Math 104
Fall 2013
Exam 2
11/08/13
Time Limit: 50 Minutes

This exam contains 5 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- If you use a theorem you must indicate this and explain why the theorem may be applied, e.g., the squeeze theorem.

- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- If a question uses the word “prove” or “show”, then you will be expected to write down a mathematical proof similar to those in the textbook or given in the homework questions. If a question uses weaker language, such as “compute” or “determine”, then a less rigorous argument will still receive full credit.

- You can assume a number of basic results, such as \( \sum n^{-p} \) converges if and only in \( p > 1 \); The triangle inequality \(|a| + |b| \geq |a + b|\); \( \lim_{n \to \infty} a^n = 0 \) when \(|a| < 1 \); \( \lim_{n \to \infty} a^n \) diverges when \(|a| > 1 \). \(|b| < a \) if and only \(-a < b < a\).

- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. (a) (5 points) Determine the interior of the set $[-1, 1] \cup \mathbb{Q}$ on the real line.

(b) (5 points) Let $(X, d)$ be a metric space. Suppose $K_1$ and $K_2$ are two compact sets of $X$, prove $K_1 \cup K_2$ is also a compact set.
2. (a) (5 points) Prove that \( f(x) = \cos \sqrt{x} \) is uniformly continuous on \([0, \infty]\).

(b) (5 points) Is \( g(x) = \sin(x^2) \) uniformly continuous on \([0, \infty]\)? Prove or disprove it.
3. (10 points) Find the radius of convergence for \( \sum_{n=1}^{\infty} \frac{x^n}{n} \). Calculate \( \sum_{n=1}^{\infty} \frac{1}{n^n} \).
4. (5 points) Let \( f(x) \) be continuous on \([0, 1]\) and \( f(1) = 0 \). Define \( g_n(x) = x^n f(x) \). Prove that \( g_n(x) \) is uniformly convergent to 0 on \([0, 1]\).

5. (5 points) Let \( f(x) \) be continuous on \([0, 1]\). Let \( 0 < x_1 < x_2 < x_3 < \cdots < x_n < 1 \). Prove that there exists some \( x_0 \in [0, 1] \) such that \( f(x_0) = \frac{1}{n} \sum_{i=1}^{n} f(x_i) \).