SOLUTION TO THE SAMPLE TEST 1

1. Let $a = 1 + \sqrt{1 + \sqrt{3}}$, then $(a - 1)^2 = 1 + \sqrt{3}$, thus $((a - 1)^2 - 1)^2 - 3 = 0$. If we expand this equation, the coefficient of the constant term is $-3$. If $a$ is rational, then $a = 1, -1, 3$ or $-3$. We can plug in these numbers to see that those are not the solution to the equation.

2. Let $c_n = a_n - b_n$, according to the assumption $c_n \geq 0$ and $\lim c_n = a - b$. Assume $a - b < 0$, then take $\epsilon = \frac{b-a}{2} > 0$, there exists $N$ such that for $n > N$, $c_n - (a - b) \leq |c_n - (a - b)| \leq \epsilon = \frac{b-a}{2}$. From this, we find that $c_n \leq \frac{a-b}{2} < 0$ for $n > N$. This is a contradiction.

3. Let $a_n = \frac{2^n}{\sqrt{n}}$, then $\lim |\frac{a_{n+1}}{a_n}| = \lim \frac{2}{\sqrt{n+1}} = 0$. By the ratio test, $\sum \frac{2^n}{\sqrt{n!}}$ is convergent.

4. Given any $\epsilon > 0$, we can find $N$ such that for $n > N$, $a_n \leq \lim sup a_n + \epsilon$, $b_n \leq \lim sup b_n + \epsilon$, thus for $n > N$, $a_n + b_n \leq \lim sup a_n + \lim sup b_n + 2\epsilon$. This implies that $\lim sup(a_n + b_n) \leq \lim sup a_n + \lim sup b_n + 2\epsilon$. But $\epsilon$ is arbitrary, $\lim sup(a_n + b_n) \leq \lim sup a_n + \lim sup b_n$.

5. Given $\epsilon > 0$, take $N = \frac{1}{4\epsilon^2}$, then for $n > N$, $|\frac{\cos n}{\sqrt{n}}| \leq \frac{1}{\sqrt{N}} < \epsilon$ (here we are using the fact that $|\cos n| \leq 1$).

6. Assume $|t_n| \leq a$ for some $a \in \mathbb{R}$. Given $M > 0$, we can find $N$ such that for any $n > N$, $s_n > M + a$, thus $s_n + t_n > M + a - |t_n| \geq M + a - a = M$. This concludes the proof.