The Atiyah-Floer Conjecture for Admissible Bundles

Student Low-Dimensional Topology Seminar
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Let Y be a closed oriented 3-manifold and $P \to Y$ an $SO(3)$ bundle.
P is determined by $w_2(P) = PD[\gamma] \in H^2(Y; \mathbb{Z}/2)$ for some embedded curve $\gamma \subset Y$.

Def^n: (Y, γ) is admissible if $\exists\, S \hookrightarrow Y$ an embedded surface with $S \cdot \gamma \equiv 1 \pmod 2$.
(Equivalently w_2 lifts to a non-torsion element of $H^2(Y; \mathbb{Z})$ (this rules out flat connections w/ ∞ isotropy gp.))

Note: Given arbitrary Y, we can form an admissible pair $Y^\# := (Y \# T^3, \{pt\} \times S^1)$.
We are interested in gauge equivalence classes of flat connections on the bundle determined by (Y, γ).
We will see these as generators for a Floer theory in two quite different ways.

§1. Instanton Floer Homology

Define the Chern-Simons functional $CS : \mathcal{A}(P) \to \mathbb{R}$ on the space of connections by:

$CS(A) = \frac{1}{8\pi^2} \int_Y tr(A \wedge F_A + \frac{2}{3} A \wedge A \wedge A)$. This descends to a $\mathbb{R}/\mathbb{Z}$-valued functional modulo action of the gauge group. The critical points of CS are exactly the flat connections.

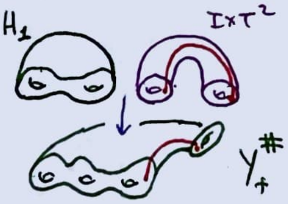
Now we can run usual Morse/Floer machinery for CS: we have homology theory generated by equiv. classes of flat connections with differential counting gradient flow lines $d/dt\, A(t) = -\nabla CS\, A(t)$.
For a choice of metric, we can interpret this flow line as a single connection A on $\mathbb{R} \times Y$ satisfying
$F_A + \star F_A = 0$. This is the anti-self dual or ASD equation, its (equiv. classes) of solutions are called "instantons". So counting the 0-dim^l moduli spaces of finite energy instantons with given asymptotics gives instanton Floer homology $I_*(Y, \gamma)$. For general Y, also have framed instanton Floer homology $I^\#(Y) := I_*(Y^\#)$. These theories are relatively $\mathbb{Z}/4$-graded and independent of choices.

Note: Originally I_* was constructed for Y a HS^3. Theory is easier there, but next theory more difficult.

§2. Symplectic Instanton Homology

Given (Y, γ) admissible, we can find splitting $Y = Y^+ \sqcup_\Sigma Y^-$ where γ (transversely) intersects each component of Σ in an odd number of pts. Note Σ is nec. disconnected.

Note: Given Y and a Heegaard splitting $Y = H_1 \sqcup_\Sigma H_2$ we may split $Y^\#$ as $H_i \natural I \times T^2$
Call pieces $Y^\#_+, Y^\#_-$.



Let Σ_g be a component we split along. γ intersects Σ in an odd set of pts. and $P|_\Sigma$ has $w_2(P|_\Sigma) = PD[\gamma \pitchfork \Sigma]$ is the non-zero element in $H^2(\Sigma; \mathbb{Z}/2)$.
The moduli space of flat connections on $P|_\Sigma$ is identified with the odd character variety

$$\mathcal{M}(\Sigma_g) := \{(A_1, \dots, A_g, B_1, \dots, B_g) \in SU(2)^{2g} \mid \prod_i [A_i, B_i] = -e\} / SU(2).$$

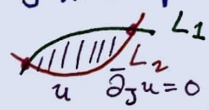
This is a smooth mfld. of dim $6g-6$. In fact it is symplectic! Space of connections $\mathcal{A}$ has symplectic form $\omega(\xi,\eta) = \int_\Sigma \langle \xi, \eta\rangle$, $\xi,\eta \in \Omega^1(\Sigma; \mathrm{ad}\,P)$. The gauge group defines a Hamiltonian action with moment map $A \mapsto F_A$ and symplectic reduction $\mathcal{A} /\!/ \mathcal{G}$ is $\mathcal{M}(\Sigma)$.

Given half of our splitting, (Y^+, γ^+) with boundary Σ', we can consider the subspace $L_{(Y^+,\gamma^+)} \subset \mathcal{M}(\Sigma')$ of flat connections that can be extended over $P|_{Y^+}$.

After a perturbation, $L_{(Y^+,\gamma^+)} \looparrowright \mathcal{M}(\Sigma')$ is an immersed Lagrangian. We will want to assume it is embedded (this holds e.g. in $Y^\#$ case, or under some topological assumption on splitting).

Now after perturbation $L_{(Y^+,\gamma^+)} \pitchfork L_{(Y^-,\gamma^-)}$, with intersections corresponding to flat connections on all of Y.

Lagrangian Floer theory defines a homology theory generated by intersections of a pair of Lagrangians with differential counting 0-dim^{l} moduli space of finite energy J-holomorphic strips with boundary on the pair of Lagrangians and asymptotic to intersection points



(these Lagrangians are monotone, so Floer analysis OK)

This will be the <u>symplectic</u> instanton homology $SI_*(Y,\gamma) = HF_*(L_{(Y^+,\gamma^+)}, L_{(Y^-,\gamma^-)})$.

There is also a framed version $SI^\#(Y) = HF_*(L_{Y^\#_+}, L_{Y^\#_-})$.

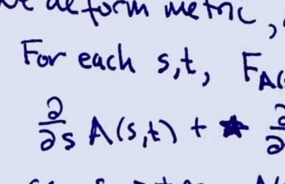
This theory is also $\mathbb{Z}/4$-graded. It is not, in any clear way, independent of our splitting.

§3. The Atiyah-Floer Conjecture

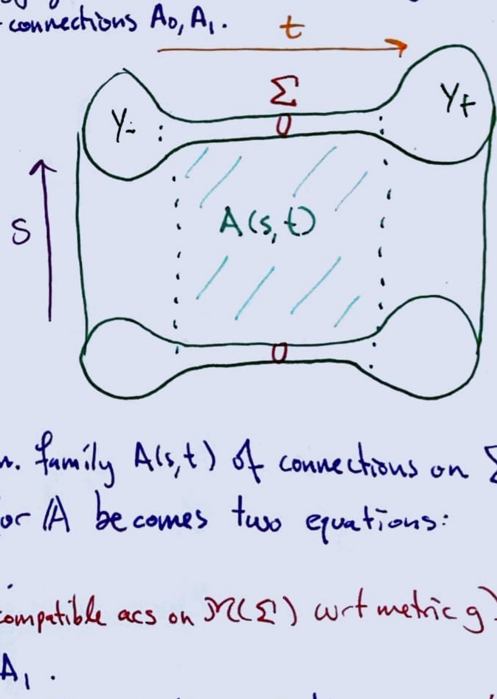
<u>Conjecture</u>: (Atiyah '88) $SI(Y,\gamma) \cong I(Y,\gamma)$ as graded homology groups (actually his conjecture was in HS^3 case)

Atiyah's Idea: Use an "adiabatic argument." Obviously the homology generators coincide, so we just need to equate differential counts. Consider differential between flat connections A_0, A_1.

On instanton side, have ASD connection $\mathbb{A}$ on $\mathbb{R}\times Y$, i.e. family $A(s)$ on Y satisfying $\frac{\partial}{\partial s} A(s) = -\star F_{A(s)}$.



Now split Y along Σ' and neck stretch



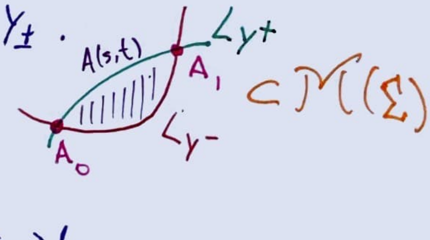
on neck, have connection on $I_s \times I_t \times \Sigma'$, i.e. 2-param. family $A(s,t)$ of connections on Σ'.

As we deform metric, and hence $\star$ operator, ASD eq^n for $\mathbb{A}$ becomes two equations:

(a) For each s,t, $F_{A(s,t)} = 0$, i.e. $A(s,t)$ is flat on Σ'.

(b) $\frac{\partial}{\partial s} A(s,t) + \star \frac{\partial}{\partial t} A(s,t) = 0$ (Note $\star$ gives an ω-compatible acs on $\mathcal{M}(\Sigma)$ wrt metric g)

Note as $s \to \pm\infty$, $A(s,t)$ approaches flat connections A_0, A_1.
as $t \to \pm\infty$, $A(s,t)$ needs to converge to flat connection on Σ' extending over $Y_\pm$.



So this neck stretched connection is really describing a $\star$-holomorphic strip inside $\mathcal{M}(\Sigma')$, which is precisely the differential on symplectic side.

By invariance under change of metric, we expect differential counts to coincide.

This heuristic was worked out rigorously in mapping tori case by Dostoglou-Salamon '94.

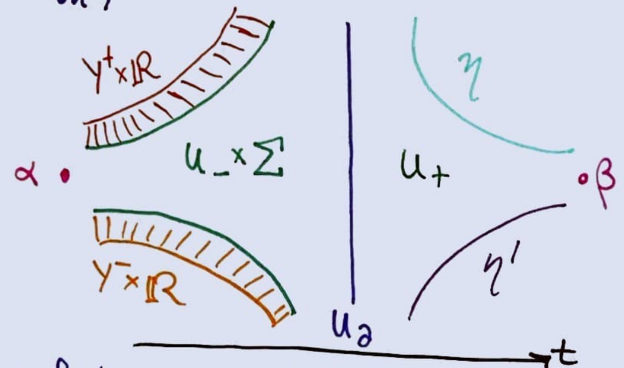
In the admissible and framed case, a proof of the conjecture has recently been published

Thm: (Daemi–Fukaya–Lipyanskiy '21) For admissible pairs $I_*(Y,\gamma) \cong SI_*(Y,\gamma)$. $I^\#(Y) \cong SI^\#(Y)$.

Idea: Quite different from adiabatic idea. We will study the quite daunting "mixed equation" moduli space.

As a domain consider a 4-times punctured disk split into two halves u_+, u_- along line u_∂.

For α, β flat connections on Y, we consider the following schematic configurations:



More precisely we are interested in the zero dimensional moduli space $\mathcal{M}_0(\alpha,\beta)$ of pairs (A,u) where:

- A is finite energy ASD connection on $Y^- \times \mathbb{R} \cup u_- \times \Sigma \cup Y^+ \times \mathbb{R}$: $F_A + * F_A = 0$
- $u : u_+ \to \mathcal{M}(\Sigma)$ is finite energy J-holomorphic curve : $\bar{\partial}_J u = 0$
- $u|_\eta \subseteq L_{(Y^+,\gamma^+)}$, $u|_{\eta'} \subseteq L_{(Y^-,\gamma^-)}$, $\lim_{t\to\infty} u = \beta$.
- $\lim_{t \to -\infty} A = \alpha$
- For $q \in u_\partial$, $A|_{\Sigma \times \{q\}}$ is flat and $[A|_{\Sigma\times\{q\}}] = u(q)$.

Modulo gauge transformations and automorphisms.

There are <u>many</u> things to show: that this is elliptic Fredholm problem, tansversality can be obtained after perturbation, compactness and gluing work, etc.

Assuming these details, we can define map $\Phi : IC_* \to SIC_*$ by $\langle \Phi(\alpha), \beta \rangle = \# \mathcal{M}_0(\alpha,\beta)$.

We deduce this is a chain map by breaking analysis at ends $t = \pm\infty$. (Need to rule out breaking near u_∂)

Our flat connections are filtered by Chern–Simons action. Since the setup is monotone, non-constant solutions of this equation have positive energy and Φ is action-decreasing.

Only zero-energy solutions are constant solutions, hence we can write

$\Phi = I + N$, where I is the identity and N is action strictly-decreasing, hence nilpotent. Thus Φ induces an isomorphism on homology.