BSO(2n); \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_n, e]/(p_n - e^2) \quad \text{where} \quad |p_i| = 4i, \ |e| = 2n.$$

For even and odd $SO$ groups, the element $p_i$ is called the *$i$th universal Pontryagin class* and for an oriented vector bundle $E$ it corresponds to a characteristic class $p_i(E)$ called *$i$th Pontryagin class of $E$.* The element $e$ in the cohomology of $BSO(2n)$ is called the *universal Euler class.*

Note we have already defined the Euler class for an oriented bundle as an element of the integral cohomology class. We will later show that this Euler class is the old Euler class reduced over $\mathbb{Q}$. The fact there is no Euler class in the rational cohomology of $BSO(2n+1)$ tells us the Euler class is torsion for odd rank bundles (in fact we will show later it is always 2-torsion in odd dimensions).

Note also that we have defined Pontryagin classes as integral classes of quaternionic bundles and rational classes of oriented real bundles. To distinguish the real and quaternionic cases, we will call the Pontryagin classes for $BSp$ *quaternionic Pontryagin classes*. They are sometimes also called *symplectic classes*.

To make matters more confusing, the usual definition of Pontryagin classes is as integral classes of any bundle. More precisely, given a real vector bundle $E \to X$, we define the *$i$th Pontryagin class of $E$* to be $p_i(E) = (-1)^i c_{2i}(\mathbb{C}E) \in H^{4i}(X;\mathbb{Z})$, where $\mathbb{C}E$ denotes the complexification of the real bundle $E$. From a classifying space perspective, there is a map $O(n) \to U(n)$ sending an $n \times n$ orthogonal matrix $A$ to the same matrix thought of as unitary (i.e. the complexification of the linear map). This gives a map $f : BO(n) \to BU(n)$. The pullback of the universal Chern class $c_{2i}$ by $f$ multiplied by $(-1)^i$ defines an element of the $4i$th cohomology group of $BO(n)$ which will be called the $i$th universal Pontryagin class $p_i$ and corresponds to the $i$th Pontryagin class of real bundles.

We will show that under pullback by the double cover map $BSO(n) \to BO(n)$ induced by inclusion the class $p_i$ reduced over $\mathbb{Q}$ will coincide with the universal Pontryagin class for oriented bundles defined above. Our argument above show that modulo torsion, the Pontryagin classes and Euler class generate the homology of $BSO(n)$ (note this is not true over $\mathbb{Z}$, see the remark below).

We summarize our findings in the following definition.

**Definition 6.21: Characteristic Classes**

Let $E \to X$ be a real rank $n$ vector bundle. To it we can associate,

$$\textit{Stiefel–Whitney classes:} \quad w_i(E) \in H^i(X;\mathbb{Z}_2),$$

$$\textit{Pontryagin classes:} \quad p_i(E) \in H^{4i}(X;\mathbb{Z}).$$

If $E$ is oriented we can also associate the

$$\textit{Euler class:} \quad e(E) \in H^n(X;\mathbb{Z}).$$

If $E$ is a complex bundle we furthermore can associate,

$$\textit{Chern classes:} \quad c_i(E) \in H^{2i}(X;\mathbb{Z}_2).$$

Finally, if $E$ is a quaternionic bundle we associate,

$$\textit{quaternionic Pontryagin classes:} \quad p_i(E) \in H^{4i}(X;\mathbb{Z}).$$

Remark 6.22. One may reasonably complain that for real and real oriented bundles we only classified characteristic classes over $\mathbb{Z}_2$ and $\mathbb{Q}$ respectively. In fact, there is no real content in passing to working over $\mathbb{Z}$. Firstly, our description of the generators of $H^*(BSO(n);\mathbb{Q})$ continues to hold if we pass to $\mathbb{Z}[1/2]$. Also,

$$H^*(BSO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_2,\dots,w_n],$$

(we will see later that $w_1$ vanishes iff a bundle is orientable).

Let $\beta : H^i(X;\mathbb{Z}_2) \to H^{i+1}(X;\mathbb{Z})$ denote the Bockstein homomorphism associated to $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2$. We define the $(i+1)$*st integral Stiefel–Whitney class* of a real bundle $E \to X$ as,

$$W_{i+1}(E) = \beta(w_i(E)) \in H^{i+1}(X;\mathbb{Z}).$$

These characteristic classes correspond to universal elements $W_i \in H^i(BO;\mathbb{Z})$ obtained by applying $\beta$ to the cohomology of $BO$. More generally, given a product $w_{i_1}\cdots w_{i_k}$ living in $H^*(BO(n);\mathbb{Z}_2)$, we can apply $\beta$ to it to obtain an integral cohomology element. Under pullback by the map induced by inclusion $BSO \to BO$, the images under $\beta$ of universal elements may also be thought of as classes in the integral homology of $BSO$.

It happens that $H^*(BSO(n);\mathbb{Z})$ is generated by the universal Pontryagin classes $p_i$ for $i \leq (n-1)/2$, the universal Euler class $e$, and the Bockstein homomorphism of a product of even degree universal Stiefel–Whitney classes: $\beta(w_{2i_1}\cdots w_{2i_k})$ for $0 < i_1 < \cdots < i_k \leq (n-1)/2$. Similarly $H^*(BO(x);\mathbb{Z})$ is generated by the universal Pontryagin classes $p_i$ for $i \leq n/2$, and the Bockstein homomorphism of a product of even degree universal Stiefel–Whitney classes possibly along with $w_1$: $\beta(w_1^{\varepsilon} w_{2i_1}\cdots w_{2i_k})$ for $\varepsilon \in \{0,1\}$, $0 < i_1 < \cdots < i_k \leq n/2$. These algebra are not free on the given generators and a complete set of relations can be given explicitly, we will see some of the relations later.

We will not need these facts and so not prove them. The arguments are not so complicated and can be found in a short paper of Brown Jr. [**B**]. For our purposes, it is enough to say that if one understands the relevant elements in Definition 6.21 and one can describe the cohomology ring and Bockstein homomorphism $\beta$ of the base, then one knows everything about the characteristic classes of the vector bundle. △

To make computations with characteristic classes it is highly useful to know their algebraic and geometric properties. In fact one often introduces the classes axiomatically as uniquely satisfying some list of properties, as in [**MS**]. We will discuss these properties/axioms in due time, as well as geometric constructions of the characteristic classes as first obstructions to extending sections. Quickly, we discuss one important property shared by most of the characteristic classes.

We say a characteristic class $c$ for $G$-bundles is *stable* if $c(V \oplus T) = c(V)$ for any vector bundle $V$ and any trivial rank one bundle $T$ in the $G$-bundle sense. Note that the maps induced by inclusion $BG(n) \hookrightarrow BG(n+1)$, for $G = O, SO, U, Sp$ induce a map from rank $n$ $G$-bundles to rank $(n+1)$ $G$-bundles given by summing with a rank one trivial $G$-bundle. Thus for a characteristic class $c \in H^*(BG(n))$ to be stable is the same as for it to be a pullback of an element in $c \in H^*(BG(\infty))$.

Note that our description of the cohomology of $BO(n), BSO(n), BU(n)$ and $BSp(n)$ extends easily to the classifying spaces of the corresponding infinite dimensional groups $BO, BSO, BU, BSp$. And the pullback maps on cohomology are the obvious inclusions (since we know the Schubert cell decomposition respects these inclusions). Thus we have proved the following corollary.

PROPOSITION 6.23. *The Stiefel–Whitney, Pontryagin, Chern, and quaternionic Pontryagin classes are all stable.*

In contrast, recall from Corollary 4.51 that the Euler class is unstable. This should be unsurprising: we know if $E \to X$ is rank $n$ that $e(E) \in H^n(X)$ and $e(E \oplus \underline{\mathbb{R}}) \in H^{n+1}(X)$, so these two cannot be the same whenever $e(E)$ is non-zero. This should not necessarily be seen as a deficit of the Euler class; it allows us to distinguish non-isomorphic but stably isomorphic bundles. For example $TS^2$ is stably trivial after direct summing with the trivial normal bundle of the embedding $S^2 \subset \mathbb{R}^3$. But $TS^2$ is non-trivial as we saw before because $2 = \chi(S^2) = \langle e(TS^2), [S^2] \rangle$.

**6.3. Characteristic Classes and First Obstructions.** Let us recall from our discussion of obstruction theory that if one has a homotopically simple bundle $E \to B$ over a CW base with $k$-connected fibre $F$, then one obtains a unique cohomology class $C(E) \in H^{k+2}(B; \pi_k(F))$ representing any attempt to extend a section of the $(k+1)$-skeleton of $B$ to the $(k+2)$-skeleton. This class is natural with respect to pullbacks, hence a functorial association of a vector bundle $\xi$ on $B$ to such a fibre bundle defines a characteristic class for $\xi$.

For example, we saw before that to an oriented rank $n$ vector bundle $\xi$ on $B$, we can associate the first obstruction $C(S\xi) \in H^n(B)$ of the sphere bundle $S\xi$. We called this the Euler class $e(\xi)$ of $\xi$ and showed it can be recovered as the restriction of the Thom class $t_\xi$ to the base, or equivalently in the smooth case as Poincaré dual to the fundamental class of the zero set of a generic section of $\xi$. We will show below that this agrees with the Euler class as defined above in the cohomology of $BSO(n)$.

*Stiefel–Whitney and Chern Classes as Obstructions.* Let $\xi \to B$ be a complex vector bundle of rank $n$. To it, we can associate the bundle $F_{n-m+1}(\xi)$ of $(n-m+1)$ frames of $\xi$ which is a fibre bundle with fibre $\mathbb{C}V(n, n-m+1)$. Consider the sequence of fibrations for $n > m$,

$$\begin{aligned}
\mathbb{C}V(n-1, n-m) &\hookrightarrow \mathbb{C}V(n, n-m+1) \to S^{2n-1} \\
\mathbb{C}V(n-2, n-m-1) &\hookrightarrow \mathbb{C}V(n-1, n-m) \to S^{2n-3} \\
&\vdots \\
\mathbb{C}V(m, 1) &\hookrightarrow \mathbb{C}V(m+1, 2) \qquad \to S^{2m+1}.
\end{aligned}$$

Note that $\mathbb{C}V(m, 1) = S^{2m-1}$. Recursively applying the homotopy long exact sequence to these fibrations we conclude that $\mathbb{C}V(n, n-m+1)$ is $(2m-2)$-connected and that $\pi_{2m-1}(\mathbb{C}V(n-m+1)) = \mathbb{Z}$. We thus obtain a first obstruction to the existence of a $(n-m+1)$-frame of $\xi$ over the $2m$-skeleton of $B$,

$$c_m(\xi) := C(F_{n-m+1}(\xi)) \in H^{2m}(B; \mathbb{Z}).$$

We show soon this agrees with the $m$th Chern class.

We point out two special cases. For $m = 1$, we are studying $n$-frames, i.e. bases of the fibre. So, $c_1(\xi)$ measures the obstruction to the existence of a trivialization of the bundle over the 2-skeleton. From this we deduce for example that any bundle $\xi$ on a 2 dimensional complex with $c_1(\xi) = 0$ is trivial.

For $m = n$, we are studying 1-frames, i.e. non-vanishing sections. So, $c_n(\xi)$ measures the obstruction to the existence of a non-vanishing section of $\xi$ over the $n$-skeleton. But this obstruction class is already familiar to us, which implies the following important corollary.

**Theorem 6.24**

Let $\xi \to X$ be a complex rank $n$ vector bundle, then $c_n(\xi) = e(\xi_{\mathbb{R}})$, where $\xi_{\mathbb{R}}$ denotes the underlying oriented real bundle.

Note that if we are dealing with a line bundle $\xi$, there is one Chern class $c_1(\xi)$ measuring the first obstruction to defining a section of $\xi$ over the two skeleton of the base. But note since the structure group of $\xi$ is $U(1)$, which has trivial higher homotopy groups, there are no further obstructions. Thus a trivialization over the two-skeleton extends to one on the entire base. And hence $c_1(\xi) = 0$ implies the bundle trivializes. This reproduces our earlier result that complex line bundles are classified by $c_1$.

The case for quaternionic bundles is exactly analogous. We redefine $p_m(\xi)$ for a quaternionic rank $n$ bundle $\xi \to B$ as the first obstruction to the existence of a quaternionic $(n-m+1)$-frame of $\xi$ over the $2m$-skeleton of $B$.

Now consider $\xi \to B$ a rank $n$ real vector bundle. To it we associate the bundle $F_{n-m+1}(\xi)$ of $n-m+1$ frame of $\xi$ which is a fibre bundle with fibre $V(n, n-m+1)$. Aw with the complex case, we have a sequence of fibrations,

$$\begin{aligned} V(n-1, n-m) &\hookrightarrow V(n, n-m+1) \to S^{n-1} \\ V(n-2, n-m-1) &\hookrightarrow V(n-1, n-m) \to S^{n-2} \\ &\vdots \\ V(m, 1) &\hookrightarrow V(m+1, 2) \qquad \to S^m. \end{aligned}$$

Note that $V(m, 1) = S^{m-1}$. We conclude recursively from the homotopy long exact sequences of the fibrations that $V(n, n-m+1)$ is $m-2$ connected. Furthermore from the last exact sequence, which is a Gysin sequence,

$$0 \to \pi_m(S^m) = \mathbb{Z} \xrightarrow{\times\chi(S^m)} \mathbb{Z} = \pi_{m-1}(V(m,1)) \to \pi_{m-1}(V(m+1,2)) \to 0 = \pi_{m-1}(S^m).$$

So we conclude,

$$\pi_{m-1}(V(n, n-m+1)) = \begin{cases} \mathbb{Z}_2 & m < n \text{ even or } m = 1, \\ \mathbb{Z} & m > 1 \text{ odd or } m = n. \end{cases}$$

The bundle $F_{n-m+1}$ may not be homotopically simple, and the obstruction class may have twisted coefficients whenever $\pi_{m-1} = \mathbb{Z}$, but we will be okay if we work over $\mathbb{Z}_2$. We thus obtain a first obstruction to the existence of a $(n-m+1)$ frame of $\xi$ over the $m$-skeleton of $B$,

$$w_m(\xi) := C(F_{n-m+1}(\xi)) \in H^m(B; \mathbb{Z}_2).$$

We will show this agrees with the $m$th Stiefel–Whitney class.

For $m = n$, $w_n(\xi)$ is the first obstruction to a 1-frame over the $n$-skeleton of $B$. Hence we have the following as with the complex case.

PROPOSITION 6.25. *Let $\xi \to X$ be a real rank $n$ vector bundle, then $w_n(\xi) = e(\xi) \pmod 2$. Here we define $e(\xi)$ in the unoriented case as the restriction of the mod 2 Thom class to the base $X$.*

For $m = 1$, $w_1(\xi)$ is the first obstruction to an $n$-frame, i.e. a trivialization, over the 1-skeleton of $B$.

*Proving An Equivalence.* Let us prove that the obstruction theoretic definitions of our characteristic classes agree with the previous cohomological definitions.

Note that by naturality of first obstructions it suffices to show that the obstruction class and cohomology class agree for the universal bundle $EG \to BG$. Actually to make things simpler we will use a version of *the splitting principle*, which we explain in generality later, that allows us to reduce to a sum of tautological bundles on $BT$, for $T$ the maximal torus of $G$.

To see what we mean, let's begin with the Stiefel–Whitney classes. Recall we had a fibration $Bi : (\mathbb{R}P^\infty)^n \to BO(n)$ induced by the inclusion $i : \mathbb{Z}_2^n \to O(n)$. The pullback of the universal bundle $EO(n)$ by $Bi$ can be seen to be a direct sum of copies of the tautological line bundle $\xi^1$ on each copy of $\mathbb{R}P^\infty$. This is because $(Bi)^*EO(n)$ is the fibration $V(\infty, n) \to (\mathbb{R}P^\infty)^n$ sending an orthonormal $n$-frame $(v_1, \dots, v_n)$ to the lines $\mathrm{span}(v_1), \dots, \mathrm{span}(v_n) \in (\mathbb{R}P^\infty)^n$.

As we showed before $H^*((\mathbb{R}P^\infty)^n; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \dots, x_n]$, $H^*(BO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots, w_n]$ and the pullback $(Bi)^*$ is injective on cohomology sending $w_i$ to the $i$th symmetric polynomial in the $x_i$. Because this pullback is injective if the obstruction class definition of $w_i(\xi^1 \oplus \cdots \oplus \xi^1)$ agrees with the cohomological class $\sigma_i(x_1, \dots, x_n)$, then we will know that the obstruction theory and cohomological definitions of Stiefel–Whitney classes agree.

To understand $w_m(\xi^1 \oplus \cdots \oplus \xi^1)$, lets evaluate it on an element of homology $[\mathbb{R}P^{\ell_1} \times \cdots \times \mathbb{R}P^{\ell_n}]$ with $\sum \ell_i = m$. We claim that,

$$\left\langle w_m(\xi^1 \oplus \cdots \oplus \xi^1), [\mathbb{R}P^{\ell_1} \times \cdots \times \mathbb{R}P^{\ell_n}] \right\rangle = \begin{cases} 1 \pmod 2 & \ell_i \le 1 \ \forall i, \\ 0 & \text{else.} \end{cases}$$

To see this note if some $\ell_i > 1$, because $\sum \ell_j = m$, there must be at least $n - m + 1$ of the $\ell_j$ which are zero. But the direct summand of the bundle $\xi^1 \oplus \cdots \oplus \xi^1$ trivializes on $[\mathbb{R}P^{\ell_1} \times \cdots \times \mathbb{R}P^{\ell_n}]$ for each factor with $\ell_j = 0$. In particular if some $\ell_i > 1$, we may find an $(n - m + 1)$ frame of the bundle on $[\mathbb{R}P^{\ell_1} \times \cdots \times \mathbb{R}P^{\ell_n}]$ and hence the obstruction $w_m(\xi^1 \oplus \cdots \oplus \xi^1)$ must vanish on this class, as claimed. Conversely, if $\ell_i \le 1$ for all $i$, then the bundle will look like $(\xi^1)^{\oplus m} \oplus (\underline{\mathbb{R}})^{\oplus(n-m)}$ on $[(\mathbb{R}P^1)^m \times \{\mathrm{pt}\}^{n-m}]$. Note $w_m((\xi^1)^{\oplus m}) =$

$e(\xi^{\oplus m}) = 1 \pmod 2$ since this is a direct sum of copies of Möbius strips. Thus $(\xi^1)^{\oplus m}$ has no non-vanishing section and $(\xi^1)^{\oplus m} \oplus (\underline{\mathbb{R}})^{\oplus(n-m)}$ does not have a $(n-m+1)$ frame on $[(\mathbb{R}P^1)^m \times \{\mathrm{pt}\}^{n-m}]$ and so the obstruction class must be the non-trivial element of $\mathbb{Z}_2$ on this class.

Note that the cohomology of $(\mathbb{R}P^\infty)^n$ is given by,

$$\langle x_1^{i_1} \cdots x_n^{i_n}, [\mathbb{R}P^{\ell_1} \times \cdots \times \mathbb{R}P^{\ell_n}]\rangle = \delta_{i_1,\ell_1} \cdots \delta_{i_n,\ell_n} \pmod 2.$$

Hence we have,

$$w_m(\xi^1 \oplus \cdots \oplus \xi^1) = \sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m} = \sigma_m(x_1, \ldots, x_n).$$

Hence the obstruction theoretic Stiefel–Whitney class agrees with the cohomological definition.

Now we move onto the complex/Chern case. As with above, we have a fibration $Bi : (\mathbb{C}P^\infty)^n \to BU(n)$ induced by the inclusion $i : T^n \to U(n)$. The pullback of the universal bundle $EU(n)$ by $Bi$ is the direct sum of copies of the tautological complex line bundle $\xi^1$ on each copy of $\mathbb{C}P^\infty$. This is because $(Bi)^*EU(n)$ is the fibration $\mathbb{C}V(\infty, n) \to (\mathbb{C}P^\infty)^n$ sending an orthonormal $n$-frame $(v_1, \ldots, v_n)$ to the complex lines $\mathrm{span}(v_1), \ldots, \mathrm{span}(v_n) \in (\mathbb{C}P^\infty)^n$. Because the induced map $(Bi)^*$ is injective on cohomology, it will suffice to show $c_i(\xi^1 \oplus \cdots \oplus \xi^1)$ defined obstruction theoretically agrees with $\sigma_i(y_1, \ldots, y_n)$ where $H^*(BU(n)) = \mathbb{Z}[y_1, \ldots, y_n]$. We claim for $\sum \ell_j = m$,

$$\left\langle c_m(\xi^1 \oplus \cdots \oplus \xi^1), [\mathbb{C}P^{\ell_1} \times \cdots \times \mathbb{C}P^{\ell_n}] \right\rangle = \begin{cases} (-1)^m & \ell_i \leq 1 \ \forall i, \\ 0 & \text{else.} \end{cases}$$

To see this, if some $\ell_i > 1$, then as before at least $n - m + 1$ of the $\ell_j$ must be zero. Then the direct summand of the bundle $\xi^1 \oplus \cdots \oplus \xi^1$ trivializes on $[\mathbb{C}P^{\ell_1} \times \cdots \times \mathbb{C}P^{\ell_n}]$ for each factor with $\ell_j = 0$ and we may find an $(n-m+1)$ frame of the bundle on $[\mathbb{C}P^{\ell_1} \times \cdots \times \mathbb{C}P^{\ell_n}]$ and hence the obstruction $c_m(\xi^1 \oplus \cdots \oplus \xi^1)$ must vanish on this class.

If all $\ell_i \leq 1$, then the bundle will look like $(\xi^1)^{\oplus m} \oplus (\underline{\mathbb{C}})^{\oplus(n-m)}$ on $[(\mathbb{C}P^1)^m \times \{\mathrm{pt}\}^{n-m}]$. Note that $\xi^1 \to \mathbb{C}P^1$ is the degree $-1$ bundle so that $c_1(\xi^1) = -1$ (the fact that $c_1$ defined cohomologically and obstruction-theoretically for a line bundle on $S^2$ can be prove easily directly). If one takes a direct sum of these bundles, the obstruction class will multiply so that

$$\begin{aligned}
\left\langle c_m((\xi^1)^{\oplus m} \oplus (\underline{\mathbb{C}})^{\oplus(n-m)}), [(\mathbb{C}P^1)^m \times \{\mathrm{pt}\}^{n-m}] \right\rangle &= \left\langle c_m((\xi^1)^{\oplus m}), [(\mathbb{C}P^1)^m] \right\rangle \\
&= \left\langle c_1(\xi^1), [\mathbb{C}P^1] \right\rangle^m \\
&= (-1)^m.
\end{aligned}$$

Hence as for the real case we have,

$$c_m(\xi^1 \oplus \cdots \oplus \xi^1) = \sum_{i_1 < \cdots < i_m} y_{i_1} \cdots y_{i_m} = \sigma_m(y_1, \dots, y_n).$$

So the obstruction theory and cohomological Chern classes agree. The quaternionic case is identical and we omit it.

Lastly, we should check that the Pontryagin classes as defined cohomologically in $H^*(BSO(n); \mathbb{Q})$ agree with their definition as $p_i(\xi) = (-1)^i c_{2i}(\mathbb{C}\xi)$.

There is a pullback map $H^*(BSO(n)) \to H^*(BT^{\lfloor n/2 \rfloor}) = \mathbb{Z}[x_1, \dots, x_{\lfloor n/2 \rfloor}]$ which is injective. Using our splitting principle it is enough to show the universal class $p_i$ pulled back to $BT^{\lfloor n/2 \rfloor}$ is $\sigma_i(x_1^2, \dots, x_i^2)$ and we should also show the universal Euler class $e$ pulled back to $BT^{\lfloor n/2 \rfloor}$ is $x_1 \cdots x_n$.

The above pullback corresponds to a map $BT^{\lfloor n/2 \rfloor} \to BSO(n)$ induced by inclusion of the maximal torus of $SO(n)$. The pullback of the universal bundle $ESO(n)$ under this map to a bundle on $(\mathbb{C}P^\infty)^{\lfloor n/2 \rfloor}$ is a direct sum of $\lfloor n/2 \rfloor$ oriented 2-plane bundles given as $\xi^1_{\mathbb{R}}$ for $\xi^1$ the tautological complex line bundle on $\mathbb{C}P^\infty$. Note that the complexification of $\xi^1_{\mathbb{R}}$ is $\xi^1 \oplus (\xi^1)^*$, where $V^*$ denotes the complex dual bundle (in general $\mathbb{C}(V_{\mathbb{R}}) = V \oplus V^*$ for any complex bundle $V$). Hence we compute,

$$p_m((\xi^1_{\mathbb{R}})^{\oplus n}) = (-1)^m [c((\xi^1 \oplus (\xi^1)^*) \oplus \cdots \oplus (\xi^1 \oplus (\xi^1)^*))]_{2m},$$

where $[\cdot]_{2m}$ denotes the part of degree $2m$. As we will show, total Chern classes are multiplicative under direct summing,

$$= (-1)^m [c(\xi^1) c((\xi^1)^*) \cdots c(\xi^1) c((\xi^1)^*)]_{2m}.$$

We have suppressed that each $\xi^1$ comes from a different copy of $\mathbb{C}P^\infty$ in the base. By above, we know $\xi^1$ over the $i$th $\mathbb{C}P^\infty$ will have $c(\xi^1) = 1 + x_i$. Since taking the complex dual multiplies the degree by minus one, $c((\xi^1)^*) = 1 - x_i$. Hence,

$$\begin{aligned} &= (-1)^m [(1 - x_1^2) \cdots (1 - x_n^2)]_{2m} \\ &= \sigma_m(x_1^2, \dots, x_n^2). \end{aligned}$$

The fact our result holds for this bundle on the maximal torus implies it for every other bundle. For a complex rank $n$ bundle, we know $e = c_{2n}$. We immediately see that

$$e((\xi^1)^{\oplus n}) = c_n((\xi^1)^{\oplus n}) = \sigma_n(x_1, \dots, x_n) = x_1 \dots x_n.$$

Again by our splitting principle, the Euler class is as claimed.

*Geometric Interpretations of Characteristic Classes.* The presentation of characteristic classes as geometric obstructions is an important perspective on the meaning and utility of them for classification problems. We now present some specific examples of how characteristic classes can be interpreted geometrically.

**Theorem 6.26**

Given a real rank $n$ bundle $\xi$ and a complex rank $n$ bundle $\eta$, one has

$$w_1(\xi) = w_1(\det \xi) \quad \text{and} \quad c_1(\eta) = c_1(\det_{\mathbb{C}} \eta),$$

where $\det \xi = \Lambda^n \xi$ and $\det_{\mathbb{C}} \eta = \Lambda^n_{\mathbb{C}} \eta$. In particular, $w_1(\xi)$ vanishes if and only if $\xi$ is orientable (reduces to structure group $SO(n)$) and $c_1$ vanishes if and only if $\eta$ reduces to structure group $SU(n)$.

Proof. We know $w_1(\xi)$ is the first obstruction for a section of the orthogonal $n$-frame bundle of $\xi$. But note this $n$-frame bundle is isomorphic to $\det \xi$ by sending a frame $(v_1, \dots, v_n)$ to $v_1 \wedge \cdots \wedge v_n$. So $w_1(\xi)$ is the first obstruction to finding a section of the line bundle $\det \xi$. But that is nothing but $e(\det \xi)$ (mod 2) which equals $w_1(\det \xi)$. The same analysis holds for $c_1$ noting that the unitary $n$ frame bundle of $\eta$ is isomorphic to $\det_{\mathbb{C}} \eta$.

We have $w_1(\xi)$ vanishes if and only if $w_1(\det \xi) = 0$, if and only if $\det \xi$ is trivial, if and only if a global orientation of $\xi$ exists. The same analysis holds for $c_1$.

Note one can also see the orientation condition more directly, as we proved many eons ago. An orientation is a reduction of structure group via a lift of classifying map $X \to BO(n)$ for the map $BSO(n) \to BO(n)$. But this fits into a fibration $\mathbb{Z}_2 \to BSO(n) \to BO(n)$. So by obstruction theory, since $\mathbb{Z}_2$ has no higher homotopy groups, the first obstruction to a lift (which is $w_1(\xi)$) is the only obstruction. And $w_1(\xi) = 0$ if and only if such a lift exists. The same goes for the complex case vis à vis the fibration $U(1) \to BSU(n) \to BU(n)$. $\square$

Proposition 6.27. *If a rank $n$ real bundle $\xi$ admits $m$ linearly independent sections, then $w_n(\xi) = w_{n-1}(\xi) = \cdots = w_{n-m+1}(\xi) = 0$. If a rank $n$ complex bundle $\eta$ admits $m$ linearly independent sections, then $w_n(\eta) = w_{n-1}(\eta) = \cdots = w_{n-m+1}(\eta) = 0$.*

Proof. A collection of $m$ linearly independent sections provides an $m$-frame of $\xi$ over the total base space. And so the obstruction classes related to $1, 2, \cdots, m$-frames, i.e. $w_n, w_{n-1}, \dots, w_{n-m+1}$, must all vanish. The same holds for the complex case. $\square$

Recall a spin structure on an oriented bundle $E \to X$ given by a classifying map $X \to BSO(n)$ is given by a lift to $X \to BSpin(n)$. A spin$^{\mathbf{c}}$ structure is a lift to $X \to BSpin^{\mathbf{c}}(n)$.

**Theorem 6.28**

A rank $n$ oriented bundle $E \to X$ has a spin structure if and only if $w_2(E) = 0$. If $E$ is spin, the set of spin structures is an affine space modelled on $H^1(X;\mathbb{Z}_2)$.

Proof. We have a fibration $\mathbb{Z}_2 \hookrightarrow BSpin(n) \to BSO(n)$ and we are looking for a lift $X \to BSO(n)$ to $X \to BSpin(n)$. Since the higher homotopy groups of $\mathbb{Z}_2$ are trivial, the only obstruction to a lift is the first obstruction, which is an element of $H^2(X;\mathbb{Z}_2)$. This obstruction must define a characteristic class (non-trivial since there are non-spin bundles) and so must be $w_2$. Hence a lift exists iff $w_2(E) = 0$.

The different possible lifts to a map $X \to BSpin(n)$ are parameterized by difference cochains. Again looking at the homotopy groups of $\mathbb{Z}_2$, the only non-vanishing difference cochain lives in $H^1(X;\mathbb{Z}_2)$ checking if two maps $X \to BSpin(n)$ are homotopic relative to $X \to BSO(n)$ on the 1-skeleton. $\square$

**Theorem 6.29**

A rank $n$ oriented bundle $E \to X$ has a spin$^{\mathbf{c}}$ structure if and only if $w_2(E)$ is the reduction mod 2 of an integral class. If $E$ is spin$^{\mathbf{c}}$, the set of spin$^{\mathbf{c}}$ structures is an affine space modelled on $H^2(X;\mathbb{Z}_2)$.

Proof. We have a fibration $U(1) \hookrightarrow BSpin^{\mathbf{c}}(n) \to BSO(n)$ and we are looking for a lift $X \to BSO(n)$ to $X \to BSpin^{\mathbf{c}}(n)$. Studying the homotopy groups of $U(1)$, the only obstruction is an element of $H^3(X;\mathbb{Z})$. This must be a characteristic class and so by the classification which we stated but did not prove, it must be of the form $aW_3(E)$ for integer $a$. In fact $W_3(E)$ is 2-torsion since $2\beta(w_2) = \beta(2w_2) = 0$. And since the obstruction class is non-trivial (since there are non-spin$^{\mathbf{c}}$ bundles) we conclude the first and only obstruction is $W_3(E)$. Recall $W_3$ comes from the Bockstein exact sequence,

$$H^2(X;\mathbb{Z}) \xrightarrow{r} H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} H^3(X;\mathbb{Z}),$$

as $W_3 = \beta(w_2)$. Since this is exact, $W_3 = 0$ if and only $w_2$ is in the image of $r$, i.e. the reduction mod 2 of an integral class.

Remark 6.30. One way to interpret this is that $w_2 \in H^2(X;\mathbb{Z}_2)$ measures the obstruction to being a spin bundle. And we can obtain spin$^{\mathbf{c}}$ bundle if and only if there is a complex line bundle $\gamma$ with first Chern class $c_1(\gamma) \in H^2(X;\mathbb{Z})$ so $r(c_1(\gamma)) = w_2$. So that adding the complex line bundles "cancels out" or mends the failure of local $Spin(n)$ charts to patch together. $\triangle$

Again analyzing the homotopy of $U(1)$, the different lifts to a map $X \to BSpin^{\mathbf{c}}(n)$ are parameterized by difference cochains, and the only non-vanishing ones live in $H^2(X;\mathbb{Z})$, checking if two maps $X \to BSpin^{\mathbf{c}}(n)$ are homotopic relative to $X \to BSO(n)$ on the 2-skeleton. $\square$

Remark 6.31. One can show for every oriented manifold of dimension less than or equal to four, the second Stiefel–Whitney class of the tangent bundle is the mod 2 reduction of an integral class and so every oriented manifold of dimension four or less is spin$^{\mathbf{c}}$. This is clear in dimensions one and two since $W_3 \in H^3(X) = 0$. If $X$ is three dimensional, then $H^3(x) = 0$ or $\mathbb{Z}$ and has no two-torsion, so again $W_3 = 0$. The case of dimension four is slightly more tricky, proofs can easily be found online.

Note that $SO(n)$ is the identity component of $O(n)$, and $Spin(n) \to SO(n)$ is the universal covering of $SO(n)$. We can continue this to form the Whitehead tower for $O(n)$,

$$\cdots \to FiveBrane(n) \to String(n) \to Spin(n) \to SO(n) \to O(n).$$

The two connected and three connected coverings of $O(n)$ turn out to be topological groups called the string group and five-brane group respectively. Unsurprisingly, these are of interest in string theory. A lift of a bundle classified by $X \to BO(n)$ via the map $BString(n) \to BO(n)$ is called a *string structure* and a further lift via the map $BFiveBrane(n) \to BO(n)$ is called a *five-brane structure*. It turns out these higher lifts are also classified by characteristic classes. More precisely, a spin bundle $E$ has a string structure if and only if $\frac{1}{2}p_1(E) = 0$ and a string bundle $E$ has a five-brane structure if and only if $\frac{1}{6}p_2(E) = 0$. One may climb even further up the Whitehead tower similarly if they are so inclined (I'll stick to climbing at Benchmark).

### 6.4. Properties of Characteristic Classes and Computations.

We now move towards some important computations of characteristic classes. We begin with one final definition of Stiefel–Whitney classes and show that they are in fact homotopical invariants.

*Axiomatic Treatment of Stiefel–Whitney Classes.* Let us see how computations of Stiefel–Whitney classes can be made from a few simple axioms.

**Theorem 6.32**

There exists characteristic classes $w_i(\xi) \in H^i(X;\mathbb{Z}_2)$ for $i \geq 0$ associated to real vector bundles $\xi \to X$ which are uniquely determined by the following properties:

**Dimension:** $w_0(\xi) = 1$ and $w_i(\xi) = 0$ for $i$ greater than the rank of $\xi$.

**Non-Triviality:** If $\gamma^1$ is the tautological line bundle on $\mathbb{R}P^1$ then $w_1(\gamma^1) \neq 0$.

**Whitney Product Formula:** Given bundles $\xi, \eta$, one has,

$$w_i(\xi \oplus \eta) = \sum_{j=0}^{i} w_j(\xi) \smile w_{i-j}(\eta).$$

Proof. We begin with uniqueness; suppose $w_i$ are characteristic classes satisfying the axioms above. Let $H^*(\mathbb{R}P^1;\mathbb{Z}_2) = \mathbb{Z}_2[a]/(a^2)$. We must have by the first two axioms that the total Stiefel–Whitney class of tautological bundle $\gamma^1$ is $w(\gamma^1) = 1 + a$.

Let $\xi^1$ be the tautological line bundle on $\mathbb{R}P^\infty$. Note the standard embedding $i : \mathbb{R}P^1 \to \mathbb{R}P^\infty$ gives $\gamma^1$ as the pullback of $\xi^1$. Since $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[b]$ and $i^*(b) = a$, we must have by naturality that $w(\xi^1) = 1 + b$.

Now consider the Cartesian product $\zeta = \xi^1 \times \cdots \times \xi^1$ over $(\mathbb{R}P^\infty)^n$. We have $H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}_2) = \mathbb{Z}_2[b_1,\ldots,b_n]$ so that the $i$th inclusion $j_i : \mathbb{R}P^\infty \to (\mathbb{R}P^\infty)^n$ has $j_i^*(b_k) = \delta_{i,k}\, b$. By the Whitney product formula, and naturality with respect to the inclusion of $\mathbb{R}P^\infty$ factors,

$$w(\zeta) = (1 + b_1)\cdots(1 + b_n).$$

Since the map $Bi : (\mathbb{R}P^\infty)^n \to BO(n)$ induces an injective pullback map in cohomology, we deduce that,

$$w(EO(n)) = 1 + w_1 + \cdots + w_n,$$

where $H^*(BO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\ldots,w_n]$ and $(Bi)^*w_i = \sigma_i(b_1,\ldots,b_n)$. But if we know the characteristic class on the universal bundle we know it on all other bundles by naturality and the fact maps to $BO(n)$ classifies vector bundles. So $w_i$ are unique.

Now we prove existence. We have already given some equivalent definitions of the Stiefel–Whitney class which we wish to show obey these axioms. The dimension axiom is really a matter of definition (since we only defined $w_i(\xi)$ for $1 \leq i \leq \mathrm{rk}(\xi)$). The non-triviality axiom follows from the fact $\gamma^1$ is the pullback of $EO(1)$ under the inclusion $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty = BO(1)$. We know what this maps does on cohomology; it will send $w_1$ to the generator of $H^1(\mathbb{R}P^1;\mathbb{Z}_2)$. The Whitney product formula can be deduced as follows. Given bundles $\xi$ and $\eta$ on $X$ classified by maps $X \to BO(n)$ and $X \to BO(m)$, the bundle $\xi \oplus \eta$ is classified by the map $X \to B(O(n) \times O(m)) \to BO(n \times m)$, where the first map combines the above two classifying maps and the second is the classified functor applied to the inclusion $O(n) \times O(m) \to O(n+m)$ of block diagonal matrices. Our usual map $(\mathbb{R}P^\infty)^{n+m} \to BO(n+m)$ must factor through $B(O(n) \times O(m))$ via the map above. It follows immediately from the combinatorics of elementary symmetric functions that the Whitney relations hold between the generators of the homology of $BO(n+m)$, $BO(n)$ and $BO(m)$. By naturality it holds when pulled back to $X$ as well. Thus all the axioms are satisfied and the Stiefel–Whitney classes as previously defined are the unique classes given by the above axioms.

Not contenting ourselves with this, we give a separate definition of the Stiefel–Whitney classes, which will also satisfy the axioms and hence agree with our previous definitions. Given a rank $n$ real vector bundle $\xi \to X$ let $t_\xi \in H^n(\xi, \xi \setminus X; \mathbb{Z}_2)$ be the unoriented Thom class. Recall cupping with the Thom class defines an isomorphism,

$$\Phi : H^i(X;\mathbb{Z}_2) \xrightarrow{\cong} H^{n+i}(\xi, \xi \setminus X; \mathbb{Z}_2).$$

**Definition 6.33**

The *ith Stiefel–Whitney class* of a rank $n$ bundle $\xi \to X$ is given by,

$$w_i(\xi) = \Phi^{-1} Sq^i(t_\xi) = \Phi^{-1} Sq^i \Phi(1).$$

The fact this is really a characteristic class, i.e. it satisfies naturality, is immediate from the fact the maps involved are natural. Let us show the axioms hold. The dimension axiom is clear from the definition of the Steenrod squares. The Whitney product formula follows from Cartan's identity for Steenrod squares and the fact the Thom isomorphism distributes over a product of bundles. For the non-triviality axiom, note that the total space of $\gamma^1$ is a Möbius strip $M$ which is homeomorphic to $\mathbb{R}P^2 \setminus D^2$ for an embedded disk $D^2$. By excision, $H^i(M, \partial M; \mathbb{Z}_2) \cong H^i(\mathbb{R}P^2, D^2; \mathbb{Z}_2) \cong H^i(\mathbb{R}P^2; \mathbb{Z}_2)$ for $i > 0$. The Thom class $t_{\gamma^1}$ certainly cannot be zero by Thom isomorphism, so under the above identification it must correspond to the generator $a \in H^1(\mathbb{R}P^2; \mathbb{Z}_2)$. But then $Sq^1(a) = a^2$. This is non-zero in $H^2(\mathbb{R}P^2; \mathbb{Z}_2)$. And so $Sq^1(t_{\gamma^1}) \neq 0$. Since $\Phi$ is an isomorphism, $w_1(\gamma^1) = \Phi^{-1} Sq^1(t_{\gamma^1}) \neq 0$, as desired. $\square$

*Wu Formula.* In the above proof we gave a new construction of Stiefel–Whitney classes as $w_i(\xi) = \Phi^{-1} Sq^i \Phi(1)$. We can refine this construction for the case of tangent bundles as follows.

**Definition 6.34**

Let $M$ be a smooth closed $n$-manifold. The *ith Wu class* $\nu_i \in H^i(M;\mathbb{Z}_2)$ is uniquely defined so that,

$$\nu_i \smile x = Sq^i(x) \quad \text{for all} \quad x \in H^{n-i}(M;\mathbb{Z}_2).$$

Note that $Sq^i \in \mathrm{Hom}(H^{n-i}(M;\mathbb{Z}_2),\mathbb{Z}_2) \cong H_{n-i}(M;\mathbb{Z}_2)$. So by Poincaré duality the Wu class is well defined.

With this definition, we obtain the following important result.

**Theorem 6.35: Wu Formula**

For a closed smooth $n$-manifold $M$, the Stiefel–Whitney classes of its tangent bundle are given by

$$w_i(TM) = \sum_{j=0}^{i} Sq^{i-j}(\nu_j).$$

The proof, following [**MS**, §11], is a little long and will require a couple lemmas. First we need to define a new operation in (co)homology.

**Definition 6.36**

Given two spaces $X, Y$, there is an operation called the *slant product* defined as the composite,

$$/ : H^*(X \times Y) \otimes H_*(Y) \xrightarrow{\text{Künneth}} H^*(X) \otimes H^*(Y) \otimes H_*(Y) \xrightarrow{\mathrm{id}\otimes\langle,\rangle} H^*(X).$$

This satisfies the identities $(a \times b)/\beta = a\langle b, \beta\rangle$ and $[(a \times 1) \smile p]/\beta = a \smile (p/\beta)$.

Let $t_\Delta$ denote the $\mathbb{Z}_2$ Thom class of the normal bundle to the diagonal embedding $\Delta : M \to M^2$. By excision and then restriction we obtain from $t_\Delta$ a class $u \in H^n(M \times M;\mathbb{Z}_2)$. It follows from local computations on a slice $\{x\} \times M$ that $u/[M] = 1$ and that $(a \times 1) \smile u = (1 \times a) \smile u$.

LEMMA 6.37. *For any homogeneous basis $b_1, \dots, b_r$ of $H^*(M;\mathbb{Z}_2)$ there is a dual basis $c_1, \dots, c_r$ so that $\langle b_i \smile c_j \rangle = \delta_{i,j}$. In this case, one has,*

$$u = \sum_i (-1)^{|b_i|} b_i \times c_i.$$

PROOF. By the Künneth formula we know that $u$ is a sum of products $u = \sum_i b_i \times e_i$ for $b_i, e_i \in H^*(M;\mathbb{Z}_2)$ of complimentary dimensions. We compute for arbitrary $a$,

$$\begin{aligned}[(1 \times a) \smile u]/[M] &= [(a \times 1) \smile u]/[M] \\ &= a \smile u/[M] \\ &= a.\end{aligned}$$

We can plug the expansion of $u$ in the product basis into the original expression,

$$\begin{aligned} a &= \sum_j (-1)^{|a|\cdot|b_j|} (b_j \times (a \smile e_j))/[M] \\ &= \sum_j (-1)^{|a|\cdot|b_j|} b_j \langle a \smile e_j, [M] \rangle \end{aligned}$$

Taking $b_i = a$, we conclude that $c_i = (-1)^{|b_i|} e_i$ satisfies the lemma. □

LEMMA 6.38. *For a closed smooth $n$-manifold $M$, one has,*

$$w_i(TM) = Sq^i(u)/[M].$$

PROOF. Note that the Stiefel–Whitney class $w_i(TM)$ is equivalently the $i$th Stiefel–Whitney class of the normal bundle to the embedding $\Delta : M \to M \times M$. From our formula for Stiefel–Whitney classes we have,

$$w_i(TM) \smile t_\Delta = Sq^i(t_\Delta).$$

Following through the isomorphism to $H^*(M^2, M^2 \setminus \Delta; \mathbb{Z}_2)$ and then the restriction to $H^*(M^2;\mathbb{Z}_2)$, this identity becomes,

$$(w_i(TM) \times 1) \smile u = Sq^i(u).$$

Taking the slant product with $[M]$, this becomes,

$$w_i(TM) = Sq^i(u)/[M].$$

□

PROOF. (of Theorem 6.35) Pick a homogeneous basis $b_1, \dots, b_r$ of $H^*(M; \mathbb{Z}_2)$. By Lemma 6.37 we may find a dual basis $c_1, \dots, c_r$. Let $\nu = 1 + \nu_1 + \cdots + \nu_n$ be the *total Wu class*. Similarly define the *total Steenrod square* $Sq = 1 + Sq^1 + \cdots + Sq^n$. We must have,

$$\nu = \sum_i b_i \langle \nu \smile c_i, [M] \rangle.$$

By definition of the Wu classes,

$$= \sum_i b_i \langle Sq(c_i), [M] \rangle.$$

Taking the total steenrod square of both sides,

$$\begin{aligned} Sq(\nu) &= \sum_i Sq(b_i) \langle Sq(c_i), [M] \rangle \\ &= \sum_i [Sq(b_i) \times Sq(c_i)]/[M]. \end{aligned}$$

By Lemma 6.37,

$$= Sq(u)/[M].$$

By Lemma 6.38,

$$= w(TM).$$

□

COROLLARY 6.39. *If $f : M \to N$ is a continuous map of closed $n$ manifolds which is an isomorphism on $\mathbb{Z}_2$-homology, then $TM$ and $TN$ have the same Stiefel–Whitney classes.*

PROOF. The Wu classes and Steenrods squares are defined intrinsically from the structure of the $\mathbb{Z}_2$ cohomology. The Wu formula shows that this suffices to recover the Stiefel–Whitney classes. □

This is a surprising result, because defining the tangent bundle explicitly requires knowing the smooth structure. Nevertheless, we learn for example that if a topological manifold $M$ admits many distinct or "exotic" smooth structures, all its smooth structures must have the same Stiefel–Whitney classes on $TM$. The Wu formula also has some nice geometric consequences, like the following.

PROPOSITION 6.40 (Stiefel). *Every closed orientable three-manifold is parallelizable*

Proof. Let $M$ be a closed oriented three-manifold. Because our manifold is orientable, a 2-frame of the tangent bundle can always be extended to a 3-frame, i.e. a trivialization. Thus it suffices to construct a section of the 2-frame bundle $F_2$ on $M$; this is a fibre bundle with fibre $V(3,2) \cong \mathbb{R}P^3$. Note that $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$ and $\pi_2(\mathbb{R}P^3) = \pi_2(S^3) = 0$.

The first obstruction to finding a section of $F_2$ on the 2-skeleton is by definition the second Stiefel–Whitney class $w_2(TM) \in H^2(M;\mathbb{Z}_2)$. If $w_2(TM) = 0$, the next obstruction to extending a section to the 3-skeleton lives in the group $H^3(M;\pi_2(V(3,2))) = 0$. Since $M$ is three dimensional, this is the last obstruction. Thus it suffices to show the first obstruction $w_2(TM) = 0$.

By the Wu formula, $w_1(TM) = \nu_1$ and since $M$ is orientable $\nu_1 = 0$. Recall $\nu_2$ is defined so that $\nu_2 \smile x = Sq^2(x)$ for any $x \in H^1(M;\mathbb{Z}_2)$. But $Sq^2 : H^1(M;\mathbb{Z}_2) \to H^3(M;\mathbb{Z}_2)$ is axiomatically the zero operation. So, $\nu_2 = 0$. Then,

$$w_2(TM) = Sq^1(\nu_1) + Sq^0(\nu_2) = \nu_1^2 + \nu_2 = 0.$$

$\square$

There is another result also called the Wu formula to compute the Steenrod squares of Stiefel–Whitney classes.

**Theorem 6.41: Wu Formula for Stiefel–Whitney Classes of Steenrod Squares**

For any bundle, the Steenrod squares of the Stiefel–Whitney classes obey the formula,

$$Sq^i(w_j) = \sum_{k=0}^{i} \binom{j+k-i-1}{k} w_{i-k} w_{j+k}.$$

Proof. Let $x$ be the generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$. We must have $Sq(x) = x + x^2$. From the Cartan formula we obtain, $Sq(x^j) = (x + x^2)^j$ and so $Sq^i(x^j) = \binom{j}{i} x^{j+1}$.

By linearity and the Cartan formula, we can extend to determine the action of $Sq$ on the cohomology of $(\mathbb{R}P^\infty)^n$. This determines the action of $Sq$ on the cohomology of $BO(n)$ using our usual splitting principle argument and we should obtain the formula above. And then by naturality, this determines the action of $Sq$ on the Stiefel–Whitney classes of any bundle. $\square$

*Splitting Principle and Other Properties.* We have informally invoked the splitting principle several times. Let us be a bit more formal about what is going on and use it to describe how characteristic classes behave under direct sums and tensor products. We will then describe a few more important properties.

Let us deal with all cases simultaneously; so let $G$ denote $O(n), U(n)$ or $Sp(n)$ and $T$ the maximal torus of $G$. For the $O(n)$ case, $\mathbb{Z}_2$ coefficients in cohomology are to be understood. As we have stated many times before $Bi : BT \to BG$ is a fibration with injective pullback in cohomology. Suppose $E \to X$ is a bundle induced by a map $g : X \to BG$, we can pull back the above fibration along $g$.

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "f"] & BT \arrow[d, "Bi"] \\ E \arrow[r] & X \arrow[r, "g"] & BG \end{tikzcd}$$

We thus obtain a space $Y$ and a $G/T$ fibre bundle $f : Y \to X$ so the above commutes. Moreover, we can pullback $E \to X$ to obtain a bundle $f^*E$. Note because the classifying map $f \to BG$ factors through $BT$, the bundle $f^*E$ must be a direct sum of $n$ line bundles.

Recall that we had an isomorphism,

$$H^*(BT) \cong H^*(BG) \otimes H^*(G/T),$$

since the Serre spectral sequence of the fibration collapsed immediately.

Using the morphism of spectral sequences induced by $g$, the spectral sequence associated to $f$ must also collapse:

$$H^*(Y) \cong H^*(X) \otimes H^*(G/T),$$

and moreover the pullback $f^* : H^*(X) \to H^*(Y)$ is the edge morphism $x \mapsto x \otimes 1$. In particular, the pullback is injective. We have thus proved the following.

**Theorem 6.42: Generalized Splitting Principle**

Given a $G$-bundle $E \to X$, there is a bundle $f : Y \to X$ with fibre $G/T$ so that $f^*E$ has structure group $T$ and $f^*$ is injective on cohomology.

One of the upshots of this, is that if we can prove some algebraic fact about the characteristic classes of the pullback bundle with reduced structure group $T$, it must hold for the original bundle in the cohomology of $X$. In particular, we deduce that *to show an algebraic relation in terms of the Stiefel–Whitney,* *Chern, or quaternionic Pontryagin classes holds for a general bundle it will suffice to do so for a direct sum of line bundles.* Some more general examples of the splitting principle can be found here in a note of May.

Let us now use this to study the behaviour of characteristic classes under direct sums.

**Theorem 6.43: Whitney Sum Formula**

Given two bundles $\xi, \eta$, the total characteristic classes of their direct sum obey a product formula. So if they are real bundles,

$$w(\xi \oplus \eta) = w(\xi) \smile w(\eta).$$

If they are oriented,

$$e(\xi \oplus \eta) = e(\xi) \smile e(\eta).$$

If they are complex,

$$c(\xi \oplus \eta) = c(\xi) \smile c(\eta).$$

The same holds for Pontryagin classes modulo 2,

$$2p(\xi \oplus \eta) = 2p(\xi) \smile p(\eta)$$

Proof. We have already proved this for Stiefel–Whitney classes, essentially using the splitting principle and the fact $(BT)^n = B(T^n)$ with the expect isomorphism in cohomology. Exactly the same idea holds for Chern classes. It also holds for Pontryagin classes provided we work in a ring with $1/2$.

Here's an alternate proof for Pontryagin classes. Note first for a real bundle that the odd Chern classes of $\mathbb{C}\xi$ are 2-torsion; we will prove this below in Corollary 6.49. We have,

$$\begin{aligned}
2p_k(\xi \oplus \eta) &= (-1)^k 2 \cdot c_{2k}(\mathbb{C}(\xi \oplus \eta)) \\
&= (-1)^k \sum_{i \leq 2k} 2 \cdot c_i(\mathbb{C}\xi) \smile c_{2k-i}(\mathbb{C}\eta).
\end{aligned}$$

Ignoring the odd Chern classes, which disappear under 2-torsion,

$$\begin{aligned}
&= (-1)^k \sum_{i \leq k} 2 \cdot c_{2i}(\mathbb{C}\xi) \smile c_{2k-2i}(\mathbb{C}\eta) \\
&= (-1)^k \sum_{i \leq k} 2(-1)^i p_i(\xi) \smile (-1)^{k-i} p_{k-i}(\eta) \\
&= 2 \sum_{i \leq k} p_i(\xi) \smile p_{k-i}(\eta)
\end{aligned}$$

For the Euler case, note that one can study the Thom class of the product bundle $\xi \times \eta \to B^2$ and it will be the cross product of the Thom classes of the two individual bundles by a local calculation. Restricting to the base and then pulling back by the diagonal embedding $\Delta : B \to B^2$ recovers on one hand the Euler class of $\xi \oplus \eta$ and on the other hand the cup product of $e(\xi)$ and $e(\eta)$. ☐

Note the following important corollary of this. Elements in the ring of $\mathbb{Z}$ or $\mathbb{Z}_2$ power series are invertible if and only if their degree zero term is one. This is a simple exercise that can be proved inductively, or by construction a germ of the reciprocal of the power series at zero using Taylor series. Since the total Stiefel–Whitney and Chern classes have this form, $w(\xi)$ **and** $c(\xi)$ **are invertible in the cohomology ring**. This means that if we have a bundle $\eta$ with simple characteristic classes and so that the sum $\xi \oplus \eta$ has simple characteristic classes then we can invert the characteristic classes of $\eta$ and use the Whitney sum formula to find the characteristic classes of $\xi$.

We will apply this later to the case where $\xi = TM$ is the tangent bundle of a manifold $M$ and $\eta = \nu_M$ is the normal bundle to an embedding into Euclidean space $\mathbb{R}^n$. Almost by definition, $TM \oplus \nu_M = T\mathbb{R}^n|_M$, the latter of which is a trivial bundle and hence has total Stiefel Whitney class $w = 1$. We thus obtain the following useful result.

PROPOSITION 6.44 (Whitney Duality Formula). *If $M$ is a smooth manifold and $f : M \to \mathbb{R}^n$ is a smooth embedding with normal bundle $\nu_M$ then,*

$$w(TM) = w(\nu_M)^{-1}.$$

Note that although the same Whiteny sum formula holds for the Euler class, the Euler class will not be a unit in the cohomology ring and may in fact be a zero divisor. So we cannot usually solve for the Euler class of a summand just using the sum formula.

As with the direct sum, we can perform similar splitting principle analysis for the tensor product of two bundles, but the result will not be as clean. We state it here just for reference.

**Theorem 6.45: Characteristic Classes of Tensor Products**

For $\xi, \eta$ real bundles of ranks $n$ and $m$,

$$w(\xi \otimes \eta) = p_{n,m}(w_1(\xi), \ldots, w_n(\xi), w_1(\eta), \ldots, w_m(\eta)).$$

And if they are complex bundles,

$$c(\xi \otimes \eta) = p_{n,m}(c_1(\xi), \ldots, c_n(\xi), c_1(\eta), \ldots, c_m(\eta)).$$

Here $p_{n,m}$ is the unique polynomial so that if $\sigma_i$ and $\sigma'_i$ are the elementary symmetric polynomials in the variables $t_1, \ldots, t_n$ and $s_1, \ldots, s_m$ respectively then,

$$p_{n,m}(\sigma_1, \ldots, \sigma_n, \sigma'_1, \ldots \sigma'_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + t_i + s_j).$$

In particular, if the bundles are real,

$$w_1(\xi \otimes \eta) = m w_1(\xi) + n w_1(\eta),$$

and if they are complex,

$$c_1(\xi \otimes \eta) = m c_1(\xi) + n c_1(\eta).$$

There is another equivalent way to find the classes of a tensor product. We will just look at the complex case, which is more commonly studied.

**Definition 6.46: Chern Character**

Suppose $\xi = L_1 \oplus \cdots \oplus L_n$ is a direct sum of complex line bundles. We define its *Chern character* to be the power series in the cohomology ring,

$$\mathrm{ch}(\xi) := e^{c_1(L_1)} + \cdots + e^{c_1(L_n)}.$$

By the splitting principle, this definition extends to an arbitrary complex vector bundle. Explicitly the first few terms of the Chern character are given by,

$$\mathrm{ch}(\xi) = \mathrm{rk} + c_1 + \frac{1}{2}(c_1^2 - c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots .$$

It satisfies the formulae,

$$\begin{aligned} \mathrm{ch}(\xi \oplus \eta) &= \mathrm{ch}(\xi) + \mathrm{ch}(\eta), \\ \mathrm{ch}(\xi \otimes \eta) &= \mathrm{ch}(\xi)\mathrm{ch}(\eta). \end{aligned}$$

Using this definition, we see that,

$$\begin{aligned} \mathrm{ch}(\xi \otimes \eta) = &(\mathrm{rk}(\xi) + c_1(\xi) + \frac{1}{2}(c_1^2(\xi) - c_2(\xi)) + \cdots) \\ &\cdot (\mathrm{rk}(\eta) + c_1(\eta) + \frac{1}{2}(c_1^2(\eta) - c_2(\eta)) + \cdots) \\ = &\mathrm{rk}(\xi)\mathrm{rk}(\eta) + \mathrm{rk}(\eta)c_1(\xi) + \mathrm{rk}(\xi)c_1(\eta) + c_1(\xi)c_1(\eta) \\ &+ \frac{\mathrm{rk}(\xi)}{2}(c_1^2(\eta) - c_2(\eta)) + \frac{\mathrm{rk}(\eta)}{2}(c_1^2(\xi) - c_2(\xi)) + \cdots . \end{aligned}$$

From this we recover our formula for the first Chern class above. We also obtain the identity,

$$c_1(\xi\otimes\eta)^2 - c_2(\xi\otimes\eta) = 2c_1(\xi)c_1(\eta) + \mathrm{rk}(\xi)\big[c_1^2(\eta) - c_2(\eta)\big] + \mathrm{rk}(\eta)\big[c_1^2(\xi) - c_2(\xi)\big].$$

Since we know $c_1(\xi\otimes\eta)$, this tells us $c_2$. In the abstract this computation is a little crazy, but for concrete $\xi$ and $\eta$ it may not be so bad.

The Chern character is important for the following $K$-theory result. Note that the properties of the Chern character and the stability of characteristic classes means it defines a ring homomorphism,

$$\mathrm{ch}_{\mathbb{Q}} : K^0(X)\otimes\mathbb{Q} \to H^{\mathrm{even}}(X;\mathbb{Q}).$$

Using stability under suspension of homology and $K$-theory, this extends in a natural way to a $\mathbb{Z}_2$-graded ring homomorphism,

$$\mathrm{ch}_{\mathbb{Q}} : K^*(X)\otimes\mathbb{Q} \to H^*(X;\mathbb{Q}).$$

**Theorem 6.47**

For $X$ a finite CW complex, the Chern character induces a rational isomorphism,

$$\mathrm{ch}_{\mathbb{Q}} : K^*(X)\otimes\mathbb{Q} \cong H^*(X;\mathbb{Q}).$$

Proof. One can check by naturality and looking at the universal case of $BU(1)$ that the Chern character is the edge morphism in the Atiyah–Hirzebruch spectral sequence for the $K$-theory of $X$. As we have discussed before, there are no non-trivial stable cohomology operations on rational homology, and hence the spectral sequence must immediately degenerate. Since $K^*(\mathrm{pt})\otimes\mathbb{Q} = \mathbb{Q}$ in degree zero, we conclude the edge map is an isomorphism. $\square$

We now discuss elementary transformations of bundles.

**Theorem 6.48**

If $\xi$ is a real bundle with dual $\xi^*$, then $\xi$ and $\xi^*$ have the same characteristic classes.

If $\xi$ is an oriented bundle, and $-\xi$ is the same bundle with opposite orientation, then $e(-\xi) = -e(\xi)$.

If $\xi$ is a complex bundle with dual $\xi^*$ and the bundle $\overline{\xi}$ is the same underlying bundle as $\xi$ equipped with the opposite complex structure, then,

$$c_i(\overline{\xi}) = c_i(\xi^*) = (-1)^i c_i(\xi).$$

PROOF. Picking a metric on a real bundle $\xi$ gives a canonical identification of $\xi$ and $\xi^*$. So these bundles are isomorphic and hence have the same characteristic classes.

Now suppose $\xi$ is oriented. Note that if we reverse the orientation of $\xi$, this by definition will reverse the sign of the fundamental class of the fibres in the Thom space of $\xi$. Hence the Thom class, which is defined to restrict to the fundamental class of each fibre, will satisfy $t_{-\xi} = -t_\xi$. Since the Euler class is the restriction of the Thom class to the base we conclude $e(-\xi) = -e(\xi)$. As an alternate proof, note that the Euler classes of $\xi$ and $-\xi$ are both defined as the primary obstruction in $H^n(X; \pi_{n-1}(S^{n-1}))$ to the existence of a section of the underlying unoriented bundle. Changing the orientation simply changes the fibrewise identification of $\pi_{n-1}(S^{n-1})$ with $\mathbb{Z}$ by a sign, and so will change the Euler class by a sign.

Lastly, let $\xi$ be complex. Note that a choice of Hermitian metric gives a canonical isomorphism of $\xi^*$ and $\overline{\xi}$. Hence it will suffice to relate $c(\overline{\xi})$ and $c(\xi)$. Note that if $\xi$ is a complex line bundle, then $c_1(\xi) = e(\xi)$. If we use the conjugate bundle $\overline{\xi}$, it will be the same underlying real bundle but with opposite orientation (since an old $\mathbb{R}$-basis $v, iv$ for a fibre of $\xi$ becomes $v, -iv$ under conjugation). By the Euler computation above we know,

$$c_1(\overline{\xi}) = e(\overline{\xi}) = -e(\xi) = -c_1(\xi).$$

Now suppose $\xi = L_1 \oplus \cdots \oplus L_n$ is a direct sum of complex line bundles. We have,

$$\begin{aligned} c(\overline{\xi}) &= \prod_{i=1}^{n}(1 + c_1(\overline{L_i})) \\ &= \prod_{i=1}^{n}(1 - c_1(L_i)) \\ &= 1 - c_1(\xi) + c_2(\xi) - \cdots \pm c_n(\xi). \end{aligned}$$

We deduce the result from the splitting principle. □

COROLLARY 6.49. *If $\xi$ is an orientable odd dimensional bundle, then $e(\xi)$ is 2-torsion.*

*If $\eta$ is a complex bundle which is self-dual (for example if $\eta$ is the complexification of a real bundle) than its odd Chern classes are 2-torsion.*

PROOF. If $\xi$ is any bundle, there is an isomorphism given by multiplication by minus one on the fibres: $(p, v) \mapsto (p, -v)$. If $\xi$ has odd rank and is orientable, this isomorphism will reverse the orientation. Thus, $\xi = -\xi$ and so $e(\xi) = e(-\xi) = -e(\xi)$. So $2e(\xi) = 0$. The same argument works for odd Chern classes in the self-dual case. $\square$

*Computations for Smooth Manifolds.* We will not do very many computations, but the canonical example is to find the Stiefel–Whitney classes of the tangent bundle of $\mathbb{R}P^n$ and the Chern class of the tangent bundle of $\mathbb{C}P^n$. We should note the convention that we refer to the characteristic classes of a manifold to mean the characteristic classes of its tangent bundle. In this case we write $w(M)$ in place of $w(TM)$.

Let us write the homology of $\mathbb{R}P^n$ as $\mathbb{Z}[a]/(a^{n+1})$ and the homology of $\mathbb{C}P^n$ as $\mathbb{Z}[x]/(x^{n+1})$ where $|a| = 1$ and $x$ is Poincaré dual to $[\mathbb{C}P^{n-1}]$.

**Theorem 6.50**

The total Stiefel–Whitney class of $\mathbb{R}P^n$ is,

$$w(\mathbb{R}P^n) = (1 + a)^{n+1}.$$

The total Chern classes of $\mathbb{C}P^n$ is,

$$c(\mathbb{C}P^n) = (1 + x)^{n+1}$$

PROOF. We claim the tangent bundle $T = T\mathbb{R}P^n$ is isomorphic to the bundle $\mathrm{Hom}(L, \underline{\mathbb{R}}^{n+1}/L)$, where $L$ is the tautological line bundle on $\mathbb{R}P^n$ and $\underline{\mathbb{R}}^{n+1}$ is a trivial bundle.

By lifting to $S^n$, we see that over a point $\ell \in \mathbb{R}P^n$ intersecting $x, -x$ on the unit circle, the fibre of $T$ consists of pairs $\{(x, v), (-x, -v)\}$ for a choice of $v \in \mathbb{R}^{n+1}$ so that $x \cdot v = 0$, i.e. $v \in \ell^\perp$. On the other hand, over $\ell$, the fibre of $\mathrm{Hom}(L, \underline{\mathbb{R}}^{n+1}/L)$ is given by the space of linear maps from the one-dimensional vector space $\ell$ to the vector space $\mathbb{R}^{n+1}/\ell$. Considering the standard metric on $\mathbb{R}^{n+1}$ (and hence on $\underline{\mathbb{R}}^{n+1}$), we may identify $\mathbb{R}^{n+1}/\ell$ with $\ell^\perp \subset \mathbb{R}^{n+1}$. Thus our fibre over $\ell$ is $\mathrm{Hom}(\ell, \ell^\perp)$. Since $\ell$ is one-dimensional, the elements $\varphi \in \mathrm{Hom}(\ell, \ell^\perp)$ are in one to one correspondence with pairs $\{(x, v), (-x, -v)\}$ above via $\varphi \mapsto \{(x, \varphi(x)), (-x, -\varphi(x))\}$. This fives the desired isomorphism.

We now use this to find the Stiefel-Whitney classes. Note that we have $\mathrm{Hom}(L, L) = \underline{\mathbb{R}}$ since the identity map provides a non-vanishing section of the line bundle $\mathrm{Hom}(L, L)$. We then compute,

$$T \oplus \underline{\mathbb{R}} = \mathrm{Hom}(L, \underline{\mathbb{R}}^{n+1}/L) \oplus \mathrm{Hom}(L, L).$$

We obviously have an equality,

$$\begin{aligned} &= \mathrm{Hom}(L, (\underline{\mathbb{R}}^{n+1}/L) \oplus L) \\ &= \mathrm{Hom}(L, \underline{\mathbb{R}}^{n+1}) \\ &= \bigoplus_{i=1}^{n+1} \mathrm{Hom}(L, \underline{\mathbb{R}}) \end{aligned}$$

Since $L$ is isomorphic to its dual,

$$= \bigoplus_{i=1}^{n+1} L.$$

We claim the line bundle $L$ has total Stiefel-Whitney class $w(L) = 1+a$. To see this, note that the embedding $i : \mathbb{R}P^1 \to \mathbb{R}P^n$ gives the Möbius bundle $M$ over the circle as the pullback $i^*L$. By naturality, we have $w_1(M) = i^*w_1(L)$. By the non-triviality axiom of the Stiefel-Whitney classes we know that $w_1(M) = 1 + i^*a$ and so $w(L) = 1 + a$.

Since the Stiefel-Whitney classes are stable,

$$\begin{aligned} w(\mathbb{R}P^n) &= w(T \oplus \underline{\mathbb{R}}) \\ &= w\left(\bigoplus_{i=1}^{n+1} L\right). \end{aligned}$$

By the properties of Stiefel-Whitney classes under direct sum,

$$\begin{aligned} &= w(L)^{n+1} \\ &= (1 + a)^{n+1}. \end{aligned}$$

The proof in the Chern case is essentially identical. We obtain as above, as identification

$$T\mathbb{C}P^n \oplus \underline{\mathbb{C}} = \bigoplus_{i=1}^{n+1} \mathrm{Hom}(\gamma, \underline{\mathbb{C}}) = \bigoplus_{i=1}^{n+1} \overline{\gamma},$$

where $\gamma$ is the tautological line bundle on $\mathbb{C}P^n$. Let $x = -c_1(\gamma) \in H^2(\mathbb{C}P^n)$. We have,

$$c(\mathbb{C}P^n) = c(\overline{\gamma})^{n+1} = (1 + x)^{n+1}.$$

To determine $x$, we note,

$$e(\mathbb{C}P^n) = c_n(\mathbb{C}P^n) = (n + 1)x^n.$$

We have,

$$\chi(\mathbb{C}P^n) = \langle e(\mathbb{C}P^n), [\mathbb{C}P^n]\rangle = (n + 1)\langle x^n, [\mathbb{C}P^n]\rangle.$$

But the Euler characteristic of $\mathbb{C}P^n$ is $n+1$, so $\langle x^n, \mathbb{C}P^n \rangle = 1$, meaning that $x$ is the compatibly oriented generator of $H^1(\mathbb{C}P^n)$, i.e. the Poincaré dual of $[\mathbb{C}P^{n-1}]$. $\square$

COROLLARY 6.51. *$\mathbb{R}P^{2^r}$ cannot be immersed in $\mathbb{R}^N$ for $N < 2^{r+1} - 1$.*

PROOF. Note if $n = 2^r$, then $w(\mathbb{R}P^n) = 1 + a + a^n$. If $f : \mathbb{R}P^n \to \mathbb{R}^N$ is an immersion with normal bundle $\nu$, the Whitney duality formula implies

$$w(\nu) = w(\mathbb{R}P^n)^{-1} = 1 + a + a^2 + \cdots + a^{n-1}.$$

But note the rank of $\nu$ is $N - n$, and since $w_{n-1}(\nu) \neq 0$ we must have $N - n \geq n - 1$, or equivalently $N \geq 2^{r+1} - 1$. $\square$

REMARK 6.52. Note the Whitney immersion theorem says any compact $n > 1$ manifold can be immersed in $\mathbb{R}^{2n-1}$. This result is proved optimal in certain dimensions by the spaces $\mathbb{R}P^{2^r}$. $\triangle$

Here is one more theoretical result.

**Theorem 6.53**

Compact non-orientable surfaces cannot be smoothly embedded in $\mathbb{R}^3$.

PROOF. Let $\Sigma$ be a compact non-orientable surface and let $f : \Sigma \to \mathbb{R}^3$ be a smooth embedding with normal bundle $\nu$. We have, by Whitney duality,

$$w(\Sigma) = w(\nu)^{-1} = 1 + w_1(\nu).$$

Since $\Sigma$ is non-orientable, $w_1(\Sigma) \neq 0$ and so $w_1(\nu) \neq 0$ as well. This implies $\nu$ is a trivial line bundle. As we discussed previously, the Euler class of a normal bundle to an embedding measures the self-intersection of a surface. Hence, for $w_1(\nu) \neq 0$ we must have that the (un-oriented) self intersection number of $\Sigma$ inside of $\mathbb{R}^3$ is non-zero. But this is clearly not true. If we perturb the image of $\Sigma$ inside of $\mathbb{R}^3$ to $\Sigma'$, then we may "drag" $\Sigma'$ away from $\Sigma$ and off to infinity. This will provide a null cobordism of $\Sigma \cap \Sigma'$. Hence $w_1(\nu) = 0$. $\square$

*Characteristic Numbers of Smooth Manifolds.* From the characteristic classes of a smooth manifold, we can extract certain numerical invariants. These allow for direct comparisons of characteristic classes.

**Definition 6.54: Characteristic Numbers**

Let $M$ be a closed $n$-manifold with $\mathbb{Z}_2$ fundamental class $[M]$. Let $r_1, r_2, \ldots, r_n$ be a sequence of non-negative integers so that $r_1 + 2r_2 +$

$\cdots + nr_n = n$. Then we define a *Stiefel–Whitney number* of $M$,

$$w_{r_1,\ldots,r_n}(M) = \langle w_1(M)^{r_1} \cdots w_n(M)^{r_n}, [M]\rangle \in \mathbb{Z}_2.$$

If $M$ is a closed oriented $4n$-manifold with oriented fundamental class $[M]$, and $r_1, r_2, \ldots, r_n$ a sequence of non-negative integers so that $r_1 + 2r_2 + \cdots + nr_n = n$ then we define a *Pontryagin number* of $M$,

$$p_{r_1,\ldots,r_n}(M) = \langle p_1(M)^{r_1} \cdots p_n(M)^{r_n}, [M]\rangle \in \mathbb{Z}.$$

If $M$ is a closed almost complex $2n$-manifold with fundamental class $[M]$, and $r_1, r_2, \ldots, r_n$ a sequence of non-negative integers so that $r_1 + 2r_2 + \cdots + nr_n = n$ then we define a *Chern number* of $M$,

$$c_{r_1,\ldots,r_n}(M) = \langle c_1(M)^{r_1} \cdots c_n(M)^{r_n}, [M]\rangle \in \mathbb{Z}.$$

EXAMPLE 6.55. Consider an odd projective space $\mathbb{R}P^{2n-1}$. Then $w(\mathbb{R}P^{2n-1}) = (1+a)^{2n} = (1+a^2)^n$. In particular, $w_i(\mathbb{R}P^{2n-1}) = 0$ for $n$ odd. But every Stiefel–Whitney number $w_{r_1,\ldots,r_{2n-1}}(\mathbb{R}P^{2n-1})$ must have some $r_i \neq 0$ for $i$ odd. Hence all the Stiefel–Whitney numbers of $\mathbb{R}P^{2n-1}$ vanish. ▲

One major use of characteristic numbers is as bordism invariants. Consider the following proposition.

PROPOSITION 6.56. *If a compact $n$-manifold $M$ is null-bordant, then its Stiefel–Whitney numbers all vanish.*

PROOF. Suppose $M = \partial W$. By the tubular neighbourhood theorem, $TW|_M$ decomposes as a direct sum of $TM$ and a trivial line bundle. Thus under the inclusion $i : M \hookrightarrow W$, $i^* w(W) = w(M)$.

Note that the $\mathbb{Z}_2$ fundamental class $[M]$ is the boundary (in the LES of a pair) of the relative fundamental class $[W, M]$. We have,

$$w_{r_1,\ldots,r_j}(M) = \langle i^* w_{r_1,\ldots,r_j}(W), \partial[M,W]\rangle = \langle w_{r_1,\ldots,r_j}(W), i_*\partial[M,W]\rangle = 0,$$

since $i_*\partial = 0$ is the composition of consecutive maps in the LES of a pair. □

This result extends to other characteristic numbers. And remarkably, the converse is also true. The proof of this is closely linked to the computation of the cobordism rings which we discussed earlier.

**Theorem 6.57: Milnor, Novikov, Thom, Wall**

Two compact $n$-manifolds are cobordant if and only if their Stiefel–Whitney numbers agree.

Two compact oriented $n$-manifolds are oriented cobordant if and only if their Stiefel–Whitney numbers and Pontryagin numbers agree.

Two compact stably almost complex $n$-manifolds are complex cobordant if and only if their Chern numbers agree.

**6.5. Thom Spectra and Pontryagin–Thom.** Earlier we described a generalized homology theory based on singular manifolds up to cobordism called the bordism groups. By Brown representability, we know the bordism groups must be representable, i.e. the groups are given by homotopy classes of maps into some sequence of spaces. A very nice geometric model for these spaces and a proof they reproduce the bordism groups is due to Pontryagin and Thom and goes by the name of the *Pontryagin–Thom construction*.

**Definition 6.58: Thom Spectrum**

Let $\nu_n \to BO(n)$ be the tautological bundle, i.e. the associated bundle of the universal bundle $EO(n) \to BO(n)$. Define $MO(n)$ to be the Thom space $T\nu_n$. Note that the map induced by inclusion $BO(n) \to BO(n+1)$ induces a map $\nu_n \oplus \underline{\mathbb{R}} \to \nu_{n+1}$ and thus a map on the corresponding Thom spaces. The Thom space of $\nu_n \oplus \underline{\mathbb{R}}$ is the same thing as the suspension of $T\nu_n$. Hence we have a map,

$$\Sigma MO(n) \to MO(n+1).$$

The sequence of spaces $\{MO(n)\}_{n\in\mathbb{N}}$ along with these suspension maps is called the *unoriented Thom spectrum*.

If $\nu_n^+$ is the tautological bundle on $BSO(n)$ then $MSO(n) = T\nu_n^+$. If $\nu_n^{\mathbb{C}}$ is the tautological bundle on $BU(n)$ then $MU(n) = T\nu_n^{\mathbb{C}}$. We get corresponding suspension maps,

$$\Sigma MSO(n) \to MSO(n+1)$$
$$\Sigma^2 MU(n) \to MU(n+1).$$

These are called the *oriented Thom spectrum* and *complex Thom spectrum* respectively. For any sufficiently nice topological group $G$ ($Sp(n), SU(n), Spin(n)$ etc.) we have a corresponding $G$-structure Thom spectrum $MG$ defined analogously.

There is a multiplicative structure on the Thom spectra. Consider the map $BO(n)\times BO(m) \to BO(n+m)$ obtained by applying the classifying functor to the inclusion of block diagonal orthogonal matrices. This classifies a map of vector bundles $\nu_n \oplus \nu_m \to \nu_{n+m}$. Taking the Thom space of both sides produces a map,

$$MO(n) \wedge MO(m) \to MO(n+m).$$

Analogous constructions will work for any Thom spectrum $MG(n)$.

Now suppose we have some singular compact $q$-manifold $f : M^q \to X$. By Whitney embedding, we can find an embedding $i : M^q \hookrightarrow \mathbb{R}^{n+q}$. By compactifying the codomain, we can turn this into an embedding into $S^{n+q}$. Let $\nu_i$ be the normal bundle of the embedding, which we identify with a tubular neighbourhood of $i(M^q) \subset S^{n+q}$. We now have the "Pontryagin–Thom collapse map." This is defined as,

$$PT : S^{n+q} \to S^{n+q}/(S^{n+q} \setminus \nu_i) \to T\nu_i,$$

where the first map is the quotient identifying the complement of the tubular neighbourhood to a point and the second is an identification of this compactified tubular neighbourhood with the Thom space of the bundle. We may extend the map $f : M^q \to X$ to the normal bundle constantly on the fibres. We thus obtain a smash product,

$$f \wedge PT : S^{n+q} \to X_+ \wedge T\nu_i.$$

The normal bundle $\nu_i$ is classified by some map $M \to BO(n)$. Looking at the map on total spaces and taking Thom spaces yields a map of $T\nu_i \to MO(n)$. Composing with the map above defines a map,

$$PT_f : S^{n+q} \to X_+ \wedge MO(n).$$

There was a choice made in constructing this map beyond just what $f$ was; namely, the embedding we used and its dimension. Recall the fact that stably, all embeddings are isotopic. To see this, given embedddings $i_1, i_2 : M \hookrightarrow \mathbb{R}^{n_1,n_2}$, we can stabilize to embeddings $\widetilde{i_1}, \widetilde{i_2} : M \hookrightarrow \mathbb{R}^{n_1+n_2}$ and then there is an isotopy,

$$\Upsilon(x,t) = \widetilde{i_1}(x)t + (1-t)\widetilde{i_2}(x).$$

If we stabilize the embedding $i : M^q \hookrightarrow \mathbb{R}^{n+q}$ we chose earlier, the map $PT_f$ is suspended to the map,

$$\Sigma PT_f : S^{n+q+1} \to X_+ \wedge MO(n+1).$$

If we have two embeddings, stabilized to be isotopic, the isotopy will define a homotopy of the two versions of the map $PT_f$. Thus taking the colimit under stabilization and considering the map only up to homotopy, our map $PT_f$ will be a canonical element independent of embedding,

$$[\Sigma^\infty PT_f] \in \lim_{n\to\infty}[S^{n+q}, X_+ \wedge MO(n)].$$

Here the colimit is defined over the maps,

$$[S^{n+q}, X_+ \wedge MO(n)] \xrightarrow{\Sigma} [S^{n+q+1}, X_+ \wedge \Sigma MO(n)] \to [S^{n+q+1}, X_+ \wedge MO(n+1)],$$

where the second map comes from the Thom spectrum map.

Note that if the map $f : M \to X$ is cobordant to another map $g : N \to X$, then this cobordism will define a homotopy between the classes $[\Sigma^\infty PT_f]$ and $[\Sigma^\infty PT_g]$. Hence this Pontryagin–Thom map descends to a map,

$$PT : \Omega_q^O(X) \to \lim_{n\to\infty}[S^{n+q}, X_+ \wedge MO(n)].$$

Actually this is a group homomorphism. Here the group structure on the right comes from the co-$H$-space structure on $S^{n+q}$ which is compatible with the diagram in the colimit. To see this, if we take the cobordism class of two singular manifolds $f, g$, the corresponding collapse map $PT_{f,g}$ up to homotopy is given by gluing the domains of the collapse maps $PT_f$ and $PT_g$ under the co-$H$-space map $S^{n+q} \to S^{n+q} \vee S^{n+q}$. It is thus immediate that the $PT$ map respects group structures.

We now construct a candidate inverse to this map. Suppose we have some map $f : S^{n+q} \to X_+ \wedge MO(n)$. Compose with projection to obtain a map $S^{n+q} \to MO(n)$. Up to homotopy this map can be made to land in $\gamma_n^r \subset MO(n)$, where $\gamma_n^r$ is the tautological $n$-plane bundle on $G(r, n)$ for sufficiently large $r$. By the Thom transversality theorem, we may homotope $f$ to obtain a smooth map $S^{n+q} \to \gamma_n^r$ which is transverse to the zero section. Hence the preimage of the zero locus will be a smooth compact $q$-manifold $M^q$ naturally embedded in $\mathbb{R}^{n+q} \subset S^{n+q}$. Restricting $f$ and projecting to $X$ then defines a singular $q$-manifold $PT_f^{-1} : M^q \to X$. Note that if we take a homotopy of our map $f$ to a map $g$, it will induce a cobordism between $PT_f^{-1}$ and $PT_g^{-1}$ by looking (after a homotopy of homotopies) at the zero locus of the induced map $S^{n+q} \times I \to \gamma_n^R$. And we may freely stabilize the map $f$ to a map $S^{n+q+1} \to X_+ \wedge MO(n+1)$ and we will obtain the same $q$-manifold $M^q$ (up to cobordism) but now embedded in $\mathbb{R}^{n+q+1}$. We conclude there is a reverse map,

$$PT^{-1} : \lim_{n\to\infty}[S^{n+q}, X_+ \wedge MO(n)] \to \Omega_q^O(X).$$

If we have two $f : S^{n+q} \to X_+ \wedge MO(n), g : S^{n+r} \to X_+ \wedge MO(n)$, then the combined map $f \vee g$ with have (up to bordism) $PT_{f\vee g}^{-1} = PT_f^{-1} \sqcup PT_g^{-1}$ for obvious reasons. Hence this inverse is also a group homomorphism.

Let us show these are indeed inverses. If $f : M^q \to X$ is a singular manifold, then $PT_f$ is a map $S^{n+q} \to X_+ \wedge MO(n)$. If we take the preimage of the zero section in $MO(n)$ under this map, we reobtain $M^q \to X$. So, $PT^{-1} \circ PT = \mathrm{id}$. To conclude these are inverse isomorphisms, it will be enough to show $PT^{-1}$ is injective. Suppose we have $f : S^{n+q} \to X_+ \wedge MO(n)$ so that $M^q = PT_f^{-1}$ is null-cobordant. After increasing the dimension $n$ and stabilizing our map, we may assume $M^q \subset S^{n+q}$ bounds a manifold $W^{q+1}$ which is embedded in $D^{n+q+1}$. As before $f$ is homotopic after projection to a map $g : S^{n+q} \to T\gamma_n^r$ Let $U = g^{-1}(\gamma_n^r)$, which can be taken after homotopy to be a tubular neighbourhood of $M^q$ in $S^{n+q}$ so that $g : U \to \gamma_n^r$ is a map of total spaces of vector bundles. This can be extended by a relative tubular neighbourhood theorem to a define a map from a tubular neighbourhood $V$ of $W$ to $\gamma_n^r$ which is also a map of total space of vector bundles. We can then extend this to a map $\widetilde{g} : D^{n+1} \to T\gamma_n^r$ by setting it equal to $\infty$ outside $W$ and then to a map $\widetilde{G} : D^{n+1} \to X_+ \wedge MO(n)$. Then $\widetilde{G}$ by its construction provides a null homotopy of $f$.

We could have done much the same construction for more complicated $G$-structures. In the forward direction the stable $G$-structure on the normal bundle of the singular manifold provides a map to $MG(n)$ instead of $MO(n)$. And conversely, taking the zero locus of a map into $S^{n+q} \to X_+ MG(n)$ would endow the resulting manifold $M \subset S^{n+q}$ with a $G$-structure on its normal bundle coming from an identification with a tubular neighbourhood of the zero section of $EG(n)$. We have thus proved the following fundamental result.

**Theorem 6.59: Pontryagin–Thom Construction**

There is an isomorphism,

$$\Omega_q^O(X) = \lim_{n\to\infty} [S^{n+q}, X_+ \wedge MO(n)].$$

Similarly, there are isomorphisms,

$$\Omega_q^{SO}(X) = \lim_{n\to\infty} [S^{n+q}, X_+ \wedge MSO(n)],$$
$$\Omega_q^{U}(X) = \lim_{n\to\infty} [S^{2n+q}, X_+ \wedge MU(n)].$$

Analogous results hold for any nice topological group $G$ (e.g. $G = Sp, Spin, SU, etc.$).

In the case where $X$ is a point, we obtain the $n$-dimensional bordism ring. And in fact we can recover the ring structure from this construction. Using the multiplication maps we defined for $MO$, there is a natural map,

$$[S^{n+q}, MO(n)] \times [S^{m+r}, MO(m)] \xrightarrow{\wedge} [S^{n+q} \wedge S^{m+r}, MO(n) \wedge MO(m)]$$
$$\to [S^{n+m+q+r}, MO(n+m)].$$

We claim this coincides with the multiplication in $\Omega_*^O$ given by Cartesian product. Indeed, if we have two manifolds $N, M$, embedded into $S^{n+q}$ and $S^{m+r}$ respectively, then the product $N \times M$ can be embedded into $S^{n+q+m+r}$. The normal bundle $\nu_{N\times M}$ of this embedding is the direct sum of the two normal bundles $\nu_N, \nu_M$. And so the Pontryagin–Thom collapse map sends $S^{n+q+m+r}$ to $T\nu_{N\times M}$ and then to $MO(n+m)$ precisely via the Thom multiplication map $MO(n) \times MO(m) \to MO(n+m)$ induced by the classifying space map corresponding to direct summing vector bundles. Thus we see we have a ring isomorphism. Of course the same holds for the homology of a point in any other Thom spectrum $MG$.

Dual to our generalized homology theory is a generalized cohomology theory, the cobordism groups. Using the Pontryagin–Thom construction, these can be defined by,

$$\begin{aligned}\Omega^n_O(X) &= \lim_{n\to\infty}[\Sigma^n X, MO(n+q)],\\ \Omega^n_{SO}(X) &= \lim_{n\to\infty}[\Sigma^n X, MSO(n+q)],\\ \Omega^n_U(X) &= \lim_{n\to\infty}[\Sigma^{2n} X, MU(n+q)],\end{aligned}$$

and analogously for other $G$-structures. Note that there is always a graded ring map on the cobordism groups using our smash map on the Thom spectra.

In the case where $X$ is a closed $k$-manifold, there is a generalized Poincaré duality for cobordism giving,

$$\Omega^O_n(X) = \Omega^{k-n}_O(X),$$

and analogously for oriented and complex cobordism if $X$ is oriented or has a stable almost complex structure. The multiplication on cohomology then becomes a map in homology analogous to the intersection product,

$$\Omega^O_n(X) \times \Omega^O_m(X) \to \Omega^O_{m+n-k}(X).$$

This map agrees with the multiplication we described earlier coming from looking generically at the preimage of the diagonal under the product of two singular manifolds in $X$.

Thom spectra are actually quite general phenomena and we describe now how a Thom-like construction recovers a familiar generalized homology theory.

**Definition 6.60**

A *normal framing* of an embedding $i : M^n \hookrightarrow \mathbb{R}^{n+m}$ of an $n$-manifold is an isomorphism of the normal bundle $\nu_i$ with a trivial rank $m$ bundle.

A *stable normal framing* of a $n$-manifold $M$ is an isomorphism class of normal framings up to homotopy of framings and stabilization. I.e. given two embeddings $i : M \to \mathbb{R}^{n+a}$ and $j : M \to \mathbb{R}^{n+b}$ with normal framings, we identify their framings if $\nu_i \oplus \underline{\mathbb{R}}^k \cong \underline{\mathbb{R}}^{a+k}$ and $\nu_j \oplus \underline{\mathbb{R}}^\ell \cong \underline{\mathbb{R}}^{b+\ell}$ are homotopic for some $k$ and $\ell$. A manifold is called *stably framed* if it is equipped with a choice of stable normal framing.

We may define the *framed bordism groups* of a space $X$,

$\Omega_n^{\mathrm{fr}}(X) = \{\text{stably framed singular } n\text{-manifold}\}/\{\text{stably framed bordism}\}.$

That is, we consider singular $n$-manifolds $M \to X$ with $M$ stably framed and we identify two such maps if there is a stably framed singular $(n+1)$-manifold with boundary $W \to X$ which restricts to the two singular manifolds with their given framings on the boundary.

Note that not all manifolds possess a normal framing. By the Whitney duality formula one needs all the characteristic classes of the manifold to vanish, in particular $M$ should be orientable and spin.

Now consider the analogous Pontryagin–Thom construction for the framed bordism group. If we begin with a stably framed singular $q$-manifold $f : M^q \to X$ and a choice of embedding $i : M^q \to \mathbb{R}^{n+q}$ with a normal framing $\nu_i \cong \underline{\mathbb{R}}^n$ , we obtain a Pontryagin–Thom collapse map,

$$PT_f : S^{n+q} \to X_+ \wedge T\nu_i$$

Because we have a preferred trivialization of $\nu_i$, we have an isomorphism $\nu_i \to M \times \mathbb{R}^n$. Taking the Thom space, this becomes an isomorphism $T\nu_i \to S^n \wedge M_+$. We may compose the above map with this one and project out the factor of $M$ to get the following,

$$PT_f : S^{n+q} \to X_+ \wedge S^n = \Sigma^n X.$$

In the same manner as we showed before, up to bordisms of $f$ and a different choice of normal framing in the same stable normal framing class, the homotopy class of the stabilized version of $PT_f$ will not change. I.e. we obtain a well defined element of,

$$\lim_{n\to\infty} [S^{n+q}, \Sigma^n X] = \lim_{n\to\infty} \pi_{n+q}(\Sigma^n X) = \pi_q^{\mathrm{st}}(X).$$

Further, we can show this map is a homomorphism and we may obtain by the same procedure an inverse map. Hence one has the following result.

**Theorem 6.61**

There is an isomorphism,

$$\Omega_n^{\mathrm{fr}}(X) = \pi_n^{\mathrm{st}}(X).$$

Stably framed bordism defines the same generalized homology theory as stable homotopy.

In general, understanding bordism groups is difficult so this is not an effective method for computing homotopy groups. Nevertheless, it does suggest deeper connections between cobordism theory and stable homotopy theory that allow for theoretical studies.

Let us at least see these two groups agree for $X = \mathrm{pt}$ in low dimensions. A normal framing of a connected 0-manifold, i.e. a point, is just an element of $GL_n(\mathbb{R})$. Up to homotopy, there are two such stable normal framings given by the determinant of the framing of the point. A line segment is a cobordism between a pair of points with opposite framings. Hence $\Omega_0^{\mathrm{fr}}(\mathrm{pt}) = \pi_0^{\mathrm{st}} = \mathbb{Z}$ counting the number of positively framed minus negatively framed points.

There are four stable normal framings of a circle. This can be seen from the fact $[S^1, GL(\mathbb{R}^\infty)] = \pi_1(GL(\mathbb{R}^\infty)) = \mathbb{Z}_2 \times \mathbb{Z}_2$. There are two positively oriented framings, one coming from the standard embedding $i : S^1 \hookrightarrow \mathbb{R}^2$ and the other coming from the left invariant framing of $S^1$ as a Lie group. The second framing is equivalently associated to the embedding $j : S^1 \hookrightarrow \mathbb{R}^3$ as a figure-eight. There are two negatively-oriented framings given by multiplying these two framings by a sign.

The stable framing from $i$ is clearly null-bordant. And the stable framing from $j$ is 2-torsion in the bordism group using a cylinder bordism. Also multiple copies of the circle are always bordant to a single copy using a pair of pants with many leg holes. We conclude that the framed bordism group consists of two elements: the identity element associated to $i$ and the non-trivial element associated to the Lie framing or $j$. Hence, $\Omega_1^{\mathrm{fr}}(\mathrm{pt}) = \pi_1^{\mathrm{st}} = \mathbb{Z}_2$.

Thanks for reading!

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