Generalized Cohomology Operations and the Atiyah-Hirzebruch Spectral Sequence

Spectra Seminar
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§1. Cohomology Operations and Steenrod Algebras

Let E be a spectrum with cohomology theory $E^*(-)$.

Def^n: A (degree n) (generalized) cohomology operation is a natural transformation $\varphi: E^i(-) \to E^{i+n}(-)$.

A stable cohomology operation is a family $\{\varphi_i\}_{i\in\mathbb{Z}}$ of cohomology operations $\varphi_i: E^i(-) \to E^{i+n}(-)$ that commute with suspension: $\Sigma\circ\varphi_i = \varphi_{i+1}\circ\Sigma$. This is the same as an endomorphism $\varphi: E \circlearrowleft$.

By Yoneda, such operations naturally correspond to $E^*(E)$.

Def^n: $E^*(E)$ is called the E-Steenrod algebra. Its dual, $E_*(E) = \pi_*(E\wedge E)$ is the dual Steenrod algebra. The dual algebra is usually better understood/behaved.

Thm: If E is a (nice) ring spectrum, the pair $(\pi_* E, E_* E)$ is a Hopf algebroid.

Remark: we will not define a Hopf algebroid; it is a many-object generalization of a Hopf algebra. Alternatively, it is a commutative algebra notion dual to an algebraic Stack.

Examples:

(i) $E = HR$ classical cohomology operations.

* over $\mathbb{Q}$, there are no non-trivial operations: $H\mathbb{Q}$ is rationalization of $\mathbb{S}$.
* Hence over $\mathbb{Z}$, all operations are Bocksteins of torsion operations: $H\mathbb{Z} \xrightarrow{\times p} H\mathbb{Z} \to H\mathbb{Z}/p \underset{[1]}{\xrightarrow{\beta}} H\mathbb{Z}$.
* Over $\mathbb{Z}/2$, we want to understand $H^*(K(\mathbb{Z}/2,*),\mathbb{Z}/2)$. This can be done inductively using Serre spectral sequence of fibration $K(\mathbb{Z}/p, n-1) \hookrightarrow * \to K(\mathbb{Z}/p, n)$. The algebra of $\mathbb{Z}/2$ cohomology operations is generated by Steenrod squares $Sq^i: H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2)$. These are uniquely characterized by: $Sq^0 = \mathrm{Id}$, $Sq^i(x) = \begin{cases} x^2 & : |x| = i \\ 0 & : |x| < i \end{cases}$, $Sq^n(x\smile y) = \sum_{i=0}^{n} Sq^i(x)\smile Sq^{n-i}(y)$. (Cartan formula)

The $\mathbb{Z}/2$-Steenrod algebra $\mathcal{A}^*$ has interesting structure determined by Adem relations:

$$Sq^i\circ Sq^j = \sum_{k=0}^{\lfloor i/2\rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k}\circ Sq^k \quad \text{for } i,j>0,\ i<2j.$$

e.g. $Sq^1Sq^2 = Sq^3$, $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$, ... Its dual $\mathcal{A}_*$ is a Hopf algebra.

* Over $\mathbb{Z}/p$, there are analogous p^{th} power operations $P^i: H^n(X;\mathbb{Z}/p) \to H^{n+2i(p-1)}(X;\mathbb{Z}/p)$.

(ii) $E = KU$. Most important operations are Adams operations $\psi^k: KU(X) \to KU(X)$ for $k\in\mathbb{Z}$. These are defined on line bundles by $\psi^k(\xi) = \xi^{\otimes k}$ and extended by splitting principle. They satisfy $\psi^k(x+y) = \psi^k(x)+\psi^k(y)$, $\psi^k(xy) = \psi^k(x)\psi^k(y)$, $\psi^k\circ\psi^\ell = \psi^{k\ell}$.

Except for $\psi^{\pm1} = \mathrm{Id}$, complex conj., these are not stable. The structure of $KU_*(KU)$ is more complicated: it is a free module over $\pi_*(KU)$ on infinitely many generators (Adams–Clarke). This implies $KU^*(KU)$ is uncountable.

(iii) (Landweber–Novikov) $MU_*(MU) = \pi_*(MU)[b_1, b_2, \dots]$ $|b_i| = 2i$.

$MU^*(MU)$ has "formal basis" of <u>Landweber–Novikov operations</u> S_ω indexed by integer partitions.

Products in this algebra are computable, e.g. $S_n \circ S_n = (n+1) S_{2n} + 2 S_{[n,n]}$

$$S_m \circ S_n = (n+1) S_{m+n} + S_{[m,n]} \quad m \neq n.$$

(iv) (Adams–Quillen) p-localizing, MU becomes a wedge of copies of BP, the <u>Brown–Peterson spectrum</u>.

$BP_*(BP) = \pi_*(BP)[t_1, t_2 \dots]$ $|t_i| = 2(p^i - 1)$.

$BP^*(BP)$ has formal basis of <u>Quillen operations</u> r_ω, but products are <u>not</u> computable.

<u>Remark</u>: The Adams spectral sequence computes homotopy groups of spheres. Its E_2 page consists of Ext groups over the classical Steenrod algebra. If instead we work over examples (iii) or (iv) above, we get Adams–Novikov spectral sequence.

Other Steenrod algebras are largely boring or intractable.

§2. <u>Postnikov Towers and k-Invariants</u>

One situation where classical cohomology operations appear will be very important to us.

Given a nice space X, we can form X_n by killing off the $(n+1)$st and higher homotopy groups.

These spaces have maps $X_n \to X_{n-1}$ which are fibrations with fibre $K(\pi_n, n)$ where $\pi_n = \pi_n(X)$.

These maps can be assembled into the <u>Postnikov tower</u> of X.

Here we assume all $\pi_i \neq 0$, otherwise remove from tower.

It is a basic theorem that X can be recovered from its tower as an inverse limit.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
K(\pi_3,3) \arrow[r] & X_3 \arrow[r, "k^3"] \arrow[d] & K(\pi_4,5) \\
K(\pi_2,2) \arrow[r] & X_2 \arrow[r, "k^2"] \arrow[d] & K(\pi_3,4) \\
K(\pi_1,1) \arrow[r, "\cong"] & X_1 \arrow[r, "k^1"] & K(\pi_2,3)
\end{tikzcd}$$

In the picture, the pink zig-zags are fibre sequences. The tower (and thus X) can be recovered from the groups π_i and the maps $k^i : X_i \to K(\pi_{i+1}, i+2)$. These are the classifying maps of the fibrations $X_{i+1} \to X_i$.

These determine classes $k^i \in H^{i+2}(X_i, \pi_{i+1})$ called the <u>Postnikov k-invariants</u>; they are the fundamental obstruction classes for the corresp. fibrations, describing how to glue the $K(\pi_i, i)$ to form X.

Note $k^1 : K(\pi_1, 1) \to K(\pi_2, 3)$ is a cohomology operation. k^2 will define a partially defined map $K(\pi_1, 1) \dashrightarrow K(\pi_3, 4)$ as follows. The isomorphism $K(\pi_1, 1) \to X_1$ lifts to a map $K(\pi_1, 1) \to X_2$ on the kernel of k^1. We may compose with k^2 to get a map $\ker(k_1) \to K(\pi_3, 4)$. This map is not unique; the element one obtains in $K(\pi_3, 4)$ is only well defined up to the image of $K(\pi_2, 2) \to X_2 \xrightarrow{k^2} K(\pi_3, 4)$.

Thus we have a partially defined map from the kernel of one cohomology operation modulo the image of another.

Such a map is called a <u>secondary cohomology operation</u>. Analogously k^i determines a partial map $K(\pi_1, 1) \dashrightarrow K(\pi_{i+1}, i+2)$ which will be a <u>i-nary cohomology operation</u>.

Examples of higher cohomology operations include Massey products and Toda brackets.

If X is now a spectrum, we can again build it as a Postnikov tower built from Eilenberg–MacLane spectra.

But note a spectrum may have no base for its tower, so we need to choose a truncation $X\langle r \rangle$.

For each r, we get a tower and thus a 2-parameter family of k-invariants.

The first invariant of each truncation will be a stable cohomology operation.

§3. Atiyah–Hirzebruch

One of the most powerful tools for computing generalized cohomology is the following.

<u>Thm</u>: (Whitehead) Let E be a spectrum and X a finite CW complex. There are spectral sequences:

$$E^2_{p,q} = H_p(X; E_q(*)) \Rightarrow E_{p+q}(X) \qquad \text{(Atiyah–Hirzebruch Spectral Sequence)}$$
$$E_2^{p,q} = H^p(X; E^q(*)) \Rightarrow E^{p+q}(X) \qquad \text{(AHSS)}$$

If E is a ring spectrum, the cohomological spectral sequence is multiplicative.

<u>Remarks</u>: (i) There is a more general Atiyah–Hirzebruch–Serre SS $H_*(B; E_*(F)) \Rightarrow E_*(X)$ for $B \hookrightarrow X \to F$ a fibration. Recover AHSS from $* \to X \to X$.

(ii) We can also generalize to the case where X is a spectrum, but we will have convergence issues if either E or X is not bounded below (we need downwards diagonals of E^2 page to be finite).

(iii) Spectral sequence arise from filtrations. There are two filtrations one can use to build the AHSS.
(a) (Cartan–Eilenberg) One can filter X by its skeleta. Then one repeats the proof of the Serre ss.
(b) (Maunder) An alternate approach uses a filtration coming from the Postnikov tower of E.

(iv) Maunder's approach implies the differentials in the cohomological AHSS for E are exactly the Postnikov k-invariants of E? In particular, the first non-trivial differential out of any row will be a stable cohomology operation.

(v) If E is a ring spectrum, the differentials must also be derivations. E.g. over $\mathbb{Z}/2$, the only elements of $\mathcal{A}^*$ which are derivations are the <u>Milnor primitives</u> defined inductively by $Q_i := [Sq^{2^i}, Q_{i-1}]$.

(vi) Note the AHSS only converges to the associated graded of $E_*(X)$. There may be extension problems.

(vii) The AHSS is natural. In particular, an isomorphism on homology induces one on all generalized cohomology (generalized Whitehead).

(viii) Every spectrum $\mathbb{Q}$-localized is a wedge sum of $H\mathbb{Q}$'s. Such a space has trivial k-invariants and so the AHSS degenerates immediately. Hence, $E^*(X) \otimes \mathbb{Q} \cong H^*(X;\mathbb{Q}) \otimes E^*(*)$. → this is likely familiar as Chern character iso. in K-theory.

<u>Moral</u>: Every interesting in stable homotopy happens p-locally!

<u>Examples</u>: For the rest of the talk, we examine some interesting topological implications of the AHSS.

(i) <u>KU</u>: The E_2-page of the AHSS for KU looks like:

$$\begin{array}{r|l} & \vdots \\ 2 & H^*(X) \\ 1 & 0 \\ 0 & H^*(X) \\ -1 & 0 \\ & \vdots \end{array} \qquad d_3$$

The first differential is on the E_3-page. It is the cohomology operation,
$d_3 = \beta \circ Sq^2 \circ r : H^*(X;\mathbb{Z}) \to H^{*+3}(X;\mathbb{Z})$ where r is mod 2 reduction and β is the mod 2 Bockstein.

(a) $KU(\mathbb{CP}^n)$ The cohomology is concentrated in even degrees, so all differentials vanish. $KU^0(\mathbb{CP}^n) \cong \mathbb{Z}^{n+1}$, $K^1(\mathbb{CP}^n) = 0$. Using multiplicativity, $KU^*(\mathbb{CP}^n) \cong \mathbb{Z}[\gamma]/\gamma^{n+1}$ where $\gamma = \xi - 1$ for ξ Hopf bundle.

(b) $KU(\mathbb{RP}^n)$ The cohomology is concentrated in even degrees, except maybe H^n. But there are no non-trivial maps $\mathbb{Z}/2 \to \mathbb{Z}$, so all differentials vanish. $KU^0(\mathbb{RP}^n) \cong \mathbb{Z} \oplus G$, $KU^1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$.
G has order $\lfloor n/2 \rfloor$. In fact it is cyclic using naturality of $\mathbb{RP}^{2n-1} \hookrightarrow \mathbb{RP}^{2n} \hookrightarrow \mathbb{CP}^n$.
$\widetilde{KU}^0(\mathbb{RP}^n) \cong \mathbb{Z}[\lambda]/\lambda^{\lfloor n/2 \rfloor}$ where $\lambda = \mathbb{C}(\xi - 1)$ for ξ the real Hopf bundle.

Exercise: Compute $KU(\mathbb{R}P^3 \times \mathbb{R}P^4)$ either with Künneth or by AHSS (d_3 will not vanish).

(ii) KO: The E_2-page looks as follows:

	$\vdots$	
8	$H^*(X;\mathbb{Z})$	$d_2^{*,8k}$
7	$H^*(X;\mathbb{Z}/2)$	$d_2^{*,8k-1}$
6	$H^*(X;\mathbb{Z}/2)$	$d_3^{*,8k+6}$
5	0	
4	$H^*(X;\mathbb{Z})$	$d_5^{*,8k+4}$
3	0	
2	0	
1	0	
0	$H^*(X;\mathbb{Z})$	
	$\vdots$	

The primary k-invariants give differentials:

$$d_2^{*,8k} = Sq^2 \circ r$$

$$d_2^{*,8k-1} = Sq^2$$

$$d_3^{*,8k+6} = \beta \circ Sq^2$$

$$d_5^{*,8k+4} = \beta \circ Sq^4 \circ r$$

(iii) MO: Fact: (Thom) MO is a wedge of Eilenberg MacLane spectra $H\mathbb{Z}/2$.

Corollary: AHSS degenerates and $\Omega^O_*(X) \cong H_*(X;\Omega^O_*(pt))$

In the AHSS, one has an edge morphism $E_*(X) = E^\infty_{*,0} \hookrightarrow E^2_{*,0} = H_*(X;\pi_0(E))$. This is the <u>generalized Hurewicz homomorphism</u> coming from map of spectra $E \to HG$ where $G = \pi_0(E)$. (one can view this as start of Postnikov tower for connective truncation.)

In the case of MO, we get a map $\Omega^O_n(X) \to H_n(X;\mathbb{Z}/2)$. Geometrically, this sends a bordism class $f: M^n \to X$ to $f_*[M]$. Note this has implications for Steenrod's problem on the realizability of homology classes by such bordism maps. We conclude that all homology classes of X are representable by manifolds (in category G) if and only if the edge morphism $\Omega^G_*(X) \to H_*(X;\pi_0(MG))$ is surjective.
i.e. iff no differentials in AHSS leaving the 0th row are non-trivial.

Thm: (Thom) All mod 2 homology classes are realized by unoriented bordisms.

(iv) MSO: We already know rationally the AHSS degenerates, so we have the following.

Thm: (Thom) An (odd) integer multiple of every integral cohomology class is realized by oriented bordism.

Fact: (Wall, BP) Localizing MSO at $p=2$ gives a wedge of $H\mathbb{Z}/2$'s & $H\mathbb{Z}_{(2)}$'s. Localizing MSO at odd primes or MU at any prime gives wedges of BP's. The k-invariants of BP were found explicitly in the original construction of Brown-Peterson.

Recall $\Omega_1^{SO} = \Omega_2^{SO} = \Omega_3^{SO} = 0$, $\Omega_4^{SO} = \mathbb{Z}$ generated by $\mathbb{C}P^2$, $\Omega_5^{SO} = \mathbb{Z}/2$ generated by $SU(3)/SO(3)$ (Wu manifold).

The E^2_{**}-page AHSS looks as follows:

5	$H_0(X;\mathbb{Z}/2)$	$\cdots$				
4	$H_0(X)$	$H_1(X)$	$\cdots$			
3	0	0	$\cdots$			
2	0	0				
1	0	0	$\cdots$			
0	$H_0(X)$	$H_1(X)$	$H_2(X)$	$H_3(X)$	$H_4(X)$	$H_5(X)$
	0	1	2	3	4	5

From the above fact, the only possible differential below degree 8 can come from the 1st k-invariant of BP at $p=3$. This gives $d_5^{*,0} = (\beta \circ P^1_{(p=3)} \circ r)^{dual}$ ↪ need to dualize map with respect to cap product since this is homological.

P^1 is trivial on classes of degree less than 2. We see, $\Omega_i^{SO}(X) = H_i(X)$ $i = 0,1,2,3$. And we have SES $0 \to \mathbb{Z} \to \Omega_4^{SO}(X) \to H_4(X) \to 0$. In fact this splits via the signature $\Omega_4^{SO}(X) \to \mathbb{Z}$.

We also see classes of dim ≤ 6 are always realized by oriented bordism. This fails for dim ≥ 7. Indeed Thom found a class x in the cohomology of lens space $B\mathbb{Z}/3^2$ with $\beta P^1(x) \neq 0$.

(v) $\mathbb{S}$: The E^2-page of the <u>reduced</u> AHSS is:

4	0	
3	$\tilde{H}_*(X;\mathbb{Z}/24)$	
2	$\tilde{H}_*(X;\mathbb{Z}/2)$	$d^2_{*,1} = (Sq^2)^{dual}$
1	$\tilde{H}_*(X;\mathbb{Z}/2)$	$d^2_{*,0} = (Sq^2 \circ r)^{dual}$
0	$\tilde{H}_*(X;\mathbb{Z})$	

We deduce, $\pi_1^{st}(X) \cong H_1(X)$. And there is SES $0 \to H_1(X;\mathbb{Z}/2) \to \pi_2^{st}(X) \to H_2(X) \to 0$.

In fact this always splits: $\pi_2^{st}(X) \cong H_1(X;\mathbb{Z}/2) \oplus H_2(X)$.

We also get SES (non-canonical)

$$0 \to H_1(X;\mathbb{Z}/2) \oplus \ker(Sq^2: H_2(X;\mathbb{Z}/2) \to H_4(X;\mathbb{Z}/2)) \to \pi_3^{st}(X) \to H_3(X;\mathbb{Z}) \to 0.$$