IS THERE AN ANALYTIC THEORY OF AUTOMORPHIC FUNCTIONS FOR COMPLEX ALGEBRAIC CURVES?

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Abstract. The geometric Langlands correspondence for complex algebraic curves differs from the original Langlands correspondence for number fields in that it is formulated in terms of sheaves rather than functions (in the intermediate case of curves over finite fields, both formulations are possible). In a recent preprint, Robert Langlands raised the possibility of developing an analytic theory of automorphic forms on the moduli of G-bundles on a complex algebraic curve. Langlands envisioned these forms as eigenfunctions of certain Hecke operators, which he attempted to define. In this note I show that Hecke operators are well-defined if G is abelian and give a complete description of their eigenfunctions and eigenvalues in this case. However, for non-abelian G, Hecke operators involve integration, which is problematic in the context of complex curves. Therefore the problem of finding Hecke eigenfunctions and eigenvalues is not well-posed. Nonetheless, I show that automorphic forms can be defined in a different way – not as eigenfunctions of Hecke operators, but rather as eigenfunctions of a commutative algebra of global differential operators on the line bundle of half-densities on the moduli of G-bundles. Conjecturally, their eigenvalues are parametrized by the opers for the Langlands dual group ${}^LG$ with real monodromy.

1. Introduction

1.1. The foundations of the Langlands Program were laid by Robert Langlands in the late 1960s [L1]. Originally, these ideas were applied in two realms: that of number fields, i.e. finite extensions of the field $\mathbb{Q}$ of rational numbers, and that of function fields, where by a function field one understands the field of rational functions on a smooth projective curve over a finite field $\mathbb{F}_q$. In both cases, the objects of interest are *automorphic forms*, which are, roughly speaking, functions on the quotient of the form $G(F)\backslash G(\mathbb{A}_F)/K$, where G is a reductive algebraic group over F, the field in question (a number field or a function field), $\mathbb{A}_F$ is the ring of adeles of F, and K is a compact subgroup of $G(\mathbb{A}_F)$. There is a family of mutually commuting *Hecke operators* acting on this space of functions, and one wishes to describe the common eigenfunctions of these operators as well as their eigenvalues. The idea is that those eigenvalues can be packaged as the "Langlands parameters" which can be described in terms of homomorphisms from a group closely related to the Galois group of F to the Langlands dual group ${}^LG$ associated to G, and perhaps some additional data.

To be more specific, let F be the field of rational functions on a curve X over $\mathbb{F}_q$ and $G = GL_n$. Let us further restrict ourselves to the unramified case, so that K is the maximal compact subgroup $K = GL_n(\mathcal{O}_F)$, where $\mathcal{O}_F \subset \mathbb{A}_F$ is the ring of integer adeles. In this case, a theorem of V. Drinfeld [Dr1, Dr2] for $n = 2$ and L. Lafforgue [Laf] for $n > 2$ states that (if we impose the so-called cuspidality condition and place a restriction on the action of the center of GL_n) the Hecke eigenfunctions on $GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O}_F)$ are in one-to-one correspondence with n-dimensional irreducible unramified representations of the Galois group of F (with a matching restriction on its determinant).

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1.2. Number fields and function fields for curves over $\mathbb{F}_q$ are two "languages" in André Weil's famous trilingual "Rosetta stone" [We], the third language being the theory of algebraic curves over the field $\mathbb{C}$ of complex numbers. Hence it is tempting to build an analogue of the Langlands correspondence in the setting of a complex curve X. Such a theory has indeed been developed starting from the mid-1980s, initially by V. Drinfeld [Dr2] and G. Laumon [La1] (and relying on ideas of an earlier work of P. Deligne), then by A. Beilinson and V. Drinfeld [BD], and subsequently by many others. See, for example, the surveys [Fr2, Gai2] for more details. However, this theory, dubbed "geometric Langlands Program," is quite different from the Langlands Program in its original formulation for number fields and function fields.

The most striking difference is that in the geometric theory the vector space of automorphic functions on the double quotient $G(F)\backslash G(\mathbb{A}_F)/K$ is replaced by a (derived) category of sheaves on an algebraic stack whose set of $\mathbb{C}$-points is this quotient. For example, in the unramified case $K = G(\mathcal{O}_F)$, this is the moduli stack Bun_G of principal G-bundles on our complex curve X. Instead of the Hecke operators of the classical theory, which act on functions, we then have *Hecke functors* acting on suitable categories of sheaves, and instead of Hecke eigenfunctions we have *Hecke eigensheaves.*

For example, in the unramified case a Hecke eigensheaf $\mathcal{F}$ is a sheaf on Bun_G (more precisely, an object in the category of D-modules on Bun_G, or the category of perverse sheaves on Bun_G) with the property that its images under the Hecke functors are isomorphic to $\mathcal{F}$ itself, tensored with a vector space (this is the categorical analogue of the statement that under the action of the Hecke operators eigenfunctions are multiplied by scalars). Furthermore, since the Hecke functors (just like the Hecke operators acting on functions) are parametrized by closed points of X, a Hecke eigensheaf actually yields a family of vector spaces parametrized by points of X. We then impose an additional requirement that these vector spaces be stalks of a local system on X for the Langlands dual group ${}^L G$ (taken in the representation of ${}^L G$ corresponding to the Hecke functor under consideration). This neat formulation enables us to directly link Hecke eigensheaves and (equivalence classes of) ${}^L G$-local systems on X, which are the same as (equivalence classes of) homomorphisms from the fundamental group $\pi_1(X, x)$ of X to ${}^L G$.

This makes sense from the point of view of Weil's Rosetta stone, because the fundamental group can be seen as a geometric analogue of the unramified quotient of the Galois group of a function field. We note that for $G = GL_n$, in the unramified case, the Hecke eigensheaves have been been constructed in [Dr1] for $n = 2$ and in [FGV, Gai1] for $n > 2$. More precisely, the following theorem has been proved: for any irreducible rank n local system $\mathcal{E}$ on X, there exists a Hecke eigensheaf on Bun_{GL_n} whose "eigenvalues" correspond to $\mathcal{E}$.[1] Many results of that nature have been obtained for other groups as well. For example, in [BD] Hecke eigensheaves on Bun_G were constructed for all ${}^L G$-local systems having the structure of an *${}^L G$-oper* (these local systems form a Lagrangian subspace in the moduli of all ${}^L G$-local systems). Furthermore, a more satisfying categorical version of the geometric Langlands correspondence has been proposed by A. Beilinson and V. Drinfeld and developed further in the works of D. Arinkin and D. Gaitsgory [AG] (see [Gai2] for a survey).

To summarize, the salient difference between the original formulation of the Langlands Program (for number fields and function fields of curves over $\mathbb{F}_q$) and the geometric formulation is that the former is concerned with functions and the latter is concerned with sheaves.

[1] Furthermore, these Hecke eigensheaves are irreducible on each connected component of Bun_{GL_n}.

What makes this geometric formulation appealing is that in the intermediate case – that of curves over $\mathbb{F}_q$ – which serves as a kind of a bridge in the Rosetta stone between the number field case and the case of curves over $\mathbb{C}$, both function-theoretic and sheaf-theoretic formulations make sense. Moreover, it is quite common that the same geometric construction works for curves over $\mathbb{F}_q$ and $\mathbb{C}$. For example, essentially the same construction produces Hecke eigensheaves on Bun_{GL_n} for an irreducible rank n local system on a curve over $\mathbb{F}_q$ and over $\mathbb{C}$ [Dr1, FGV].[2]

Furthermore, in the realm of curves over $\mathbb{F}_q$, the function-theoretic and sheaf-theoretic formulations are connected to each other by Alexander Grothendieck's "functions-sheaves dictionary." This dictionary assigns to a (ℓ-adic) sheaf $\mathcal{F}$ on a variety (or an algebraic stack) V over $\mathbb{F}_q$, a function on the set of closed points of V whose value at a given closed point v is the alternating sum of the traces of the Frobenius (a generator of the Galois group of the residue field of v) on the stalk cohomologies of $\mathcal{F}$ at v (see [La2], Sect. 1.2 or [Fr2], Sect. 3.3 for details). Thus, for curves over $\mathbb{F}_q$ the geometric formulation of the Langlands Program may be viewed as a *refinement* of the original formulation: the goal is to produce, for each ${}^LG$-local system on X, the corresponding Hecke eigensheaf on Bun_G, but at the end of the day we can always go back to the more familiar Hecke eigenfunctions by taking the traces of the Frobenius on the stalks of the Hecke eigensheaf at the $\mathbb{F}_q$-points of Bun_G. Thus, the function-theoretic and the sheaf-theoretic formulations go hand-in-hand for curves over $\mathbb{F}_q$.

1.3. In the case of curves over $\mathbb{C}$ there is no Frobenius, and hence no direct way to get functions out of Hecke eigensheaves on Bun_G. However, since a Hecke eigensheaf is a D-module on Bun_G, we could view its sections as analogues of automorphic functions of the analytic theory. The problem is that for non-abelian G, these D-modules – and hence their sections – are known to have complicated singularities. Outside of the singularity locus, a Hecke eigensheaf is a holomorphic vector bundle with a holomorphic flat connection, but its horizontal sections undergo non-trivial monodromies as we move around the multiple components of the singularity locus. So instead of functions we get multi-valued sections of a vector bundle. On top of that, in the non-abelian case the rank of this vector bundle grows exponentially as a function of the genus of X, and furthermore, the components of the singularity locus have a rather complicated structure. Thus, it is the D-modules themselves, rather than their sections, that are more meaningful objects of study, and that's why traditionally, in the geometric formulation of the Langlands Program for curves over $\mathbb{C}$, we focus on these D-modules rather than their multi-valued sections. For this reason, the geometric theory in the case of complex curves becomes inherently sheaf-theoretic. It appears to be far away from the more familiar world of automorphic functions (though, as we will see later on, this appearance is somewhat misleading).

1.4. Against this backdrop, in a recent preprint [L2] Robert Langlands asked whether it is possible to develop a function-theoretic version of the Langlands correspondence for complex algebraic curves. He suggested considering the space of L_2 functions on Bun_G (with respect to some unspecified integration measure) and defining Hecke operators acting on it. By analogy with the Satake isomorphism of the original formulation, he anticipated the eigenvalues of these Hecke operators to give rise to functions on the curve X with values in conjugacy classes of ${}^LG$. He proposed that these functions could be expressed as holonomies

[2]The term "local system" has different meanings in the two cases: it is an ℓ-adic sheaf in the first case and a bundle with a flat connection in the second case, but what we do with these local systems to construct Hecke eigensheaves (in the appropriate categories of sheaves) is essentially the same in both cases.

of some (projectively) flat ${}^LG$-connections on X. By taking the monodromy representation of these flat connections, he argued, one would obtain a link between Hecke eigenfunctions (in some space of L_2 functions on Bun_G) and homomorphisms from the fundamental group of X to ${}^LG$, and furthermore, he argued that they should take values in a maximal compact subgroup of ${}^LG$. Langlands presented some computations in the case that X is an elliptic curve and $G = GL_1$ or GL_2.

In the present paper I discuss this proposal. In the case of GL_1, the Hecke operators do not involve integration and are well-defined on the space of L_2 functions on the Picard variety of a complex curve X (it is a good substitute for Bun_{GL_1}, see Section 2.1). The question of finding their eigenfunctions and eigenvalues is indeed well-posed. Moreover, I give a complete answer to this question in Section 2: first for elliptic curves in Sections 2.1 and 2.2 and then for curves of an arbitrary genus in Section 2.3. Furthermore, in Section 2.4 I generalize these results to the case of an arbitrary torus T instead of GL_1. In particular, I show that Hecke eigenfunctions are labeled by $H^1(X, \Lambda^*(T))$, the first cohomology group of X with coefficients in the lattice of cocharacters of T, and give an explicit formula for the corresponding eigenvalues. The construction uses the Abel–Jacobi map.

The results presented in Section 2 do not agree with Langlands' expectations from [L2]. He had suggested in [L2] that Hecke eigenfunctions in the case of the group GL_1 (and X being an elliptic curve) should be in one-to-one correspondence with one-dimensional representations of the fundamental group $\pi_1(X, x)$ of X with finite image. He suggested to establish this link by expressing the eigenvalues as holonomies of a flat connection on a line bundle on X and then taking the monodromy representation of this connection. I show in Section 2 that the Hecke eigenvalues may indeed be written as holonomies of flat connections; namely, flat connections on the trivial line bundle on X (this is so not only for elliptic curves but for curves of an arbitrary genus). But, by construction, the monodromy representation of each of these connections is *trivial* (indeed, the collection of Hecke eigenvalues, labeled by $x \in X$, on a specific eigenfunction gives rise to a *single-valued* function on X, and this function is, by definition, a horizontal section of the connection). Therefore, one cannot possibly obtain in this way a correspondence between Hecke eigenfunctions and non-trivial one-dimensional representations of $\pi_1(X, x)$ (see Remark 1 for more details). Rather, we obtain a one-to-one correspondence between the Hecke eigenvalues and certain flat connections on the trivial line bundle on X with trivial monodromy, which are in fact harmonic one-forms corresponding to integral cohomology classes (note, however, that in Section 3.7 I will discuss a different interpretation). This one-to-one correspondence can be generalized to the case of an arbitrary torus T (and a curve of an arbitrary genus).

A special feature of the Hecke operators in the case of GL_1 (which also holds for arbitrary tori instead of GL_1), and one that is crucial for the Hecke operators being well-defined in this case, is that they may be expressed as pull-backs of functions under a map on the moduli space of line bundles on X, which is the Picard variety of X (there is an equivalent reformulation, explained in Section 2, in which the Hecke operators act by pull-backs of functions on the neutral component of the Picard variety). In other words, the action of Hecke operators on functions in the abelian case is given by the same formula in the case of curves over $\mathbb{C}$ as in the case of curves over $\mathbb{F}_q$. In particular, no integration is needed, and hence no measure of integration has to be defined. This is why the Hecke operators are well-defined and the question of finding their eigenfunctions and eigenvalues is well-posed in the abelian case (and furthermore, has a simple answer for a general curve and a general abelian group G, as I show in Section 2).

However, in the non-abelian case, in order to define Hecke operators, one cannot avoid integration, and this necessitates defining a measure of integration on the group $G(\mathbb{C}((t)))$. This group is *not* locally compact (unlike the group $G(\mathbb{F}_q((t)))$, for example), and therefore it does not have a Haar measure, which is defined for locally compact groups. The question of defining the measure of integration on $G(\mathbb{C}((t)))$ is not addressed in [L2]. Instead, in [L2] an attempt is made to define Hecke operators acting on an L_2 space of Bun_{GL_2} of an elliptic curve.

I discuss all this in detail in Section 3. First, I explain the difficulties involved in defining a measure of integration on $G(G((t)))$ that would give rise to a meaningful spherical Hecke algebra having an analogue of the Satake isomorphism. I then discuss whether one could define Hecke operators directly, as operators acting on some space of functions associated to Bun_G. Here again we face what appear to be insurmountable obstacles, which I illustrate with concrete examples in Sections 3.3 and 3.4. From this analysis, it appears that for non-abelian G (in fact, already for $G = GL_2$ and an elliptic curve) it is not possible to give a meaningful definition of Hecke operators, and therefore the question of finding their eigenfunctions and eigenvalues is not well-posed.

1.5. There is, however, another possibility: rather than looking for the eigenfunctions of Hecke operators, let's look for the eigenfunctions of global differential operators on Bun_G. These eigenfunctions and the corresponding eigenvalues have been recently studied for $G = SL_2$ in the framework of conformal field theory by Joerg Teschner [T]. In an ongoing joint work with David Kazhdan [FK], we are attempting to extend this analysis to other groups.

According to a theorem of Beilinson and Drinfeld [BD], there is a large commutative algebra of global holomorphic differential operators acting on sections of a square root $K^{1/2}$ of the canonical line bundle K on Bun_G (this square root always exists, and is unique if G is simply-connected [BD]). The complex conjugates of these differential operators are anti-holomorphic and act on sections of the complex conjugate line bundle $\overline{K}^{1/2}$ on Bun_G. The tensor product of these two algebras is therefore a commutative algebra that naturally acts on sections of the line bundle $K^{1/2} \otimes \overline{K}^{1/2}$ which we refer to as the bundle of half-densities on Bun_G.

The space of global sections of the line bundle $K^{1/2} \otimes \overline{K}^{1/2}$ on Bun_G (or rather, on its open subspace of stable G-bundles) has a natural norm. Taking the completion of the space of sections with finite norm, we obtain a natural Hilbert space. Our differential operators act on this space (in fact, one can show that they are normal), and we can ask what are their eigenfunctions and eigenvalues. In Section 3.6, as a preview of [FK], we give some more details of this construction. We then look at the abelian case of $G = GL_1$ in Section 3.7.

In the abelian case, we have both the Hecke operators and the global differential operators, which are nothing but polynomials in the shift vector fields, holomorphic and anti-holomorphic, on the neutral component Pic^0 of Picard variety. These operators commute with each other and therefore have joint eigenfunctions, the standard Fourier harmonics. What about the eigenvalues? The spectrum of the commutative algebra of global holomorphic differential operators can be identified with the space of holomorphic connections on the trivial line bundle. Hence every eigenvalue of this algebra can be encoded by a point in this space. It turns out that the points corresponding to the eigenvalues of this algebra on the space of L_2 functions on Pic^0 are precisely those connection that give rise to the homomorphisms $\pi_1(X, x) \to \mathbb{C}^\times$ with the image in $\mathbb{R}^\times \subset \mathbb{C}^\times$. In other words, these are the connections with *real monodromy*. This agrees with the conjecture of Teschner [T] in the case of $G = SL_2$. We expect an analogous statement to hold for a general reductive group G.

Suppose for simplicity that G is simply-connected. Then, according to a theorem of Beilinson and Drinfeld [BD], the spectrum of the algebra of global holomorphic differential operators on Bun_G is canonically identified with the space of ${}^L G$-opers on X. In the case $G = SL_2$, ${}^L G = PGL_2$ and PGL_2-opers are the same as projective connections. Teschner [T] proposed that in this case, the eigenvalues correspond to the projective connections with real monodromy (these projective connections have been described by Goldman [Gol]). For general G, we expect the joint eigenvalues of the global holomorphic differential operators on Bun_G to correspond to those ${}^L G$-opers that have monodromy in the split real form of ${}^L G$ (up to conjugation). If so, then the spectra of the global differential operators on Bun_G can be described by analogues of the Langlands parameters of the classical theory: namely, certain homomorphisms from the fundamental group of X to the Langlands dual group ${}^L G$. A somewhat surprising element is that the homomorphisms that appear here are the ones whose image is in the real form of ${}^L G$ (rather than the compact form). More details will appear in [FK].

1.6. Thus, we see that we can develop a meaningful analytic theory of automorphic forms on Bun_G in which the role of Hecke operators is played by global differential operators. This raises the question: is there a connection between this theory and the geometric theory?

Important insights into this question may be gleaned from two-dimensional conformal field theory (CFT). In CFT, one has two types of correlation functions. The first type is chiral correlation functions, also known as conformal blocks. They form a vector space for fixed values of the parameters of the CFT. Hence we obtain a vector bundle of conformal blocks on the space of parameters. In addition, the data of conformal field theory give rise to a projectively flat connection on this bundle. The conformal blocks are *multi-valued* horizontal sections of this bundle. The second type is the "true" correlation functions. They are sesquilinear combinations of conformal blocks and their complex conjugates (anti-conformal blocks), chosen so that the combination is a *single-valued* function of the parameters (see, e.g., [Gaw]).[3]

As we discussed above, the automorphic D-modules on Bun_G arising in the geometric Langlands theory may be defined as sheaves of conformal blocks of a certain two-dimensional conformal field theory [Fr2]. Away from a singularity locus, these sheaves are vector bundles with a connection, and conformal blocks are their multi-valued horizontal sections (see Section 1.3 above). On the other hand, by analogy with CFT, we may construct combinations of these conformal blocks and their complex conjugates which are single-valued functions of the parameters. That's how we could interpret the automorphic forms arising in the analytic theory, as predicted in [Fr4] and [T]. A crucial difference with the CFT is that whereas in CFT the monodromy of conformal blocks is typically unitary, here we expect the monodromy to be real.

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[3]A useful analogy is the exponentials of harmonic functions, which may be written as products of holomorphic and anti-holomorphic functions.

2. The abelian case

2.1. The case of an elliptic curve. Let's start with the case of an elliptic curve E_τ with complex parameter τ. Let's choose, once and for all, a reference point p_0 on this curve. Then we can identify it with

$$E_\tau \simeq \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau). \tag{2.1}$$

Next, consider the Picard variety $\mathrm{Pic}(E_\tau)$ of E_τ. This is the (fine) moduli space of line bundles on E_τ (note that the corresponding moduli stack $\mathrm{Bun}_{GL_1}(E_\tau)$ of line bundles on E_τ is the quotient of $\mathrm{Pic}(E_\tau)$ by the trivial action of the multiplicative group $\mathbb{G}_m = GL_1$, which is the group of automorphisms of every line bundle on E_τ). It is a disjoint union of connected components $\mathrm{Pic}^d(E_\tau)$ corresponding to line bundles of degree d. Using the reference point p_0, we can identity $\mathrm{Pic}^d(E_\tau)$ with $\mathrm{Pic}^0(E_\tau)$ by sending a line bundle $\mathcal{L}$ of degree d to $\mathcal{L}(-d \cdot p_0)$. Furthermore, we can identify the degree 0 component $\mathrm{Pic}^0(E_\tau)$, which is the Jacobian variety of E_τ, with E_τ itself using the Abel–Jacobi map; namely, we map a point $p \in E_\tau$ to the degree 0 line bundle $\mathcal{O}(p - p_0)$.

Now we define the Hecke operators H_p. They are labeled by points p of the curve E_τ. The operator H_p is the pull-back of functions with respect to the geometric map

$$\begin{aligned} T_p : \mathrm{Pic}^d(E_\tau) &\to \mathrm{Pic}^{d+1}(E_\tau) \\ \mathcal{L} &\mapsto \mathcal{L}(p) \end{aligned} \tag{2.2}$$

These operators commute with each other.

Formula (2.2) implies that if f is a joint eigenfunction of the Hecke operators $H_p, p \in E_\tau$, on $\mathrm{Pic}(E_\tau)$, then f is uniquely determined by its restriction f_0 to the connected component $\mathrm{Pic}^0(E_\tau)$, which is an eigenfunction of the operators

$$_{p_0}H_p = H_{p_0}^{-1} H_p,$$

where p_0 is our reference point.

Conversely, given an eigenfunction f_0 of $_{p_0}H_p, p \in X$, on $\mathrm{Pic}^0(X)$ and $\mu_{p_0} \in \mathbb{C}^\times$, there is a unique extension of f_0 to an eigenfunction f of $H_p, p \in X$, such that the eigenvalue of H_{p_0} on f is equal to μ_{p_0}. Namely, any line bundle $\mathcal{L}$ of degree $\mathcal{L}$ may be represented uniquely as $\mathcal{L}_0(d \cdot p_0)$, where $\mathcal{L}_0$ is a line bundle of degree 0. We then set

$$f(\mathcal{L}) = (\mu_{p_0})^d \cdot f_0(\mathcal{L}_0).$$

By construction, the eigenvalue μ_p of H_p on f is equal to $\lambda_p \cdot \mu_{p_0}$, where λ_p is the eigenvalue of $_{p_0}H_p$ on f_0.

Therefore, from now on we will consider the eigenproblem for the operators $_{p_0}H_p$ acting on the space $L_2(\mathrm{Pic}^0(E_\tau))$. Using the above isomorphism between $\mathrm{Pic}^0(E_\tau)$ and E_τ as above, we identify $L_2(\mathrm{Pic}^0(E_\tau))$ with $L_2(E_\tau)$ (with respect to the standard measure). The Hecke operator $_{p_0}H_p$ acting on $L_2(E_\tau)$ is given by the formula

$$(_{p_0}H_p \cdot f)(q) = f(q + p). \tag{2.3}$$

In other words, it is simply the pull-back under the shift by p with respect to the (additive) abelian group structure on E_τ, which can be described explicitly using the isomorphism (2.1). The subscript p_0 serves as a reminder that this operator depends on the choice of the reference point p_0.

Now we would like to describe the joint eigenfunctions and eigenvalues of the operators $_{p_0}H_p$ on $L_2(E_\tau)$.

To be even more concrete, let's start with the case $\tau = i$, so $E_\tau = E_i$ which is identified with $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ as above. Thus, we have a measure-preserving isomorphism between E_i and the product of two circles $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ corresponding to the real and imaginary parts of $z = x + iy$. The space of L_2 functions on the curve E_i is therefore the completed tensor product of two copies of $L_2(\mathbb{R}/\mathbb{Z})$, and so it has the standard orthogonal Fourier basis:

$$f_{m,n}(x, y) = e^{2\pi imx} \cdot e^{2\pi iny}, \qquad m, n \in \mathbb{Z}. \tag{2.4}$$

Let us write $p = x_p + y_p i \in E_i$, with $x_p, y_p \in [0, 1)$. The operator ${}_{p_0}H_p$ corresponds to the shift of z by p (with respect to the abelian group structure on E_i):

$$({}_{p_0}H_p \cdot f)(x, y) = f(x + x_p, y + y_p), \qquad f \in L_2(E_i). \tag{2.5}$$

It might be instructive to consider first the one-dimensional analogue of this picture, in which we have $L_2(S^1)$, where $S_1 = \mathbb{C}/\mathbb{Z}$ with coordinate ϕ. Then the role of the family $\{{}_{p_0}H_p\}_{p \in E_i}$ is played by the family $\{H'_\alpha\}_{\alpha \in S^1}$ acting by shifts:

$$(H'_\alpha \cdot f)(x) = f(\phi + \alpha), \qquad f \in L_2(S^1). \tag{2.6}$$

Then the functions $f_n(x) = e^{2\pi in\phi}$ form an orthogonal eigenbasis of the operators $H'_\alpha, \alpha \in S^1$. The eigenvalue of H'_α on f_n is $e^{2\pi in\alpha}$.

Likewise, in the two-dimensional case of the elliptic curve E_i, the functions $f_{m,n}$ form an orthogonal basis of eigenfunctions of the operators ${}_{p_0}H_p, p \in E_i$, in $L_2(E_i)$:

$${}_{p_0}H_p \cdot f_{m,n} = e^{2\pi i(mx_p + ny_p)} f_{m,n}. \tag{2.7}$$

Thus, we see that the eigenvalue of ${}_{p_0}H_p$ on $f_{m,n}$ is $e^{2\pi i(mx_p+ny_p)}$.

In [L2], Langlands suggested representing the eigenvalues of the Hecke operators on a specific eigenfunction as holonomies of a connection on the curve (in this case, E_i) with values in the dual group (in this case, GL_1). Indeed, this can be done rather explicitly in this case.

Consider the unitary flat connection $\nabla^{(m,n)}$ on the trivial line bundle over E_i given by the formula

$$\nabla^{(m,n)} = d - 2\pi im\, dx - 2\pi in\, dy \tag{2.8}$$

(since the line bundle is trivial, a connection on it is the same as a one-form on the curve). In other words, the corresponding first order differential operators along x and y are given by the formulas

$$\nabla_x^{(m,n)} = \frac{\partial}{\partial x} - 2\pi im, \tag{2.9}$$

$$\nabla_y^{(m,n)} = \frac{\partial}{\partial y} - 2\pi in. \tag{2.10}$$

The horizontal sections of this connection are the solutions of the equations

$$\nabla_x^{(m,n)} \cdot \Phi = \nabla_y^{(m,n)} \cdot \Phi = 0. \tag{2.11}$$

They have the form

$$\Phi_{m,n}(x, y) = e^{2\pi i(mx+ny)}$$

up to a scalar. The function $\Phi_{m,n}$ is the unique solution of (2.11) normalized so that its value at the point $0 \in E_i$, corresponding to our reference point $p_0 \in E_i$, is equal to 1. The value of this function $\Phi_{m,n}$ at $p = x_p + iy_p \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ is indeed equal to the eigenvalue of the Hecke operator ${}_{p_0}H_p$ on the harmonic $f_{m,n}$.

In other words, this eigenvalue can be represented as the holonomy of the connection $\nabla^{(m,n)}$ over a path connecting our reference point $p_0 \in E_i$, which corresponds to $0 \in \mathbb{C}/(\mathbb{Z}+\mathbb{Z}i)$, with the point $p \in E_i$. Since the connection is flat, it does not matter which path we choose.

However, and this is a crucial point, the connection $\nabla^{(m,n)}$ has *trivial monodromy* on E_i. Indeed,

$$\Phi_{m,n}(x+1,y) = \Phi_{m,n}(x,y+1) = \Phi_{m,n}(x,y)$$

for all $m,n \in \mathbb{Z}$. Herein lies an important difference between the analytic theories of automorphic forms for curves over $\mathbb{C}$ and over $\mathbb{F}_q$, as explained in the following Remark.

Remark 1. Recall that in the classical unramified Langlands correspondence for a curve over $\mathbb{F}_q$, to each joint eigenfunction of the Hecke operators we assign a Langlands parameter. In the case of $G = GL_n$, this is an equivalence class of ℓ-adic homomorphisms from the étale fundamental group of X to GL_n (and more generally, one considers homomorphisms to the Langlands dual group ${}^L G$ of G). Given such a homomorphism σ, to each closed point x of X we can assign an ℓ-adic number, the trace of $\sigma(\mathrm{Fr}_x)$, where Fr_x is the Frobenius conjugacy class, so we obtain a function from the set of closed points of X to the set of conjugacy classes in $GL_n(\overline{\mathbb{Q}}_\ell)$.

In the geometric Langlands correspondence for curves over $\mathbb{C}$, the picture is different. Now the role of the étale fundamental group is played by the topological fundamental group $\pi_1(X,p_0)$. Thus, the Langlands parameters are the equivalence classes of homomorphisms $\pi_1(X,p_0) \to GL_n$ (or, more generally, to ${}^L G$). The question then is: how to interpret such a homomorphism as a Hecke "eigenvalue" on a Hecke eigensheaf?

The point is that for a Hecke eigensheaf, the "eigenvalue" of a Hecke operator (or rather, Hecke functor) is not a number but an n-dimensional *vector space*. As we move along a closed path on our curve (starting and ending at the point p_0 say), this vector space will in general undergo a non-trivial linear transformation, thus giving rise to a non-trivial homomorphism $\pi_1(X,p_0) \to GL_n$.

Note that over $\mathbb{C}$ we have the Riemann–Hilbert correspondence, which sets up a bijection between the set of equivalence classes of homomorphisms $\pi_1(X,p_0) \to GL_n$ (or, more generally, $\pi_1(X,p_0) \to {}^L G$) and the set of equivalence classes of pairs $(\mathcal{P},\nabla)$, where $\mathcal{P}$ is a rank n bundle on X (or, more generally, an ${}^L G$-bundle) and ∇ is a flat connection on $\mathcal{P}$. The map between the two data is defined by assigning to $(\mathcal{P},\nabla)$ the monodromy representation of ∇ (corresponding to a specific a trivialization of $\mathcal{P}$ at p_0). We may therefore take equivalence classes of the flat bundles $(\mathcal{P},\nabla)$ as our Langlands parameters instead of equivalence classes of homomorphisms $\pi_1(X,p_0) \to GL_n$. As explained in the previous paragraph, these flat bundles $(\mathcal{P},\nabla)$ will in general have non-trivial monodromy.

However, in this section we consider (in the case of GL_1 and a curve X) the *eigenfunctions* of the Hecke operators ${}_{p_0}H_p, p \in X$, on $\mathrm{Pic}^0(X)$. Their eigenvalues are *numbers*, not vector spaces. Therefore they cannot undergo any transformations as we move along a closed path on our curve. In other words, these numbers give rise to a *single-valued* function from X to $GL_1(\mathbb{C})$ (it actually takes values in $U_1 \subset GL_1(\mathbb{C})$). Because the function is single-valued, if we represent this function as the holonomy of a flat connection on a line bundle on X, then this connection necessarily has trivial monodromy. And indeed, we have seen above that each collection of joint eigenvalues of the Hecke operators ${}_{p_0}H_p, p \in E_i$, on functions on $\mathrm{Pic}^0(E_i)$ can be represented as holonomies of a specific (unitary) connection $\nabla^{m,n}$ with trivial monodromy. The same is true for other curves, as we will see below. □

2.2. General elliptic curve. The above computation can be easily generalized to an arbitrary elliptic curve $E_\tau \simeq \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, where $\operatorname{Im}\tau > 0$. We again identify every component of $\operatorname{Pic}(E_\tau)$ with E_τ using the reference point p_0. Then we obtain the Hecke operators ${}_{p_0}H_p$ labeled by $p \in E_\tau$ corresponding to the shift by p naturally acting on E_τ. The Hecke eigenfunctions and their eigenvalues are given by the following statement.

Theorem 1. *The joint eigenfunctions of the Hecke operators ${}_{p_0}H_p, p \in E_\tau$, on $L_2(E_\tau)$ are*

$$f^\tau_{m,n}(z,\overline{z}) = e^{2\pi i m(z\overline{\tau}-\overline{z}\tau)/(\overline{\tau}-\tau)} \cdot e^{2\pi i n(z-\overline{z})/(\tau-\overline{\tau})}, \qquad m,n \in \mathbb{Z}. \tag{2.12}$$

The eigenvalues are given by the right hand side of the following formula:

$${}_{p_0}H_p \cdot f^\tau_{m,n} = e^{2\pi i m(p\overline{\tau}-\overline{p}\tau)/(\overline{\tau}-\tau)} \cdot e^{2\pi i n(p-\overline{p})/(\tau-\overline{\tau})} f^\tau_{m,n}. \tag{2.13}$$

In the next subsection we will give an alternative formula for these eigenfunctions (for an arbitrary smooth projective curve instead of E_τ).

Finally, let us express the Hecke eigenvalues as holonomies of a flat connection on the trivial line bundle.

Introduce a flat unitary connection ${}_\tau\nabla^{(m,n)}$ on E_τ (which becomes the connection $\nabla^{(m,n)}$ from Section 2.1 when we specialize $\tau = i$):

$${}_\tau\nabla^{(m,n)} = d - 2\pi i\frac{n - m\overline{\tau}}{\tau-\overline{\tau}}dz - 2\pi i\frac{m\tau - n}{\tau - \overline{\tau}}d\overline{z}. \tag{2.14}$$

The first order operators corresponding to z and $\overline{z}$ are

$${}_\tau\nabla^{(m,n)}_z = \frac{\partial}{\partial z} - 2\pi i\frac{n - m\overline{\tau}}{\tau - \overline{\tau}}, \tag{2.15}$$

$${}_\tau\nabla^{(m,n)}_{\overline{z}} = \frac{\partial}{\partial \overline{z}} - 2\pi i\frac{m\tau - n}{\tau - \overline{\tau}}. \tag{2.16}$$

Just as in the case $\tau = i$, for every $p \in E_\tau$, the holonomy of the connection ${}_\tau\nabla^{(m,n)}$ over a path connecting $p_0 \in E_\tau$ and $p \in E_\tau$ is equal to the eigenvalue of ${}_{p_0}H_p$ on $f^\tau_{m,n}$ given by the right hand side of formula (2.13). Again, all connections ${}_\tau\nabla^{(m,n)}$ yield the trivial monodromy representation $\pi_1(E_\tau) \to GL_1$, as explained at the end of Section 2.1.

2.3. Higher genus curves. Let X be a smooth projective connected curve over $\mathbb{C}$. Denote by $\operatorname{Pic}(X)$ the Picard variety of X, i.e. the moduli space of line bundles on X (as before, the moduli stack $\operatorname{Bun}_{GL_1}(X)$ of line bundles on X is the quotient of $\operatorname{Pic}(X)$ by the trivial action of $\mathbb{G}_m = GL_1$). We have a decomposition of $\operatorname{Pic}(X)$ into a disjoint union of connected components $\operatorname{Pic}^d(X)$ corresponding to line bundles of degree d. The Hecke operator $H_p, p \in X$, is the pull-back of functions with respect to the map (see formula (2.2) for $X = E_\tau$):

$$\begin{aligned} T_p : \operatorname{Pic}^d(X) &\to \operatorname{Pic}^{d+1}(X) \\ \mathcal{L} &\mapsto \mathcal{L}(p) \end{aligned} \tag{2.17}$$

The Hecke operators H_p with different $p \in X$ commute with each other, and it is natural to consider the problem of finding joint eigenfunctions and eigenvalues of these operators on functions on $\operatorname{Pic}(X)$. In the same way as in Section 2.1, we find that this problem is equivalent to the problem of finding joint eigenfunctions and eigenvalues of the operators ${}_{p_0}H_p = H^{-1}_{p_0}H_p$ on functions on $\operatorname{Pic}^0(X)$, where p_0 is a reference point on X that we choose once and for all. The operator ${}_{p_0}H_p$ is the pull-back of functions with respect to the map ${}_{p_0}T_p : \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X)$ sending a line bundle $\mathcal{L}$ to $\mathcal{L}(p - p_0)$.

Now, $\mathrm{Pic}^0(X)$ is the Jacobian of X, which is a $2g$-dimensional torus (see, e.g., [GH])

$$\mathrm{Pic}^0(X) \simeq H^0(X, \Omega^{1,0})^*/H_1(X, \mathbb{Z}),$$

where $H_1(X, \mathbb{Z})$ is embedded into the space of linear functionals on the space $H^0(X, \Omega^{1,0})$ of holomorphic one-forms on X by sending $\beta \in H_1(X, \mathbb{Z})$ to the linear functional

$$\omega \in H^0(X, \Omega^{1,0}) \mapsto \int_\beta \omega. \tag{2.18}$$

Motivated by Theorem 1, it is natural to guess that the standard Fourier harmonics in $L_2(\mathrm{Pic}^0(X))$ form an orthogonal eigenbasis of the Hecke operators. This is indeed the case.

To see that, we give an explicit formula for these harmonics. They can be written in the form $e^{2\pi i\varphi}$, where $\varphi : H^0(X, \Omega^{1,0})^* \to \mathbb{R}$ is an $\mathbb{R}$-linear functional such that $\varphi(\beta) \in \mathbb{Z}$ for all $\beta \in H_1(X, \mathbb{Z})$. To write them down explicitly, we use the Hodge decomposition

$$H^1(X, \mathbb{C}) = H^0(X, \Omega^{1,0}) \oplus H^0(X, \Omega^{0,1}) = H^0(X, \Omega^{1,0}) \oplus \overline{H^0(X, \Omega^{1,0})}$$

to identify $H^0(X, \Omega^{1,0})$, viewed as an $\mathbb{R}$-vector space, with $H^1(X, \mathbb{R})$ by the formula

$$\omega \in H^0(X, \Omega^{1,0}) \mapsto \omega + \overline{\omega}. \tag{2.19}$$

In particular, for any class $c \in H^1(X, \mathbb{R})$, there is a unique holomorphic one-form ω_c such that c is represented by the real-valued harmonic one-form $\omega_c + \overline{\omega}_c$,

$$H^1(X, \mathbb{R}) \ni c \quad = \quad \omega_c + \overline{\omega}_c \qquad \omega_c \in H^0(X, \Omega^{1,0}). \tag{2.20}$$

Viewed as a real manifold,

$$\mathrm{Pic}^0(X) \simeq H^1(X, \mathbb{R})^*/H_1(X, \mathbb{Z}),$$

where $H_1(X, \mathbb{Z})$ is embedded into $H^1(X, \mathbb{R})^*$ by sending $\beta \in H_1(X, \mathbb{Z})$ to the linear functional on $H^1(X, \mathbb{R})$ given by the formula (compare with formulas (2.18) and (2.19))

$$H^1(X, \mathbb{R}) \ni c \quad \mapsto \quad \int_\beta c = \int_\beta (\omega_c + \overline{\omega}_c). \tag{2.21}$$

Now, to each $\gamma \in H^1(X, \mathbb{Z})$ we attach the corresponding element of the vector space $H^1(X, \mathbb{R})$, which can be viewed as a linear functional φ_γ on the dual vector space $H^1(X, \mathbb{R})^*$,

$$\varphi_\gamma : H^1(X, \mathbb{R})^* \to \mathbb{R}.$$

It has the desired property: $\varphi_\gamma(\beta) \in \mathbb{Z}$ for all $\beta \in H_1(X, \mathbb{Z})$. The corresponding functions

$$e^{2\pi i\varphi_\gamma}, \qquad \gamma \in H^1(X, \mathbb{Z}), \tag{2.22}$$

are the Fourier harmonics that form an orthogonal basis of the Hilbert space $L_2(\mathrm{Pic}^0(X))$.

We claim that each of these functions is an eigenfunction of the Hecke operators ${}_{p_0}H_p, p \in X$, so that together they give us a sought-after orthogonal eigenbasis of the Hecke operators. To see that, we use the Abel–Jacobi map.

For $d > 0$, let $X^{(d)}$ be the dth symmetric power of X, and $p_d : X^{(d)} \to \mathrm{Pic}^d(X)$ the Abel–Jacobi map

$$p_d(D) = \mathcal{O}(D), \qquad D = \sum_{i=1}^d [x_i], \quad x_i \in X. \tag{2.23}$$

We can lift the map T_p to a map

$$\begin{aligned} \widetilde{T}_p : & X^{(d)} \to X^{(d+1)} \\ & D \mapsto D + [p] \end{aligned} \tag{2.24}$$

so that we have a commutative diagram

$$\begin{array}{ccc} X^{(d)} & \xrightarrow{\widetilde{T}_p} & X^{(d+1)} \\ {\scriptstyle p_d}\big\downarrow & & \big\downarrow{\scriptstyle p_{d+1}} \\ \mathrm{Pic}^d(X) & \xrightarrow{T_p} & \mathrm{Pic}^{d+1}(X) \end{array} \tag{2.25}$$

Denote by $\widetilde{H}_p$ the corresponding pull-back operator on functions.

Now let f_0 be a function on $\mathrm{Pic}^0(X)$. Identifying $\mathrm{Pic}^d(X)$ with $\mathrm{Pic}^0(X)$ using the reference point p_0:

$$\mathcal{L} \mapsto \mathcal{L}(-d \cdot p_0), \tag{2.26}$$

we obtain a function f_d on $\mathrm{Pic}^d(X)$ for all $d \in \mathbb{Z}$. Let $\widetilde{f}_d$ the pull-back of f_d to $X^{(d)}$ for $d > 0$. Suppose that these functions satisfy

$$\widetilde{H}_p(\widetilde{f}_{d+1}) = \lambda_p \widetilde{f}_d, \qquad p \in X, \quad d > 0, \tag{2.27}$$

where $\lambda_p \neq 0$ for all p and $\lambda_{p_0} = 1$. This is equivalent to the following factorization formula for $\widetilde{f}_d$:

$$\widetilde{f}_d\left(\sum_{i=1}^{d}[x_i]\right) = c\prod_{i=1}^{d}\lambda_{x_i}, \qquad c \in \mathbb{C}, \qquad d > 0. \tag{2.28}$$

The commutativity of the diagram (2.25) then implies that

$$H_p(f_{d+1}) = \lambda_p f_d, \qquad p \in X, \qquad d > 0. \tag{2.29}$$

But then f_0 is an eigenfunction of the operators ${}_{p_0}H_p = H_{p_0}^{-1} H_p$ with the eigenvalues $\lambda_p = \widetilde{f}_1([p])$.

This observation gives us an effective way to demonstrate that a given function f_0 on $\mathrm{Pic}^0(X)$ is a Hecke eigenfunction.

Let us use it in the case of the function $f_0 = e^{2\pi i \varphi_\gamma}, \gamma \in H^1(X, \mathbb{Z})$, on $\mathrm{Pic}^0(X)$ given by formula (2.22). For that, denote by

$${}_d e^{2\pi i \varphi_\gamma}, \qquad \gamma \in H^1(X, \mathbb{Z}), \tag{2.30}$$

the corresponding functions f_d on $\mathrm{Pic}^d(X)$ obtained via the identification (2.26). We claim that for any $\gamma \in H^1(X, \mathbb{Z})$, the pull-backs of ${}_d e^{2\pi i \varphi_\gamma}$ to $X^{(d)}, d > 0$, via the Abel–Jacobi maps have the form (2.28), and hence $e^{2\pi i \varphi_\gamma}$ is a Hecke eigenfunction on $\mathrm{Pic}^0(X)$.

To see that, we recall an explicit formula for the composition

$$X^{(d)} \to \mathrm{Pic}^d(X) \to \mathrm{Pic}^0(X) \simeq H^0(X, \Omega^{1,0})^*/H_1(X, \mathbb{Z}), \tag{2.31}$$

where the second map is given by formula (2.26) (see, e.g., [GH]). Namely, the composition (2.31) maps $\sum_{i=1}^{d}[x_i] \in X^{(d)}$ to the linear functional on $H^0(X, \Omega^{1,0})$ sending

$$\omega \in H^0(X, \Omega^{1,0}) \mapsto \sum_{i=1}^{d} \int_{p_0}^{x_i} \omega.$$

Composing the map (2.31) with the isomorphism $H^0(X,\Omega^{1,0}) \simeq H^1(X,\mathbb{R})$ defined above, we obtain a map

$$
{}_{p_0}\Phi_d : X^{(d)} \to H^1(X,\mathbb{R})^*/H_1(X,\mathbb{Z}), \tag{2.32}
$$

which maps $\sum_{i=1}^d [x_i] \in X^{(d)}$ to the linear functional ${}_{p_0}\Phi_d\left(\sum_{i=1}^d [x_i]\right)$ on $H^1(X,\mathbb{R})$ given by the formula

$$
{}_{p_0}\Phi_d\left(\sum_{i=1}^d [x_i]\right) : c \in H^1(X,\mathbb{R}) \mapsto \sum_{i=1}^d \int_{p_0}^{x_i} (\omega_c + \overline{\omega}_c) \tag{2.33}
$$

(see formula (2.20) for the definition of ω_c).

If $c \in H^1(X,\mathbb{R})$ is the image of an *integral* cohomology class

$$
\gamma \in H^1(X,\mathbb{Z}),
$$

we will write the corresponding holomorphic one-form ω_c as ω_γ.

Let ${}_{p_0}\widetilde{f}_{d,\gamma}$ be the pull-back of the function ${}_d e^{2\pi i \varphi_\gamma}$ (see formula (2.30)) to $X^{(d)}$. Equivalently, ${}_{p_0}\widetilde{f}_{d,\gamma}$ is the pull-back of the function $e^{2\pi i \varphi_\gamma}$ under the map ${}_{p_0}\Phi_d$. It follows from the definition of ${}_{p_0}\Phi_d$ that the value of ${}_{p_0}\widetilde{f}_{d,\gamma}$ at $\sum_{i=1}^d [x_i]$ is equal to

$$
\exp\left(2\pi i \;\; {}_{p_0}\Phi_d\left(\sum_{i=1}^d [x_i]\right)(\gamma)\right) = \exp\left(2\pi i \;\; \sum_{i=1}^d \int_{p_0}^{x_i} (\omega_\gamma + \overline{\omega}_\gamma)\right).
$$

Thus, we obtain that ${}_{p_0}\widetilde{f}_{d,\gamma}$ is given by the formula

$$
{}_{p_0}\widetilde{f}_{d,\gamma}\left(\sum_{i=1}^d [x_i]\right) = \exp\left(2\pi i \;\; \sum_{i=1}^d \int_{p_0}^{p} (\omega_\gamma + \overline{\omega}_\gamma)\right) = \prod_{i=1}^d \lambda_\gamma(x_i), \tag{2.34}
$$

where

$$
\lambda_\gamma(p) = e^{2\pi i \int_{p_0}^{p} (\omega_\gamma + \overline{\omega}_\gamma)}. \tag{2.35}
$$

We conclude that the functions ${}_{p_0}\widetilde{f}_{d,\gamma}$ indeed satisfy the factorization property (2.28). Therefore the function $e^{2\pi i \varphi_\gamma}$ on $\mathrm{Pic}^0(X)$ is indeed an eigenfunction of ${}_{p_0}H_p$, with the eigenvalue λ_p given by formula (2.35), which is what we wanted to prove.[4]

Thus, we have proved the following theorem.

Theorem 2. *The joint eigenfunctions of the Hecke operators ${}_{p_0}H_p, p \in X$, on $L_2(\mathrm{Pic}^0(X))$ are the functions $e^{2\pi i \varphi_\gamma}, \gamma \in H^1(X,\mathbb{Z})$. The eigenvalues of ${}_{p_0}H_p$ are given by formula* (2.35), *so that we have*

$$
{}_{p_0}H_p \cdot e^{2\pi i \varphi_\gamma} = e^{2\pi i \int_{p_0}^{p} (\omega_\gamma + \overline{\omega}_\gamma)} e^{2\pi i \varphi_\gamma}. \tag{2.36}
$$

As in the case of elliptic curve discussed in Section 2.2, the eigenvalues (2.35) can be interpreted as the holonomies of the flat unitary connections

$$
\nabla_\gamma = d - 2\pi i(\omega_\gamma + \overline{\omega}_\gamma), \qquad \gamma \in H^1(X,\mathbb{Z})
$$

---

[4]Note that Abel's theorem implies that each function ${}_{p_0}\widetilde{f}_{d,\gamma}, \gamma \in H^1(X,\mathbb{Z})$, is constant along the fibers of the Abel–Jacobi map $X^{(d)} \to \mathrm{Pic}^d$ and therefore descends to Pic^d. This suggests another proof of Theorem 2: we start from the functions ${}_{p_0}\widetilde{f}_{d,\gamma}$ on $X^{(d)}, d > 0$. Formula (2.34) shows that they combine into an eigenfunction of the operators $\widetilde{H}_p$. Hence the function on $\mathrm{Pic}^d(X), d \geq g$, to which ${}_{p_0}\widetilde{f}_{d,\gamma}$ descends, viewed as a function on $\mathrm{Pic}^0(X)$ under the identification (2.26), is a Hecke eigenfunction. One can then show that this function is equal to $e^{2\pi i \varphi_\gamma}$.

on the trivial line bundle on X, taken along (no matter which) path from p_0 to p. As in the case of elliptic curves, the monodromy representation of each of these connections is *trivial*, ensuring that the Hecke eigenvalues λ_p, viewed as functions of $p \in X$, are single-valued.

2.4. General torus. Let now T be a connected torus over $\mathbb{C}$, and $Bun_T(X)$ the moduli space of T-bundles on X (note that the moduli stack $\mathrm{Bun}_T(X)$ is the quotient of $Bun_T(X)$ by the trivial action of T). In Section 2.3 we find the joint eigenfunctions and eigenvalues of the Hecke operators in the case of $Bun_T(X)$ where $T = \mathbb{G}_m$; in this case $Bun_{\mathbb{G}_m}(X) = \mathrm{Pic}(X)$. Here we generalize these results to the case of an arbitrary T.

Let $\Lambda^*(T)$ and $\Lambda_*(T)$ be the lattices of characters and cocharacters of T, respectively. Any $\mathcal{P} \in Bun_T(X)$ is uniquely determined by the $\mathbb{G}_m$-bundles (equivalently, line bundles) $\mathcal{P} \underset{\mathbb{G}_m}{\times} \chi$ associated to the characters $\chi : T \to \mathbb{G}_m$ in $\Lambda^*(T)$. This yields a canonical isomorphism

$$Bun_T(X) \simeq \mathrm{Pic}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_*(T) = \bigsqcup_{\check{\nu} \in \Lambda_*(T)} Bun_T^{\check{\nu}}(X).$$

The neutral component

$$Bun_T^0(X) = \mathrm{Pic}^0(X) \underset{\mathbb{Z}}{\otimes} \Lambda_*(T)$$

is non-canonically isomorphic to $\mathrm{Pic}^0(X)^r$, where r is the rank of the lattice $\Lambda_*(X)$.

The Hecke operators $H_p^{\check{\mu}}$ are now labeled by $p \in X$ and $\check{\mu} \in \Lambda_*(T)$. The operator $H_p^{\check{\mu}}$ corresponds to the pull-back under the map

$$\begin{aligned} T_p^{\check{\mu}} : &Bun_T^{\check{\nu}}(X) \to Bun_T^{\check{\nu}+\check{\mu}}(X) \\ &\mathcal{P} \mapsto \mathcal{P}(\check{\mu} \cdot p) \end{aligned} \tag{2.37}$$

where $\mathcal{P}(\check{\mu} \cdot p)$ is defined by the formula

$$\mathcal{P}(\check{\mu} \cdot p) \underset{\mathbb{G}_m}{\times} \chi = (\mathcal{P} \underset{\mathbb{G}_m}{\times} \chi)(\langle \chi, \check{\mu} \rangle \cdot p), \qquad \chi \in \Lambda^*(T).$$

As in the case of $T = \mathbb{G}_m$, we choose, once and for all, a reference point $p_0 \in X$.

As in the case of $T = \mathbb{G}_m$, finding eigenfunctions and eigenvalues of the commuting operators $H_p^{\check{\mu}}$ on functions on $Bun_T(X)$ is equivalent to finding eigenfunctions and eigenvalues of ${}_{p_0}H_p^{\check{\mu}} = (H_{p_0}^{\check{\mu}})^{-1} H_p^{\check{\mu}}$ on functions on $Bun_T^0(X)$. As in the case of $T = \mathbb{G}_m$, we represent $Bun_T^0(X)$ as

$$Bun_T^0(X) \simeq H^1(X, \mathfrak{t}_{\mathbb{R}}^*)^*/H_1(X, \Lambda_*(T)) = (H^1(X, \mathbb{R}) \underset{\mathbb{Z}}{\times} \Lambda^*(T))^*/(H_1(X, \mathbb{Z}) \underset{\mathbb{Z}}{\times} \Lambda_*(T)), \tag{2.38}$$

where $\mathfrak{t}_{\mathbb{R}} = \mathbb{R} \underset{\mathbb{Z}}{\times} \Lambda_*(T)$ is the split real form of the complex Lie algebra $\mathfrak{t}$ of T.

As in Section 2.3, for any

$$\gamma \in H^1(X, \Lambda^*(T)) \tag{2.39}$$

the image of γ in $H^1(X, \mathfrak{t}_{\mathbb{R}}^*)$ is represented by a unique $\mathfrak{t}_{\mathbb{R}}^*$-valued one-form on X of the form

$$\omega_\gamma + \overline{\omega}_\gamma, \tag{2.40}$$

where $\omega_\gamma \in H^0(X, \Omega^{1,0}) \underset{\mathbb{C}}{\otimes} \mathfrak{t}^*$ is a holomorphic $\mathfrak{t}^*$-valued one-form.

On the other hand, the image of γ in $H^1(X, \mathfrak{t}_{\mathbb{R}}^*)$ gives rise to a linear functional $\varphi_\gamma : H^1(X, \mathfrak{t}_{\mathbb{R}}^*)^* \to \mathbb{R}$ satisfying $\varphi_\gamma(\beta) \in \mathbb{Z}$ for all $\beta \in H_1(X, \Lambda_*(T))$. Therefore, according to formula (2.38), $e^{2\pi i \varphi_\gamma}$ is a well-defined function on $Bun_T^0(X)$. These are the Fourier harmonics on $Bun_T^0(X)$.

In the same way as in Section 2.3, we prove the following result.

Theorem 3. *The functions $e^{2\pi i\varphi_\gamma}, \gamma \in H^1(X, \Lambda^*(T))$ form a basis of joint eigenfunctions of the Hecke operators ${}_{p_0}H_p^{\check{\mu}}, p \in X, \check{\mu} \in \Lambda_*(X)$, on $L_2(Bun_T^0(X))$. The eigenvalues of ${}_{p_0}H_p^{\check{\mu}}$ are given by the right hand side of the formula*

$$ {}_{p_0}H_p^{\check{\mu}} \cdot e^{2\pi i\varphi_\gamma} = \check{\mu}\left(e^{2\pi i\int_{p_0}^{p}(\omega_\gamma+\overline{\omega}_\gamma)}\right)\, e^{2\pi i\varphi_\gamma}. \tag{2.41} $$

3. Non-abelian case

In this section we try to generalize to the case of a non-abelian group G the results obtained in the previous section for abelian G.

3.1. Spherical Hecke algebra for groups over $\mathbb{F}_q(\!(t)\!)$. In the case of the function field of a curve X over a finite field, the Hecke operators attached to a closed point x of X generate the spherical Hecke algebra $\mathcal{H}(G(\mathbb{F}_q(\!(t)\!)), G(\mathbb{F}_q[[t]]))$. As a vector space, it is the space of $\mathbb{C}$-valued functions on the group $G(\mathbb{F}_q(\!(t)\!))$ that are bi-invariant with respect to the subgroup $G(\mathbb{F}_q[[t]])$ (here $\mathbb{F}_q$ is the residue field of x). This vector space is endowed with the convolution product defined by the formula

$$ (f_1 \star f_2)(g) = \int f_1(gh^{-1})f_2(h)dh, \tag{3.1} $$

where dh stands for the Haar measure on $G(\mathbb{F}_q(\!(t)\!))$ normalized so that the volume of the subgroup $G(\mathbb{F}_q[[t]])$ is equal to 1 (in this normalization, the characteristic function of $G(\mathbb{F}_q[[t]])$ is the unit element of the convolution algebra). The Haar measure can be defined because $G(\mathbb{F}_q(\!(t)\!))$ is *locally compact*.

The resulting convolution algebra $\mathcal{H}(G(\mathbb{F}_q(\!(t)\!)), G(\mathbb{F}_q[[t]]))$ is commutative and we have the Satake isomorphism between this algebra and the complexified representation ring $\operatorname{Rep} {}^LG$ of the Langlands dual group ${}^LG$.

3.2. Is there a spherical Hecke algebra for groups over $\mathbb{C}(\!(t)\!)$? In contrast to the group $G(\mathbb{F}_q(\!(t)\!))$, the group $G(\mathbb{C}(\!(t)\!))$ is *not* locally compact. Therefore it does not carry a Haar measure. Indeed, the field $\mathbb{C}(\!(t)\!)$ is an example of a two-dimensional local field, in the terminology of [Fe1], more akin to $\mathbb{F}_q(\!(z)\!)(\!(t)\!)$ or $\mathbb{Q}_p(\!(t)\!)$ than to $\mathbb{F}_q(\!(t)\!)$ or $\mathbb{Q}_p$.

Ivan Fesenko has developed integration theory for the two-dimensional local fields [Fe1, Fe2], and his students have extended it to algebraic groups over such fields [Mo1, Mo2, Wa1], but this theory is quite different from the familiar case of $G(\mathbb{F}_q(\!(t)\!))$.

First of all, integrals over $\mathbb{C}(\!(t)\!)$ and $G(\mathbb{C}(\!(t)\!))$ take values not in real numbers, but in formal Laurent power series $\mathbb{R}(\!(X)\!)$, where X is a formal variable. Under certain restrictions, the value of the integral is a polynomial in X, and so one could set X to be equal to a real number. But this way one may lose some properties of the integration that one normally takes for granted.

Second, if S is a Lebesque measurable subset of $\mathbb{C}$, then according to [Fe1, Fe2], the measure of a subset of $\mathbb{C}(\!(t)\!)$ of the form

$$ St^i + t^{i+1}\mathbb{C}[[t]], \tag{3.2} $$

is equal to $\mu(S)X^i$, where $\mu(S)$ is the usual Lebesque measure of S. In particular, this means that the measure of the subset $\mathbb{C}[[t]]$ of $\mathbb{C}(\!(t)\!)$ is equal to 0, as is the measure of the subset $t^n\mathbb{C}[[t]]$ for any $n \in \mathbb{Z}$. Contrast this with the fact that under a suitably normalized Haar measure on $\mathbb{F}_q(\!(t)\!)$, the measure of $t^n\mathbb{F}_q[[t]]$ is equal to q^{-n}. Thus, if we take as G the additive group, it's not even clear how to define a unit element in the would-be spherical

Hecke algebra (which would be the characteristic function of the subset $\mathbb{F}_q[[t]]$ in the case of the field $\mathbb{F}_q((t))$). The situation is similar in the case of a general group G.

For this reason, according to Waller [Wa2], from the point of view of the two-dimensional integration theory it would make more sense to consider functions (or rather distributions) on $G((t))$ that are bi-invariant not with respect to $G(\mathbb{C}[[t]])$, but its subgroup K consisting of those elements $g(t) \in G(\mathbb{C}[[t]])$ for which $g(0)$ belongs to a compact subgroup of $G(\mathbb{C})$. Perhaps, the resulting convolution algebra is meaningful, but it is no longer isomorphic to $\operatorname{Rep} {}^L G$, nor do its elements act on functions on Bun_G (which is what we require).

Another option to consider is motivic integration theory. A motivic version of the Haar measure has been defined by Julia Gordon [Gor] for the group $G(\mathbb{C}((t)))$ (it may also be obtained in the framework of the general theories of Cluckers–Loeser or Hrushovsky–Kazhdan). Also, in a recent paper [CCH] it was shown that the spherical Hecke algebra $\mathcal{H}(G(\mathbb{F}_q((t))), G(\mathbb{F}_q[[t]]))$ can be obtained by a certain specialization from its version in which the ordinary integration with respect to the Haar measure on $G(\mathbb{F}_q((t)))$ is replaced by the motivic integration with respect to the motivic Haar measure. Presumably, one could carry some of the results of [CCH] over to the case of $\mathbb{C}((t))$.

However, this does not seem to give us much help, for the following reason: the motivic integrals over a ground field k take values in a certain algebra $\mathcal{M}_k$, which is roughly speaking a localization of the Grothendieck ring of algebraic varieties over k. In the case of the ground field $\mathbb{F}_q$, ordinary integrals may be recovered from the motivic ones by taking a homomorphism from $\mathcal{M}_{\mathbb{F}_q}$ to $\mathbb{R}$ sending the class of the affine line over $\mathbb{F}_q$ to q. But in the case of the ground field $\mathbb{C}$ this does not seem to work. In fact, it appears that there are very few (if any) known homomorphisms from $\mathcal{M}_{\mathbb{C}}$ to positive real numbers, besides the Euler characteristic.[5] Perhaps, taking the Euler characteristic, one can obtain a non-trivial convolution algebra structure on the space of $G(\mathbb{C}[[t]])$ bi-invariant functions on $G(((t)))$ (one may wonder whether it could be interpreted as a kind of $q \to 1$ limit of $\mathcal{H}(G(\mathbb{F}_q((t))), G(\mathbb{F}_q[[t]])))$, but it is not clear that this algebra could be useful in any way for defining an analytic theory of automorphic forms on Bun_G for complex algebraic curves.

3.3. An attempt to define Hecke operators. What we are interested in here, however, is not the spherical Hecke algebra itself, but rather the action of the corresponding Hecke operators on automorphic functions. After all, in Section 2 we were able to define Hecke operators without any reference to a convolution product on the group $\mathbb{C}((t))^\times$. As we will see below though, the abelian case was an exception in that the action of the Hecke operators did not require integration. In the non-abelian case, it appears that integration is necessary, and this creates serious problems.

We recall that for curves over $\mathbb{F}_q$, the unramified automorphic functions are functions on the double quotient

$$G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_F), \tag{3.3}$$

where $F = \mathbb{F}_q(X)$, and X is a curve over $\mathbb{F}_q$. The action of Hecke operators on functions on this double quotient can be defined by means of certain correspondences, and we can try to imitate this definition for complex curves.

To this end, we take the same double quotient (3.3) with $F = \mathbb{C}(X)$, where X is a curve over $\mathbb{C}$. As in the case of $\mathbb{F}_q$, this is the set of equivalence classes of principle G-bundles on X. The Hecke correspondences can be conveniently defined in these terms.

[5] I learned this from David Kazhdan (private communication).

For instance, consider the case of GL_2 and the first Hecke operator (for a survey of the general case, see, e.g., [Fr2], Sect. 3.7). Then we have the Hecke correspondence $\mathcal{H}ecke_{1,x}$, where x is a closed point of X:

$$\begin{array}{ccc} & \mathcal{H}ecke_{1,x} & \\ {\scriptstyle h_{\ell,x}}\swarrow & & \searrow{\scriptstyle h_{r,x}} \\ \mathrm{Bun}_{GL_2} & & \mathrm{Bun}_{GL_2} \end{array} \tag{3.4}$$

Here $\mathcal{H}ecke_{1,x}$ is the moduli stack classifying the quadruples

$$(\mathcal{M}, \mathcal{M}', \beta : \mathcal{M}' \hookrightarrow \mathcal{M}),$$

where $\mathcal{M}$ and $\mathcal{M}'$ are points of Bun_{GL_2}, which means that they are rank two vector bundles on X, and β is an embedding of their sheaves of (holomorphic) sections $\beta : \mathcal{M}' \hookrightarrow \mathcal{M}$ such that $\mathcal{M}/\mathcal{M}'$ is supported at x and is isomorphic to the skyscraper sheaf $\mathcal{O}_x = \mathcal{O}_X/\mathcal{O}_X(-x)$. The maps are defined by the formulas $h_{\ell,x}(\mathcal{M}, \mathcal{M}') = \mathcal{M}$, $h_{r,x}(\mathcal{M}, \mathcal{M}') = \mathcal{M}'$.

It follows that the points of the fiber of $\mathcal{H}ecke_{1,x}$ over $\mathcal{M}$ in the "left" Bun_{GL_2} correspond to all locally free subsheaves $\mathcal{M}' \subset \mathcal{M}$ such that the quotient $\mathcal{M}/\mathcal{M}'$ is the skyscraper sheaf $\mathcal{O}_x$. Defining such $\mathcal{M}'$ is the same as choosing a line L in the dual space $\mathcal{M}_x^*$ to the fiber of $\mathcal{M}$ at x (which is a two-dimensional complex vector space). The sections of the corresponding sheaf $\mathcal{M}'$ (over a Zariski open subset of X) are the sections of $\mathcal{M}$ that vanish along L, i.e. sections s which satisfy the equation $\langle v, s(x)\rangle = 0$ for a non-zero $v \in L$.

Thus, the fiber of $\mathcal{H}ecke_{1,x}$ over $\mathcal{M}$ is isomorphic to the projectivization of the two-dimensional vector space $\mathcal{M}_x^*$, i.e. to $\mathbb{CP}^1$. We conclude that $\mathcal{H}ecke_{1,x}$ is a $\mathbb{CP}^1$-fibration over over the "left" Bun_{GL_2} in the diagram (3.4) . Likewise, it is also easy to see that $\mathcal{H}ecke_{1,x}$ is a $\mathbb{CP}^1$-fibration over the "right" Bun_{GL_2} in (3.4).

In the geometric theory, we use the correspondence (3.4) to define a *Hecke functor* $\mathrm{H}_{1,x}$ on the (derived) category of D-modules on Bun_{GL_2}:

$$\mathrm{H}_{1,x}(\mathcal{K}) = h_{\ell,x*}h_{r,x}^*(\mathcal{K})[1]. \tag{3.5}$$

A D-module $\mathcal{K}$ is called a Hecke eigensheaf if we have isomorphisms

$$\imath_{1,x} : \mathrm{H}_{1,x}(\mathcal{K}) \xrightarrow{\sim} \mathbb{C}^2 \boxtimes \mathcal{K} \simeq \mathcal{K} \oplus \mathcal{K}, \qquad \forall x \in X, \tag{3.6}$$

and in addition have similar isomorphisms for the second set of Hecke functors $\mathrm{H}_{2,x}, x \in X$. These are defined similarly to the Hecke operators for GL_1, as the pull-backs with respect to the morphisms sending a rank two bundle $\mathcal{M}$ to $\mathcal{M}(x)$. (As explained in [FGV], Sect. 1.1, the second Hecke eigensheaf property follows from the first together with a certain S_2-equivariance condition.)

Thus, if $\mathcal{K}$ is a Hecke eigensheaf, we obtain a family of isomorphisms (3.6) for all $x \in X$, and similarly for the second set of Hecke functors. We then generalize them by requiring that the two-dimensional vector spaces appearing on the right hand side of (3.6) as "eigenvalues" can be organized as stalks of a single a rank two local system $\mathcal{E}$ on X (and similarly for the second set of Hecke functors, where the eigenvalues should be the stalks of the rank one local system $\wedge^2\mathcal{E}$ on X; this is, however, automatic if we impose the S_2-equivariance condition from [FGV], Sect. 1.1). If that's the case, we say that $\mathcal{K}$ is a *Hecke eigensheaf* with the *eigenvalue* $\mathcal{E}$. This is explained in more detail, e.g., in [Fr2], Sect. 3.8.

The first task of the geometric theory (in the case of $G = GL_2$) is to show that such a Hecke eigensheaf on Bun_{GL_2} exists for every irreducible rank two local system $\mathcal{E}$. This has been accomplished by Drinfeld in [Dr1], in a work that was the starting point of the geometric theory. We now know that the same is true for $G = GL_n$ [FGV] and in many other cases.

Now we try to adapt the diagram (3.4) to functions. In other words, given a function f on the set of $\mathbb{C}$-points, we wish to define the action of the first Hecke operator $H_{1,x}$ on it by the formula

$$(H_{1,x} \cdot f)(\mathcal{M}) = \int\limits_{\mathcal{M}' \in h_{\ell,x}^{-1}(\mathcal{M})} f(\mathcal{M}')\, d\mathcal{M}'. \tag{3.7}$$

Thus, we see that the result must be an integral over the complex projective line $h_{\ell,x}^{-1}(\mathcal{M})$. The key question is: what is the measure $d\mathcal{M}'$?

This represents a crucial difference with the abelian case considered in Section 2: in the abelian case every Hecke operator acted by pull-back of a function, so no integration was needed. But in the non-abelian case, already for the first Hecke operators $H_{1,x}$ in the case of $G = GL_2$, we must integrate functions over the projective lines $h_{\ell,x}^{-1}(\mathcal{M})$, where $\mathcal{M} \in \mathrm{Bun}_{GL_2}(\mathbb{C})$.

Note that if our curve were over a finite field, this integration is in fact a summation over a finite set of $q+1$ elements, the number of points of $\mathbb{P}^1$ over $\mathbb{F}_q$, where $\mathbb{F}_q$ is the residue field of the closed point x at which we take the Hecke operator. The terms of this summation correspond to points of the fibers $h_{\ell,x}^{-1}(\mathcal{M})$.

3.4. The case of an elliptic curve. Let's look at the case of an elliptic curve. If it is defined over a finite field, the fibers $h_{\ell,x}^{-1}(\mathcal{M})$ appearing in the Hecke operators have been described explicitly in [Lo, Al], using the classification of rank two bundles on elliptic curves due to Atiyah [At].

For a complex elliptic curve, the fibers $h_{\ell,x}^{-1}(\mathcal{M})$ have been described explicitly in [Bo]. In [L2], Langlands attempted to describe them in the language of adeles, which is more unwieldy than the vector bundle language used in [Bo]. As the result of his computations, he stated on p.18 of [L2]:

"The dimension $\dim(g\Delta_1/G(\mathcal{O}_x))$ [which is is our $h_{\ell,x}^{-1}(\mathcal{M})$ if $\mathcal{M}$ is the rank two bundle corresponding the adele $g \in GL_2(\mathbb{A}_F)$] is always equal to 0... Hence the domain of integration in [adelic version of our formula (3.7) above] is a finite set."

This statement appears to be incorrect. First of all, as we show below, there are rank two bundles $\mathcal{M}$ on an elliptic curve for which there are infinitely many non-isomorphic bundles in the fiber $h_{\ell,x}^{-1}(\mathcal{M})$ (in fact, we have a continuous family of non-isomorphic bundles parametrized by the points of $h_{\ell,x}^{-1}(\mathcal{M})$). Second, even if there are finitely many isomorphism classes among those $\mathcal{M}'$ which appear in the fiber $h_{\ell,x}^{-1}(\mathcal{M})$ for a fixed $\mathcal{M}$, this does not mean that we are integrating over a finite set.

In fact, according to formula (3.7) (whose adelic version is formula (10) of [L2]), for every $\mathcal{M}$, the fiber $h_{\ell,x}^{-1}(\mathcal{M})$ of the Hecke correspondence over which we are supposed to integrate is always isomorphic to $\mathbb{CP}^1$ if we take into account the automorphism groups of the bundles involved (or equivalently, the stabilizers of the corresponding points in the adelic quotient (3.3)).

To illustrate this point, consider the case that $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two line bundles of degrees d_1 and d_2 such that $d_1 > d_2 + 1$. Then the vector bundles $\mathcal{M}'$ that appear in the fiber $h_{\ell,x}^{-1}(\mathcal{M})$ are isomorphic to either $\mathcal{M}'_1 = \mathcal{L}_1(-x) \oplus \mathcal{L}_2$ or $\mathcal{M}'_2 = \mathcal{L}_1 \oplus \mathcal{L}_2(-x)$.

However, the groups of automorphisms of these bundles are different: each of them is a semi-direct product of the group $(\mathbb{C}^\times)^2$ of rescalings of the two line bundles appearing in a direct sum decomposition and an additive group, which is $\operatorname{Hom}(\mathcal{L}_2, \mathcal{L}_1)$ for $\mathcal{M}$; $\operatorname{Hom}(\mathcal{L}_2, \mathcal{L}_1(-x))$ for $\mathcal{M}'_1$; and $\operatorname{Hom}(\mathcal{L}_2(-x), \mathcal{L}_1)$ for $\mathcal{M}'_2$.

Under our assumption that $d_1 > d_2 + 1$, we find that these groups are isomorphic to $\mathbb{C}^{d_1-d_2-1}$, $\mathbb{C}^{d_1-d_2-2}$, and $\mathbb{C}^{d_1-d_2}$, respectively. Thus, the automorphism group of $\mathcal{M}'_1$ is "smaller" by one copy of the additive group $\mathbb{G}_a$ than that of $\mathcal{M}$, whereas the automorphism group of $\mathcal{M}'_2$ is larger than that of $\mathcal{M}$ by the same amount.

This implies that the fiber $h^{-1}_{\ell,x}(\mathcal{M})$ is the union of an affine line worth of points corresponding to $\mathcal{M}'_1$ and a single point corresponding to $\mathcal{M}'_2$.

If we worked over $\mathbb{F}_q$, we would find that

$$(H_{1,x} \cdot f)(\mathcal{M}) = q f(\mathcal{M}'_1) + f(\mathcal{M}'_2),$$

with the factor of q being the number of points of the affine line (see [Lo, Al]). Over $\mathbb{C}$, we obtain instead an *integral* of a function on $\mathbb{CP}^1$ which is constant on an open dense subset isomorphic to the affine line, where the function takes value $f(\mathcal{M}'_1)$, and takes another value $f(\mathcal{M}'_2)$ at the remaining point. Which measure of integration should we use?

If we use a standard integration measure on $\mathbb{CP}^1$, then the answer would be $f(\mathcal{M}'_1)$ multiplied by the measure of the affine line. The second bundle would drop out, as it would correspond to a subset (a point) of measure zero. If we want to include the second bundle, then the measure of this point would have to be non-zero. But we also expect to have an invariant measure (for how could we *a priori* distinguish a special point on this projective line?). Therefore the measure of every point of $\mathbb{CP}^1$ would have to be non-zero. But then our integral would diverge. It seems unlikely that one could somehow regularize this divergent integral in a meaningful way.

Next, we give an example in which the fiber $h^{-1}_{\ell,x}(\mathcal{M})$ is a continuous family of *non-isomorphic* vector bundles (thus directly contradicting the above statement from [L2]). This is that case if $\mathcal{M}$ is the indecomposable rank two bundle degree 1 vector bundle $F_2(x)$ (in the notation of [L2]) which is a unique, up to an isomorphism, non-trivial extension

$$0 \to \mathcal{O}_X \to F_2(x) \to \mathcal{O}_X(x) \to 0 \tag{3.8}$$

In this case, as shown in [Bo], Sect. 4.3, the fiber $h^{-1}_{\ell,x}(F_2(x))$ may be described in terms of a canonical two-sheeted covering $\pi : \operatorname{Pic}^0(X) \to \mathbb{CP}^1 = h^{-1}_{\ell,x}(F_2(x))$ ramified at 4 points such that (1) if $a \in h^{-1}_{\ell,x}(F_2(x))$ is outside of the ramification locus, then $\pi^{-1}(a) = \{\mathcal{L}_a, \mathcal{L}_a^{-1}\}$, where $\mathcal{L}_a$ is a line bundle on X; and (2) the fibers over the 4 ramification points are the four square roots $\mathcal{L}_i, i = 1, \ldots, 4$, of the trivial line bundle on X.

Namely, the vector bundle $\mathcal{M}'(a)$ corresponding to a point $a \in h^{-1}_{\ell,x}(F_2(x))$ is described in terms of π as follows (note that in [Bo] the bundle $F_2(x)$ is denoted by $G_2(x)$):

- if $a \in h^{-1}_{\ell,x}(F_2(x))$ is outside of the ramification locus, then $\mathcal{M}'(a) = \mathcal{L}_a \oplus \mathcal{L}_a^{-1}$;
- if a is a ramification point corresponding to the line bundle $\mathcal{L}_i$, then $\mathcal{M}'(a) = \mathcal{L}_i \otimes F_2$, where F_2 is the unique, up to an isomorphism, non-trivial extension of $\mathcal{O}_X$ by itself.

According to the Atiyah's classification, the bundles $\mathcal{M}'(a)$ and $\mathcal{M}'(b)$ corresponding to different points $a \neq b$ in $h^{-1}_{\ell,x}(F_2(x))$ are non-isomorphic. Thus, there is an infinite (in fact, continuous) family of non-isomorphic vector bundles appearing in the fiber $h^{-1}_{\ell,x}(F_2(x))$ in this case.

One gets a similar answer for $\mathcal{M} = F_2(x) \otimes \mathcal{L}$, where $\mathcal{L}$ is an arbitrary line bundle on X (note that unlike the vector bundles discussed in the previous example, all of the bundles $F_2(x) \otimes \mathcal{L}$ are *stable*). This means that the value of $H_{1,x} \cdot f$ at bundles $\mathcal{M}$ of this form depends in a crucial way on the choice of a measure of integration on $h_{\ell,x}^{-1}(\mathcal{M})$.

It seems unlikely that one could define these measures for different $\mathcal{M}$ and different $x \in X$ consistently, so that they would not only yield well-defined integrals but that the corresponding operators $H_{1,x}, x \in X$, would commute with each other and with the second set of Hecke operators $H_{2,x}, x \in X$ (whose definition is given above; it does not involve integration, as in the abelian case).

In [L2], Langlands assumes that Hecke operators $H_{1,x}, x \in X$, satisfying these properties can somehow be defined. Furthermore, he suggests considering their action on the direct sum of the spaces of L^2 functions on two subsets, denoted by $\mathfrak{D}$ and $\mathfrak{U}$. The former parametrizes rank two vector bundles on X decomposable as direct sums of line bundles. The latter parametrizes indecomposable rank two vector bundles. Both can be explicitly described using Atiyah's classification results [At].

However, treating the moduli of rank two bundles on X as a disjoint union of the subsets $\mathfrak{D}$ and $\mathfrak{U}$ appears to be problematic. Indeed, on the moduli stack Bun_{GL_2} they are "glued" together in a non-trivial way.[6] If we tear them apart, we are at the same time tearing apart the projective lines $h_{\ell,x}^{-1}(\mathcal{M})$ appearing as the fibers of the Hecke correspondences.

Let me illustrate this with a concrete example: let $\mathcal{M} = \mathcal{O}_X \oplus \mathcal{O}_X(x)$. Then, as explained in [Bo], there are two points in the fiber $h_{\ell,x}^{-1}(\mathcal{M})$, corresponding to $\mathcal{M}_1' = \mathcal{O}_X \oplus \mathcal{O}_X$ and $\mathcal{M}_2' = \mathcal{O}_X(-x) \oplus \mathcal{O}_X(x)$, and each point in the complement (which is isomorphic to $\mathbb{C}^\times$) corresponds to the indecomposable bundle F_2. Thus, in particular, we see that there exist continuous families of rank two vector bundles on X over an affine line $\mathbb{A}^1$ which are isomorphic to F_2 away from $0 \in \mathbb{A}^1$ and to $\mathcal{O}_X \oplus \mathcal{O}_X$ or to $\mathcal{O}_X(-x) \oplus \mathcal{O}_X(x)$ at the point $0 \in \mathbb{A}^1$.

Now, if we were to treat $\mathcal{M}_1' = \mathcal{O}_X \oplus \mathcal{O}_X$ and $\mathcal{M}_2' = \mathcal{O}_X(-x) \oplus \mathcal{O}_X(x)$ as belonging to a different connected component of Bun_{GL_2} than F_2, then what to make of the integral (3.7)? It would seemingly break into the sum of two points and an integral over their complement. That would be fine in the case of a curve over $\mathbb{F}_q$: we would simply obtain the formula

$$(H_{1,x} \cdot f)(\mathcal{M}) = f(\mathcal{M}_1') + f(\mathcal{M}_2') + (q-1) f(F_2),$$

with the factor of $(q-1)$ being the number of points of $\mathbb{P}^1$ without two points (see [Lo, Al]). But over complex numbers we have to integrate over $\mathbb{C}^\times$. We would therefore have to somehow combine summation over two points and integration over their complement. As in another example of this nature that we considered above, it seems unlikely that there exists an integration measure that could achieve this in a consistent and meaningful fashion.

Similarly, if $\mathcal{M} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-isomorphic line bundles of degree 0, then the fiber of the Hecke correspondence $h_{\ell,x}^{-1}(\mathcal{M})$ over $\mathcal{M}$ has two points corresponding $\mathcal{M}_1' = \mathcal{L}_1(-x) \oplus \mathcal{L}_2$ and $\mathcal{M}_2' = \mathcal{L}_1 \oplus \mathcal{L}_2(-x)$, and every point in the complement of these two points (which is isomorphic to $\mathbb{C}^\times$) corresponds to the indecomposable vector bundle

[6] If we were to consider instead the moduli space of semi-stable bundles, then, depending on the stability condition we choose, some of the decomposable bundles in $\mathfrak{D}$ would have to be removed, or identified with the indecomposable ones in $\mathfrak{U}$. Considering the moduli space of semi-stable bundles is, however, problematic for a different reason: it is *not preserved by the Hecke correspondences*, and therefore it is not clear how to define the Hecke operators.

$F_2(x) \otimes \mathcal{L}(-x)$, where $\mathcal{L}^{\otimes 2} \simeq \mathcal{L}_1 \otimes \mathcal{L}_2$. Again, it is not clear how one could possibly integrate over $h^{-1}_{\ell,x}(\mathcal{M})$ in this case.[7]

In light of this discussion, it is not clear what to make of the statement on p.20 of [L2] that the presumed Hecke operators are well-defined, and moreover, act separately on the spaces of L_2 functions on the loci $\mathfrak{D}$ and $\mathfrak{U}$ of decomposable and indecomposable vector bundles. This statement is a key assumption made in [L2] about the Hecke operators, on which the analysis of the GL_2 case is based (in the case of elliptic curves). It is not clear that such operators could be defined in a meaningful way.

3.5. A toy model. At this point, it may be worthwhile to consider a toy model of the questions we have been discussing. Over $\mathbb{F}_q$, there is a well-understood finite-dimensional analogue of the spherical Hecke algebra of $G(\mathbb{F}_q((t)))$; namely, the Hecke algebra $\mathcal{H}_q(G)$ of $B(\mathbb{F}_q)$ bi-invariant $\mathbb{C}$-valued functions on the group $G(\mathbb{F}_q)$, where B is a Borel subgroup of a simple algebraic group G.

As a vector space, this algebra has a basis labeled by the characteristic functions c_w of the Bruhat-Schubert cells $B(\mathbb{F}_q)wB(\mathbb{F}_q)$, where w runs over the Weyl group of G. The convolution product on $\mathcal{H}_q(G)$ is defined using the constant measure on the finite group $G(\mathbb{F}_q)$ normalized so that the measure of $B(\mathbb{F}_q)$ is equal to 1. Then the function c_1 is a unit element of $\mathcal{H}_q(G)$.

It is convenient to describe the convolution product on $\mathcal{H}_q(G)$ as follows: identify the B bi-invariant functions on G with B-invariant functions on G/B and then with G-invariant on $(G/B) \times (G/B)$ (with respect to the diagonal action). Given two G-invariant functions f_1 and f_2 on $(G/B) \times (G/B)$, we define their convolution product by the formula

$$(f_1 \star f_2)(x, y) = \int\limits_{G/B} f_1(x, z) f_2(z, y) dz. \tag{3.9}$$

Under this convolution product, the algebra $\mathcal{H}_q(G)$ is generated by the functions c_{s_i}, where the s_i are the simple reflections in W. They satisfy the well-known relations.

Observe also that the algebra $\mathcal{H}_q(G)$ naturally acts on the space $\mathbb{C}[G(\mathbb{F}_q)/B(\mathbb{F}_q)]$ of $\mathbb{C}$-valued functions on $G(\mathbb{F}_q)/B(\mathbb{F}_q)$. It acts on the right and commutes with the natural left action of $G(\mathbb{F}_q)$. Unlike the spherical Hecke algebra, $\mathcal{H}_q(G)$ is non-commutative. Nevertheless, we can use the decomposition of the space $\mathbb{C}[G(\mathbb{F}_q)/B(\mathbb{F}_q)]$ into irreducible representations of $\mathcal{H}_q(G)$ to describe it as direct sum of irreducible representations of $G(\mathbb{F}_q)$.

Now suppose that we wish to generalize this construction to the complex case. Thus, we consider the group $G(\mathbb{C})$, its Borel subgroup $B(\mathbb{C})$, and the quotient $G(\mathbb{C})/B(\mathbb{C})$, which is the set of $\mathbb{C}$-points of the flag variety G/B over $\mathbb{C}$. A naive analogue of $\mathcal{H}_q(G)$ would be the space $\mathcal{H}_{\mathbb{C}}(G)$ of $B(\mathbb{C})$ bi-invariant functions on $G(\mathbb{C})$. Therefore we have the following analogues of the questions that we discussed above in the case of the spherical Hecke algebra: Is it possible to define a measure of integration on $G(\mathbb{C})$ that gives rise to a meaningful convolution product on $\mathcal{H}_{\mathbb{C}}(G)$? Is it possible to use the resulting algebra to decompose the space of L_2 functions on $G(\mathbb{C})/B(\mathbb{C})$?

For example, consider the case of $G = SL_2$. Then $G/B = \mathbb{P}^1$. The Hecke algebra $\mathcal{H}_q(SL_2)$ has a basis consisting of two elements, c_1 and c_s, which (in its realization as G-invariant

[7] Incidentally, this example shows that there are families of rank two vector bundles on X over an affine line $\mathbb{A}^1$ that are isomorphic to $F_2(x) \otimes \mathcal{L}$ away from $0 \in \mathbb{A}^1$ and to $\mathcal{L}_1(x) \oplus (\mathcal{L}^{\otimes 2} \otimes \mathcal{L}_1^{-1})$ at the point $0 \in \mathbb{A}^1$, for any degree zero line bundle $\mathcal{L}_1$. This and the previous example illustrate the intricate topology of Bun_{GL_2} for an elliptic curve.

functions on $(G/B) \times (G/B)$ explained above) correspond to the characteristic functions of the two SL_2-orbits in $\mathbb{P}^1 \times \mathbb{P}^1$: the diagonal and its complement, respectively. Applying formula (3.9), we obtain that

$$c_1 \star c_1 = c_1, \qquad c_1 \star c_s = c_s, \tag{3.10}$$

$$c_s \star c_s = qc_1 + (q-1)c_s. \tag{3.11}$$

The two formulas in (3.10) follow from the fact that for each x and y, in formula (3.9) there is either a unique value of z for which the integrand is non-zero, or no such values. The coefficients in formula (3.11) have the following meaning: $q = \mu_q(\mathbb{A}^1), q-1 = \mu_q(\mathbb{A}^1 \backslash 0)$.

Now, if we try to adopt this to the case of $\mathbb{P}^1$ over $\mathbb{C}$, we quickly run into trouble. Indeed, if we want c_1 to be the unit element, we want to keep the two formulas in (3.10). But in order to reproduce the second formula in (3.10), we need a measure dz on $\mathbb{CP}^1$ that would give us $\int \chi_u dz = 1$ for every point $u \in \mathbb{CP}^1$, where χ_u is the characteristic function of u. However, then the integral of this measure over the affine line inside $\mathbb{CP}^1$ would diverge, rendering the convolution product $c_s \star c_s$ meaningless.

Likewise, we run into trouble if we attempt to define an action of $\mathcal{H}_{\mathbb{C}}(G)$ on the space of functions on $G(\mathbb{C})/B(\mathbb{C})$. It thus appears that the questions we asked above do not have satisfactory answers, and the reasons for that are similar to those we discussed above, concerning the spherical Hecke algebra and the action of Hecke operators on functions on Bun_G.

However, there are two natural variations of these questions that do have satisfactory answers. The first possibility is to consider a *categorical* version of the Hecke algebra, i.e., instead of the space of B-invariant functions, the category $D(G/B)^B$-mod of B-equivariant D-modules on G/B. According to a theorem of Beilinson and Bernstein [BB], we have an exact functor of global sections (as $\mathcal{O}$-modules) from this category to the category of modules over the Lie algebra $\mathfrak{g}$ of G, which is an equivalence with the category of those $\mathfrak{g}$-module which have a fixed character of the center of $U(\mathfrak{g})$ (the character of the trivial representation of $\mathfrak{g}$). This is the category that appears in the Kazhdan–Lusztig theory, which gives rise, among other things, to character formulas for irreducible $\mathfrak{g}$-modules from the category $\mathcal{O}$. Furthermore, instead of the convolution product on functions, we now have convolution functors on a derived version of $D(G/B)^B$-mod. This is the categorical Hecke algebra which has many applications. For example, Beilinson and Bernstein have defined a categorical action of this category on the derived category of the category $\mathcal{O}$ (which may be viewed as the category of $(\mathfrak{g}, B)$ Harish-Chandra modules). This is a special case of a rich theory.

Note also that a closely related category of perverse sheaves may also be defined over $\mathbb{F}_q$. Taking the traces of the Frobenius on the stalks of those sheaves, we obtain the elements of the original Hecke algebra $\mathcal{H}_q(G)$. This operation transforms convolution product of sheaves into convolution product of functions. Thus, we see many parallels with the geometric Langlands Program (for more on this, see [Fr3], Sect. 1.3.3). In particular, the spherical Hecke algebra has a categorical analogue, for which a categorical version of the Satake isomorphism has been proved [Gin, Lu, MV]. In other words, the path of categorification of the Hecke algebra $\mathcal{H}_q(G)$ is parallel to the path taken in the geometric Langlands theory.

But there is also a second option: We can consider the space of L_2 functions on $G(\mathbb{C})/B(\mathbb{C})$ with respect to the natural measure of integration coming from a symplectic structure. Or, alternatively, we can define the $L_2(G(\mathbb{C})/B(\mathbb{C}))$ as the completion of the space of half-densities on $G(\mathbb{C})/B(\mathbb{C})$ with respect to the natural Hermitian inner product.

Both are meaningful objects, but we no longer have an action of a Hecke algebra on it. However, and *this is a key point*, there is a meaningful substitute for it: differential operators on G/B.

The Lie algebra $\mathfrak{g}$ acts on $L_2(G(\mathbb{C})/B(\mathbb{C}))$ by holomorphic vector fields, and we have a commuting action of another copy of $\mathfrak{g}$ by anti-holomorphic vector fields. Therefore, the tensor product of two copies of the center of $U(\mathfrak{g})$ acts by mutually commuting differential operators. As we mentioned above, both holomorphic and anti-holomorphic ones act according to the central character of the trivial representation. However, the center of $U(\mathfrak{g}_c)$, where $\mathfrak{g}_c$ is a compact form of the Lie algebra $\mathfrak{g}$, also acts on $L_2(G(\mathbb{C})/B(\mathbb{C}))$ by commuting differential operators, and this action is non-trivial. It includes the Laplace operator, which corresponds to the Casimir element of $U(\mathfrak{g}_c)$.

Instead of asking for the eigenfunctions and eigenvalues of the (likely, non-existent) Hecke operators, we then ask for the eigenfunctions and eigenvalues of these commuting differential operators. This question has a meaningful answer. Indeed, using the isomorphism $G/B \simeq G_c/T_c$, where T_c is a maximal torus of the compact form of G, and the Peter-Weyl theorem, we obtain that $L_2(G(\mathbb{C})/B(\mathbb{C}))$ can be decomposed as a direct sum of irreducible finite-dimensional representations of $\mathfrak{g}_c$ which can be exponentiated to the group G_c of adjoint type, each representation appearing with multiplicity one. Therefore the combined action of the center of $U(\mathfrak{g}_c)$ and the Cartan subalgebra $\mathfrak{t}_c$ of T_c (acting by vector fields from the right) has as eigenspaces various weight components of the irreducible finite-dimensional representations of $\mathfrak{g}_c$. All of these eigenspaces are therefore finite-dimensional.

For instance, for $G = SL_2$ every eigenspace is one-dimensional, and so we find that these differential operators have simple spectrum. In fact, suitably normalized joint eigenfunctions of the center of $U(\mathfrak{g}_c)$ and $\mathfrak{t}_c$ are in this case the standard spherical harmonics (note that in this case $G(\mathbb{C})/B(\mathbb{C}) \simeq S^2$).

This discussion suggests that an analytic theory of automorphic forms on Bun_G can be build if, rather than looking for the eigenfunctions of Hecke operators (which are problematic in the non-abelian case), we look for the eigenfunctions of a commutative algebra of global differential operators on Bun_G. It turns out that we are in luck: there exists a large commutative algebra of differential operators on the line bundle of half-densities on Bun_G. We can use this algebra instead of the Hecke operators in order to build a meaningful theory of automorphic forms for complex algebraic curves.

3.6. Global differential operators on Bun_G. In this subsection and the next one we report on a joint work in progress with David Kazhdan [FK]. Let us assume for simplicity that G is a connected, simply-connected, simple algebraic group over $\mathbb{C}$. In [BD], Beilinson and Drinfeld have described the algebra D_G of global holomorphic differential operators on Bun_G acting on the square root $\mathcal{K}^{1/2}$ of a canonical line bundle (which exists for any reductive G and is unique under our assumptions). They have proved that D_G is commutative and is isomorphic to the algebra of functions on the space $\mathrm{Op}_{^LG}(X)$ of ^LG-opers on X. For a survey of this construction and the definition of $\mathrm{Op}_{^LG}(X)$, see, e.g., [Fr2], Sects. 8 and 9. Under the above assumptions on G, the space $\mathrm{Op}_{^LG}(X)$ may be identified with the space of all holomorphic connections on a particular holomorphic G-bundle $\mathcal{F}_0$ on X. In particular, it is an affine space of dimension equal to $\dim \mathrm{Bun}_G$.

The construction of these global differential operators is similar to the construction outlined in Section 3.5 above. Namely, they are obtained in [BD] from the central elements of the completed enveloping algebra of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level, using the realization of Bun_G as a double quotient of the formal loop group $G(\mathbb{C}((t)))$ and the Beilinson–Bernstein type localization functor. The critical level of $\widehat{\mathfrak{g}}$ corresponds to the square root of the canonical line bundle on Bun_G. A theorem of Feigin and myself [FF, Fr1, Fr3] identifies the center of this enveloping algebra with the algebra of functions on the space of ${}^L G$-opers on the formal punctured disc. This is a local statement that Beilinson and Drinfeld use in the proof of their theorem.

Now, we can use the same method to construct the algebra $\overline{D}_G$ of global anti-holomorphic differential operators on Bun_G acting on the square root $\overline{K}^{1/2}$ of the anti-canonical line bundle. The theorem of Beilinson and Drinfeld implies that $\overline{D}_G$ is isomorphic to the algebra of functions on the complex conjugate space to the space of opers, which we denote by $\overline{\mathrm{Op}}_{^LG}(X)$. Under the above assumptions on G, it can be identified with the space of all anti-holomorphic connections on the G-bundle $\overline{\mathcal{F}}_0$ that is the complex conjugate of the G-bundle $\mathcal{F}_0$. While $\mathcal{F}_0$ carries a holomorphic structure, i.e. a $(0,1)$-connection, $\overline{\mathcal{F}}_0$ carries a $(1,0)$-connection (which one could call an "anti-holomorphic structure" on $\overline{\mathcal{F}}_0$). Just as a $(1,0)$, i.e. holomorphic, connection on $\mathcal{F}_0$ completes its holomorphic structure to a flat connection, a $(0,1)$, i.e. anti-holomorphic, connection $\overline{\mathcal{F}}_0$ completes its $(1,0)$-connection to a flat connection.

Both $\mathrm{Op}_{^LG}(X)$ and $\overline{\mathrm{Op}}_{^LG}(X)$ may be viewed as Lagrangian subspaces of the moduli stack of flat ${}^L G$-bundles on X, and it turns out that it is their intersection that is relevant to the eigenfunctions of the global differential operators.

Indeed, we have a large commutative algebra $D_G \otimes \overline{D}_G$ of global differential operators on the line bundle $K^{1/2} \otimes \overline{K}^{1/2}$ of half-densities on Bun_G. This algebra is isomorphic to the algebra of functions on $\mathrm{Op}_{^LG}(X) \times \overline{\mathrm{Op}}_{^LG}(X)$.

On the other hand, there is a natural L_2 norm on the space of sections of $K^{1/2} \otimes \overline{K}^{1/2}$ on Bun_G. We can therefore define the space $L_2(\mathrm{Bun}_G)$ as the completion of the subspace of sections with well-defined norm. These sections are uniquely determined by their restriction to an open dense subset $\mathrm{Bun}_G^{\mathrm{st}}$ of stable G-bundles[8] (note that the issue of this subset not being preserved by Hecke correspondence is now moot). Thus, we can define $L_2(\mathrm{Bun}_G)$ as the completion of the space of sections of $K^{1/2} \otimes \overline{K}^{1/2}$ with the finite norm on $\mathrm{Bun}_G^{\mathrm{st}}$. It is a Hilbert space with respect to the standard Hermitian inner product.

The elements of the algebra $D_G \otimes \overline{D}_G$ are well-defined linear operators on $L_2(\mathrm{Bun}_G)$. Furthermore, we expect that these operators are normal. Thus, we get a nice set-up for the problem of finding eigenfunctions and eigenvalues of these operators. It is natural to call these eigenfunctions the automorphic forms on Bun_G (or $\mathrm{Bun}_G^{\mathrm{st}}$) for a complex algebraic curve. The construction generalizes to arbitrary connected reductive complex groups G.

The joint eigenvalues of $D_G \otimes \overline{D}_G$ on $L_2(\mathrm{Bun}_G)$ correspond to points in $\mathrm{Op}_{^LG}(X) \times \overline{\mathrm{Op}}_{^LG}(X)$, i.e. pairs (χ, ρ), where $\chi \in \mathrm{Op}_{^LG}(X)$ and $\rho \in \overline{\mathrm{Op}}_{^LG}(X)$. The question then is to describe the set of those pairs that occur as eigenvalues.

As far as I know, this spectral problem was first considered by Teschner [T], in the case of $G = SL_2$. In this case, ${}^L G$-opers are PGL_2-opers, or equivalently, projective connections on X. Teschner conjectured in [T] that the eigenvalues of the algebra $D_{SL_2} \otimes \overline{D}_{SL_2}$ on $L_2(\mathrm{Bun}_{SL_2})$ are in one-to-one correspondence with those projective connections that have *real monodromy*. Given such a projective connection $\chi \in \mathrm{Op}_{PGL_2}(X)$, the corresponding

[8] If X is an elliptic curve, we need to include semi-stable bundles.

point in $\mathrm{Op}_{PGL_2}(X) \times \overline{\mathrm{Op}}_{PGL_2}(X)$ is $(\chi, \overline{\chi})$, where $\overline{\chi}$ is determined by χ (it also has real monodromy).

Projective connections with real monodromy have been described by Goldman [Gol]. If the genus of X is greater than 1, then among them there is a special one, corresponding to the uniformization of X. But there are many others ones as well, and they have been the subject of interest for many years. It is fascinating that they now show up in the context of the Langlands correspondence for complex curves.

We expect that a similar description holds for other groups as well, i.e. the eigenvalues of the algebra $D_G \otimes \overline{D}_G$ on $L_2(\mathrm{Bun}_G)$ are in one-to-one correspondence with the ${}^L G$-opers on X whose monodromy takes values in the split real form of ${}^L G$. The details will be discussed in [FK].

In the next subsection we will illustrate how these opers appear in the abelian case $G = GL_1$.

3.7. The spectra of global differential operators or $G = GL_1$. For simplicity, we restrict ourselves to the elliptic curve $X = E_i = C/(\mathbb{Z} + \mathbb{Z}i)$ discussed in Section 2.1 and the neutral component $\mathrm{Pic}^0(X)$ of the corresponding Picard variety. We identify $\mathrm{Pic}^0(X)$ with X using a reference point p_0, as in Section 2.1. Then the algebra D_{GL_1} (resp. $\overline{D}_{GL_1}$) coincides with the algebra of constant holomorphic (resp. anti-holomorphic) differential operators on X:

$$D_{GL_1} = \mathbb{C}[\partial_z], \qquad \overline{D}_{GL_1} = \mathbb{C}[\partial_{\overline{z}}].$$

The eigenfunctions of these operators are precisely the Fourier harmonics $f_{m,n}$ given by formula (2.4):

$$f_{m,n} = e^{2\pi imx} \cdot e^{2\pi iny}, \qquad m, n \in \mathbb{Z}.$$

If we rewrite it in terms of z and $\overline{z}$:

$$f_{m,n} = e^{\pi z(n+im)} \cdot e^{-\pi\overline{z}(n-im)}$$

then we find that the eigenvalues of ∂_z and $\partial_{\overline{z}}$ on $f_{m,n}$ are $\pi(n + im)$ and $-\pi(n - im)$ respectively. Let us recast these eigenvalues in terms of the corresponding GL_1-opers.

By definition, a GL_1-oper is a holomorphic connection on the trivial line bundle on X (see [Fr2], Sect. 4.5). The space of such connections is canonically isomorphic to the space of holomorphic one-forms on X which may be written as $-\lambda dz$, where $\lambda \in \mathbb{C}$. An element of the space of GL_1-opers may therefore be represented as a holomorphic connection on the trivial line bundle, which together with its $(0, 1)$ part $\partial_{\overline{z}}$ yields the flat connection

$$\nabla = d - \lambda\, dz, \qquad \lambda \in \mathbb{C}. \tag{3.12}$$

Under the isomorphism $\mathrm{Spec}\, D_{GL_1} \simeq \mathrm{Op}_{GL_1}(X)$, the oper (3.12) corresponds to the eigenvalue λ of ∂_z (this is why we included the sign in (3.12)).

Likewise, an element of the complex conjugate space $\overline{\mathrm{Op}}_{GL_1}(X)$ is an anti-holomorphic connection on the trivial line bundle, which together with its $(1, 0)$ part ∂_z yields the flat connection

$$\overline{\nabla} = d - \mu\, d\overline{z}, \qquad \mu \in \mathbb{C}. \tag{3.13}$$

Under the isomorphism $\mathrm{Spec}\, \overline{D}_{GL_1} \simeq \overline{\mathrm{Op}}_{GL_1}(X)$, the oper (3.13) corresponds to the eigenvalue μ of $\partial_{\overline{z}}$.

We have found above that the eigenvalues of ∂_z and $\partial_{\overline{z}}$ on $L_2(\mathrm{Bun}_{GL_1})$ are $\pi(n + im)$ and $-\pi(n - im)$, respectively, where $m, n \in Z$. The following lemma, which is proved by a direct computation, links them to GL_1-opers with real monodromy.

Lemma 4. *The connection* (3.12) (*resp.* (3.13)) *on the trivial line bundle on* $E_i = \mathbb{C}(\mathbb{Z}+\mathbb{Z}i)$ *has real monodromy* (*i.e. its monodromy takes values in* $\mathbb{R}^\times \subset \mathbb{C}^\times$) *if and only if* $\lambda = \pi(n+im)$ (*resp.* $\mu = -\pi(n-im)$), *where* $m, n \in \mathbb{Z}$.

This Lemma generalizes in a straightforward fashion to arbitrary curves and arbitrary abelian groups. Thus, the conjectural description of the spectra of global differential operators in terms of opers with real monodromy, as described at the end of Section 3.6, holds in the abelian case.

Recall that in the abelian case we also have well-defined Hecke operators. It is interesting to note that they commute with the global differential operators and hence share the same eigenfunctions. Furthermore, the eigenvalues of the Hecke operators may be expressed in terms of the eigenvalues of the global differential operators. For non-abelian G, the definition of Hecke operators is problematic, as we have argued in this section. But the global differential operators are well-defined. Their eigenvalues are given by ${}^L G$-opers satisfying some special conditions. Each oper gives rise to a homomorphism $\pi_1(X,x) \to {}^L G$. Thus, we obtain a parametrization of automorphic functions on Bun_G by homomorphisms $\pi_1(X,x) \to {}^L G$ of special kind; namely, we expect their monodromy to take values in the split real form of ${}^L G$. One could think of these homomorphisms as the Langlands parameters of the automorphic forms for curves over $\mathbb{C}$.

The details will appear in [FK].

References

[Al] R. Alvarenga, *Hall algebra and graphs of Hecke operators for elliptic curves*, arXiv:1805.00567.

[AG] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, Selecta Math. **21** (2015) 1–199.

[At] M. Atiyah, *Vector bundles over an elliptic curve*, Proc. Lond. Math. Soc. **7** (1957) 414–452.

[BB] A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, Advances in Soviet Mathematics **16**, Part 1, pp. 1–50, AMS, 1993.

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, available at `http://www.math.uchicago.edu/~mitya/langlands/QuantizationHitchin.pdf`

[Bo] D. Boozer, *Moduli spaces of Hecke modifications for rational and elliptic curves*, arXiv:1805.11184.

[Be] J. Bernstein, *Algebraic theory of D–modules*, available at `http://www.math.uchicago.edu/~mitya/langlands/Bernstein/Bernstein-dmod.ps`

[CCH] W. Casselman, J.E. Cely, T. Hales, *The Spherical Hecke algebra, partition functions, and motivic integration*, arXiv:1611.05773.

[Dr1] V.G. Drinfeld, *Two-dimensional ℓ–adic representations of the fundamental group of a curve over a finite field and automorphic forms on $GL(2)$*, Amer. J. Math. **105** (1983) 85–114.

[Dr2] V.G. Drinfeld, *Langlands conjecture for $GL(2)$ over function field*, Proc. of Int. Congress of Math. (Helsinki, 1978), pp. 565–574; *Moduli varieties of F–sheaves*, Funct. Anal. Appl. **21** (1987) 107–122; *The proof of Petersson's conjecture for $GL(2)$ over a global field of characteristic p*, Funct. Anal. Appl. **22** (1988) 28–43.

[FF] B. Feigin and E. Frenkel, *Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras*, Int. J. Mod. Phys. **A7**, Suppl. 1A (1992) 197–215.

[Fe1] I. Fesenko, *Analysis on arithmetic schemes I*, available at `https://www.maths.nottingham.ac.uk/plp/pmzibf/a1.pdf`

[Fe2] I. Fesenko, *Measure, integration and elements of harmonic analysis on generalized loop spaces*, available at `https://www.maths.nottingham.ac.uk/plp/pmzibf/aoh.pdf`

[Fr1] E. Frenkel, *Wakimoto modules, opers and the center at the critical level*, Adv. Math. **195** (2005) 297–404 (math.QA/0210029).

[Fr2] E. Frenkel, *Lectures on the Langlands Program and Conformal Field Theory*, in Frontiers in Number Theory, Physics and Geometry II, eds. P. Cartier, e.a., pp. 387-536, Springer, 2007, arXiv:hep-th/0512172.

[Fr3] E. Frenkel, *Langlands Correspondence for Loop Groups*, Cambridge University Press, 2007.

[Fr4] E. Frenkel, *Gauge Theory and Langlands Duality*, talk given at MSRI, Berkeley, in September 2014, available at `https://youtu.be/564NQQrFUA0?t=1h8m45s`.

[FGV] E. Frenkel, D. Gaitsgory and K. Vilonen, *On the geometric Langlands conjecture*, Journal of AMS **15** (2001) 367–417.

[FK] E. Frenkel and D. Kazhdan, in preparation.

[Gai1] D. Gaitsgory, *On a vanishing conjecture appearing in the geometric Langlands correspondence*, Ann. Math. **160** (2004) 617–682.

[Gai2] D. Gaitsgory, *Recent progress in geometric Langlands theory*, Séminaire Bourbaki talk, arXiv:1606.09462.

[Gaw] K. Gawedzki, *Lectures on conformal field theory*, in *Quantum fields and strings: a course for mathematicians*, Vol. 2, pp. 727–805, AMS, 1999.

[Gin] V. Ginzburg, *Perverse sheaves on a loop group and Langlands duality*, Preprint alg-geom/9511007.

[Gol] W. Goldman, *Projective structures with Fuchsian holonomy*, J. Diff. Geom. **25** (1987) 297–326.

[Gor] J. Gordon, *Motivic Haar Measure on Reductive Groups*, Canad. J. Math. **58** (2006) 93–114.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, vol. 1, Wiley, 1978.

[Laf] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002) 1–241.

[L1] R.P. Langlands, *Problems in the theory of automorphic forms*, in Lect. Notes in Math. **170**, pp. 18–61, Springer Verlag, 1970.

[L2] R.P. Langlands, *Об аналитическом виде геометрической теории автоморфных форм*, Preprint available at `http://publications.ias.edu/rpl/paper/2678`

[La1] G. Laumon, *Correspondance de Langlands géométrique pour les corps de fonctions*, Duke Math. J. **54** (1987) 309-359.

[La2] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionelles et conjecture de Weil*, Publ. IHES **65** (1987) 131–210.

[Lo] O. Lorscheid, *Automorphic forms for elliptic function fields*, Math. Z.**272** (2012) 885–911.

[Lu] G. Lusztig, *Singularities, character formulas, and a q–analogue of weight multiplicities*, Astérisque **101** (1983) 208–229.

[MV] I. Mirković, K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Preprint math.RT/0401222.

[Mo1] M. Morrow, *Integration on Valuation Fields over Local Fields*, Tokyo J. Math. **33** (2010) 235–281.

[Mo2] M. Morrow, *Integration on product spaces and GL_n of a valuation field over a local field* Comm. Number Theory and Phys. **2** (2008) 563–592.

[T] J. Teschner, *Quantisation conditions of the quantum Hitchin system and the real geometric Langlands correspondence*, arXiv:1707.07873.

[Wa1] R. Waller, *Measure and Integration on non-Locally Compact Spaces and Representation Theory*, Ph.D. Thesis, 2018.

[Wa2] R. Waller, Letter to the author, Nov. 2018.

[We] A. Weil, *Collected Papers*, vol. I, p. 244–255, Springer, 1979. (English translation in: *A 1940 letter of André Weil on analogy in mathematics*, Translated by Martin H. Krieger, Notices of AMS **52** (2005) 334–341.)

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