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A Mathematical Romance

Jim Holt

Love and Math: The Heart of Hidden Reality
by Edward Frenkel
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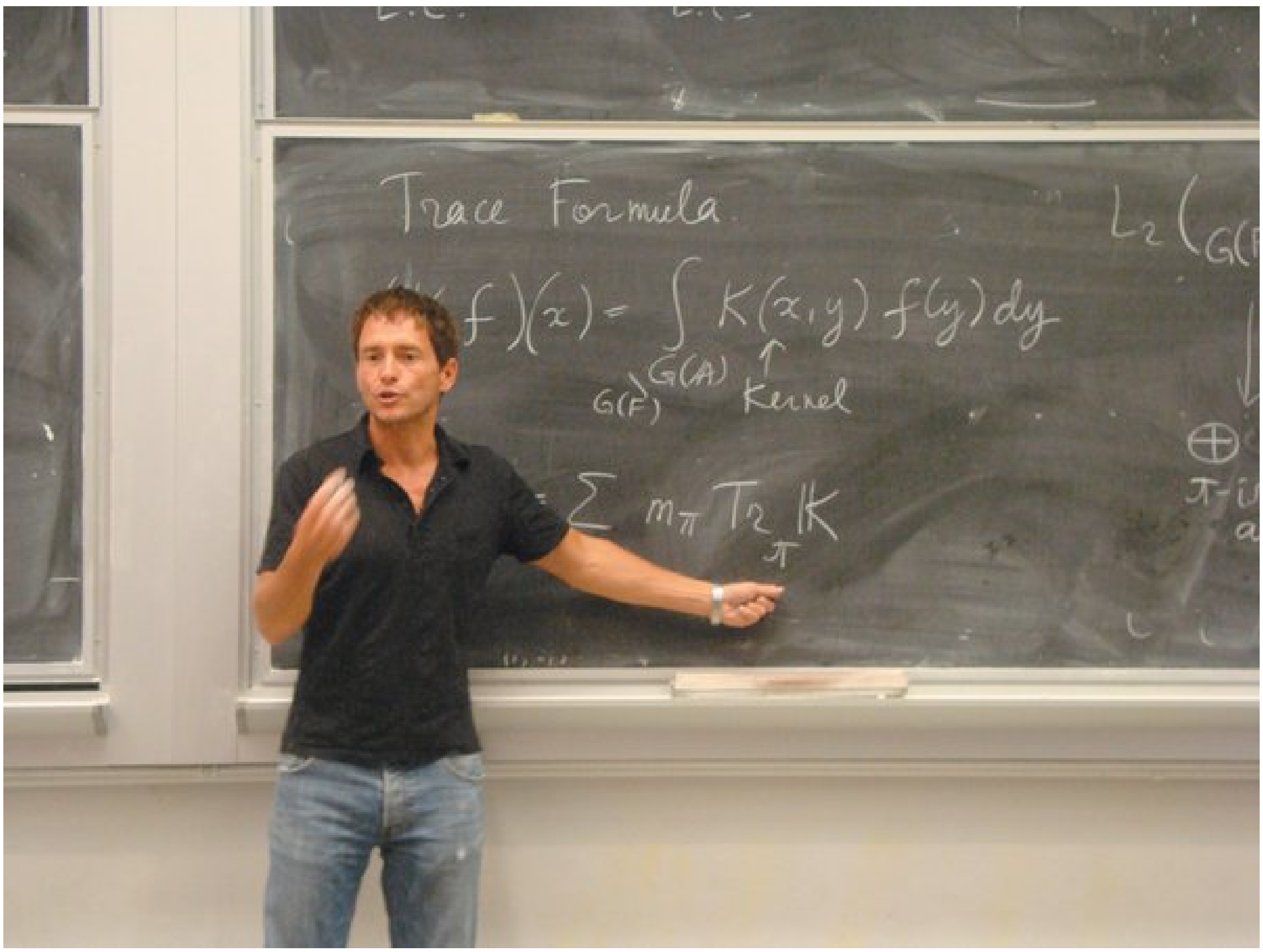

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Edward Frenkel, Berkeley, California, September 2010

For those who have learned something of higher mathematics, nothing could be more natural than to use the word "beautiful" in connection with it. Mathematical beauty, like the beauty of, say, a late Beethoven quartet, arises from a combination of strangeness and inevitability. Simply defined abstractions disclose hidden quirks and complexities. Seemingly unrelated structures turn out to have mysterious correspondences. Uncanny patterns emerge, and they remain uncanny even after being underwritten by the rigor of logic.

So powerful are these aesthetic impressions that one great mathematician, G.H. Hardy, declared that beauty, not usefulness, is the true justification for mathematics. To Hardy, mathematics was first and foremost a creative art. “The mathematician’s patterns, like the painter’s or the poet’s, must be *beautiful*,” he wrote in his classic 1940 book, *A Mathematician’s Apology*. “Beauty is the first test: there is no permanent place in the world for ugly mathematics.”

And what is the appropriate reaction when one is confronted by mathematical beauty? Pleasure, certainly; awe, perhaps. Thomas Jefferson wrote in his seventy-sixth year that contemplating the truths of mathematics helped him to “beguile the wearisomeness of declining life.” To Bertrand Russell—who rather melodramatically claimed, in his autobiography, that it was his desire to know more of mathematics that kept him from committing suicide—the beauty of mathematics was “cold and austere, like that of sculpture…sublimely pure, and capable of a stern perfection.” For others, mathematical beauty may evoke a distinctly warmer sensation. They might take their cue from Plato’s *Symposium*. In that dialogue, Socrates tells the guests assembled at a banquet how a priestess named Diotima initiated him into the mysteries of Eros—the Greek name for desire in all its forms.

One form of Eros is the sexual desire aroused by the physical beauty of a particular beloved person. That, according to Diotima, is the lowest form. With philosophical refinement, however, Eros can be made to ascend toward loftier and loftier objects. The penultimate of these—just short of the Platonic idea of beauty itself—is the perfect and timeless beauty discovered by the mathematical sciences. Such beauty evokes in those able to grasp it a desire to reproduce—not biologically, but intellectually, by begetting additional “gloriously beautiful ideas and theories.” For Diotima, and presumably for Plato as well, the fitting response to mathematical beauty is the form of Eros we call love.[1]

Edward Frenkel, a Russian mathematical prodigy who became a professor at Harvard at twenty-one and who now teaches at Berkeley, is an unabashed Platonist. Eros pervades his winsome new memoir, *Love and Math*. As a boy, he was hit by the beauty of mathematics like a *coup de foudre*. When, while still in his teens, he made a new mathematical discovery, it was "like the first kiss." Even when his career hopes seemed blighted by Soviet anti-Semitism, he was sustained by the "passion and joy of doing mathematics." And he wants everybody to share that passion and joy.

Therein lies a challenge. Mathematics is abstract and difficult; its beauties would seem to be inaccessible to most of us. As the German poet Hans Magnus Enzensberger has observed, mathematics is "a blind spot in our culture—alien territory, in which only the elite, the initiated few have managed to entrench themselves." People who are otherwise cultivated will proudly confess their philistinism when it comes to mathematics. The problem, says Frenkel, is that they have never been introduced to its masterpieces. The mathematics taught in school, and even at university (through, say, introductory calculus), is mostly hundreds or thousands of years old, and much of it involves routine problem-solving by tedious calculation.

That bears scant resemblance to what most mathematicians do today. Around the middle of the nineteenth century, a sort of revolution occurred in mathematics: the emphasis shifted from science-bound calculation to the free creation of new structures, new languages. Mathematical proofs, for all their rigorous logic, came to look more like narratives, with plots and subplots, twists and resolutions. It is this kind of mathematics that most people never see. True, it can be daunting. But great works of art, even when difficult, often allow the untutored a glimpse into their beauty. You don't have to know the theory of counterpoint to be moved by a Bach fugue.

To give the reader a similar glimpse into the beauty of higher mathematics, Frenkel goes straight to the most exciting mathematical development of the last half-century: the Langlands Program. Conceived in the 1960s by Robert Langlands, a Canadian mathematician at the Institute for Advanced Study in Princeton (and the inheritor of Einstein's old office there), the Langlands Program aims at being a grand unifying theory of mathematics. Yet it is little known outside the mathematical community. Indeed, most professional mathematicians were unaware of the Langlands Program as late as the 1990s, when it figured in the headline-making resolution of Fermat's Last Theorem.[2]

Since then, its scope has expanded beyond pure mathematics to the frontiers of theoretical physics. As far as I know, Frenkel is the first to try to explain the Langlands Program—for him, "the source code of all mathematics"—to readers without any mathematical background. His book, then, is three things: a Platonic love letter to mathematics; an attempt to give the layman some idea of its most magnificent drama-inprogress; and an autobiographical account, by turns inspiring and droll, of how the author himself came to be a leading player in that drama.

Frenkel grew up during the Brezhnev era in an industrial town called Kolomna, about seventy miles outside of Moscow. "I hated math when I was at school," he tells us. "What really excited me was physics—especially quantum physics." In his early teens he avidly read popular physics books that contained titillating references to subatomic particles like "hadrons" and "quarks." Why, he wondered, did the fundamental particles of nature come in such bewildering varieties? Why did they fall into families of certain sizes? It was only when his parents (both industrial engineers) arranged for him to meet with an old friend of theirs, a mathematician, that Frenkel was enlightened. What brought order and logic to the building blocks of matter, the mathematician explained to him, was something called a "symmetry group"—a mathematical beast that Frenkel had never encountered in school. "This was a moment of epiphany," he recalls, a vision of "an entirely different world."

To a mathematician, a "group" is a set of actions or operations that hang together in a nice way.[3] One kind of group—the kind Frenkel first encountered—is a symmetry group. Suppose you have a square card table sitting in the middle of a room. Intuitively, this piece of furniture is symmetrical in certain ways. How can this claim be made more precise? Well, if you rotate the table about its center by exactly 90 degrees, its appearance will be unchanged; no one who was out of the room when the table was rotated will notice any difference upon returning (assuming there are no stains or scratches on its surface). The same is true if you rotate the card table by 180 degrees, or by 270 degrees, or by 360 degrees—the last of which, since it takes the card table in a complete circle, is equivalent to no rotation at all.

These actions constitute the symmetry group of the card table. Since there are only four of them, the group is finite. If the table were circular, by contrast, its symmetry group would be infinite, since any rotation at all—by 1 degree, by 45 degrees, by 132.32578 degrees, or whatever—would leave its appearance unchanged. Groups are thus a way of measuring the symmetry of an object: a circular table, with its infinite symmetry group, is more symmetrical than a square table, whose symmetry group contains just four actions.

But (fortunately) it gets more interesting than that. Groups can capture symmetries that go beyond the merely geometrical—like the symmetries hidden in an equation, or in a family of subatomic particles. The real power of group theory was first demonstrated in 1832, in a letter that a twenty-year-old Parisian student and political firebrand named

Évariste Galois hastily scrawled to a friend late the night before he was to die in a duel (over the honor of a woman, and quite possibly at the hand of a government agent provocateur).[4]

What Galois saw was a truly beautiful way to extend the symmetry concept into the realm of numbers. By his *théorie des groupes*, he was able to resolve a classical problem in algebra that had bedeviled mathematicians for centuries—and in an utterly unexpected way. ("Galois did not *solve* the problem," Frenkel writes. "He *hacked* the problem.") The significance of Galois's discovery far transcended the problem that inspired it. Today, "Galois groups" are ubiquitous in the literature, and the group idea has proved to be perhaps the most versatile in all mathematics, clarifying many a deep mystery. "When in doubt," the great André Weil advised, "look for the group!" That's the *cherchez la femme* of mathematics.

Once smitten, the young Frenkel became obsessed with learning as much of mathematics as he could. ("This is what happens when you fall in love.") When he reached the age of sixteen, it was time to apply to a university. The ideal choice was obvious: Moscow State University, whose department of mechanics and mathematics, nicknamed *Mekh-Mat*, was one of the great world centers for pure mathematics. But it was 1984, a year before Gorbachev came to power, and the Communist Party still reached into all aspects of Russian life, including university admissions. Frenkel had a Jewish father, and that, apparently, was enough to scupper his chances of getting into Moscow State. (The unofficial rationale for keeping Jews out of physics-related academic areas was that they might pick up nuclear expertise and then emigrate to Israel.) But the appearance of fairness was maintained. He was allowed to sit for the entrance exam—which turned into a sadistic five-hour ordeal out of *Alice in Wonderland*. (Interrogator: "What is the definition of a circle?" Frenkel: "A circle is the set of points on the plane equidistant from a given point." Interrogator: "Wrong! It's the set of *all* points on the plane equidistant from a given point.")

Frenkel's consolation prize was a place at the Moscow Institute of Oil and Gas (cynically nicknamed *Kerosinka*), which had become a haven for Jewish students. But such was his craving for pure mathematics, he tells us, that he would scale a twenty-foot fence at the heavily guarded *Mekh-Mat* to get into the seminars there. Soon his extraordinary ability was recognized by a leading figure in Moscow mathematics, and he was put to work on an unsolved problem, which engrossed him for weeks to the point of insomnia. "And then, suddenly, I had it," he recalls. "For the first time in my life, I had in my possession something that *no one else in the world* had." The problem he had solved concerned yet another species of abstract group, called "braid groups" because they arise from systems of entwined curves that look quite literally like braided hair.

Granger Collection

French mathematician Évariste Galois; engraving by Alfred Galois, 1848

Despite this and other breakthroughs Frenkel made while still in his late teens, his academic prospects as a quasi Jew were dim. But his talent had come to the attention of mathematicians abroad. In 1989, the mail brought an unexpected letter from Derek Bok, the president of Harvard. The letter addressed Frenkel as "Doctor" (though he did not yet possess so much as a bachelor's degree) and invited him to come to Harvard as a prize fellow. "I had heard about Harvard University before," Frenkel writes, "though I must admit I did not quite realize at the time its significance in the academic world." At the age of twenty-one, he would be a visiting professor of mathematics at Harvard, with no formal obligations except to give occasional lectures about his work. And to his equal amazement, he received a Soviet exit visa in a month, becoming one of the first in what would be an exodus of Jewish mathematicians in the age of *perestroika*.

Frenkel's adjustment to American life was reasonably smooth. He marveled at the "abundance of capitalism" in the aisles of a Boston supermarket; he bought "the hippest jeans and a Sony Walkman"; he struggled to learn the ironic nuances of English by devotedly watching the David Letterman show on TV every night. Most important, he met another Russian Jewish émigré at Harvard who introduced him to the Langlands Program.

As with Galois theory, the Langlands Program had its origins in a letter. It was written in 1967 by Robert Langlands (then in his early thirties) to one of his colleagues at the Institute for Advance Study, André Weil. In his letter, Langlands proposed the possibility of a deep analogy between two theories that seemed to lie at opposite ends of the mathematical cosmos: the theory of Galois groups, which concerns symmetries in the realm of numbers; and "harmonic analysis," which concerns how complicated waves (e.g., the sound of a symphony) are built up from simple harmonics (e.g., the individual instruments). Certain structures in the harmonic world, called automorphic forms, somehow "knew" about mysterious patterns in the world of numbers. Thus it might be possible to use the methods of one world to reveal hidden harmonies in the other—so Langlands conjectured. If Weil did not find the intuitions in the letter persuasive, Langlands added, "I am sure you have a waste basket handy."

But Weil, a magisterial figure in twentieth-century mathematics (he died in 1998 at the age of ninety-two), was a receptive audience. In a letter that he had written in 1940 to his sister, Simone Weil, he had described in vivid terms the importance of analogy in mathematics. Alluding to the *Bhagavad-Gita* (he was also a Sanskrit scholar), André explained to Simone that, just as the Hindu deity Vishnu had ten different avatars, a seemingly simple mathematical equation could manifest itself in dramatically different abstract structures. The subtle analogies between such structures were like "illicit liaisons," he wrote; "nothing gives more pleasure to the connoisseur." As it happens, Weil was writing to his sister from prison in France, where he had been temporarily confined for desertion from the army (after nearly being executed as a spy in Finland).

The Langlands Program is a scheme of conjectures that would turn such hypothetical analogies into sturdy logical bridges, linking up diverse mathematical islands across the surrounding sea of ignorance. Or it can be seen as a Rosetta stone that would allow the mathematical tribes on these various islands—number theorists, topologists, algebraic geometers—to talk to one another and pool their conceptual resources. The Langlands conjectures are largely unproved so far.[5] Are they even true? There is an almost Platonic confidence among mathematicians that they must be. As Ian Stewart has remarked, the Langlands Program is "the sort of mathematics that ought to be true because it was so beautiful." The unity it could bring to higher mathematics could usher in a new golden age, in which we may finally discover, as Frenkel puts it, "what mathematics is really about."

Since Frenkel had no graduate degree, he had to undergo a temporary "demotion" from Harvard professor to graduate student while he wrote a Ph.D. thesis—which he wrapped up in a single year. (At his 1991 graduation, he was pleased to be personally congratulated by the commencement speaker, Eduard Shevardnadze, one of the architects of *perestroika*.) In his thesis, Frenkel proved a theorem that helped open a new chapter in the Langlands Program, extending it from the realm of numbers into the geometrical realm of curved surfaces, like the surface of a ball or a donut.[6] This meant twisting, even shattering, many familiar mathematical ideas—ideas as basic as the counting numbers.

Consider the number 3. It's boring; it has no internal structure. But suppose you replace the number 3 with a "vector space" of three dimensions—that is, a space in which each point represents a trio of numbers, with its own rules for addition and multiplication. Now you've got something interesting: a structure with more symmetries than a Greek temple. "In modern math, we create a new world in which numbers come alive as vector spaces," Frenkel writes. And other basic concepts are enriched too. "Functions" that you may have run into in high school mathematics (as in $y=f(x)$) are transformed into exotic creatures called "sheaves."[7]

The next move was to extend the Langlands Program beyond the borders of mathematics itself. In the 1970s, it had been noticed that one of its key ingredients—the "Langlands dual group"—also crops up in quantum physics. This came as a surprise. Could the same patterns that can be dimly glimpsed in the worlds of number and geometry also have counterparts in the theory that describes the basic forces of nature? Frenkel was struck by the potential link between quantum physics and the Langlands Program, and set about to investigate it—aided by a multimillion-dollar grant that he and some colleagues received in 2004 from the Department of Defense, the largest grant to date for research in pure mathematics. (In addition to being clean and gentle, pure mathematics is cheap: all its practitioners need is chalk and a little travel money. It is also open and transparent, since there are no inventions to patent.)

This brought him into a collaboration with Edward Witten, widely regarded as the greatest living mathematical physicist (and, like Langlands himself, a member of the Institute for Advanced Study in Princeton). Witten is a virtuoso of string theory, an ongoing effort by physicists to unite all the forces of nature, including gravity, in one neat mathematical package. He awed Frenkel with his "unbreakable logic" and his "great taste." It was Witten who saw how the "branes" (short for "membranes") postulated by string theorists might be analogous to the "sheaves" invented by mathematicians. Thus opened a rich dialogue between the Langlands Program, which aims to unify mathematics, and string theory, which aims to unify physics. Although optimism about string theory has faded somewhat with its failure (thus far) to deliver an effective description of our universe, the Langlands connection has yielded deep insights into the workings of particle physics.

This is not the first time that mathematical concepts studied for their pure beauty have later turned out to illumine the physical world. "How can it be," Einstein asked in wonderment, "that mathematics, being after all a product of human thought independent of experience, is so admirably appropriate to the objects of reality?" Frenkel's take on this is very different from Einstein's. For Frenkel, mathematical structures *are* among the "objects of reality"; they are every bit as real as anything in the physical or mental world.

Moreover, they are not the product of human thought; rather, they exist timelessly, in a Platonic realm of their own, waiting to be discovered by mathematicians. The conviction that mathematics has a reality that transcends the human mind is not uncommon among its practitioners, especially great ones like Frenkel and Langlands, Sir Roger Penrose and Kurt Gödel. It derives from the way that strange patterns and correspondences unexpectedly emerge, hinting at something hidden and mysterious. Who put those patterns there? They certainly don't seem to be of our making.

The problem with this Platonist view of mathematics—one that Frenkel, going on in a *misterioso* vein, never quite recognizes as such—is that it makes mathematical knowledge a miracle. If the objects of mathematics exist apart from us, living in a Platonic heaven that transcends the physical world of space and time, then how does the human mind "get in touch" with them and learn about their properties and relations? Do mathematicians have ESP? The trouble with Platonism, as the philosopher Hilary Putnam has observed, "is that it seems flatly incompatible with the simple fact that we think with our brains, and not with immaterial souls."

Perhaps Frenkel should be allowed his Platonic fantasy. After all, every lover harbors romantic delusions about his beloved. In 2009, while Frenkel was in Paris as the occupant of the *Chaire d'Excellence* of the Fondation Sciences Mathématiques, he decided to make a short film expressing his passion for mathematics. Inspired by Yukio Mishima's *Rite of Love and Death*, he titled it *Rites of Love and Math*. In this silent Noh-style allegory, Frenkel plays a mathematician who creates a formula of love. To keep the formula from falling into evil hands, he hides it away from the world by tattooing it with a bamboo stick on the body of the woman he loves, and then prepares to sacrifice himself for its protection.

Upon the premiere of *Rites of Love and Math* in Paris in 2010, *Le Monde* called it "a stunning short film" that "offers an unusual romantic vision of mathematicians." The "formula of love" used in the film was one that Frenkel himself discovered (in the course of investigating the mathematical underpinnings of quantum field theory). It is beautiful, yet forbidding. The only numbers in it are zero, one, and infinity. Isn't love like that?

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1 In one of those pointless but amusing coincidences, G.H. Hardy tells us near the end of *A Mathematician's Apology* that the Cambridge don who first opened his eyes to the beauty of mathematics was "Professor Love." ↩

2 Fermat's Last Theorem is a simple-sounding conjecture about numbers that Pierre de Fermat, a seventeenth-century councilor to the Parlement of Toulouse, penned in the margin of a treatise he was reading; namely, that the equation a n + b n = c n has no whole-number solutions when the exponent n is greater than 2. Although nothing important turned on its truth, the 350-year-long effort to prove it gave rise to rich new fields of mathematics, like algebraic number theory. ↩

3 What is meant by "in a nice way" is spelled out in the four axioms of group theory, which define the algebraic structure of a group. One of the axioms, for example, says that for any action in the group, there is another action in the group that undoes it. ↩

4 So, at any rate, goes the legend, as established in E.T. Bell's *Men of Mathematics* and repeated by Frenkel. Historians of mathematics have raised doubts over whether the timing of Galois's letter was really that dramatic. ↩

5 An exception is the Taniyama-Shimura conjecture, framed in the 1950s by a pair of Japanese mathematicians and proved in the 1990s by the Englishman Andrew Wiles, who thereby established the truth of Fermat's Last Theorem. ↩

6 These are called "Riemann surfaces," after the nineteenth-century mathematician Bernhard Riemann. ↩

7 The man most responsible for this reinvention of the language of mathematics is Alexandre Grothendieck, generally considered to be the greatest mathematician of the latter half of the twentieth century. Grothendieck, also a political radical, flourished in Paris in the 1960s and now lives hidden from the world in the Pyrénées. ↩