SFT compactness theorem.

Moral: We would like to understand the degeneracies of our moduli spaces. and "compactify" them with "broken" buildings of "nodal" things

①. Most J-curve theories involve "counting J-curve". which requires compactness to guarantee finiteness

②. The degeneracies often become key to "$\partial^2=0$" in our chain complex and sometimes also ruin "$\partial^2=0$".

§ Compact Case.

Recall: If J_ν is a sequence of A.C.S. on W^{2n} ~~compact~~ compact. with $J_\nu \to J$ in the C^∞ topology, then any uniformly C^1-bounded sequence of J_ν-holomorphic curves $u_\nu : \mathbb{D} \to W$ has a subsequence convergent in C^∞_{loc} on $\operatorname{int}(\mathbb{D})$.

Rmk: Sobolev embedding: $W^{1,p}(\mathbb{D}) \hookrightarrow C^1(\mathbb{D})$ for $p>2$. But then an energy bound only gives us a $W^{1,2}$-bound: $E(u) = \int_{\mathbb{D}} u^*\omega = \frac{1}{2}\int_{\mathbb{D}} |du|^2 dv$

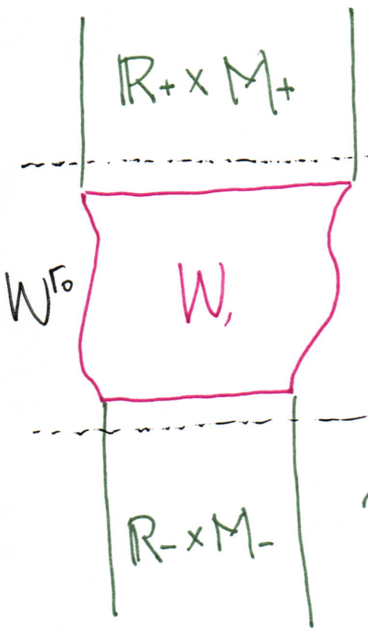
WHICH IS BOARDERLINE.

What could happen is $\|du\|_J$ blows up to ∞ at a point while u can still be $W^{1,2}$, and so $\exists\, z_\nu$ s.t. $\|du_\nu\|_J(z_\nu) \to \infty$ with $z_\nu \to z_\infty$.

If this is the only point at which $\|du_\nu\|_J$ "blows up" then we have $\Omega_\nu \subseteq \mathbb{D}$ exhausting $\mathbb{D}$ s.t. $u_\nu|_{\Omega_\nu}$ is C^1 actually, so there is a C^∞_{loc} convergence on $\mathbb{D}-\{z_\infty\}$ to $u_\infty : \mathbb{D}-z_\infty \to W$ J-holomorphic

Then a rescaling of a nbhd of z_∞ (into $\mathbb{C}$ with ∞ mapping to $p \in W$ that connects with $u_\infty(\mathbb{D}-z_\infty)$ at the removable singularity is our familiar bubbling example.

§ Cobordism Completion:



$\mathcal{H}_+ = (\omega_+, \lambda_+)$

$J_+ = J|_{[r_0,\infty)\times M_+}$

ω

$J_- = J|_{(-\infty,-r_0]\times M_-}$

$\mathcal{H}_- = (\omega_-, \lambda_-)$

Hyperplane distribution: $\xi_\pm = \ker(\omega_\pm)$

Reeb vector field $R_\pm$

* We will always pick a metric invariant under translation on the completion ends.

$$W^{r_0} := ([-r_0, 0]\times M_-) \cup W \cup ([0,r_0]\times M_+)$$

* We will always pick J "compatible with the completion" that is invariant under translation in $\hat{W} - W^{r_0}$.

Def: $\mathcal{J}(h, r_0) = \{ f \in C^\infty(\mathbb{R}, (-\varepsilon,\varepsilon)) \mid f' > 0,\ f \equiv h \text{ near } [-r_0, r_0] \}$

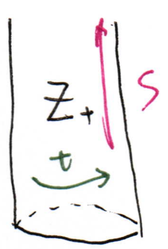
where $h: \mathbb{R} \to \mathbb{R}$ is C^0-small and $h(r) = r$ near $r = 0$.

There is a symplectic form ω_h on $\hat{W}$ for h sufficiently C^0-small

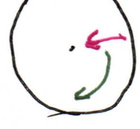
$$\omega_h = \begin{cases} d(h(r)\cdot\lambda_+) + \omega_+ & \text{on } [0,\infty)\times M_+ \\ \omega & \text{on } W. \\ d(h(r)\lambda_-) + \omega_- & \text{on } (-\infty, 0]\times M_- \end{cases}$$

Def: $E(u) = \sup_{f\in\mathcal{J}(h,r_0)} \int_{\dot{\Sigma}} u^*\omega_f$ for $u: \dot{\Sigma} \to \hat{W}$ J-curve.

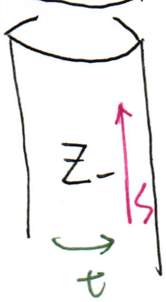
§ Biholomorphism between half cylinders and disks



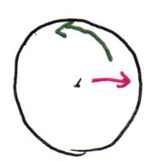
$$[0,\infty)_s \times S^1_t \xrightarrow{\varphi_+} \dot{\mathbb{D}}$$
$$(s,t) \longmapsto e^{-2\pi(s+it)}$$



Rmk: Energy is preserve under this biholomorphism.



$$(-\infty, 0]_s \times S^1_t \xrightarrow{\varphi_-} \dot{\mathbb{D}}$$
$$(s,t) \longmapsto e^{2\pi(s+it)}$$



Obs: The translation on $Z = \mathbb{R}\times S^1$, $T_c: (s,t) \mapsto (s+c, t)$ under this biholomorphism, is a RESCALING around $(0,0)$. In some sense, we are again zooming in at $(0,0)$ with this translation.

§ Non Compactness causes additional Issues.

We have uniformly C^1-bounded ($W^{1,p}$, $p>0$) as a condition for our convergence <u>theorem</u>. W there being <u>compact</u> guarantees C^0-uniform bound

So all we needed to worry about is when $\|du\|_J(z)$ exploding. Now, we need to also worry about failure of C^0-uniform bound! i.e. portion of our curve ~~as~~ escaping to $\{\pm\infty\}\times M_\pm$ inside of $\hat{W}$. But a uniform Energy bound again will help us rule out a lot of "unwanted situations"

<u>Thm</u>: (9.10) $J \in \mathcal{J}_\tau(\omega_H, r_0, \mathcal{H}_+, \mathcal{H}_-)$. $u:(\dot{\mathbb{D}}, i) \to (\hat{W}, J)$ is a J-curve, $E(u)<\infty$

Either $0 \in \mathbb{D}$ is removable (i.e. u extend over $\mathbb{D}$). or 0 is a positive or negative puncture, with a well-defined charge $Q = \lim_{\varepsilon\to 0^+} \int_{\partial \mathbb{D}_\varepsilon} u^* \lambda_\pm$. $\pm Q > 0$.

Moreover $(u_{\mathbb{R}}(s,t), u_M(s,t)) := u \circ \varphi_\pm(s,t) \in \mathbb{R}\times M_\pm$ for (s,t) near $s=\pm\infty$ satisfies the following:

① $u_{\mathbb{R}}(s,\cdot) - Ts \longrightarrow c$ in $C^\infty(S^1)$ as $s \to \pm\infty$ for $T = |Q|$, $c \in \mathbb{R}$

② For $s_k \to \pm\infty$, $u_M(s_k,\cdot) \to \gamma(T\cdot)$ in $C^\infty(S^1, M_\pm)$ as $k\to\infty$ for γ a T-periodic Reeb orbit. If γ is nondegenerate or Morse-Bott then in fact $u_M(s,\cdot) \to \gamma(T\cdot)$ in $C^\infty(S^1, M_\pm)$ as $s\to\infty$.

<u>Lemma</u>: $J \in \mathcal{J}(\mathcal{H})$ for some SHS on M, $u:(\dot{\Sigma}, j)\to(\mathbb{R}\times M, J)$ J-curve

$E(u)<\infty$, $\int_{\dot{\Sigma}} u^*\omega = 0$ either $\dot{\Sigma} = \mathbb{C}$, $u = $ const.

or $\dot{\Sigma} = \dot{\mathbb{C}}$, $u = $ constant or holomorphic cylinder that is ~~...~~ $\mathbb{R}\times\{\gamma\}$

<u>Rmk</u>: Trivial in the contact case.

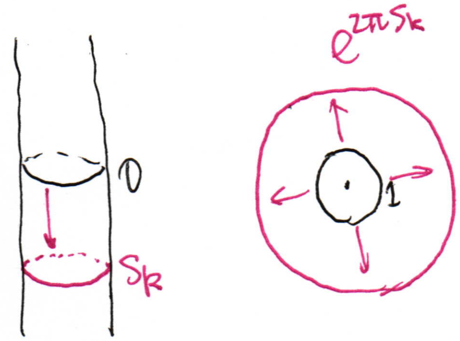
<u>Lemma</u>. $\exists C>0$ so that $\|du_+(s,t)\|_J \le C$ $\forall (s,t)\in Z_+$ with $u_+ = u\circ\varphi_+$

We sketch the outline of <u>Thm</u>

§ Proof of Asyptotic Limit.

Define $u_k : [-s_k, \infty) \times S^1 \to \hat{W}$ $\quad (s,t) \mapsto u_+(s+s_k, t)$ $\quad s_k \to \infty$.

Geometrically what this look like is



Again, by a black-boxed Lemma, $\|du_k\|$ is uniformly bounded. But there is no bound in C^0.

Case 1: $u_k(0,0)$ has a bounded subsequence

Case 2: $u_k(0,0)$ has a subsequence diverging to $\{\infty\} \times M_+$

Case 3: ——— // ——— $\{-\infty\} \times M_-$

<u>Case 1</u>: u_k has a now uniform C^1-bound on Ω_k exhausting $\mathbb{R} \times S^1$, so u_k converge in a further subsequence to $u_\infty : \mathbb{R} \times S^1 \to \hat{W}$ in C^∞_{loc}.

For any $f \in \mathcal{T}(h, r_0)$, $c > 0$, we have

~~(domain going to a meager set)~~

$$\int_{[-c,c]\times S^1} u_\infty^* \omega_f = \lim_{k\to\infty} \int_{[-c,c]\times S^1} u_k^* \omega_f \le \lim_{k\to\infty} \int_{[-\frac{s_k}{2}, \infty)\times S^1} u_k^* \omega_f = \lim_{k\to\infty} \int_{[\frac{s_k}{2}, \infty)\times S^1} u_+^* \omega_f = 0$$

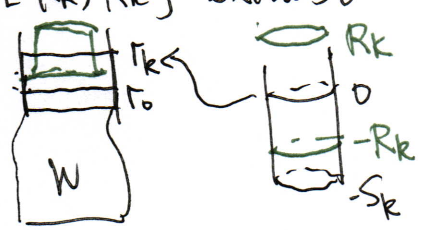
(integrate over a subdomain EVENTUALLY) $\quad$ (translation back up)

Since $\int_{Z_+} u_+^* \omega_f < \infty$. $[\frac{s_k}{2}, \infty) \times S^1$ as $k \to \infty$ become small $\Rightarrow \int_{\mathbb{R}\times S^1} u_\infty^* \omega_f = 0$

So u_∞ is a constant map. Hence $u_+(s_k, \cdot) \to p$ in $C^\infty(S^1, \hat{W})$ as $k \to \infty$.

<u>Case 2</u>: Assume $u_k(0,0) \in \{r_k\} \times M_+$ with $r_k \to \infty$ since $|du_k|$ bounded uniformly locally, find $[-R_k^-, R_k^+] \subseteq [-s_k, \infty)$ so $u_k([-R_k^-, R_k^+] \times S^1) \subseteq [r_0, \infty) \times M_+$

$[-R_k, R_k]$ exhaust $\mathbb{R}$. $[r_0, \infty) \times M_+$ as J invariant under translation there.



$$T_{-r_k} \circ \left(u_k\big|_{[-R_k^-, R_k^+]}\right) : [-R_k^-, R_k^+] \times S^1 \to \mathbb{R} \times M^+$$

so that $(0,0)$ mapped to $\{0\} \times M^+$

is C^1_{loc}-uniformly bounded. (translation back give C^0_{loc}). So that sequence has a subsequence C^∞_{loc}-converge to J_+ curve $E(u_\infty) < \infty$, $\int_{\mathbb{R}\times S^1} u_\infty^* \omega = 0$ as before

Lemma $\Rightarrow$ u_∞ is a trivial cylinder.

§ Degenerations of Holomorphic Curves

$u_k := [(\Sigma_k, j_k, \Gamma_k^+, \Gamma_k^-, \Theta, u)] \in \mathcal{M}_{g,m}(J_k, A_k, \gamma^+, \gamma^-)$ where $\gamma^\pm$ are Reeb orbits that the punctures Γ_k^+, Γ_k^- asymptotes to. $A_k \in H_2(W, \gamma^+ \cup \gamma^-)$ is the relative homology class that u lives in. Σ_k a genus g surface with m-marked points Θ.

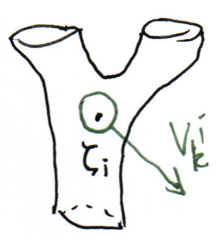
Rmk: $E(u)$ depends only on $\gamma^\pm$ and A_k. if A_k is fixed $E(u_k)$ is uniformly bounded.

Q: What could happen if u_k has no limiting subsequence?

- If there is some $\zeta_0 \in \dot{\Sigma} = \Sigma - \Gamma$ so $u_k(\zeta_0) \subseteq K$ for some K compact
- $(\Sigma_k, j_k, \Gamma_k^+ \cup \Gamma_k^- \cup \Theta)$ converges. assume after biholomorphism, $\Sigma_k = \Sigma$; $\Gamma_k^\pm = \Gamma^\pm$; $\Theta_k = \Theta$. and $j_k \to j$ in C^∞ topology
- $u_k : \dot{\Sigma} \to \hat{W}$ locally C^1-bounded outside $\Gamma' = \{\zeta_1, \dots, \zeta_N\} \subseteq \dot{\Sigma}$

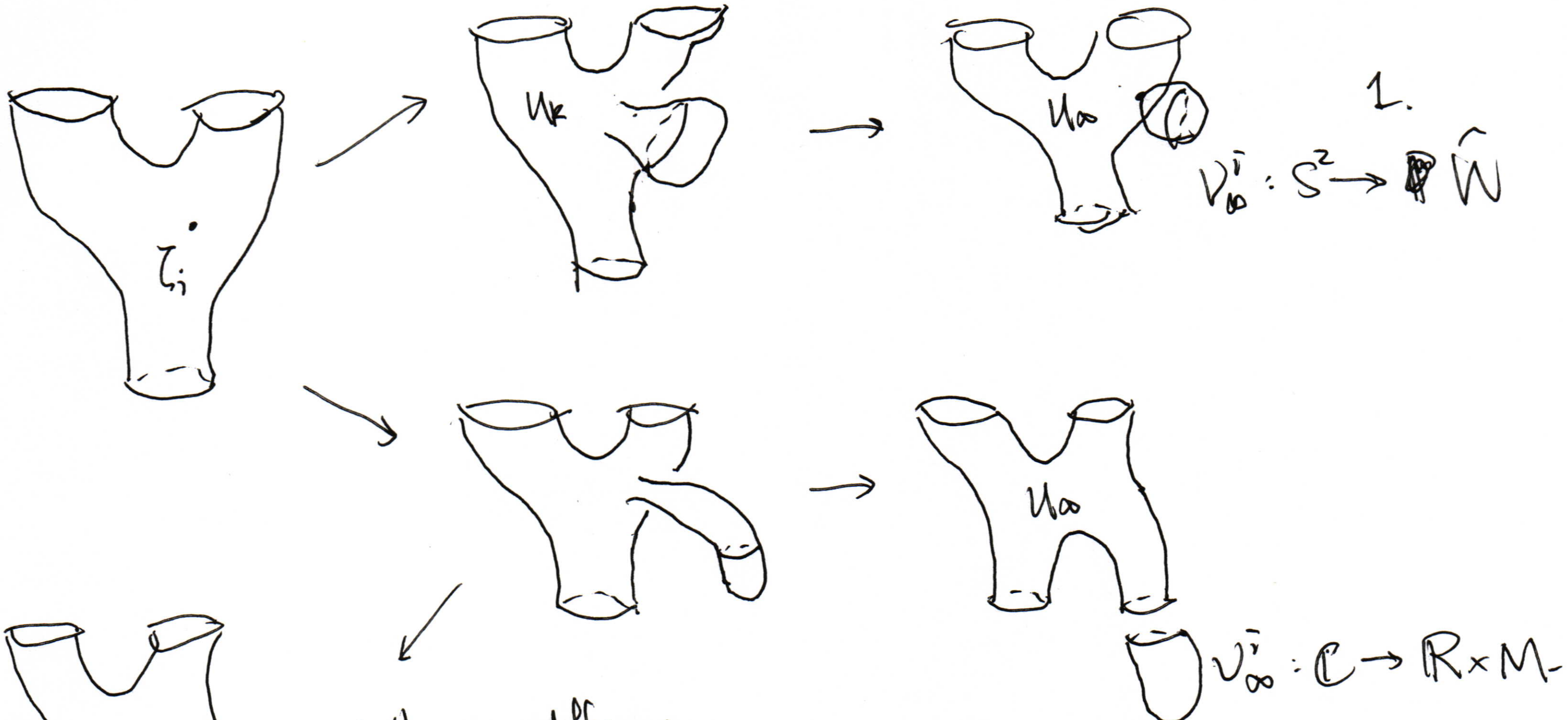
So u_k converges to $u_\infty : \dot{\Sigma} - \Gamma' \to \hat{W}$ in $C^\infty_{loc}(\dot{\Sigma} - \Gamma')$. $E(u_\infty) \le \limsup_{k \to \infty} E(u_k)$

$|du_k|$ blows up at Γ'.



rescaling at $\zeta_i \Rightarrow$ if $\|du_k(z_k^i)\| \to \infty$ then suppose

1. $u_k(z_k^i)$ has a subsequence in a compact set.
   the rescaled maps v_k^i near ζ_i converges in C^∞_{loc} to v_∞^i
   $v_\infty^i : \mathbb{C} \to \hat{W}$ with finite energy. $\mathbb{C} = S^2 - \{\infty\}$
   Removal of singularities + connect bubbles $\Rightarrow$ familiar bubbles
2. $u_k(z_k^i)$ has a subsequence $\to \{\pm\infty\} \times M_\pm$. A translation of v_k^i by $\mathbb{R}$ action we get a plane bubbles off into the symplectization. (One can only translate if we have the image inside the completion ends pass $|r_0|$ where $\hat{W}$ and $\mathbb{R} \times M_\pm$ shares symplectic structure and $J_\pm$.)

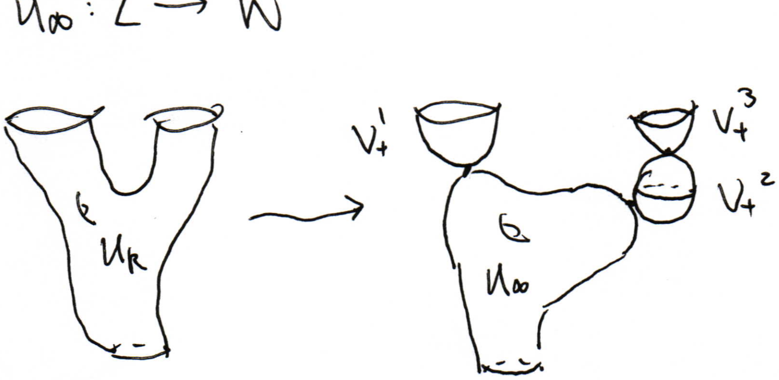


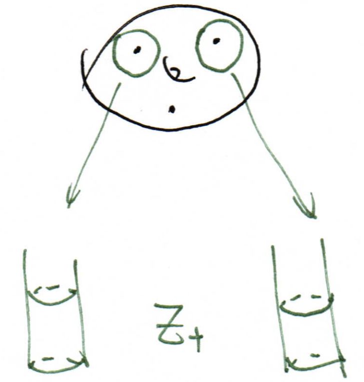
With some different reparameterization/ translation we can get bubble tree bc rate of divergence may differs

貳.

- $(\Sigma_k, j_k, \Gamma_k^+ \cup \Gamma_k^- \cup \Theta_k)$ converges as before
- $\|du_k\| < C \quad \forall z \in \dot\Sigma = \Sigma - \Gamma$.
- $u_k(\zeta_0) \subseteq K$ for some ζ_0 as well. We get C^∞_{loc} conv to u_∞

$u_\infty : \dot\Sigma \to \hat W$





$s_k \to \infty$, $r_k \to \infty$ } different rate

$u_k(s+s_k,t) \mapsto V_+^3$
$u_k(s+r_k,t) \mapsto V_+^2$

$s_k \to \infty$ s.t.

$[-s_k,\infty)\times S^1 \to \hat W$, $(s,t) \mapsto u_k(s+s_k,t)$ uniformly C^1 bounded converges in $C^\infty_{loc}(\mathbb{R}\times S^1)$ to a cylinder V_+^1 with one removable one puncture

"Deligne-Mumford Space"

$M_{g,l} = \{(\Sigma_g, j, \Theta)\}/\sim$ stable if ~~2g-2+l~~ $2-2g-l \le -1$ $\Rightarrow$ finite $\mathrm{Aut}(\Sigma, \Theta)$.

$\dim M_{g,l} = 6g-6+2l$ with compactification $\overline{M}_{g,l}$

Thm. Every (Σ, j), $\pi_1(\Sigma) = \{e\}$, is biholomorphic equivalent to $\mathbb{CP}^1$, $\mathbb{C}$, $\mathbb{H}$.

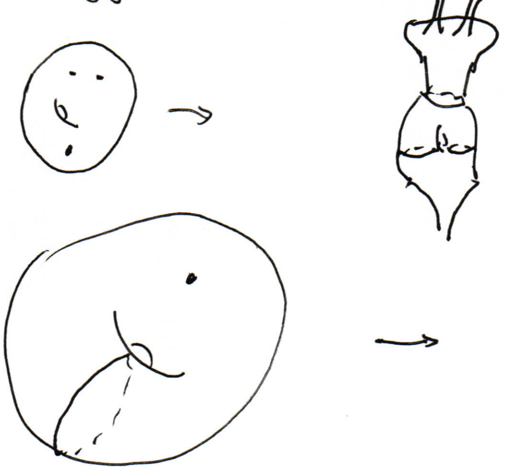
Thm. On $(\mathbb{H}, i)$ $\exists$ a Riemannian metric with constant curvature -1 that defines the same conformal structure as i. s.t. all conformal transformation are isometries

Obs 1: Only (punctured) Riemann surface whose ~~un~~ universal cover is S^2 is S^2 itself. (Aut S^2 has fixed points)

Obs 2: All $\mathrm{Aut}(\mathbb{C}, i)$ that has no fixed points are translations. this can only quotient into $S^1 \times \mathbb{R}$ / T^2

So all other punctured R.S. has ~~D~~ $(\mathbb{H}, i)$ as its universal cover

Cor: All pointed R.S. (Σ, j, Θ) ~~has~~ $\chi(\Sigma - \Theta) < 0$, has a complete Riemannian metric g_j. const. curvature -1. (same conformal structure, conformal transformations are isometries).



Each class in $\pi_1(\Sigma - \Theta)$ has a ~~shortest~~ unique geodesics loop

C = collection of geodesics loops

$\Sigma - C$ into p.o.p. decomp.

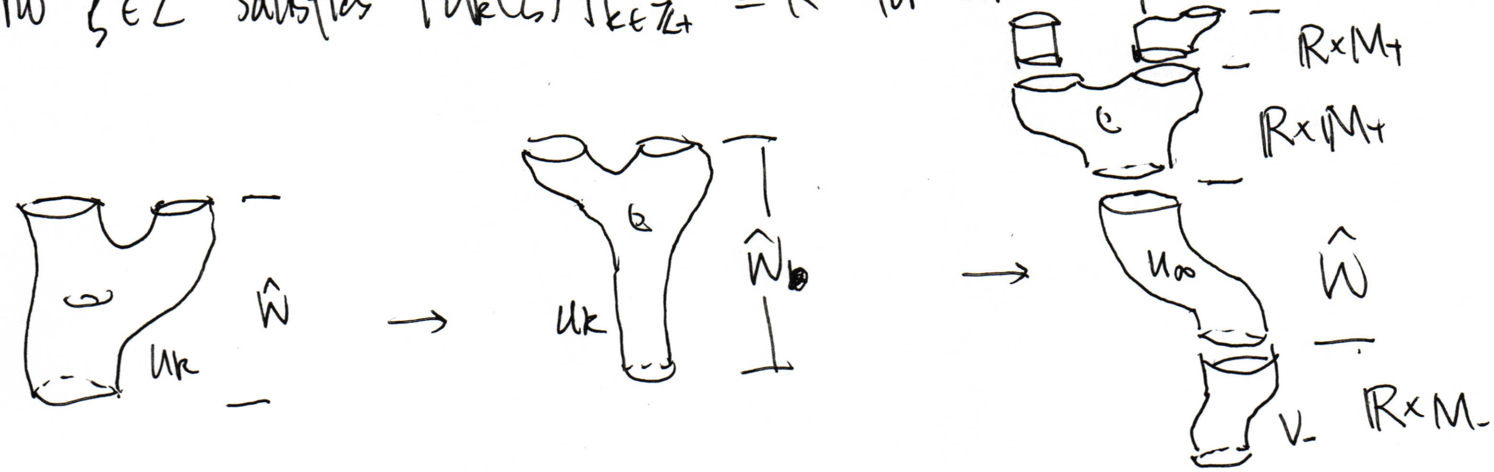
(constant -1 curvature probably only sketchable in 4d ?.)

There are $(2g-2+l)$ geodesics pair of pants $\Rightarrow$ $(3(2g-2+l)-l)/2 = \underline{3g-3+l}$ geodesics. each has a length and gluing parameter $(0, C] \times S^1$.

参:

Riemann Surface converges $\|du_k\| < C \quad \forall z \in \dot{\Sigma}$

But we drop C^0_{loc}-bound. meaning that we can have a situation where no $\zeta \in \dot{\Sigma}$ satisfies $\{u_k(\zeta)\}_{k \in \mathbb{Z}_+} \subseteq K$ for some compact K.



A translation of things diverging to ∞ as before yield the curve one the right. of height 1 | 1 | 2 |

These are the potential things that could happen to "prevent" convergence. $\angle$ 7.1.

§ Deligne-Mumford. Space $\mathcal{M}_{g,l} = \{(\Sigma, j, \Theta)\}/\sim$ if $\chi(\Sigma - \Theta) < 0$.

or $2g + l \geq 3$, (Σ, j, Θ) stable, therefore $\mathcal{M}_{g,l}$ is an orbifold $6g-6+2l$ dimensional (Because Automorphism group is finite.)

Def: Nodal Riemann surface with $l \geq 0$ marked points and $N > 0$ nodes is a tuple (S, j, Θ, Δ)

S, j R.S. of genus g closed but probably not connected

Θ ordered set of marked points. l

Δ 2N unordered points paired by an involution that fixes no point $\{z, r(z)\}$ is a node

It is stable if each component is

Equivalences consist of biholomorphism $S \to S'$ preserving ordered marke points & nodes.

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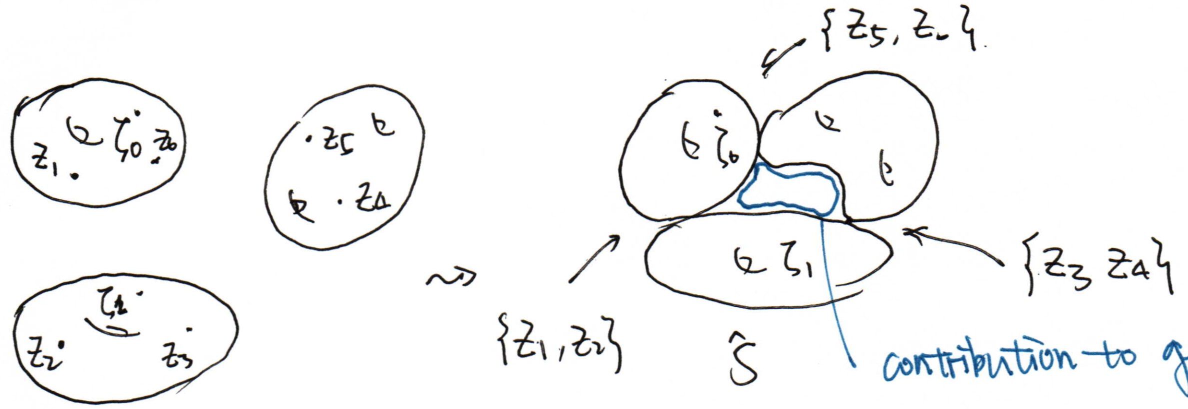
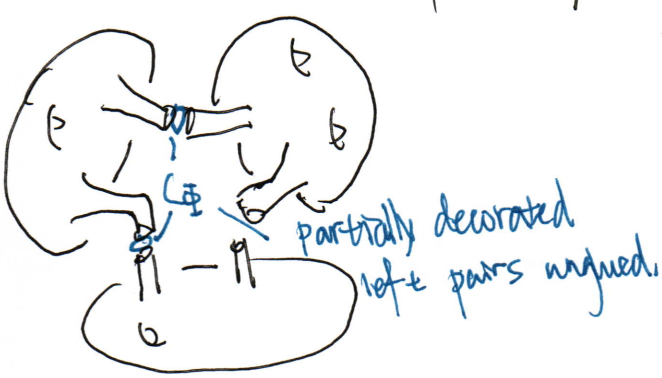


Arithmetic genus $AG(\hat S) = 5$ here

Rmk:
$AG = \sum_{\text{component}} g(\Sigma) + (N - \#\text{pieces} + 1)$. if $\hat S$ connected

Circle compactification: If $z \in \Gamma$ then let $\delta_z = T_z\Sigma - \{0\}/\mathbb{R}^* \simeq S^1$ and gluing δ_z back to $\dot\Sigma$ compactifies $\dot\Sigma$ at z

To glued surfaces compactified by S^1 we need a
Decoration: $\Phi : \delta_{z_+} \to \delta_{z_-}$ orientation reversing.
for a pair of nodes $\{z_+, z_-\}$ $\hat S_\Phi = \bar S/\sim$ (circle compactified S mod Φ's)
after they are glued $\boxed{C_\Phi = \sqcup \text{ gluing circles}}$



$(\Sigma_k, j_k, \Theta_k) \longrightarrow (S, j, \Theta, \Delta)$ if
① $\exists$ homeo $\varphi_k : \Sigma_k \to \hat S_\Phi$ smooth outside of C_Φ
② $\varphi^* j_k \to j$ in $C^\infty_{loc}(\hat S_\Phi - C_\Phi)$

$\overline{M}_{g,1}$ are these nodal curves compactifying $M_{g,1}$.

§ Nodal Curves.

A nodal curve consists of $(S, j, \Gamma^+ \sqcup \Gamma^- \sqcup \Theta, \overset{u}{\Delta})$ with $|\Theta| = m$ and $u : (\dot{S}, j) \to (\hat{W}, J)$ $\dot{S} = S - (\Gamma^+ \sqcup \Gamma^-)$ asymptotically cylindrical J-curve mapping Γ^+, Γ^- to Reeb orbits so that if $\{z_+, z_-\} \in \Delta$, $u(z_+) = u(z_-)$.

Two nodal curves are equivalent $u_1 \sim u_2$ if $\exists$ equivalence of domain given by $\varphi : S_1 \to S_2$ biholomorphism s.t. $u_1 = u_2 \circ \varphi$.

* u is stable if all component of S on which $u \equiv$ const has negative χ.

This is enough to compactify $\mathcal{M}_{g,m}(\Sigma, W, J)$ for compact / closed W.

But this is not enough for us as curves "escapes" to symplectization.

§ Buildings.

Given $g, m, N_+, N_- > 0$. A building of height $N_- | 1 | N_+$ with arithmetic genus g & m marked points is:

$u = (S, j, \Gamma^+, \Gamma^-, \Theta, \Delta^{nd}, \Delta^{br}, L, \Phi, u)$

- $(S, j, \Gamma^+ \sqcup \Gamma^- \sqcup \Theta, \Delta^{nd} \sqcup \Delta^{br})$ connected nodal curve (not necessarily stable) or arithmetic genus g. $|\Theta| = m$. And Δ^{nd} is identified w/ Δ^{nd}, so is Δ^{br}. Δ^{nd} are nodal points, Δ^{br} are breaking points. $\Theta, \Gamma^+, \Gamma^-$ as before
- Level structure $L : S \to \{-N_-, \cdots, 1, 0, 1, \cdots, N_+\}$ locally constant L surject onto $\{-N, \cdots, -1, 1, \cdots, N\}$ w/ $L(z^+) = L(z^-)$ for $(z^+, z^-) \in \Delta^{nd}$
  $L(z^+) - L(z^-) = 1$ for $\{z^+, z^-\} \in \Delta^{br}$
  $L(\Gamma^+) = N_+$, $L(\Gamma^-) = N_-$
- Decoration $\Phi : \delta_{z_+} \to \delta_{z_-}$ for each breaking pair $\{z_+, z_-\}$

- $u : (\dot{S} = S - (\Gamma^+ \cup \Gamma^- \cup \Delta^{br}), j) \longrightarrow \bigsqcup_{N_- \le N \le N_+} (\hat{W}_N, J_N)$. asymptotically cylind.

  sending $\dot{S} \cap L^{-1}(N)$ to W_N N-th level

  With $u(z^+) = u(z^-)$ for each $\{z^+, z^-\} \subseteq \Delta^{nd}$

  For each $\{z^+, z^-\} \subseteq \Delta^{br}$; $L(z_+) - L(z_-) = 1$ then u has a $+$-puncture at z^+, negative puncture at z_-. If $u_\pm : \delta_{z_\pm} \longrightarrow M_?$ asymptotical param. of the orbit, $u_+ = u_- \circ \Phi$.

u is stable if

- constant component has negative χ
- No level w/ unmarked trivial cylinder without nodes.

Two buildings are equivalent if φ: equivalence of domain gives $u_1^\circ \varphi = u_1$ ($\hat{W}$ level) and $u_1^N \circ \varphi = u_0^N + C$ (translation allowed on symplectization level).

§ Convergence

Assume Non Degenerate orbits $g, m \ge 0$. For $[(\Sigma_k, j_k, \Gamma_k^+, \Gamma_k^-, \Theta_k, u_k)] \in \mathcal{M}_{g,m}(J_k, A_k, \gamma^+, \gamma^-)$ with uniformly bounded energy $\exists\, u_\infty = [(S, j, \Gamma^+, \Gamma^-, \Theta, \Delta^{nd}, \Delta^{br}, L, \Phi, u_\infty)] \in \overline{\mathcal{M}}_{g,m}(J_\infty, A, \gamma^+, \gamma^-)$ so a subsequence of u_k converges to u_∞ in the following sense.

- Φ extend to Δ^{nd} & $\hat{S}_\Phi \xrightarrow{\varphi} \Sigma_k$ homeomorphism smooth outside of C_Φ, preserve $\Gamma^+ \cup \Gamma^- \cup \Theta$. and $\varphi_k^* j_k \to j$ in C^∞_{loc}
- $u_k^N := u_k \circ \varphi|_{\ddot{S}_N} : \ddot{S}_N \to \hat{W}$ $\ddot{S}_N := (S - (\Gamma^+ \cup \Gamma^- \cup \Delta^{nd} \cup \Delta^{br}) \cap L^{-1}(N))$ then $u_k^N \to u_\infty^N$ in $C^\infty_{loc}(\ddot{S}_N, \hat{W})$ & For $\pm N > 0$, u_k^N has image in cylindrical end of $\hat{W}$ for all k large enough s.t. $r_k^N \to \pm\infty$ so $T_{-r_k^N} \circ u_k^N \to u_\infty^N$ in $C^\infty_{loc}(\ddot{S}_N, \mathbb{R} \times M_\pm)$ & $r_k^{N+1} - r_k^N \to \infty$ $\forall N < N_+$

• $\overline{S}_{\Phi}$ circle compactification of $\hat{S}_{\Phi} - (\Gamma^+ \cup \Gamma^-)$, $\overline{\Sigma}_k$ circle compactification of $\Sigma - (\Gamma^+_{\Phi} \cup \Gamma^-)$ $\forall$ k large enough φ_k extends to a continuous map $\varphi_k : \overline{S}_{\Phi} \longrightarrow \overline{\Sigma}_k$ s.t. $\overline{u}_k \circ \varphi_k \longrightarrow \overline{u}_{\infty}$ in $C^0(\overline{S}_{\Phi}, \hat{W})$