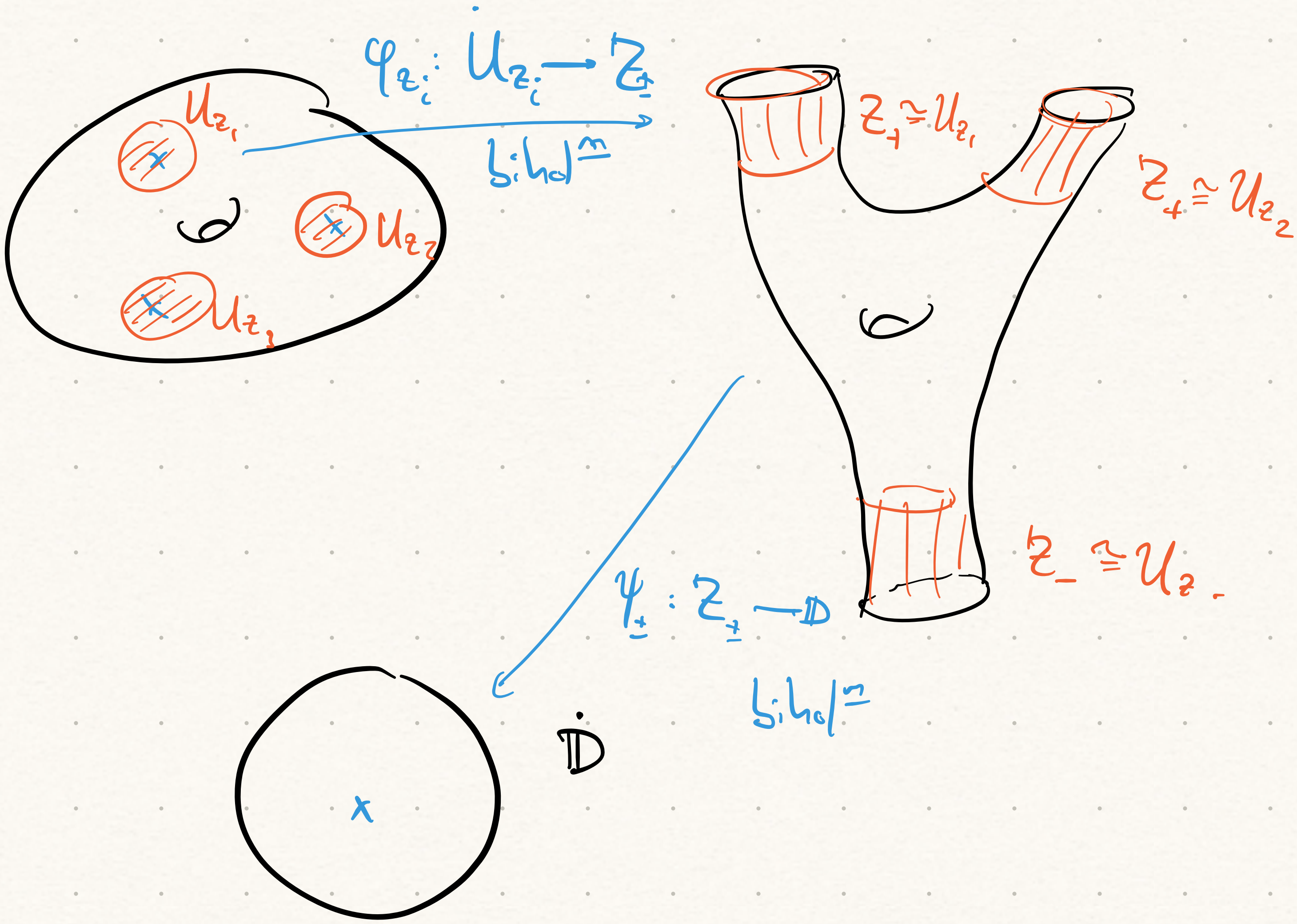
Cauchy-Riemann type operators in SFT

Setup:

> Punctures $\Gamma^\pm \subset \Sigma$



Choose φ_{z_i} s.t. $\psi_\pm \circ \varphi_{z_i} : \dot{U}_{z_i} \to \dot{\mathbb{D}}$ extends holomorphically to a map

$$U_z \to \mathbb{D} \text{ with } z \mapsto 0.$$

Def$^{\underline{n}}$:

abbreviate a.H.

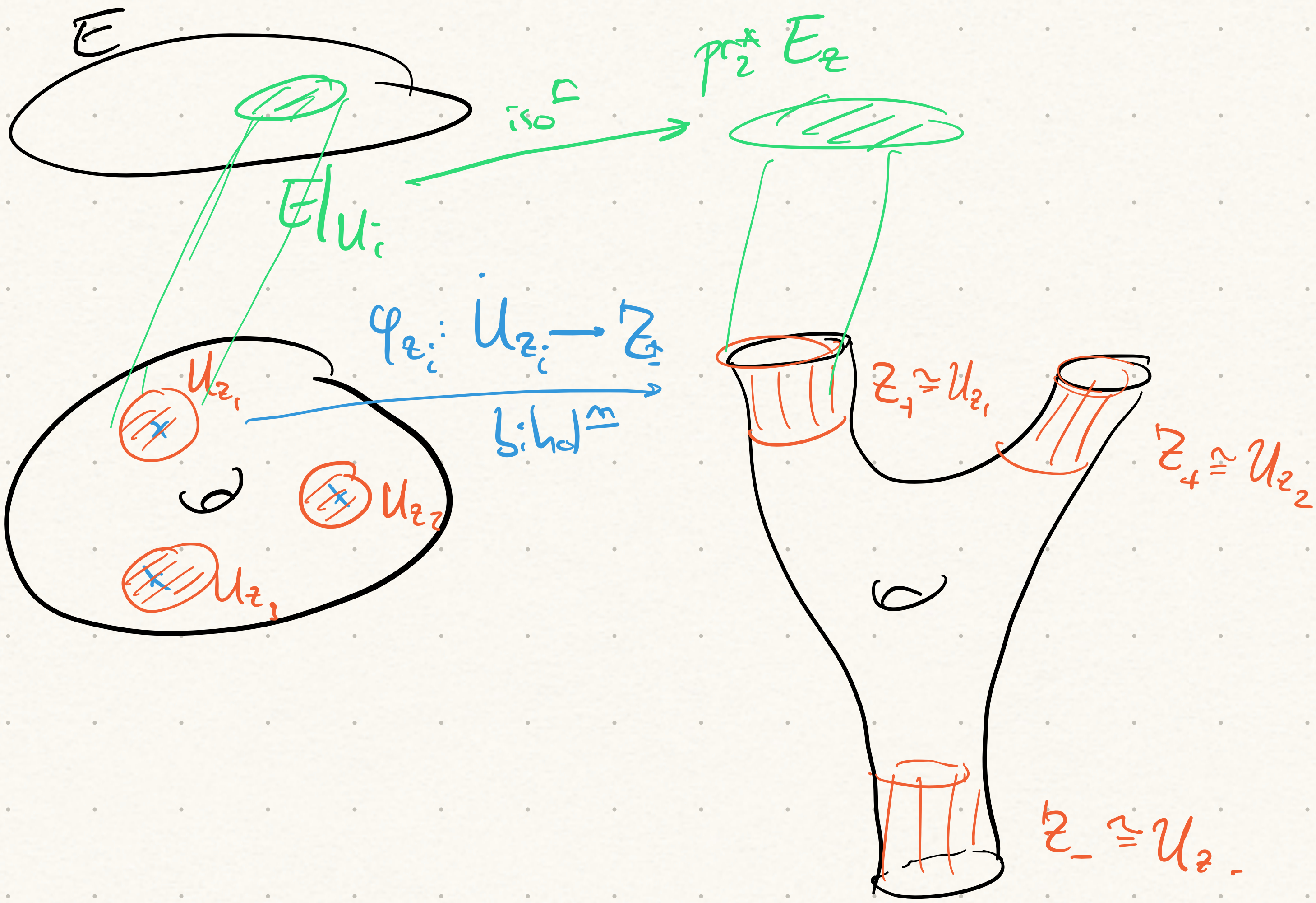
> $(E, J) \longrightarrow (\dot{\Sigma}, j)$ a rank n $\mathbb{C}$-v.b.

An asymptotically Hermitian structure on E is the following:

↳ Call (E, J) asymptotically Hermitian

- $\forall z \in \Gamma^{\pm}$ a choice of rank n $\mathbb{C}$-v.b. $(E_z, J_z, \omega_z) \longrightarrow S^1$ with choice of $\mathbb{C}$-v.b. iso$^{\underline{m}}$s

$$E|_{\dot{U}_z} \longrightarrow pr_2^* E_z .$$



> A unitary trivialisation τ of (E_z, J_z, ω_z), i.e. choice of iso^{m}

$$E_z \underset{\tau}{\cong} S^1 \times \mathbb{R}^{2n}.$$

induces trivialisation $\tau : E|_{\dot{U}_z} \longrightarrow Z_{\pm} \times \mathbb{R}^{2n}$.

identifies J with $J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

call this asymptotic trivialisation. of $E|_{\dot{U}_z}$ near z.

Def^n: Cauchy-Riemann type operator

> Assume $E \to \dot{\Sigma}$ is of class C^{m+1}, $m \in \mathbb{N} \cup \{\infty\}$. A Cauchy-Riemann type operator of class C^m on E is a first-order differential operator

$$D : C^{m+1}(\dot{\Sigma}, E) \longrightarrow C^m(\dot{\Sigma}, \underbrace{\Lambda^{0,1} T^*\dot{\Sigma} \otimes E}_{\text{call } F.})$$

s.t. $D = \bar{\partial} + S$ in local trivialisations and the zeroth order term S is of class C^m.

> Now need some kind of conditions if we have asymptotic Hermitian structures at punctures:

Using the asymptotic trivialisation τ of $E|_{\dot{U}_z}$, we also get trivialisation of

$$F|_{\dot{U}_z} \to Z_\pm \times \mathbb{R}^{2n}$$
$$\lambda \longmapsto \tau(\lambda(\partial_s))$$

↳ $[0,\infty)$ coordinate.

As λ is $\mathbb{C}$-antilinear, this also determines $\lambda(\partial_t)$

In this trivialisation $D: \Gamma(\dot{\Sigma}, E) \to \Gamma(\Lambda^{0,1}T^*\dot{\Sigma} \otimes E)$ over $\dot{U}_z$ is a linear map on $C^\infty(Z_\pm, \mathbb{R}^{2n})$ of the form

$\in \Gamma(E|_{\dot{U}_z})$

$$D\eta(s,t) = \bar{\partial}\eta(s,t) + S(s,t)\eta(s,t) \quad (+)$$

$\left(\text{where } \bar{\partial} = \partial_s + J_0\partial_t \text{ and } S \in C^\infty(Z_\pm, \mathrm{End}(\mathbb{R}^{2n}))\right).$

> Recall Hessian: $\gamma: S^1 \to Y$ a closed Reeb orbit. then asymptotic operator associated to γ / Hessian of γ $A_\gamma: \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi)$ is given by

$$\underline{A}_\gamma \eta = -J(\nabla_t \eta - T \cdot \nabla_\eta R_\alpha).$$

> In general, let $\dot{E} \to \dot{\Sigma}$ an asymptotically Hermitian, with $\underline{A}_z$ asymptotic operators on (E_z, J_z, ω_z), and D C.R. type operator of class C^m on E. D is C^m- asymptotic to $\underline{A}_z$ if D is of the form $(\dagger)$ with respect to an asymptotic trivialisation near z with

$$\| S - S_\infty \|_{C^k(Z_\pm^R)} \longrightarrow 0 \quad \text{as } R \to \infty. \qquad \forall k \leq m.$$

$= \max_{|\beta| \leq k} \sup_{x \in Z_\pm^R} |\partial^\beta (S - S_\infty)|$

$\hookrightarrow [R, \infty) \times S^1$ for Z_+^R.

Here $S_\infty(s,t) := S_\infty(t)$ is a C^m- smooth loop of symmetric matrices s.t. in the corresponding unitary trivialisation of (E_z, J_z, ω_z) as

$$\underline{A}_z = -J_0 \partial_t - S_\infty.$$

$E_z \underset{\tau}{\cong} S^1 \times \mathbb{R}^{2n}.$

Th^{m} (4.5 Wendl):

asymptotically Hermitian

> (E, J) a.H. of class C^{m+1}
$$\downarrow$$
$(\dot{\Sigma}, j)$, $\underline{A}_z$ nondegenerate asymptotic operators on (E_z, J_z, ω_z) $\forall z \in \Gamma$

D a linear C.R. type operator of class C^m, that is C^m-asymptotic to A_z $\forall z$.

Then $\forall k \in \{1, \ldots, m+1\}$ $p \in (1, \infty)$

$$D : W^{k,p}(\dot{\Sigma}, E) \longrightarrow W^{k-1,p}(F)$$
$$= W^{k-1,p}(\dot{\Sigma}, \Lambda^{0,1} T^* \dot{\Sigma} \otimes E)$$

is Fredholm. Also, ind D and ker D are independent of k and p.

i.e. Space of C^m sections whose derivatives up to order m decay exponentially fast to 0 on cylindrical ends.

Th^{m}: (index formula) (Schwarz '95)

> In same setting
$$\operatorname{ind} D = n\,\chi(\dot{\Sigma}) + 2c_1^{\tau}(E) + \sum_{z \in \Gamma^{\pm}} \mu_{CZ}^{\tau}(\underline{A}_z)$$

$\hookrightarrow$ 2-2g − # punctures

$= \operatorname{rank}_{\mathbb{C}} E$

relative 1st Chern class w.r.t. τ.

$$- \sum_{z \in \Gamma^-} \mu_{CZ}^{\tau}(\underline{A}_z))$$

Note :

- Nondegeneracy of $\underline{A}_z$ is required for Fredholm property: if D was Fredholm but A_z degenerate for some z, D can be perturbed to make A_z nondegenerate with ≥ 2 possible values of $\mu_{CZ}(\underline{A}_z)$
  $\Rightarrow$ two different Fredholm indices for small perturbations $\lightning$ D cannot be Fredholm.
- Doesn't happen on closed surfaces: difference of any two CR type operators is order zero, which is compact as inclusion $W^{k,p}(E) \hookrightarrow W^{k-1,p}(E)$ is compact.

In our case $\dot{\Sigma}$ is not compact and so inclusion $W^{k,p}(E) \longrightarrow W^{k-1,p}(E)$ is not compact.

$\therefore$ Zeroth order terms can affect Fredholm property and index.

Compact oriented surface with boundary

Def$\underline{^n}$:

> The relative first Chern number of $(E,J) \to S$ in the trivialisation τ of $E|_{\partial S}$ is the unique integer $c_1^\tau(E) \in \mathbb{Z}$ s.t.

i) If $(E,J) \to S$ is a line bundle, $c_1^\tau(E)$ is the signed count of zeroes of a

generic section $\eta \in \Gamma(E)$ that is a nonzero constant at ∂S w.r.t. τ.

ii) If (E_i, J_i) has trivialisation τ_i over ∂S,
$$c_1^{\tau_1 \oplus \tau_2}(E_1 \oplus E_2) = c_1^\tau(E_1) + c_2^\tau(E_2).$$

Facts:

> Given distinct choices of asymptotic trivializations τ_1, τ_2 for a.H. E of rank n,

$$c_1^{\tau_2}(E) = c_1^{\tau_2}(E) - \underbrace{\deg(\tau_2 \circ \tau_1^{-1})}_{\in \mathbb{Z}}$$

Sum over all punctures of winding numbers of the determinant of the transition maps $S^1 \to U(n)$,

↳ $\tau_2 \circ \tau_1^{-1}(t, v) = (t, g(t)v)$

> $$\mu_{CZ}^{\tau_2}(A_z) = \mu_{CZ}^{\tau_1}(A_z) + 2\deg(\tau_2 \circ \tau_1^{-1}).$$

$$\mu_{CZ}^{\tau_2}(A_z) = \mu_{CZ}^{\tau_z}(\gamma)$$

circle coordinate

vector in E_z.

> $\therefore$ RHS of index formula is independent of τ.