THE ADHM CONSTRUCTION

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1. Introduction

The Yang-Mills equations are an important class of non-linear PDEs in physics and geometry. The *instantons* or anti-self dual connections are a special family of minimal energy four-dimensional solutions which Simon Donaldson used to prove many novel results in the classification of smooth four-manifolds (see [6] for a broad summary of the achievements of mathematical gauge theory).

In 1977, Atiyah, Drinfield, Hitchin, and Manin classified all SU(2) ASD-connections on $\mathbb{R}^4$ and S^4 in terms of simple linear-algebraic data [3]. This well-known classification is now referred to as the *ADHM construction.* We will recount their construction and the algebro-geometric ingredients that go into proving their classification is complete.

Our account will largely follow that of a lecture series by Atiyah [2] and in parts a note by Donaldson [7]. Both are based on the original account of Atiyah, Drinfield, Hitchin, and Manin, using twistor theory, the Ward correspondence, and algebro-geometric results of Horrocks and Barth. Other approaches to the same result exist; see [8, chapter 3] for a different proof.

2. Preliminaries

We will assume the reader is familiar with the formulation of the Yang-Mills equations, an account of which can be read in [8, chapter 2]. Recall our basic setup is a vector bundle E with structure group G, a (matrix) Lie group, over X, a four-dimensional Riemannian manifold. A connection A on this bundle is represented by a covariant derivative operator ∇_A which locally has components,

$$\nabla_\mu = \partial_\mu + A_\mu,$$

where A_μ is matrix-valued. The curvature F_A of the connection locally has components,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The curvature $F_A = F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is globally a two-form with values in the adjoint bundle $\mathfrak{g}_E$. The metric on X along with the Killing form on $\mathfrak{g} = \mathrm{Lie}(G)$ determines a Hodge-star operator $\star : \Omega^2_X(\mathfrak{g}_E) \to \Omega^2_X(\mathfrak{g}_E)$ and so we may define the *anti-self dual (ASD) equations* for our connection A as,

$$\star F_A = -F_A.$$

Solutions of this equation are called *instantons* (we will sometimes be imprecise about distinguishing self dual and anti-self dual solutions, as they are interchanged by switching orientations). On S^4, instantons are the absolute minima of the Dirichlet energy, $\|F_A\|^2_{L^2}$; their value on this functional will be $8\pi^2 c_2(E)$, where $c_2(E) \in \mathbb{Z}$ is the second Chern class of the bundle evaluted on $[S^4] \in H_4(S^4;\mathbb{Z})$. We will speak of k-instanons on S^4 to refer to instantons defined on the bundle E with $c_2(E) = k$ (note the value of the second Chern class totally determines the isomorphism class of the bundle).

Uhlenbeck's removable singularities theorem (see [12]) implies that instantons on S^4 correspond to finite energy instantons on $\mathbb{R}^4$ via stereographic projection (or in the inverse direction, via conformal compactification). On $\mathbb{R}^4$, one may think of the k-instantons as those consisting of k localized particles (or solitons in PDE terminology).

An early explicit solution due to Belavin, Polyakov, Schwartz, and Tyupkin in 1975 was the *BPST instanton*, a one-instanton in the SU(2)-theory [5]. Explicitly this is given by,

$$A_\mu(x) = \frac{2x_\nu}{x^2+\lambda^2}\eta_{\mu\nu} \quad \text{and} \quad F_{\mu\nu}(x) = \frac{4\lambda^2}{(x^2+\lambda^2)^2}\eta_{\mu\nu},$$

where $\lambda \in \mathbb{R}^+$ is the *scale*, parameterizing how localized the curvature is at zero. This was extended by 't Hooft in 1976 to a $5N$-parameter family of SU(2) N-instantons. The $5N$ parameters are N scales $\lambda_i \in \mathbb{R}^+$ and N centres $x_i \in \mathbb{R}^4$ which describe the localization and location of the N instantons [13, p. 87]. This was slightly extended to an explicit family of $5N + 4$ solutions, but the Atiyah-Singer index theorem implies the space of solutions should actually have dimension $8N-3$ [13]. This larger family and its analogue for other structure groups is what was classified in the ADHM construction.

Thus, the problem we address in the remainder of this document is to completely classify the k-instantons on a SU(2) bundle over S^4 for $k \geq 1$ (or more generally bundles for other classical Lie groups).

Let us quickly summarize our path to a solution. We will first explain how twistor theory allowed Ward to express instantons on S^4 as equivalent to certain complex-geometric data. Then we discuss the work of Horrocks, who constructs certain such bundles using linear-algebraic data. Then, we summarize how one proves using some technical tools of algebraic geometry that Horrocks' construction is exhaustive and so classifies all instantons.

3. Penrose's Twistor Theory and the Ward Correspondence

We begin by re-expressing the problem of classifying instantons in terms of finding certain holomorphic vector bundles. This is known as the Ward correspondence, which we establish by way of ideas from twistor theory.

3.1. **Twistor Geometry.** Twistor theory, as developed by Roger Penrose, is a method of studying four-dimensional physical theories by converting problems to complex projective space (called *twistor space* in the physicst theory). See Chapter 7 of [9] for an introduction to the subject.

For our context, we use twistor theory to export the instanton problem from S^4 to $\mathbb{C}P^3$ in the following manner. Consider homogeneous coordinates $[z_1 : z_2 : z_3 : z_4]$ for $\mathbb{C}P^3$. Identifying $\mathbb{C} \subset \mathbb{H}$ in the usual way, we may define a map,

$$\mathbb{C}P^3 \to \mathbb{H}P^1 \quad \text{by} \quad [z_1 : z_2 : z_3 : z_4] \mapsto [z_1 + jz_2 : z_3 + jz_4].$$

One can easily see this is well-defined and each point of $\mathbb{H}P^1$ has preimage $\mathbb{C}P^1$. We have a diffeomorphism $\mathbb{H}P^1 \cong S^4$. Hence the map defines a fibre bundle $\pi : \mathbb{C}P^3 \to S^4$ with fibres $\mathbb{C}P^1$, which we call the *real lines* of $\mathbb{C}P^3$.

Identifying $\mathbb{C}^4$ with $\mathbb{H}^2$, there is an anti-holomorphic map σ given by left multiplication by j. On the projectivization, this defines an involution given by, $\sigma([z_1 : z_2 : z_3 : z_4]) = [-\overline{z_2} : \overline{z_1} : -\overline{z_4} : \overline{z_3}]$. This map preserves the fibration π and acts on the fibres S^1 by the antipodal map. We call σ a *real structure*, hence the term real lines for the preserved fibres.

Given a holomorphic vector bundle $E \to \mathbb{C}P^3$, suppose we have a linear isomorphism $\sigma : E \to E^*$ covering σ on $\mathbb{C}P^3$. If $\sigma^2 = 1$, we say E has a *real structure*, if $\sigma^2 = -1$, we say E has a *symplectic structure*. We then have the following.

Theorem 3.1 (Ward Correspondence [4])**.** *Instantons on S^4 with structure group* SU(2) *are in correspondence with rank 2 holomorphic vector bundles $E \to \mathbb{C}P^3$ which are trivial when restricted to any real line and are equipped with a symplectic structure. Gauge transformations of the instantons correspond to σ-preserving isomorphisms of holomorphic vector bundles.*

This result is helpful because it moves us away from having to consider some of the difficulties of Yang-Mills theory. Instead, we require ideas from the classification of holomorphic vector bundles. This is a natural and well-studied (but still challenging) theory through which we will proceed. Some comments are warranted here.

Remark 3.2. There are analogous versions of this result for any compact structure group. Note $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. To extend the construction to groups $\mathrm{Sp}(n)$ we consider rank n holomorphic bundles E instead, with the same conditions imposed. To consider the groups $\mathrm{O}(n)$, we require a rank n holomorphic bundle with a real, as opposed to symplectic, structure [3]. Any compact Lie group has a faithful real orthogonal representation and so we may represent instantons for any such group by placing restrictions on the holomorphic bundles of the orthogonal theory [3].

Remark 3.3. The algebraic 2-dimensional bundles on $\mathbb{C}P^1$ can always be described (uniquely) as direct sums of two line bundles (see section 4.1 below). For a given holomorphic vector bundle $E \to \mathbb{C}P^3$, an embedded $\mathbb{C}P^1 \subset \mathbb{C}P^3$ will be called a *jumping line* if the two line bundles constituting the restriction of E to $\mathbb{C}P^1$ are both the trivial line bundle. The condition that our bundles are trivial on real lines is equivalent to the fact the real lines are not jumping lines for E. It also implies that the first Chern class $c_1(E) = 0$. By semi-continuity we also deduce that E restricted to a generic $\mathbb{C}P^1$ in $\mathbb{C}P^3$ is trivial [4, p. 119].

3.2. **Explaining the Ward Correspondence.** Let us now establish how instantons can be related to these complex bundles, following the approach of [7]. Given a complex three-manifold Z, we can consider submanifolds which are "lines" (embedded copies of $\mathbb{C}P^1$). Some complex geometry implies that the moduli space $\mathcal{M}$ of lines in Z is a four-dimensional complex manifold with a natural holomorphic conformal structure [7, pp. 4]. Given $p \in Z$, let Σ_p be the submanifold of lines in $\mathcal{M}$ which intersect p. Then Σ_p will be isotropic with respect to the conformal structure, i.e. the restriction to $T\Sigma_p$ of the conformal structure is zero [7, p. 4].

Given a non-degenerate quadratic form on $\mathbb{C}^4$, we may decompose $\Lambda^2\mathbb{C}^4$ into self-dual and anti-self-dual parts with respect to the form and the induced Hodge-star operator. One can check that a 2-form will be self dual or anti-self dual if and only if it vanishes on all isotropic subspaces of $\mathbb{C}^4$ [7, p. 5]. We conclude by looking at the tangent spaces of $\mathcal{M}$ that a connection A on a bundle over $\mathcal{M}$ will have a self-dual or anti-self dual curvature 2-form precisely when the curvature vanishes when restricted to any Σ_p. That is, the instantons of $\mathcal{M}$ are the connections which are flat on each Σ_p.

We now want to show how such connections correspond to holomorphic vector bundles. Given a holomorphic vector bundle $E \to Z$ which is trivial on lines $L \subset Z$, we may define a vector bundle $\mathcal{E} \to \mathcal{M}$ whose fibre over L are the holomorphic sections of $E|_L$. We explain later in Lemma 5.2 how sheaf cohomology implies that a trivialization of E over L extends to one over the first formal neighbourhood $L_{(1)}$ defined as $\mathcal{O}_Z/\mathcal{I}_L^2$, the sheaf of functions on Z quotiented by the square of the vanishing ideal of L. This gives a connection on $\mathcal{E}$. For any Σ_p, we may trivialize $\mathcal{E}|_{\Sigma_p}$ by evaluating sections at p; this trivialization respects the connection on $\mathcal{E}$ and so our connection is flat on submanifolds Σ_p. By above, we conclude that $E \to Z$ determines an instanton on $\mathcal{M}$.

On the other hand, given an instanton A on a bundle $\mathcal{E} \to \mathcal{M}$, we can define a homolomorphic bundle $E \to Z$ whose fibre over p is the space sections of $\mathcal{E}|_{\Sigma_p}$ whose covariant derivative vanishes. Given a line $L \subset Z$ and $p \in L$, we can use the fact the covariant derivative vanishes on Σ_p for these sections to canonically identify the fibre E_p with E_q for any other $q \in L$ via parallel transport. Hence E is trivial on lines.

This establishes the complex Ward correspondence: that instantons on $\mathcal{M}$ are equivalent to holomorphic vector bundles $E \to Z$, trivial on lines.

We can simplify things by supposing Z is equipped with an anti-holomorphic involution σ. The lines of Z preserved by σ gives a real manifold $M \subset \mathcal{M}$; $\mathcal{M}$ is called the *complexification* of M. Assume σ has no fixed points when restricted to a point of M; then it defines a Riemannian conformal structure on M. By the same manner as above, we get a correspondence of instantons on M to holomorphic vector bundles $E \to Z$ which are trivial restricted to the lines of M, subject to the condition that E has a linear involution $E \to E^*$ which squares to minus one. We require this extra condition, which we call the "reality condition," so that the structure group of the instanton is SU(2) and not its complexification $\mathrm{SL}(2, \mathbb{C})$ [4, p. 119].

Now we can complete the proof of the theorem. Taking $Z = \mathbb{C}P^3$ with real structure σ considered before, its moduli space of lines is the complex Grassmannians $\mathrm{Gr}_{\mathbb{C}}(2,4)$ which is defined by a quadric equation in $\mathbb{C}P^5$. This space is a complexification of S^4 which parameterizes the real lines fixed by σ [7, p. 5]. We conclude that instantons on S^4 are in correspondence with holomorphic vector bundles over $\mathbb{C}P^3$, trivial on real lines, and equipped with a symplectic structure. This is what we claimed above.

4. Horrocks' Construction: The Data of the ADHM theory

We now give an algebraic method, due to Horrocks, to construct certain holomorphic vector bundles satisfying the criteria of our Ward correspondence.

4.1. **Bundles on Projective Space.** First we need some preliminaries. On any projective space $\mathbb{C}P^n$ there is a *tautological line bundle* defined as,

$$\mathcal{O}(-1) = \{(\ell, z) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} : [z] = \ell\}.$$

We denote, for $p \in \mathbb{Z}$, the $-p$th tensor power of this complex line bundle as $\mathcal{O}(p)$ (we have $\mathcal{O}(1) = \mathcal{O}(-1)^*$ is the inverse of $\mathcal{O}(-1)$ under $\otimes$ and $\mathcal{O}(0)$ is trivial). It is a general fact that vector bundles on $\mathbb{C}P^1$ are all isomorphic to a direct sum of bundles $\mathcal{O}(p_i)$ [7, p. 4]. Also, any line bundle on $\mathbb{C}P^n$ is isomorphic to some $\mathcal{O}(p)$ [10].

For $p > 0$, a global holomorphic section of $\mathcal{O}(p)$ is given by a homogeneous polynomial of degree p. For $p < 0$, local holomorphic sections of $\mathcal{O}(p)$ are given by certain rational functions and there are no non-zero global holomorphic sections [2, pp. 60].

4.2. **The Construction.** Fix $k \in \mathbb{N}$. We begin with some necessary data for the construction.

Data 4.1. *We suppose we have the following structures:*

- *Complex vector spaces V and W of dimensions $2k+2$ and k respectively.*
- *An non-degenerate skew-symmetric form on V.*

- *Anti-linear maps $\sigma : W \to W$ and $\sigma : V \to V$ with $\sigma^2 = 1$ on W and $\sigma^2 = -1$ on V. We require σ on V be compatible with the skew form on V:*

$$(\sigma v_1, \sigma v_2) = \overline{(v_1, v_2)}.$$

We also ask the induced Hermitian form on V:

$$\langle u, v\rangle = (u, \sigma v)$$

be positive.

- *A family of linear maps $A(z) : W \to V$ depending linearly on $z \in \mathbb{C}^4$.*

The last datum means we can write $A(z) = \sum_i A_i z_i$ for fixed maps $A_i : V \to W$. We impose the following condition:

Data 4.2. *The image space $U_z = A(z)W \subset V$ is k-dimensional (i.e. full rank) and isotropic with respect to the form on V for all $z \neq 0$.*

We can define then for each $z \neq 0$ the quotient space $E_z = U_z^\circ/U_z$, where U_z° is the annihilator of U_z with respect to the skew form. Note $E_z = E_{\lambda z}$ for $\lambda \in \mathbb{C}^\times$ and the condition 4.2 implies that E_z is of dimension two for any non-zero z. Since E_z depends algebraically on z, we obtain an algebraic vector bundle $E \to \mathbb{C}P^3$ of rank two whose fibre over $[z]$ is E_z. The skew form on V induces one on E, and so E has structure group $\mathrm{SL}(2, \mathbb{C})$.

Since U_z is k-dimensional for non-zero z, the image of a basis vector of W under $A(z)$ determines a bundle isomorphic to $\mathcal{O}(1)$ and so U is isomorphic to $\mathcal{O}(-1)^{\oplus k}$. We claim for $x, y \in \mathbb{C}P^3$ that $U_x \cap U_y = 0$. If not, a vector in $U_x \cap U_y$ would give a non-zero holomorphic section for the restriction of U to the line joining x and y. But since U is a sum of line bundles, such a section would give a non-zero section of $\mathcal{O}(-1)$, which does not exist as we stated above.

We now are interested in the jumping lines of E, i.e. lines ℓ for which $E|_\ell$ is non-trivial. We claim these jumping lines are those connecting points $x, y \in \mathbb{C}P^3$ with $U_x^\circ \cap U_y \neq 0$. Suppose we have a line ℓ so that any two points $x, y \in \mathbb{C}P^3$ it connects satisfy $U_x^\circ \cap U_y = 0$. Fix such a pair x, y. Then for any $z \in \ell$ we have,

$$U_x^\circ \cap U_y^\circ \cap U_z = 0.$$

Since $R = U_x^\circ \cap U_y^\circ$ is two dimensional and contained in U_z°, we see that $U_z^\circ = R \oplus U_z$. Hence we may trivialize $E|_\ell = R$. Conversely, if ℓ joins some pair $x, y \in \mathbb{C}P^2$ so that $U_x^\circ \cap U_y \neq 0$, we have $U_x^\circ \cap U_y^\circ \cap U_y \neq 0$ and so contains a vector v. Since $U_x \cap U_y = 0$ by our argument above, $v \notin U_x$ and so can be used to define an algebraic section of $E|_\ell$ zero at y but not x. This means $E|_\ell$ is not trivial.

We now focus on the reality conditions to reduce the structure group to $\mathrm{SU}(2)$ and make one final assumption on $A(z)$. Recall our data from 4.1 gave linear maps σ on V and W, compatible with the skew form on V.

Data 4.3. *The map $A(z)$ is compatible with our three σ maps:*

$$\sigma(A(z)w) = A(\sigma z)(\sigma w)\ \forall\ z \in \mathbb{C}^4,\ w \in W.$$

Here, σ represents both the linear maps on V and W as well as the map we have previously consider on $\mathbb{C}^4 \cong \mathbb{H}^2$ given by left multiplication by j; which is which is clear from context.

From the compatibility condition we clearly infer that $U_{\sigma z} = \sigma(U_z)$ and $U_z^\circ = U_{\sigma z}^\perp$, where $\perp$ is orthogonal complement with respect to the positive Hermitian form on V. We can thus form a orthogonal decomposition,

$$V = U_z \oplus (U_z^\circ \cap U_{\sigma z}^\circ) \oplus U_{\sigma z}. \tag{4.1}$$

Note the middle term depends only on the point $x \in S^4$ which parameterizes the real line containing z and σz in the fibration of $\mathbb{C}P^3$; as such we denote $U_z^\circ \cap U_{\sigma z}^\circ$ as R_x.

Since $U_z^\circ = U_{\sigma z}^\perp$, we have $U_z^\circ \cap U_{\sigma z} = 0$. So, the real line connecting z and σz is not a jumping line for E. Any real line is of this form and so E is trivial when restricted to the real lines. Furthermore, the map σ on V descends to a map σ on E with $\sigma^2 = -1$ via the fact $U_{\sigma z} = \sigma(U_z)$. So, E has a symplectic structure.

From Ward's theorem, we can conclude that E corresponds to an instanton on S^4. We can go further and make this explicit. The construction above determined a vector bundle $R \to S^4$ with fibre R_x over $x \in S^4$. R is clearly realized as a subbundle of $S^4 \times V$. We can then give R a connection ∇ induced by orthogonal projection π. Namely, if ι is the inclusion $R \hookrightarrow S^4 \times V$ and d is the flat connection on $S^4 \times V$, we let ∇ act on a section s by,

$$\nabla s = \pi \circ \mathrm{d} \circ \iota(s).$$

One has that this connection is anti-self dual and defines a k-instanton (recall $k = \dim(W)$) [3].

Our claim is that we have now found *all* $\mathrm{SU}(2)$ instantons on S^4.

Theorem 4.4 (The ADHM Construction [3])**.** *The data of 4.1,4.2, and 4.3 considered up to isomorphism (i.e. a transformation $A(z) \mapsto PA(z)Q$ for $P \in \mathrm{Sp}(k+1), Q \in \mathrm{GL}(k, \mathbb{R})$) is in correspondence with a gauge equivalence class of $\mathrm{SU}(2)$ k-instantons on S^4 via the procedure outlined above.*

The remainder of this essay will outline how one proves this correspondence via algebro-geometric methods.

Remark 4.5. As with our previous work, this generalizes to other compact Lie groups. For the larger groups $\mathrm{Sp}(n)$, we would instead take V to have dimension $2k + 2n$. If we were dealing with orthogonal groups $\mathrm{O}(n)$, we would take V to be $2k + 2n$ dimensional with a symmetric instead of skew form, and take W to be symplectic: $\sigma^2 = -1$ (this corresponds to a real instead of symplectic structure on the corresponding holomorphic bundle over $\mathbb{C}P^3$). We can treat $\mathrm{SU}(n) \subset \mathrm{O}(2n)$ by asking all our data be compatible with an orthogonal matrix J such that $J^2 = 1$. Other matrix subgroups of the orthogonal group are handled similarly by adding additional structure.

Remark 4.6. The condition from 4.2 that U_z be isotropic is equivalent to the equation,

$$A(z)^\intercal J A(z) = 0 \quad \text{for all} \quad z \neq 0,$$

where J is the matrix of the skew form. This can be solved as certain quadratic equations for the coefficients of $A_1, \ldots, A_4$. The full-rank condition from 4.2 can also be phrased as some algebraic equation for the determinant of minors of A. The reality condition 4.3 is also straightforward. We thus see that all the data of the ADHM construction can be made explicit and algebraic. In [2, V.III] a nice rephrasing of the conditions is given in terms of quaternions.

In [3], the 't Hooft solutions discussed earlier are explicitly given in terms of the ADHM construction. Consider a k-instanton with scales $\lambda_i \in \mathbb{R}^+$ and centres $y_i \in \mathbb{R}^4$, $i = 1, \ldots, k$. We may interpret each y_i as a quaternion and $z \in \mathbb{C}^4$ as a quaternion pair $(p, q) \in \mathbb{H}^2$. Then using quaternion multiplication, we can write $A(z)$ as the $(k+1) \times k$ quaternion matrix,

$$A(z) = \begin{pmatrix} \lambda_1 p & \lambda_2 p & \cdots & \lambda_k p \\ y_1 p - q & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_k p - q \end{pmatrix}.$$

If the ADHM data is given in terms of quaternions, then the connection potential A_μ can be recovered algebraically from the quaternion matrix form of $A(z)$ [3].

5. Proving the Construction

We now embark on a sketch of the proof of theorem 4.4. This will involve some important ideas from algebraic geometry that allow us to show that a certain class of bundles are classified by the ADHM construction and that all instantons belong to this class.

5.1. **A Non-Technical Discussion of Sheaf Cohomology.** The proof will require techniques from sheaf cohomology which we only somewhat explain. These groups are, for us, defined for the sheaf of algebraic/holomorphic/smooth sections of a vector bundle.

Let $E \to X$ be such a bundle, and $\mathcal{E}$ its sheaf of sections. Then $\Gamma : \mathcal{E} \to \Gamma(X, E)$ is a functor from sheaves of abelian groups to abelian groups defined by considering the global sections of $\mathcal{E}$. This is left exact and so has right derived functors for $i \in \mathbb{N}$ determining the *sheaf cohomology groups* $H^i(X, \mathcal{E})$ [10, p. 207]. We will also denote the groups $H^i(X, E)$ or simply $H^i(E)$ if the context makes their meaning clear. One should think of these groups as measuring the failure to extend local sections of the sheaf; this is motivated by the fact that $H^i(X, \mathcal{E})$ is zero for all positive integers i when $\mathcal{E}$ is flabby (e.g. the sheaf of smooth sections of a smooth bundle) [10, p. 208]. Also useful to note is that a short exact sequence of sheaves on X,

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

induces a long exact sequence of sheaf cohomology groups,

$$\cdots \to H^{i-1}(X, \mathcal{C}) \to H^i(X, \mathcal{A}) \to H^i(X, \mathcal{B}) \to H^i(X, \mathcal{C}) \to \cdots$$

by standard homological algebra [10, p. 203].

We give something more explicit for the zeroth and first groups, which will be of primary interest. From the definition, one finds that $H^0(X, \mathcal{E})$ is simply the group of global sections of E [2, p. 66]. We will define the first group via Čech cohomology. Consider a compact complex manifold X with a finite cover of open sets $\{U_\alpha\}$. We define the space of 1-cochains to be the space of collections of sections $g_{\alpha\beta}$ of $\mathcal{E}$ on each $U_\alpha \cap U_\beta$. The sections $g_{\alpha\beta}$ form a 1-cocyle if, for any U_γ, we may write

$$g_{\alpha\beta} = g_{\alpha\gamma} + g_{\gamma\beta}.$$

The sections form a 1-coboundary if there is a collection of sections h_α on the open sets U_α so that $g_{\alpha\beta} = h_\alpha - h_\beta$ on any $U_\alpha \cap U_\beta$. The quotient of the co-cyles by the co-boundaries defines an abelian group $H^1(X, \mathcal{E}, \{U_\alpha\})$. For sufficiently small and well-chosen sets in the covering (e.g. consider taking a direct limit), this will be independent of the cover and reproduce the first sheaf cohomology group $H^1(X, \mathcal{E})$ [2, p. 66].

We also give explicitly the sheaf cohomology of our line bundles on $\mathbb{C}P^1$:

$$H^1(\mathbb{C}P^1, \mathcal{O}(-m)) \cong \begin{cases} \mathbb{C}^{m-1} & m > 1, \\ 0 & m \leq 1, \end{cases}$$

which can be computed from the Čech definition above [2, p. 65].

Lastly, we state for completeness a foundational theorem in our cohomology theory. We will not apply this directly, but it is needed for some of the details we skip in proving theorem 5.5.

Theorem 5.1 (Serre Duality [10, Corollary 7.7])**.** *Let X be an n-dimensional compact complex manifold and E a holomorphic vector bundle over X. Let ω_X denote the space of sections of the canonical line bundle Ω^n_X, i.e. the top power of the cotangent bundle. Then for each $0 \leq i \leq n$, there is a canonical isomorphism,*

$$H^i(X, \mathcal{E}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{E}^*)^*.$$

5.2. **Computing Some Cohomology Groups.** Given some holomorphic vector bundle $E \to \mathbb{C}P^n$ let $E(p)$ denote the bundle $E \otimes \mathcal{O}(p)$, where $\mathcal{O}(p) \to \mathbb{C}P^n$ is a tensor power of our tautological bundle as before.

Suppose we begin with an SU(2) instanton on a bundle F over S^4. We then obtain, via the Ward correspondence, a rank 2 holomorphic vector bundle $E \to \mathbb{C}P^3$ which is trivial on real lines and carries a symplectic strucuture. We want to show that this data implies the sheaf cohomology group $H^1(\mathbb{C}P^3, E(-2)) = 0$.

First we need a lemma, which we have already used in section three.

Lemma 5.2. *If a holomorphic vector bundle $E \to \mathbb{C}P^3$ is trivial along a projective line L, then it is also trivial along its first formal neighbourhood $L_{(1)}$.*

Our proof follows [2, pp. 71–72]. Let $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}P^3$ and $\mathcal{J}$ be the vanishing ideal of L. We have an exact sequence of sheaves defined by taking a quotient,

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}/\mathcal{J}^2 \to \mathcal{O}/\mathcal{J} \to 0.$$

We can clearly identity $\mathcal{O}/\mathcal{J}$ with the holomorphic functions of our line L. Similarly, $\mathcal{O}/\mathcal{J}^2$ is identified with holomorphic functions on $L_{(1)}$, i.e. first order Taylor approximations to functions on L. The sheaf $\mathcal{J}/\mathcal{J}^2$ is the space of these first order Taylor terms, or more formally, the conormal bundle N^* of $L \subset \mathbb{C}P^3$. One can check that the normal bundle decomposes into a direct sum of two copies of the tautological bundle and so $N^* = \mathcal{O}(1) \oplus \mathcal{O}(1)$. We know that $\mathcal{O}(1)$ has no global sections and vanishing first sheaf cohomology, so we infer that,

$$H^0(N^*) = H^1(N^*) = 0.$$

Tensoring the above sequence with E gives an exact sequence,

$$0 \to E|_L \otimes N^* \to E|_{L_{(1)}} \to E|_L \to 0.$$

Since $E|_L$ is trivial, its sheaf cohomology vanishes, hence,

$$H^0(N^* \otimes E|_L) = H^1(N^* \otimes E|_L) = 0.$$

From the above exact sequence, we obtain an exact sequence,

$$0 \to \overset{0}{\cancel{H^0(N^* \otimes E|_L)}} \to H^0(E|_{L_{(1)}}) \to H^0(E|_L) \to \overset{0}{\cancel{H^1(N^* \otimes E|_L)}} \to \cdots .$$

This establishes an isomorphism of the zeroth cohomologies of $E|_{L_{(1)}}$ and $E|_L$, i.e. there is an isomorphism between the global sections of the two bundles. Hence, a global trivialization of $E|_L$ induces a global trivialization of $E|_{L_{(1)}}$.

We come to the question of computing $H^1(\mathbb{C}P^3, E(-2))$. The following construction again comes from twistor theory. Suppose we have a element $\Phi \in H^1(\mathbb{C}P^3, E(-2))$. We have a fibre bundle $\mathbb{C}P^3 \to S^4$ parameterizing the real lines of $\mathbb{C}P^3$. Given $x \in S^4$, we obtain a corresponding real line $L_x \subset \mathbb{C}P^3$. We may restrict $E(-2)$ to this line and obtain a corresponding restriction,

$$\varphi_x \in H^1(L_x, E(-2)|_{L_x}) \cong F_x \otimes H^1(L_x, \mathcal{O}(-2)),$$

where the isomorphism follows from the triviality of $E|_{L_x}$ (recall here $F \to S^4$ is the instanton bundle associated to E). As we stated earlier, the cohomology group $H^1(\mathbb{C}P^1, \mathcal{O}(-2))$ is one-dimensional and so, varying x, defines some line bundle $W \to S^4$ with fibre $W_x = H^1(L_x, \mathcal{O}(-2))$ (one can identify W with a 1/4-density bundle [2, p. 70]). Thus, we may associate to Φ a section ϕ of $F \otimes W$ over S^4.

It is a consequence of twistor theory that one should be able to associate the sections ϕ originating from a cohomology element Φ with the solutions of certain differential equations on $F \otimes W$. For example, if we had instead taken $E(-3)$ to begin with, we would obtain sections solving the Dirac equation in a suitable background [2, p. 73]. Identifying and proving a correspondence is rather technical. One uses machinery of sheaf cohomology and is able to make computations about φ and its derivatives using the spaces $F_x \otimes H^1(L_x, \mathcal{O}(-2))$ and $F_{x,(1)} \otimes H^1(L_{x,(1)}, \mathcal{O}(-2))$ (here is where we use the triviality of E on the first formal neighbourhood of the line). One obtains:

Theorem 5.3 (Hitchin [11, 3.1])**.** *The map,*

$$H^1(\mathbb{C}P^3, E(-2)) \to \Gamma(S^4, F \otimes W)$$

sending Φ to ϕ, is an isomorphism onto the space of solutions to a Laplace equation on S^4,

$$\left(\nabla^*\nabla + \frac{1}{6}R\right)\phi = 0,$$

where ∇ is the covariant derivative on F and R is the (constant positive) scalar curvature of S^4.

Note then that if ϕ solves the above Laplace equation *globally* on S^4, we conclude,

$$0 = \int_{S^4} \left\langle \left(\nabla^*\nabla + \frac{R}{6}\right)\phi, \phi \right\rangle = \int_{S^4} \langle \nabla\phi, \nabla\phi \rangle + \frac{R}{6}\langle \phi, \phi \rangle \geq 0,$$

where the inequality follows from $R > 0$ on the sphere. We get equality on the right only when ϕ vanishes. We conclude the only global solution is zero and so we finally obtain that:

$$H^1(\mathbb{C}P^3, E(-2)) = 0.$$

Remark 5.4. It is important to note that the statement $H^1(\mathbb{C}P^3, E(-2)) = 0$ is not true for a general vector bundle on $\mathbb{C}P^3$. To prove this correspondence with our twistor method requires using the fact E comes from a unitary bundle F on S^4 [2, p. 74].

5.3. **Barth's Criterion.** We have arrived at the final important ingredient of the ADHM construction. We rely critically on the following result of Barth and Huleck.

Theorem 5.5. *Let $E \to \mathbb{C}P^3$ be a symplectic vector bundle satisfying the following:*

(1) There is a line $\ell \subset \mathbb{C}P^3$ so that $E|_\ell$ is trivial.

(2) The cohomology group $H^1(\mathbb{C}P^3, E(-2)) = 0$.

Under these conditions, E arises via Horrocks' construction from a triple $(A(z), W, V)$ unique up to isomorphism.

We will not prove the full theorem, but will discuss the idea with some details skipped. First we construct a certain diagram from the data of Horrocks. Abusing notation, let V, W denote the trivial bundles on $\mathbb{C}P^3$ with fibre the vector spaces V and W

respectively. Since $A(z)$ depends linearly on z, we can use the tautological bundle $\mathcal{O}(-1)$ to interpret A as a bundle homomorphism,

$$A : W(-1) \to V,$$

where the map $A_{[z]}$ for $[z] \in \mathbb{C}P^3$ is given by $A(w, \ell) = A(\ell)w$ for $\ell \in [z] \subset \mathbb{C}^4$ and $w \in W_{[z]}$. Note the bundle $U \to \mathbb{C}P^3$ with fibres our image spaces U_z is then isomorphic via A to $W(-1)$. Dualizing this gives an isomorphism $V/U^\circ \cong W^*(1)$. Set $Q = V/U, Q^* = U^\circ$. Recall also we had E as a quotient U°/U. We thus obtain the following diagram of interlacing short exact sequences.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W(-1) & \longrightarrow & Q^* & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W(-1) & \xrightarrow{A} & V & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle A^*} & & \downarrow & & \\
 & & & & W^*(1) & = & W^*(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{5.1}
$$

Note the skew forms give identifications $V \cong V^*, E \cong E^*$ and so this diagram is self-dual under reflection through the diagonal. The data of this diagram is called a *monad* and goes back to the original construction of Horrocks [7, p. 6].

The proof of Barth's criterion then amounts to reconstructing (5.1) from only a symplectic bundle $E \to \mathbb{C}P^3$ satisfying our constructions.

Let us take such a bundle E. Define $W^* = H^1(\mathbb{C}P^3, E(-1))$. It is a general fact that the holomorphic vectors D with subbundle D_1 and quotient $D_2 = D/D_1$, i.e. bundles D fitting in the exact sequence,

$$0 \to D_1 \to D \to D_2 \to 0,$$

are canonically classified by elements of $H^1(D_2^* \otimes D_1)$ [2, p. 67]. Thus the identity element,

$$1 \in \mathrm{End}(W) \cong W \otimes W^* \cong H^1(W \otimes E(-1)),$$

defines an exact sequence (after tensoring with $\mathcal{O}(1)$),

$$0 \to E \to Q \to W^*(1) \to 0.$$

This is the second column of (5.1). Dualizing defines the first row of (5.1) and in particular our space W. Some constructions with homology [2, p. 88] allow us to repeat the idea we used to product Q to determine V via a compatible exact sequence,

$$0 \to Q^* \to V \to W^*(1) \to 0.$$

In this step we use the homology condition on E. After dualizing this, we have reconstructed all of (5.1). Some further work shows that the bundle V we have built is trivial (so we obtain our vector space V) and that the induced isomorphism $V \cong V^*$ from dualizing (5.1) is skew [2, pp. 88–89]. In this step we use the triviality of E on some line. This gives us the full data of the monad, and in particular the original map $A(z) : V \to W$. The reconstruction of Horrocks' data was all canonical, which implies uniqueness.

With more work, one can also reconstruct V more explicitly as,

$$V \cong H^1(E \otimes \Omega^1_{\mathbb{C}P^3}).$$

The map A's dual can be found explicitly from the cohomology exact sequence associated to the dual of the exact Euler sequence on $\mathbb{C}P^3$ [2, pp. 91–92].

5.4. **Completing the Construction.** We now finally put the pieces together of the ADHM theory.

Consider the complete data of the ADHM construction: the triplet $(A(z), V, W)$ and the actions σ on V and W. We know these give us a unique $SU(2)$-instanton on S^4 and we want to show this gives all instantons.

We know by the Ward correspondence that every instanton corresponds uniquely to a holomorphic rank 2 vector bundle $E \to \mathbb{C}P^3$ which has a symplectic structure and is trivial on real lines. With twistor theory we showed that any such E must satisfy $H^2(\mathbb{C}P^3, E(-2)) = 0$. Combining this with the fact that E is trivial on real lines, in particular trivial on some line $\ell \subset \mathbb{C}P^3$, we see that E meets Barth's criterion and so corresponds to a triple $(A(z), W, V)$. We just need to construct the maps σ on V, W to show E comes from the ADHM construction.

The symplectic structure means that E carries a linear isomorphism $\sigma : E \to E^*$ with $\sigma^2 = -1$. This will then induce a map on V. Pulling back E by σ^* and using the uniqueness of Barth's criterion, we conclude σ acts in the desired way on V with respect to the skew form. The condition of data 4.3 then implies the action of σ on W.

All that remains is to show the induced Hermitian form on V of data 4.1 is positive. For $[z] \in \mathbb{C}P^3$, recall from 4.1 we have an orthogonal decomposition,

$$V = U_z \oplus E_z \oplus U_{\sigma z},$$

and compatability conditions imply $U_{\sigma z}^{\perp} = U_z^{\circ} = U_z \oplus E_z$. We know the form is positive on E_z by definition. We know $\sigma(U_z) = U_{\sigma z}$ and so the form will have the same sign on U_z and $U_{\sigma z}$. If it were negative on these two spaces, each E_z would lie in the positive cone of the form. Using the trivialization of V, we could deform the bundle E to a fixed positive subspace [2, p. 90]. This contradicts the fact that E is not topologically trivial which is implied by $c_2(E) \neq 0$. Hence our induced Hermitian form is definite as claimed.

We conclude that the ADHM data precisely characterizes the SU(2) instantons on S^4 and we have completed the construction.

Remark 5.6. As we have remarked before, all of this theory can be applied to other compact structure groups. We just need to show compatibility with the maps σ and any extra structures we use; everything is similar to our reasoning above.

6. Conclusion

The ADHM construction is important to differential geometry, where for example it plays a role in Donaldson theory for the classification of four-manifolds (see [8]). It also has significance in physics, where classical Euclidean instantons describe tunneling between vacuum states in the quantized Minkowski theory (see [1]).

Beyond its applications, the ADHM theory is a wonderfully elegant classification of an important class of solutions of the Yang-Mills equations. Despite the seeming simplicity, there is a vast amount of interesting and deep mathematics underlying the original two-page paper of Atiyah, Drinfield, Hitchin, and Manin, which we have explored only some of.

The merits of the construction are well expressed by a quote of Simon Donaldson [7, p. 4], which we conclude with.

> "The ADHM work is significant as one of the first applications of sophisticated modern geometry to physics, playing a large part in opening up a dialogue which has of course flourished mightility in the near half century since."

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