Hessians in SFT

Ward Veltman

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SFT is an infinite-dimensional analogue of Morse theory where the Morse critical points are replaced with closed nondegenerate Reeb orbits. To each closed orbit we will assign a self-adjoint differential operator along the contact bundle, and this operator will play the role of the Hessian. Before we construct these operators in the SFT case, we will briefly revisit an anlogous construction in Hamiltonian Floer theory.

All of the material below taken from Wendl's lecture notes [1].

1 Review of the "Hessian" in Hamiltonian Floer theory

The action functional for a time-dependent Hamiltonian $H = \{H_t : M \to \mathbb{R}\}_{t\in S^1}$ on a symplectic manifold (M^{2n}, ω) is a functional $\mathcal{A}_H : C^\infty_{\text{contr}}(S^1, M) \to \mathbb{R}$. $\mathcal{A}_H$ assigns to a contractible loop $\gamma \in C^\infty_{\text{contr}}(S^1, M)$ the value

$$\mathcal{A}_H(\gamma) = -\int_{\mathbb{D}^2} \bar{\gamma}^*\omega + \int_{S^1} H_t(\gamma(t))\, dt, \tag{1.0.1}$$

where $\bar{\gamma} : \mathbb{D} \to M$ is a disc with $\bar{\gamma}(e^{2\pi i t}) = \gamma(t)$. The tangent space to a contractible loop is $T_\gamma C^\infty_{\text{contr}}(S^1, M) := \Gamma(\gamma^*TM)$. The variation of the action functional is then given by

$$d\mathcal{A}_H(\gamma)\eta = \int_{S^1} \omega(\dot{\gamma} - X_t(\gamma), \eta)\, dt, \tag{1.0.2}$$

where X_t is the Hamiltonian vector field associated to H_t, and $\eta \in \Gamma(\gamma^*TM)$. The critical points of $\mathcal{A}_H$ are then precisely the contractible orbits of X_t with period 1. Choosing a family of ω-compatible almost complex structures $J = \{J_t \in \mathcal{J}(M, \omega\}_{t\in S^1}$, we can define an L^2-inner product on $\Gamma(\gamma^*TM)$ by setting

$$\langle \eta, \eta' \rangle_{L^2} = \int_{S^1} \omega(\eta(t), J_t\eta'(t))\, dt, \tag{1.0.3}$$

for $\eta, \eta' \in \Gamma(\gamma^*TM)$. We may then express the variation of the action functional by the formula

$$d\mathcal{A}_H(\gamma)\eta = \langle J_t(\dot{\gamma} - X_t(\gamma)), \eta \rangle_{L^2}. \tag{1.0.4}$$

Mirroring the definition of the gradient on Riemannian manifolds, we define the *unregularised gradient* of $\mathcal{A}_H$ by

$$\operatorname{grad} \mathcal{A}_H(\gamma) := J_t(\dot{\gamma} - X_t(\gamma)) \in \Gamma(\gamma^*TM). \tag{1.0.5}$$

Using this definition, critical points of $\mathcal{A}_H$ are precisely loops that satisfy $\operatorname{grad} \mathcal{A}_H(\gamma) = 0$. Now assume that $\gamma \in \operatorname{Crit}(\mathcal{A}_H))$ is such a loop. The Hessian should is morally the linearisation of $\operatorname{grad} \mathcal{A}_H$ at γ, and should be a linear operator

$$\mathbf{A}_\gamma := \nabla \operatorname{grad} \mathcal{A}_H(\gamma) : \Gamma(\gamma^* TM) \to \Gamma(\gamma^* TM). \tag{1.0.6}$$

It is defined using a connection ∇ on M in the following way: for $\eta \in \Gamma(\gamma^* TM)$ pick a smooth family $\{\gamma_\rho : S^1 \to M\}_{\rho \in (-\varepsilon, \varepsilon)}$ with $\gamma_0 = \gamma$ and $\partial_\rho \gamma_\rho|_{\rho=0} = \eta$, and setting

$$\mathbf{A}_\gamma \eta := \nabla_\rho (\operatorname{grad} \mathcal{A}_H(\gamma_\rho))|_{\rho=0} \tag{1.0.7}$$

The "Hessian" is independent of the choice of connection since the difference of two connections is a tensor and $\operatorname{grad} \mathcal{A}_H(\gamma) = 0$.

Proposition 1.0.1. *If ∇ is a torsion-free connection on M, then*

$$\mathbf{A}_\gamma \eta = J_t(\nabla_t \eta - \nabla_\eta X_t). \tag{1.0.8}$$

Proof. By the Leibniz rule,

$$\begin{aligned} \nabla_\rho (\operatorname{grad} \mathcal{A}_H(\gamma_\rho)) &= \nabla_\rho (J_t(\gamma_\rho)(\dot\gamma_\rho - X_t(\gamma_\rho))) & (1.0.9)\\ &= (\nabla_\rho J_t(\gamma_\rho))(\dot\gamma_\rho - X_t(\gamma_\rho)) + J_t(\gamma_\rho) \nabla_\rho (\dot\gamma_\rho - X_t(\gamma_\rho)). & (1.0.10) \end{aligned}$$

Evaluating at $\rho = 0$, the first term vanishes since $\dot\gamma - X_t(\gamma) = 0$ as $\operatorname{grad} \mathcal{A}_H(\gamma) = 0$. Since ∇ is torsion free, the torsion tensor on M is identically zero, and

$$0 \equiv T(\partial_\rho, \partial_t) := \nabla_\rho \partial_t - \nabla_t \partial_\rho - [\partial_\rho, \partial_t]. \tag{1.0.11}$$

The commutator is zero, and hence $\nabla_\rho \partial_t = \nabla_t \partial_\rho$. Then

$$\nabla_\rho (\dot\gamma_\rho - X_t(\gamma_\rho))|_{\rho=0} = \nabla_t \partial_\rho \gamma_\rho|_{\rho=0} - \nabla_{\partial_\rho \gamma_\rho} X_t|_{\rho=0} = \nabla_t \eta - \nabla_\eta X_t. \tag{1.0.12}$$

Therefore, $\mathbf{A}_\gamma \eta = J_t(\nabla_t \eta - \nabla_\eta X_t)$. □

2 The "Hessian" in SFT

Let (Y^{2n-1}, ξ) be a contact manifold with contact form α, Reeb vector field R_α, and an almost complex structure $J : \xi \to \xi$ compatible with the symplectic structure $d\alpha|_\xi$. Let $\pi_\xi : TM \to \xi$ be the projection along R_α. The *contact action functional* is the functional $\mathcal{A}_\alpha : C^\infty(S^1, Y) \to \mathbb{R}$ defined by

$$\mathcal{A}_\alpha(\gamma) := \int_{S^1} \gamma^* \alpha. \tag{2.0.1}$$

Similarly, if $\eta \in \Gamma(\gamma^* TM) =: T_\gamma C^\infty(S^1, Y)$, the variation of $\mathcal{A}_\alpha$ is given by

$$d\mathcal{A}_\alpha(\gamma)\eta = \int_{S^1} d\alpha(\eta, \dot\gamma)\, dt = -\int_{S^1} d\alpha(\pi_\xi \dot\gamma, \eta)\, dt. \tag{2.0.2}$$

Here we have used that $d\alpha(R_\alpha, -) = 0$, so we may project $\dot\gamma$ onto the contact structure along the Reeb vector field via π_ξ. Unlike the action functional in Hamiltonian Floer theory, the variation of the contact action functional has a built-in degeneracy. This is a result of the fact that the Reeb vector field is time-independent, and so closed Reeb orbits come in S^1-families related by reparametrisation. It is clear that $d\mathcal{A}_\alpha(\gamma)\eta = 0$, whenever η points in the direction of the Reeb vector field R_α.

It is also clear from the variation of $\mathcal{A}_\alpha$ that a loop $\gamma : S^1 \to M$ is critical if and only if $\pi_\xi \dot{\gamma} = 0$, i.e. $\dot{\gamma}$ is everywhere tangent to the Reeb vector field R_α. There are infinitely many ways to parametrise the same closed orbit that is everywhere tangent to the Reeb vector field. We define a *preferred parametrisation* of a closed orbit γ, as one where $\dot{\gamma}$ is a constant multiple of $\mathbb{R}_\alpha$. So, a preferred parametrisation of a critical loop γ is one where

$$\dot{\gamma} = T \cdot R_\alpha(\gamma), \qquad T := \mathcal{A}_\alpha(\gamma). \tag{2.0.3}$$

It will become clear in what follows, why this is a sensible parametrisation. Now we run into a slight problem if we try to define the Hessian of the contact action functional in a similar way to the Hessian in Hamiltonian Floer theory. The Hessian in Hamiltonian Floer theory had a trivial kernel, because ω (which appears in the first variation, and the L^2-metric) is nondegenerate. If we did an analogous construction in the contact case, the resulting Hessian will always have a nontrivial kernel since $d\alpha$ is degenerate in the Reeb direction. To avoid this degeneracy, we assume that every closed orbit has the preferred parametrisation, and we only allow perturbations (for tangent vectors) in directions tangent to ξ. In other words, we will consider preferred trivialisations, and restrict the tangent space to sections $\eta \in \Gamma(\gamma^*\xi) \subset \Gamma(\gamma^*Y)$. For $\eta \in \Gamma(\gamma^*\xi)$, we then have

$$d\mathcal{A}_\alpha(\gamma)\eta = \int_{S^1} d\alpha(-J\pi_\xi\dot{\gamma}, J\eta)\, dt = \langle -J\pi_\xi\dot{\gamma}, \eta\rangle_{L^2}, \tag{2.0.4}$$

with L^2-inner product

$$\langle \eta, \eta' \rangle_{L^2} := \int_{S^1} d\alpha(\eta, J\eta')\, dt. \tag{2.0.5}$$

In this way, the L^2-metric on sections of $\gamma^*\xi$ is nondegenerate. Now we can proceed in exactly the same way as in the Hamiltonian Floer case. We define the gradient

$$\operatorname{grad} \mathcal{A}_\alpha(\gamma) := -J\pi_\xi\dot{\gamma} \in \Gamma(\gamma^*\xi). \tag{2.0.6}$$

The Hessian is the linearisation of $\operatorname{grad} \mathcal{A}_\alpha$ in the ξ directions, and should be a differential operator

$$\nabla \operatorname{grad} \mathcal{A}_\alpha(\gamma) : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi). \tag{2.0.7}$$

To define it, let $\eta \in \Gamma(\gamma^*\xi)$, pick a smooth family $\{\gamma_\rho : S^1 \to M\}_{\rho\in(-\varepsilon,\varepsilon)}$ with $\gamma_0 = \gamma$ and $\partial_\rho\gamma_\rho|_{\rho=0} = \eta$ and pick a torsion-free connection on Y. We want to define the analogue of the Hessian as

$$\mathbf{A}_\gamma\eta := \nabla_\rho(\operatorname{grad} \mathcal{A}_H(\gamma_\rho))|_{\rho=0}. \tag{2.0.8}$$

This will be independent of the choice of connection if γ is a closed Reeb orbit, but it will assume that γ has the preferred parametrisation. As was mentioned before, the operator $\mathbf{A}_\gamma$ may have a nontrivial kernel, but we will come back to that.

Proposition 2.0.1. *If the connection ∇ on Y is torsion-free, the operator $\mathbf{A}_\gamma$ defined above is given by the equation*

$$\mathbf{A}_\gamma\eta = -J(\nabla_t\eta - T \cdot \nabla_\eta R_\alpha). \tag{2.0.9}$$

Proof. The proof is slightly more involved than in the Hamiltonian Floer theory case, but is simplified by the preferred parametrisation. In order to calculate the covariant derivative of the gradient, note that the projection of $\dot{\gamma}$ onto the contact structure is given by

$$\pi_\xi \dot{\gamma}_\rho = \partial_t \gamma_\rho - \alpha(\partial_t \gamma_\rho) R_\alpha(\gamma_\rho), \tag{2.0.10}$$

since $\alpha(R_\alpha) = 1$. By the Leibniz rule, and a calculation similar to that of **??**, we get that

$$\nabla_\rho(\pi_\xi \dot{\gamma}_\rho)|_{\rho=0} = \nabla_t \eta - \alpha(\dot{\gamma}) \nabla_\eta - \partial_\rho(\alpha(\partial_t \gamma_\rho))|_{\rho=0} \cdot R_\alpha(\gamma). \tag{2.0.11}$$

This first step requires that ∇ is torsion-free, but we haven't used the preferred parametrisation yet. Note that this covariant derivative is a section of $\Gamma(\gamma^*\xi)$, since $\pi_\xi \dot{\gamma}_\rho$ is in the subspace $\Gamma(\gamma_\rho^*\xi)$. Using the preferred parametrisation ($\dot{\gamma} = T \cdot R_\alpha$), we get that

$$0 = T \cdot d\alpha(\eta, R_\alpha(\gamma)) = d\alpha(\rho\gamma_\rho \partial_t \gamma)|_{\rho=0} = \partial_\rho(\alpha(\dot{\gamma}_\rho))|_{\rho=0} - \partial_t(\alpha(\partial_\rho \gamma_\rho))|_{\rho=0}. \tag{2.0.12}$$

In the last step we've used the standard formula for evaluating an exact 2-form on two vector fields, and used that $[\partial_\rho, \partial_t] = 0$. Now the last term in this equation vanishes since $\alpha(\eta) = 0$, since $\eta \in \Gamma(\gamma^*\xi)$ and $\xi = \ker \alpha$. Hence the last term in equation 2.0.11 vanishes, and

$$\nabla_\rho(\pi_\xi \dot{\gamma}_\rho)|_{\rho=0} = \nabla_t \eta - T \cdot \nabla_\eta R_\alpha \in \Gamma(\gamma^*\xi). \tag{2.0.13}$$

Using that J and ∇_ρ commute when evaluated on critical loops ($\pi_\xi \dot{\gamma} = 0$), we obtain

$$\nabla_\rho(-J\pi_\xi \dot{\gamma}_\rho)|_{\rho=0} = -J(\nabla_t \eta - T \cdot \nabla_\eta R_\alpha) \in \Gamma(\gamma^*\xi) \tag{2.0.14}$$

This concludes the proof. □

Motivated by the previous proposition, we make the following definition.

Definition 2.0.2. Let $\gamma : S^1 \to Y$ be a loop parametrising a closed Reeb orbit in $(Y, \xi = \ker \alpha)$ of period $T = \alpha(\dot{\gamma})$. Then the *asymptotic operator associated to* γ is the first order differential operator $\mathbf{A}_\gamma : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi)$ defined by

$$\mathbf{A}_\gamma \eta := -J(\nabla_t \eta - T \cdot \nabla_\eta R_\alpha). \tag{2.0.15}$$

Proposition 2.0.3. $\mathbf{A}_\gamma$ *is symmetric with respect to the* L^2*-inner product on* $\Gamma(\gamma^*\xi)$*.*

Proof. Let $\eta, \eta' \in \Gamma(\gamma^*\xi)$. Then

$$\langle \mathbf{A}_\gamma \eta, \eta' \rangle_{L^2} = \int_{S^1} d\alpha(\mathbf{A}_\gamma \eta, J\eta')\, dt = \int_{S^1} (-d\alpha(\nabla_t \eta, \eta') + d\alpha(T \cdot \nabla_\eta R_\alpha, \eta'))\, dt. \tag{2.0.16}$$

Since the Reeb flow preserves the contact structure ξ and the symplectic structure $d\alpha|_\xi$, i.e. $\mathcal{L}_{R_\alpha} \alpha = 0$ and $\mathcal{L}_{R_\alpha} d\alpha = 0$, we have

$$0 = (\mathcal{L}_{R_\alpha} d\alpha)(\eta, \eta') = R_\alpha(d\alpha(\eta, \eta')) - d\alpha([R_\alpha, \eta], \eta') - d\alpha(\eta, [R_\alpha, \eta']). \tag{2.0.17}$$

Using that ∇ is torsion free, $[R_\alpha, \eta] = \nabla_{R_\alpha} \eta - \nabla_\eta R_\alpha$, and similarly for η'. Substituting these relation into equation 2.0.16, and integrating by parts then shows $\langle \mathbf{A}_\gamma \eta, \eta' \rangle_{L^2} = \langle \eta, \mathbf{A}_\gamma \eta' \rangle_{L^2}$. □

Proposition 2.0.4. *A Reeb orbit* γ *is nondegenerate (i.e. the linearisation of the Reeb flow* ϕ_α^t *has the property that its linearisation along the contact bundle* $d\phi_\alpha^T(\gamma(0))|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ *does not have* 1 *as an eigenvalue) if and only if* $\ker \mathbf{A}_\gamma$ *is trivial.*

This proposition justifies the terminology that the asymptotic operator $\mathbf{A}_\gamma$ is nondegenerate if its kernel is trivial.

Remark 2.0.5. The Hermitian vector bundle $(\gamma^*\xi, J, d\alpha)$ over S^1 is globally trivialisable, since Hermitian vector bundles over S^1 are classified by $[S^1, BU(k)] = \pi_1(BU(k)) = \pi_0(U(k)) = 0$.

Let τ be a global trivialisation of $\gamma^*\xi$. There are equivalent ways of defining the Conley-Zehnder index of a nondegenerate Reeb orbit γ in a trivialisation. One way is to set $\mu^\tau_{\mathrm{CZ}}(\gamma) := \mu^\tau_{\mathrm{CZ}}(\mathbf{A}_\gamma)$, and define the difference of Conley-Zehnder indices $\mu^\tau_{\mathrm{CZ}}(\mathbf{A}_\gamma) - \mu^\tau_{\mathrm{CZ}}(\mathbf{A}_{\gamma'})$ to be the spectral flow $\mu^{\mathrm{spec}}(\mathbf{A}_\gamma, \mathbf{A}_{\gamma'})$. This definition agrees with the definition for a path of symplectic matrices determined by the time t linearised Reeb flow.

References

[1] Chris Wendl. Lectures on symplectic field theory, 2016.