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LIE ALGEBRAS AND EQUATIONS OF KORTEWEG—DE VRIES TYPE

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The survey contains a description of the connection between the infinite-dimensional Lie algebras of Kats—Moody and systems of differential equations generalizing the Korteweg—de Vries and sine-Gordon equations and integrable by the method of the inverse scattering problem. A survey of the theory of Kats—Moody algebras is also given.

INTRODUCTION

Among nonlinear differential equations integrable by the method of the inverse scattering problem the Korteweg—de Vries (KdV) equation $u_t = u_{xxx} + 6uu_x$ and the sine-Gordon equation $v_{\tau x} = \sin v$ are especially popular. Both these equations are connected with the modified Korteweg—de Vries (mKdV) equation $w_t = w_{xxx} + 6w^2w_x$. The connection between KdV and mKdV is realized by the Miura transformation $u = iw_x + w^2$ taking solutions of mKdV into solutions of KdV. The connection between the mKdV and sine-Gordon equations is as follows. After the substitution $w = v_x/2$ the mKdV equation can be written in the form

$$v_t = v_{xxx} + \frac{1}{2} v_x^3. \tag{0.1}$$

It is found that if the function $v(x, t, \tau)$ satisfies Eq. (0.1) and $v_{\tau x} = \sin v$ for $t = 0$, then $v_{\tau x} = \sin v$ for all (this assertion is formulated in a more rigorous way in Sec. 10 of the present work).

Systems of equations usually called two-dimensional Toda lattices have been intensively studied recently (see [30, 65, 59, 46, 55, 72]). These systems have the form

$$\frac{\partial^2 u_i}{\partial x \partial \tau} = \exp \sum_{j=1}^{n} A_{ij} u_j, \quad i = 1, 2, \ldots, n,$$

where (A_{ij}) is the Cartan matrix of a Kats—Moody algebra. To the simplest Kats—Moody algebra $\mathfrak{sl}(2, \mathbf{C}[\lambda, \lambda^{-1}])$ there corresponds the system

$$\frac{\partial^2 u_1}{\partial x \partial \tau} = \exp(2u_1 - 2u_2),$$
$$\frac{\partial^2 u_2}{\partial x \partial \tau} = \exp(2u_2 - 2u_1),$$

which is essentially equivalent to the sine-Gordon equation. It turns out that for each Kats—Moody algebra there exist systems of evolution equations connected with a corresponding Toda lattice in exactly the same way as KdV and mKdV are connected with the sine-Gordon equation. For arbitrary Kats—Moody algebras these analogues of the KdV and mKdV equations are constructed in [12]. Some of the equations considered in this work were investigated earlier (see [49, 54, 13, 42, 41, 60] and also many works devoted to the scalar Lax equation), but their connection with Kats—Moody algebras was apparently not recognized.

Together with a detailed exposition of the results of the note [12], the present paper contains a survey of the theory of Kats—Moody algebras. Moreover, some general questions of algebraic character connected with the method of the inverse scattering problem are treated.

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Here for the equations investigated we study mainly local conservation laws, symmetries, and the Hamiltonian formalism. All assertions are presented with complete proofs. Section 5 devoted to Kats–Moody algebras is an exception. We proceed to a detailed statement of the contents of the work.

In Sec. 1 for the example of the well-studied [32, 15, 16] equation of N-waves the methods applied subsequently (see Secs. 3, 4, and 6) in more complex situations are demonstrated. Proposition 1.2 seems to us methodologically important; it may be considered an algebraic version of the dressing method [19, 20]. At the end of Sec. 1 we discuss the connection of the approach based on Proposition 1.2 with the traditional approach based on considering formal eigenfunctions.

In Sec. 2 in the language of the theory of fractional powers (see [5, 34, 67]) we present the well-known [7, 26, 27, 44, 70] facts concerning the scalar Lax equation, i.e., the equation $dL/\partial t = AL - LA$, where L and A are scalar differential operators. This equation is of interest to us, since, as will be apparent later, it is an equation of KdV type connected with the Kats–Moody algebra $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$, where k is the order of L. We note that for one of the basic results of Sec. 2 – the proposition on the equivalence of two methods of constructing local conservation laws for the Lax equation – we were unable to find a simple proof. Apparently, we are not alone (see [51, 71]).

In Sec. 3 we essentially explain the connection between the scalar Lax equation and the Kats–Moody algebra $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$. Of course, it is not difficult to find for the scalar Lax equation a representation of the form

$$\frac{d\mathfrak{L}}{dt} = \mathscr{A}\mathfrak{L} - \mathfrak{L}\mathscr{A}, \tag{0.2}$$

where

$$\mathfrak{L} = \frac{d}{dx} + \Lambda + q(x), \tag{0.3}$$

$$\Lambda \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & \ldots & 0 & \lambda \\ 1 & 0 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \quad q(x) = \begin{pmatrix} 0 & 0 & \ldots & u_1(x) \\ 0 & 0 & \ldots & u_2(x) \\ & & \ldots & \\ 0 & 0 & \ldots & u_k(x) \end{pmatrix}. \tag{0.4}$$

However, an operator $\mathfrak{L}$ of this form at first glance has no natural analogue in the case where $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$ is replaced by an arbitrary Kats–Moody algebra. This difficulty is overcome in the following manner which again emphasizes the importance of the concept of gauge equivalence (see [17, 18]) in the theory of integrable nonlinear equations. We assume that in formula (0.3) q(x) is an upper triangular matrix of general form. From the condition of self-consistency of Eq. (0.2) $\mathscr{A}$ is then determined only up to the addition of an upper triangular matrix with zeros on the diagonal. The indeterminacy of the system of equations arising in this manner is compensated by its invariance relative to gauge transformations of the form $\tilde{\mathfrak{L}} = N\mathfrak{L}N^{-1}$, where N is a function with values in the group of upper triangular matrices with ones on the diagonal.[1] Condition (0.4) on q(x) is only one of the possible gauges. This manner of viewing Eq. (0.2) makes it possible to construct analogues of the KdV equation for any Kats–Moody algebra.

The approach to scalar Lax equations described above, aside from the possibility of generalization, possesses the merit that it makes it possible to obtain a natural group-theoretic interpretation of the so-called second Hamiltonian structure of Gel'fand–Dikii [7]. For this it suffices to combine Theorem 3.22, which is one of the main results of [12], with the construction of the work [39]. For details see part 6.5 of the present survey.

In Sec. 3 we also define, following [42, 60], analogues of mKdV and the Miura transformation for the algebra $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$.

Section 4 is devoted to a generalization of the results of Sec. 1 to the case where the operator

$$L = \frac{d}{dx} + \lambda a + q(x), \quad a, \ q(x) \in \mathrm{Mat}(k, \mathbf{C}), \tag{0.5}$$

contained in Sec. 1 is replaced by an operator of the form (0.5), where a and q(x) belong to an arbitrary Lie algebra.

[1] A. V. Mikhailov brought to our attention the importance of gauge transformations of this type.

Section 5 is an elementary introduction to the theory of semisimple Lie algebras and Kats–Moody algebras. Its purpose is purely utilitarian — communication of facts used in Secs. 6-10. The choice of material and character of exposition in Sec. 5 is therefore somewhat unusual. A detailed exposition of the theory of Kats–Moody algebras is contained in [21, 50, 62].

In Sec. 6, finally, we define analogues of the KdV and mKdV equations corresponding to an arbitrary Kats–Moody algebra. It turns out that to each algebra G there corresponds a series of equations of mKdV type and several series of equations of KdV type (roughly speaking, these series correspond to the vertices of the Dynkin scheme for G).

We mention that in Sec. 3 the scalar Lax equation is interpreted as the equation of KdV type corresponding to $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}]$. In Sec. 7 for classical Kats–Moody algebras distinct from $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$ we solve the converse problem in a certain sense: scalar (L, A)-pairs are found for equations of KdV type corresponding to such algebras. The answer is very curious (see part 7.1).

In Sec. 8 we consider questions connected with the Hamiltonian formalism for equations of KdV type.

Section 9 is devoted to examples of equations of KdV and mKdV types.

In Sec. 10 we present some facts concerning two-dimensional Toda lattices including their connection with equations of KdV type and an assertion to the effect that the orders found in [65] of the conservation laws for Toda lattices are the exponents of the corresponding Kats–Moody algebra. The answer to the question of the orders of the conservation laws (see [65]) has been obtained in a very simple manner thanks in final analysis to Proposition 1.2.

In conclusion we wish to thank Yu. I. Manin who brought a preprint of the work [60] to our attention and thus stimulated the writing of the note [12]. Moreover, we wish to thank Wilson who acquainted us with the contents of the papers [51, 59, 71, 72] before their publication, B. A. Magadeev who took part in writing Sec. 9, and also O. B. Sokolov for helping with the manuscript.

List of Basic Notation

If V is a vector space over $\mathbf{C}$, then

$$V(\lambda) \overset{\text{def}}{=} \left\{ \sum_{i=0}^{m} v_i \lambda^i \mid v_i \in V,\ m \geqslant 0 \right\},$$

$$V[\lambda, \lambda^{-1}] \overset{\text{def}}{=} \left\{ \sum_{i=n}^{m} v_i \lambda^i \mid v_i \in V,\ n,\ m \in \mathbf{Z} \right\},$$

$$V[[\lambda^{-1}]] \overset{\text{def}}{=} \left\{ \sum_{i=-\infty}^{0} v_i \lambda^i \mid v_i \in V \right\},\quad V((\lambda^{-1})) \overset{\text{def}}{=} \left\{ \sum_{i=-\infty}^{m} v_i \lambda^i \mid v_i \in V,\ m \in \mathbf{Z} \right\}.$$

For any $P = \sum_{i=-\infty}^{m} p_i \lambda^i \in V((\lambda^{-1}))$ we set $P_+ \overset{\text{def}}{=} \sum_{i \geqslant 0} p_i \lambda^i,\ P_- \overset{\text{def}}{=} \sum_{i<0} p_i \lambda^i,\ \operatorname{res} P \overset{\text{def}}{=} p_{-1}$.

The set of smooth mappings from M to N is denoted by $C^\infty(M, N)$. For brevity we use the notation $B \overset{\text{def}}{=} C^\infty(\mathbf{R}, \mathbf{C})$, $B_0 \overset{\text{def}}{=} C^\infty(\mathbf{R}/\mathbf{Z},\ \mathbf{C})$. All functions not specified to belong to a particular class are assumed smooth.

We often denote the operator d/dx by D. We set

$$B[D] \overset{\text{def}}{=} \left\{ \sum_{i=0}^{n} a_i D^i \mid a_i \in B,\ n \geqslant 0 \right\},\quad B((D^{-1})) \overset{\text{def}}{=} \left\{ \sum_{i=-\infty}^{n} a_i D^i \mid a_i \in B,\ n \in \mathbf{Z} \right\}.$$

The notation $B_0[D]$, $B_0((D^{-1}))$ has an analogous meaning. Elements of $B((D^{-1}))$ are called pseudodifferential symbols, while elements of B[D] are differential operators. B[D] and $B((D^{-1}))$ are algebras over $\mathbf{C}$. Multiplication in $B((D^{-1}))$ is defined by the formula

$$D^n a - a D^n = \sum_{i=1}^{\infty} n(n-1) \ldots (n-i+1) \frac{d^i a}{dx^i} D^{n-i},$$

where $n\in Z$, $a\in B$. If $L=\sum_{i=-\infty}^{n} a_i D^i \in B((D^{-1}))$, then $L_+ \stackrel{\text{def}}{=} \sum_{i\geq 0} a_i D^i$, $L_- \stackrel{\text{def}}{=} \sum_{i<0} a_i D^i$, $\operatorname{res} L \stackrel{\text{def}}{=} a_{-1}$. The symbol *

denotes the operation of forming the formal adjoint in $B((D^{-1})):\left(\sum a_i D^i\right)^* \stackrel{\text{def}}{=} \sum (-1)^i D^i a_i$ (we emphasize that * acts identically on B and not as complex conjugation).

Mat (k, F) denotes the set of square matrices of dimension k with coefficients in F, E denotes the identity matrix, and e_{ij} denotes the matrix having a one at the intersection of the i-th row and j-th column and zeros elsewhere. If $A=(a_{ij})\in \mathrm{Mat}(k, F)$, then $A^t \stackrel{\text{def}}{=} (a_{ji})$. Suppose that F is equipped with an antiautomorphism * such that $(a^*)^* = a$ for any $a\in F$, then $A^T \stackrel{\text{def}}{=} (a^*_{k+1-j,\, k+1-i})$. In particular, if the ring F is commutative and * is the identity automorphism, then A^T is obtained from A by transposition relative to the secondary diagonal. The trace of a matrix A is denoted by tr A (while Tr denotes the Adler functional; see part 2.3). By definition $\operatorname{diag}(a_1, a_2 \ldots, a_k)$ is the matrix $\begin{pmatrix} a_1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_k \end{pmatrix}$. For any matrix $A = (a_{ij})$, $A_{\mathrm{diag}} \stackrel{\text{def}}{=} \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{kk})$. Diag denotes the set of all diagonal matrices in Mat (k, **C**). The set Mat (k, **C**) considered as a Lie algebra is written $\mathfrak{gl}(k)$.

1. THE METHOD OF ZAKHAROV–SHABAT (THE ALGEBRAIC ASPECT)

The title of the section is somewhat tentative: we treat only some questions, and not the deeper ones, connected with the method of Zakharov–Shabat (see [19, 20]).

1.1. We consider the relation

$$\frac{dL}{dt}=[A,\ L], \tag{1.1}$$

where $L=\frac{d}{dx}+q-\lambda a$, $A=\sum_{i=0}^{m} A_i\lambda^i$; q and A_i are functions of x, t with values in Mat (k, **C**), a is a constant matrix of order k whose eigenvalues are distinct,[2] and λ is a spectral parameter.

In order that Eq. (1.1) be satisfied identically in λ it is necessary that the commutator [A, L] not depend on λ. The following problem arises in this connection: for a given L find the set Ω_L of all matrix polynomials A such that [A, L] does not depend on λ. In solving this problem t plays the role of a parameter; we can therefore temporarily forget that L and A depend on t. Moreover, we may assume with no loss of generality that the matrix a is diagonal. We set $Z_L \stackrel{\text{def}}{=} \{M\in \mathrm{Mat}(k,\ B((\lambda^{-1})))\mid [L,\ M]=0\}$. The first step in solving our problem is the following observation.

<u>LEMMA 1.1.</u> If $M\in Z_L$, then $M_+\in\Omega_L$. 2) Let $M\in Z_L$. Then $[M_+, L] = [\operatorname{res} M, a]$.

<u>Proof.</u> We note that $[M_+, L] = -[M_-, L]$. It is clear that the left side of this equality is a polynomial in λ, while the right side has the form $[\operatorname{res} M,\ a]+\sum_{i=1}^{\infty} p_i\lambda^{-i}$. ∎

We shall now describe Z_L. The next result plays a key role in this.

<u>Proposition 1.2.</u> There exists a formal series T of the form $E+\sum_{i=1}^{\infty} T_i(x)\lambda^{-i}$ such that $L_0 \stackrel{\text{def}}{=} TLT^{-1}$ has the form $\frac{d}{dx}-\lambda a+\sum_{i=0}^{\infty} h_i(x)\lambda^{-i}$, where the matrices h_i are diagonal. T is uniquely determined up to multiplication on the left by an arbitrary series with diagonal coefficients and can be chosen so that the matrices T_i are differential polynomials in q with zero free terms.[3]

[2]The method used in the present section is applicable also in the case where a is an arbitrary matrix for which not all eigenvalues coincide.
[3]The words "the matrix T is a differential polynomial in the matrix q" here and henceforth means that each element of T is a differential polynomial in the elements of q.

Proof. Equating coefficients of λ^{-n} in the equality $TL = L_0T$, we obtain the recurrence relation

$$h_n+[T_{n+1}, a]=T_nq-T_n'-\sum_{i=0}^{n-1} h_iT_{n-i}. \tag{1.2}$$

Since a has distinct eigenvalues, any matrix can be represented (nonuniquely) in the form $X + [Y, a]$, where X is diagonal. Therefore, relation (1.2) makes it possible to find the coefficients h_n and T_{n+1}, knowing the coefficient with the preceding indices. It is hereby possible to require that the diagonal elements of the matrices T_i be equal to zero; the remaining elements of them are then found to be uniquely determined differential polynomials in q with zero free terms.

Suppose that the coefficients of the series $\tilde{L}_0 = \tilde{T}L\tilde{T}^{-1}$ are also diagonal. We set $S = \tilde{T}T^{-1}$. Then $\tilde{L}_0S = SL_0$, whence it follows easily that the coefficients of the series S are diagonal. ■

To find Z_L we need the following result.

LEMMA 1.3. Let $M=\sum_{i=-\infty}^{n} b_i\lambda^i$, $N=\frac{d}{dx}-\lambda a+\sum_{i=-\infty}^{0} s_i\lambda^i$, $\deg[M, N]<n$. Then b_n is a diagonal constant matrix.

Proof. We have $[M, N]=[a, b_n]\lambda^{n+1}+([b_n, s_0]+[a, b_{n-1}]-b_n')\lambda^n+\ldots$. By equating to zero the coefficient of λ^{n+1} we see that b_n is diagonal. Equating now the coefficient of λ^n to zero, we obtain $b_n' = 0$. ■

Let T and L_0 be the same as in Proposition 1.2. It follows from Lemma 1.3 that $Z_{L_0} = \mathrm{Diag}((\lambda^{-1}))$. Therefore, $Z_L = T^{-1}\,\mathrm{Diag}((\lambda^{-1}))T$.

For any $u\in\mathrm{Diag}((^{-1}))$ we set $\varphi(u)=T^{-1}uT$. It follows from Proposition 1.2 that φ is well defined and that the coefficients of the formal series $\varphi(u)$ are differential polynomials in q. According to Lemma 1.1 $\varphi(u)_+\in\Omega_L$ for any $u\in\mathrm{Diag}((\lambda^{-1}))$ [actually $\varphi(u)_+$ depends only on the coefficients of negative powers of λ in the series for u]. Moreover, it is obvious that any function in $C^\infty(\mathbf{R}, \mathrm{Diag})$ belongs to Ω_L. It follows from Lemma 1.3 that the vector space Ω_L is generated by functions and elements of the form $\varphi(b\lambda^n)_+$, where $b\in\mathrm{Diag}$, $n\in\mathbf{N}$. Below we shall consider the relation (1.1), where

$$A=\sum_{i=0}^{m}\varphi(b_i\lambda^i)_+,\quad b_i\in\mathrm{Diag} \tag{1.3}$$

as an equation for q and call it for brevity Eq. (1.1).

Example. The Equation of N-Waves. Let $L=\frac{d}{dx}+q-\lambda a$, $a=\mathrm{diag}(a_1,\ldots,a_k)$, $A=\varphi(b\lambda)_+$, $b=\mathrm{diag}(b_1,\ldots,b_k)$. The corresponding equation has the form $q_t = p' + [q, p]$, where $p_{ij}=\frac{b_i-b_j}{a_i-a_j}q_{ij}$. We note that for $b = a$ it becomes the equation $q_t = q'$.

We have shown that Eq. (1.1) is an evolution equation whose right side is a differential polynomial. We shall find the order of this polynomial.

Proposition 1.4. If we set $A=\varphi(b\lambda^n)_+$, where $b\in\mathrm{Diag}$, then Eq. (1.1) has the form $\partial q/\partial t = Pq^{(n)} + f(q, q',\ldots,q^{(n-1)})$, where P is a linear operator in the space of matrices which annihilates diagonal matrices and is such that its restriction to the space of matrices with zeros on the main diagonal is equal to $\mathrm{ad}\,b(\mathrm{ad}\,a)^{-n}$ (we recall that $\mathrm{ad}\,a(y)\overset{\mathrm{def}}{=}[a, y]$). If we assume that $q^{(i)}$ has degree of homogeneity i + 1, then f is a homogeneous polynomial of degree of homogeneity n + 1.

Proof. According to Lemma 1.1, Eq. (1.1) has the form $\partial q/\partial t = [\mathrm{res}\,M, a]$, where $M = T^{-1}b\lambda^nT$, $T=E+\sum_{i=1}^{\infty}T_i\lambda^{-i}$ is the formal series of Proposition 1.2. We recall that T is uniquely determined if we require that the diagonal elements of the matrices T_i be equal to zero. It is evident from formula (1.2) that with this choice of T, T_i is a homogeneous polynomial in q of degree of homogeneity i. Therefore, res M has degree of homogeneity n + 1.

From this it follows that $[\mathrm{res}\,M, a]=P\overset{(n)}{q}+f(q, q',\ldots,q^{(n-1)})$, where P is a linear operator, and f does not contain linear terms.

We shall find P. For this we set $L(\varepsilon) = d/dx + \varepsilon q - \lambda a$ and differentiate with respect to ε the relation $T(\varepsilon)L(\varepsilon) = L_0(\varepsilon)T(\varepsilon)$; we then set ε = 0. We obtain $S'-\lambda[a, S]=q-\frac{\partial L_0(\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$, where $S=\sum_{i=1}^{\infty}S_i\lambda^{-i}\overset{\text{def}}{=}\frac{\partial T(\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$. Recalling that the S_i have zero diagonal elements while the coefficients of the series $L_0(\varepsilon)$ have zero off-diagonal elements, we obtain

$$S_i'-[a, S_{i+1}]=0 \text{ for } i\geqslant 1,\quad [a, S_1]=q_{\text{diag}}-q. \tag{1.4}$$

From the relation $M(\varepsilon) = T^{-1}(\varepsilon)b\lambda^n T(\varepsilon)$ it follows that $\frac{\partial M(\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}=[b\lambda^n, S]$. Therefore, the linear part of the expression $[\operatorname{res} M, a]$ is equal to $-[a, [b, S_{n+1}]] = -[b, [a, S_{n+1}]]$. From this and formula (1.4) it follows that $P = \operatorname{ad} b(\operatorname{ad} a)^{-n}$. ■

Remark. From part 2) of Lemma 1.1 it follows that if L satisfies Eq. (1.1), then $(\partial/\partial t)\times q_{\text{diag}} = 0$. Therefore, if desired we can set $q_{\text{diag}} = 0$ without violating the self-consistency of the equation.

1.2. We shall show that Eq. (1.1) possesses an infinite series of polynomial conservation laws.

Proposition 1.5. Let h_i be the same as in Proposition 1.2. Then the elements of the matrices h_i are densities of conservation laws for Eq. (1.1). Here $h_0 = q_{\text{diag}}$; if i > 0, $h_i = \operatorname{diag}(h_i^1, h_i^2,\ldots,h_i^k)$, then h_i^r is a differential polynomial in q whose linear part is a total derivative and whose quadratic part, up to total derivatives, is equal to $-\sum_{s\neq r}\frac{q_{rs}^{(i-1)}q_{sr}}{(a_r-a_s)^i}$, where $a = \operatorname{diag}(a_1,\ldots,a_k)$.

Clarification. It is easy to see that in spite of the nonuniqueness in the choice of T in Proposition 1.2 the h_i are uniquely determined up to total derivatives, and in giving the density of a conservation law such arbitrariness is admissible.

Proof of Proposition 1.5. Equation (1.1) can be written in the form

$$\left[\frac{d}{dt}-A, L\right]=0. \tag{1.5}$$

Let T and L_0 be the same as in Proposition 1.2. Then $\left[T\left(\frac{d}{dt}-A\right)T^{-1}, L_0\right]=0$, i.e.,

$$\left[\frac{d}{dt}-\tilde{A}, L_0\right]=0,\quad \tilde{A}\overset{\text{def}}{=}TAT^{-1}+\frac{dT}{dt}T^{-1}. \tag{1.6}$$

We recall that $L_0=\frac{d}{dx}+H$, where $H=-\lambda a+\sum_{i=0}^{\infty}h_i\lambda^{-i}$ is a diagonal matrix. From (1.6) it is therefore easy to deduce the diagonality of $\tilde{A}$. From this, in turn, it follows that relation (1.6) can be rewritten in the form

$$\frac{\partial H}{\partial t}+\frac{\partial\tilde{A}}{\partial x}=0. \tag{1.7}$$

This implies that h_i are the densities of conservation laws.

We normalize T so that $(T_i)_{\text{diag}} = 0$. From formula (1.2) it then follows that $h_0 = q_{\text{diag}}$, $h_i = (T_i q)_{\text{diag}}$ for i > 0. Therefore, the linear part of h_i is equal to zero, while to find the quadratic part it suffices to know the linear part of T_i which was denoted by S_i in the proof of Proposition 1.4. From (1.4) we obtain the desired formula for the quadratic part. ■

Remark. For any i > 0 the expression $\sum_{r=1}^{k}h_i^r$ is a total derivative. Indeed, from relation $T\left(\frac{d}{dx}+q-\lambda a\right)T^{-1}=\frac{d}{dx}+H$ it follows that $\operatorname{tr}(H+\lambda a-q)=-\operatorname{tr}\left(\frac{dT}{dx}T^{-1}\right)=-\frac{d}{dx}\ln\det T$. From the formula for the quadratic part of h_i^r it is evident that the densities h_i^r, where i > 0, r ≠ 1, are linearly independent modulo total derivatives.

1.3. The purpose of this subsection is to prove that the flows defined by relation (1.1) for different A commute.

LEMMA 1.6. Let $M = T^{-1}uT$, where $u \in \mathrm{Diag}((\lambda^{-1}))$, and T is the same as in Proposition 1.2. If L satisfies Eq. (1.1), then $dM/dt = [A, M]$.

Proof. It is given that $[d/dt - A, L] = 0$. We must show that $[d/dt - A, M] = 0$. After conjugating these equalities with T the first of them becomes (1.6), while the second takes the form

$$\left[\frac{d}{dt} - \tilde{A}, u\right] = 0. \tag{1.8}$$

As noted in the proof of Proposition 1.5, it follows from (1.6) that $\tilde{A}$ is diagonal, and this, in turn, gives Eq. (1.8). ■

We consider the equation

$$\frac{\partial L}{\partial t} = [M_+, L], \quad M = T^{-1}uT, \quad u \in \mathrm{Diag}[\lambda], \tag{1.9}$$

$$\frac{\partial L}{\partial \tau} = [\tilde{M}_+, L], \quad \tilde{M} = T^{-1}\tilde{u}T, \quad \tilde{u} \in \mathrm{Diag}[\lambda]. \tag{1.10}$$

Proposition 1.7. $\frac{\partial^2 L}{\partial t \partial \tau} = \frac{\partial^2 L}{\partial \tau \partial t}$, where the derivatives are computed by means of Eqs. (1.9), (1.10).

Proof. We have $\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial t}\right) = \frac{\partial}{\partial \tau}[M_+, L] = \left[\frac{\partial M_+}{\partial \tau}, L\right] + \left[M_+, \frac{\partial L}{\partial \tau}\right]$. According to Lemma 1.6, $\frac{\partial M_+}{\partial \tau} = [\tilde{M}_+, M]_+$. Thus, $\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial t}\right) = [[\tilde{M}_+, M]_+, L] + [M_+, [\tilde{M}_+, L]]$. Similarly, $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \tau}\right) = [[M_+, \tilde{M}]_+, L] + [\tilde{M}_+, [M_+, L]]$. Using the Jacobi identity, we obtain $\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \tau}\right) = [[\tilde{M}_+, M]_+ - [M_+, \tilde{M}]_+ + [M_+, \tilde{M}_+], L]$, but $[\tilde{M}_+, M]_+ = [M, \tilde{M}_-]_+ = [M_+, \tilde{M}_-]_+$, so that $[\tilde{M}_+, M]_+ - [M_+, \tilde{M}]_+ + [M_+, \tilde{M}_+] = 0$. ■

1.4. It is well known that Eqs. (1.1) are Hamiltonian. More precisely, there exist at least two Hamiltonian structures such that the conservation laws obtained in Proposition 1.5 are Hamiltonians for Eqs. (1.1). In this subsection we recall the explicit form of these structures.

A Hamiltonian structure on a finite-dimensional smooth manifold M is determined by giving the Poisson brackets on the set F of smooth functions on M which converts F into a Lie algebra and possesses the following property: if $f, g \in F$ and the differential of f at a point $\eta \in M$ is equal to zero, then $\{f, g\}(\eta) = 0$. In our situation a class of functions of x with values in Mat $(k, \mathbf{C})$ plays the role of M, while the role of F is a class of functionals on M. For M we take the set of all smooth functions $q: \mathbf{R}/\mathbf{Z} \to \mathrm{Mat}(k, \mathbf{C})$ while for F we take the set of all functionals $l: M \to \mathbf{C}$ of the form

$$l(q) = \int_{x \in \mathbf{R}/\mathbf{Z}} f(x, q(x), q'(x), \ldots, q^{(n)}(x))\, dx, \tag{1.11}$$

where f is a polynomial in $q, q', \ldots, q^{(n)}$ whose coefficients are smooth functions of x. The following considerations possibly clarify our rather clumsy choice of M and F. The reason for choosing as M a class of periodic functions (and not, say, rapidly decreasing functions) is as follows. It is natural to require that the class F contain all functionals l of the form

$$l(q) = \int f(q(x), q'(x), \ldots, q^{(n)}(x))\, dx, \tag{1.12}$$

where f is a polynomial. However, in the case where M consists of rapidly decreasing functions this requirement is inadmissible, since the integral in formula (1.12) becomes meaningless if the free term of f is not equal to zero. If we assume that F contains only functionals of the form (1.12) corresponding to polynomials f with zero free terms, then F turns out not to be closed with respect to some reasonable Poisson brackets. Having chosen for M a class of periodic functions, it would be natural to include in F all analytic functionals on M. However, on this path difficulties arise connected with the fact that the differential of an analytic functional on M can be any generalized function of x (not necessarily smooth). If in the periodic case we include in F only functionals of the form (1.12), then functionals in F will not separate points of M. Recognizing the shortcomings of our approach to the Hamiltonian formalism as compared, say, with [8-11], we have chosen it in striving to emphasize the analogy with the finite-dimensional case.

For any functions u, $v \in M$ we set

$$(u, v)=\int \operatorname{tr}(u(x) v(x))\, dx.$$

If $l \in F$, $q \in M$, then we define the function $\operatorname{grad}_q l \in M$ from the condition $\frac{d}{d\varepsilon} l(q+\varepsilon h)|_{\varepsilon=0}=(\operatorname{grad}_q l, h)$ which should be satisfied for any $h \in M$. It is easy to see that grad l is nothing other than the variational derivative of the density f of formula (1.11).

Following the works [32, 38] we define the first and second Hamiltonian structures on M by the formulas

$$\{\varphi, \psi\}_1(q)=(\operatorname{grad}_q \varphi, [\operatorname{grad}_q \psi, a]), \tag{1.13}$$

$$\{\varphi, \psi\}_2(q)=\left(\operatorname{grad}_q \varphi, \left[\operatorname{grad}_q \psi, \frac{d}{dx}+q\right]\right), \tag{1.14}$$

where $\varphi, \psi \in F$, $q \in M$. In order to see that these formulas actually define Poisson brackets, it is necessary to verify that a) $\{\varphi, \psi\}_1 \in F$ and $\{\varphi, \psi\}_2 \in F$ for any $\varphi, \psi \in F$; b) the brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are skew symmetric and satisfy the Jacobi identity. Assertion a) follows from the explicit formula for the variational derivative. In place of b) we prove the following stronger assertion.

Proposition 1.8. For any $\lambda, \mu \in \mathbf{C}$ the formula $\{\varphi, \psi\}=\lambda_1\{\varphi, \psi_1\}_1+\mu\{\varphi, \psi\}_2$ gives a Poisson bracket.

Proof. We have $\{\varphi, \psi\}(q)=(\operatorname{grad}_q \varphi, [\operatorname{grad}_q \psi, \mathfrak{L}])$, where $\mathfrak{L} \overset{\text{def}}{=} \lambda a+\mu\left(\frac{d}{dx}+q\right)$. It is easy to verify the following properties of the scalar product on M:

$$(u, [v, w])=-(v, [u, w]), \quad (u, v')=-(v, u'). \tag{1.15}$$

From these formulas it follows that $(u, [v, \mathfrak{L}])=-(v, [u, \mathfrak{L}])$ for any $u, v \in M$ and such that $\{\varphi, \psi\}$ is skew-symmetric.

Standard considerations show that it suffices to verify the Jacobi identity for functionals $\varphi_1, \varphi_2, \varphi_3$ of the form $\varphi_i(q)=(u_i, q)$, $u_i \in M$. Since $\operatorname{grad} \varphi_i=u_i$, it follows that $\{\varphi_1, \varphi_2\}(q)=(u_1, [u_2, \mathfrak{L}])=\left(u_1, \left[u_2, \lambda a+\mu \frac{d}{dx}\right]\right)+\mu(u_1, [u_2, q])=\left(u_1, \left[u_2, \lambda a+\mu \frac{d}{dx}\right]\right)+\mu([u_1, u_2], q)$ (we have used formula (1.15)]. Hence, $\operatorname{grad}_q\{\varphi_1, \varphi_2\}=\mu[u_1, u_2]$, so that $\{\{\varphi_1, \varphi_2\}, \varphi_3\}=\mu([u_1, u_2], [u_3, \mathfrak{L}])$. Again using (1.15), we obtain $\{\{\varphi_1, \varphi_2\}, \varphi_3\}=\mu(u_1, [u_2, [u_3, \mathfrak{L}]])$. In exactly the same way $\{\{\varphi_1, \varphi_3\}, \varphi_2\}=\mu(u_1, [u_3, [u_2, \mathfrak{L}]])$. Therefore, $\{\{\varphi_1, \varphi_2\}, \varphi_3\}-\{\{\varphi_1, \varphi_3\}, \varphi_2\}=\mu(u_1, [[u_2, u_3], \mathfrak{L}])=\{\varphi_1, \{\varphi_2, \varphi_3\}\}$ which is equivalent to the Jacobi identity. ■

Remark 1. Generally speaking, a linear combination of two Poisson brackets does not satisfy the Jacobi identity. If any linear combination of two Poisson brackets is again a Poisson bracket, then it is said that the original two brackets are coordinated. A general scheme is known which makes it possible on the basis of two coordinated Hamiltonian structures and some additional data to construct an infinite sequence of functionals commuting relative to both structures (see [8, 63]).

Remark 2. A beautiful group-theoretic interpretation of the second Hamiltonian structure on M is proposed in [39]. We shall recall it in part 4.4.

Let $u \in \operatorname{Diag}[\lambda]$, $u=\sum_{l=0}^{m} b_l \lambda^l$. We define a functional H_u: M → **C** by the formula $H_u(q)=\sum_{l=0}^{m}(b_l, h_l)$, where the h_i are the same as in Proposition 1.2. Because of Proposition 1.5, the functionals H_u are conservation laws for Eq. (1.1).

Proposition 1.9. Equation (1.1) in which A is defined by formula (1.3) is the Hamiltonian H_u and the second Hamiltonian structure. 2) This same equation is the Hamiltonian equation corresponding to the Hamiltonian $H_{\lambda u}$ and the first Hamiltonian structure.

Proof. We recall that q satisfies the Hamiltonian equation corresponding to a Hamiltonian H if $\frac{d\varphi(q)}{dt}=\{\varphi, H\}(q)$ for any $\varphi \in F$. Since $\frac{d\varphi(q)}{dt}=(\operatorname{grad}_q \varphi, \dot{q})$, the Hamiltonian equation corresponding to the Hamiltonian H and the first Hamiltonian structure has the form

$$\dot{q}=[\operatorname{grad}_q H, a]. \tag{1.16}$$

The analogous equation for the second Hamiltonian structure is given by the formula

$$\dot{q}=\left[\operatorname{grad}_q H, \frac{d}{dx}+q\right]. \tag{1.17}$$

LEMMA. $\operatorname{grad}_q H_u = A_0$, where A_0 is the free term of A.

Proof. Let $h \in M$. We shall find $\frac{d}{d\varepsilon} H_u(q+\varepsilon h)|_{\varepsilon=0}$. We set $L(\varepsilon)=\frac{d}{dx}+q+\varepsilon h-\lambda a$. Let T and L_0 be the same as in Proposition 1.2. We define $T(\varepsilon)$ and $L_0(\varepsilon)$ in the obvious way. It is easy to see that $H_u(q)$ is equal to the free term of $\left(u, L_0-\frac{d}{dx}\right)$; therefore, $\frac{d}{d\varepsilon} H_u(q+\varepsilon h)$ is equal to the free term of $\left(u, \frac{dL_0(\varepsilon)}{d\varepsilon}\right)$. Differentiating with respect to ε the relation $L_0(\varepsilon) = T(\varepsilon)L(\varepsilon)T^{-1}(\varepsilon)$, we obtain $\frac{dL_0(\varepsilon)}{d\varepsilon}=T(\varepsilon)hT^{-1}(\varepsilon)+\left[\frac{dT(\varepsilon)}{d\varepsilon}T^{-1}(\varepsilon), L_0(\varepsilon)\right]$, whence $\left(u, \frac{dL_0(\varepsilon)}{d\varepsilon}\right)=(u, T(\varepsilon)hT^{-1}$ × $(\varepsilon))+\left(u, \left[\frac{dT(\varepsilon)}{d\varepsilon}T^{-1}(\varepsilon), L_0(\varepsilon)\right]\right)=(u, T(\varepsilon)hT^{-1}(\varepsilon))=(T^{-1}(\varepsilon)uT(\varepsilon), h)$. We have used formulas (1.15) and the fact that due to the diagonality of $L_0(\varepsilon)$, $[u, L_0(\varepsilon)] = 0$. Thus, $\frac{d}{d\varepsilon} H_u(q+\varepsilon h)|_{\varepsilon=0}=$ the free term of $(T^{-1}uT, h) = (A_0, h)$ [we recall that $A = (T^{-1}uT)_+$]. ■

Formula (1.17) and the lemma show that for the proof of the first part of Proposition 1.9 it suffices to verify the equality $[A, L] = [A_0, d/dx + q]$; now this is obvious, since $[A, L]$ does not depend on λ, and hence in computing this commutator it is possible to set $\lambda = 0$. It follows from the lemma that $\operatorname{grad}_q H_{\lambda u} = \operatorname{res}(T^{-1}uT)$. Therefore, the second part of the proposition follows from Lemma 1.1 and formula (1.16). ■

Since the functionals H_u are conservation laws for Eqs. (1.1), the next result follows from Proposition 1.9.

COROLLARY. For any $u, \tilde{u} \in \operatorname{Diag}[\lambda]$, $\{H_u, H_{\tilde{u}}\}_1=\{H_u, H_{\tilde{u}}\}_2=0$. ■

If the Hamiltonians commute, then the corresponding flows also commute, so that we have proved Proposition 1.7 anew.

1.5. In this subsection we clarify the connection between the method of obtaining conservation laws for Eq. (1.1) described in Proposition 1.5 and the well-known formulas expressing conservation laws in terms of a formal scattering matrix.

We assume that q is a smooth, compactly supported function $\mathbf{R} \to \operatorname{Mat}(k, \mathbf{C})$. It is well-known (see [24], Chap. 6) that the equation $L\psi = 0$ has exactly one formal matrix solution of the form $\psi(x, \lambda)=f(x, \lambda)\exp(\lambda a x)$, where $f(x, \lambda)=\sum_{i=0}^{\infty} f_i(x)\lambda^{-i}$, such that $f(x, \lambda) = E$ for $x \ll 0$. It is clear that $f(x, \lambda)$ does not depend on x for $x \gg 0$. We set $S(\lambda) = f(x, \lambda)$ for $x \gg 0$. We call $S(\lambda)$ a formal scattering matrix. The connection between the formal S-matrix and the conservation laws of Proposition 1.5 is as follows.

Proposition 1.10.

$$S(\lambda)=\exp\left(-\sum_{i=0}^{\infty}\lambda^{-i}\int_{-\infty}^{\infty}h_i(x)dx\right),$$

where h_i are the same as in Proposition 1.2.

Proof. Let T and L_0 be the same as in Proposition 1.2, whereby the coefficients of the formal series T are chosen in the form of differential polynomials with zero free terms. Then $T = E$ for $|x| \gg 0$. Since $L_0 = TLT^{-1}$, the equation $L\psi = 0$ is equivalent to the equation $L_0T\psi = 0$. Therefore, the solution of the equation $L\psi = 0$ of interest to us is equal to $T^{-1}\varphi$, where φ is a matrix solution of the equation $L_0\varphi=0$ such that $\varphi(x, \lambda)=e^{\lambda a x}$ for $x \ll 0$. Since $L_0=\frac{d}{dx}-\lambda a+\sum_{i=0}^{\infty}h_i\lambda^{-i}$, it follows that $\varphi(x, \lambda)=\exp\left(\lambda a x-\sum_{i=0}^{\infty}\lambda^{-i}\int_{-\infty}^{x}h_i(y)dy\right)$, so that for sufficiently large x we have $\psi(x, \lambda)=\exp\left(\lambda a x-\sum_{i=0}^{\infty}\lambda^{-i}\int_{-\infty}^{\infty}h_i(y)dy\right)$. ■

2.1. We consider the relation

$$\frac{dL}{dt}=[A, L], \tag{2.1}$$

where $L=D^k+\sum_{i=0}^{k-1}u_iD^i$, $D\overset{\text{def}}{=}\frac{d}{dx}$, $A=\sum_{i=0}^{m}v_iD^i$; here u_i, v_i are functions of x, t with values in **C**. The left side of Eq. (2.1) is a differential operator of order no greater than k − 1, while the right side, generally speaking has order m + k − 1. Therefore, the coefficients of the operator A are connected with the coefficients of the operator L by means of m relations. The theory of fractional powers (see [5, 34, 67]) shows that the operator A is determined by these relations on the basis of the operator L up to m constants and one arbitrary function.[4] Here t plays the role of a parameter; we therefore temporarily forget that the coefficients of our operators depend on t.

We are interested in the structure of the set Ω_L of differential operators $A\in B[D]$ such that ord [A, L] ≤ k − 1. We set $Z_L=\{M\in B((D^{-1}))\,|\,[M, L]=0\}$.

LEMMA 2.1. 1) If $M\in Z_L$, then $M_+\in\Omega_L$, and, moreover, $\operatorname{ord}[M_+, L]\leqslant k-2$; 2) $B\subset\Omega_L$.

Proof. If $M\in Z_L$, then $[M_+, L] = -[M_-, L]$, whence $\operatorname{ord}[M_+, L]\leqslant \operatorname{ord}M_-+\operatorname{ord}L-1\leqslant k-2$. The second part of the lemma is obvious.

We shall now find Z_L. We hereby consider a more general case than we now require: we shall assume that L is a pseudodifferential symbol with leading term D^k. It is easy to see that there exists exactly one pseudodifferential symbol M of the form $D+\sum_{i=0}^{\infty}a_iD^{-i}$ such that $M^k = L$; the coefficients of M are here differential polynomials in the coefficients of L. The symbol M is written $L^{1/k}$. For any $r\in\mathbf{Z}$ we set $L^{r/k} = (L^{1/k})^r$.

Proposition 2.2. Z_L is the set of all series of the form

$$\sum_{i=-\infty}^{m}\gamma_iL^{i/k},\quad \gamma_i\in\mathbf{C}. \tag{2.2}$$

Proof. It is clear that series of the form (2.2) belong to Z_L. We shall show that any element $P\in Z_L$ has the form (2.2). Let $P=\sum_{i=-\infty}^{m}p_iD^i$. Equating to zero the coefficient of D^{m+k+1} in the symbol [P, L], we find that $p_m\in\mathbf{C}$. Therefore, $p_mL^{m/k}\in Z_L$, and hence the previous argument is applicable to the symbol $P - p_mL^{m/k}$. Repeating this process, we obtain the representation of P in the form (2.2). ■

Proposition 2.3. As a vector space over **C** Ω_L is generated by B and operators of the form $(L^{r/k})_+$, $r\in\mathbf{N}$.

Proof. According to Lemma 2.1, $(L^{r/k})_+\in\Omega_L$, $B\subset\Omega_L$. We shall show that any operator $P\in\Omega_L$ has the form $\sum_{i=1}^{m}\gamma_i(L^{i/k})_++f$, where $f\in B$, $\gamma_i\in\mathbf{C}$. The proof is carried out by induction on the order of P. Let $P=\sum_{i=0}^{m}p_iD^i$, $m>0$. Equating to zero the coefficient of D^{m+k-1} in the operator [P, L], we find $p_m\in\mathbf{C}$. Therefore, $p_m(L^{m/k})_+\in\Omega_L$, and hence the induction hypothesis can be applied to the operator $P - p_m(L^{m/k})_+$. ■

Below we shall consider relation (2.1) where

$$A=\sum_{i=0}^{m}c_i(L^{i/k})_+,\quad c_i\in\mathbf{C} \tag{2.3}$$

as a system of equations for the coefficients of the differential operator L and call it the Lax equation. It is clear that the right sides of this system are differential polynomials.

[4] In [26, 43] formal eigenfunctions of the operator L are used in place of its fractional powers. In final analysis these approaches to the investigation of Eq. (2.1) are equivalent.

Remark 1. It may be assumed with no loss of generality that in formula (2.3) $c_i = 0$ if i is divisible by k. Indeed, if $i/k \in \mathbf{N}$, then $(L^{i/k})_+ = L^{i/k}$ commutes with L.

Remark 2. It follows from assertion 1) of Lemma 2.1 that the right side of the Lax equation (2.1) is a differential operator of order no higher than $k - 2$. Therefore, if $L = D^k + \sum_{i=0}^{k-1} u_i D^i$ satisfies the Lax equation, then $\frac{\partial}{\partial t} u_{k-1} = 0$. Hence, without violating the self-consistency of the equation, it may be assumed that $u_{k-1} = 0$, as is usually done.

Example 1. Let $L = D^2 + u$. If we set $A = (L^{3/2})_+ = D^3 + \frac{3}{2} uD + \frac{3}{4} u'$, then Eq. (2.1) is the Korteweg–de Vries equation (KdV) $u_t = \frac{1}{4}(u''' + 6uu')$. The equations corresponding to $A = (L^{n+1/2})_+$, where $n \in \mathbf{N}$, $n > 1$, are called higher KdV.

Example 2. If $L = D^3 + uD + v$, $A = (L^{2/3})_+$, then the system (2.1) has the form

$$\begin{cases} u_t = -u'' + 2v', \\ v_t = v'' - \frac{2}{3} u''' - \frac{2}{3} uu'. \end{cases}$$

Eliminating v from the system, it is easy to see that u satisfies the Boussinesque equation [14].

We shall now prove that the flows determined by the Lax equations commute with one another. For this we need the following result.

LEMMA 2.4. If $\frac{dL}{dt} = [A, L]$, then $\frac{d}{dt}(L^{r/k}) = [A, L^{r/k}]$.

Proof. We set $M = L^{r/k}$. It is given that

$$\left[\frac{d}{dt} - A, L\right] = 0. \tag{2.4}$$

It is necessary to prove that $\left[\frac{d}{dt} - A, M\right] = 0$. Since $M^k = L^r$, from (2.4) it follows that

$$\left[\frac{d}{dt} - A, M^k\right] = 0. \tag{2.5}$$

On the other hand,

$$\left[\frac{d}{dt} - A, M^k\right] = \sum_{i=1}^{k} M^{i-1} \left[\frac{d}{dt} - A, M\right] M^{k-i}. \tag{2.6}$$

It is easy to see that the leading coefficient on the right side of (2.6) is k times larger than the leading coefficient of $\left[\frac{d}{dt} - A, M\right]$. Therefore, the assumption that $\left[\frac{d}{dt} - A, M\right] \neq 0$ contradicts (2.5). ■

We consider the equation

$$\frac{\partial L}{\partial t} = [M_+, L], \quad M = \sum_{i=0}^{m} c_i L^{i/k}, \quad c_i \in \mathbf{C}, \tag{2.7}$$

$$\frac{\partial L}{\partial \tau} = [\tilde{M}_+, L], \quad \tilde{M} = \sum_{i=0}^{n} \tilde{c}_i L^{i/k}, \quad \tilde{c}_i \in \mathbf{C}. \tag{2.8}$$

Proposition 2.5. $\frac{\partial^2 L}{\partial t \partial \tau} = \frac{\partial^2 L}{\partial \tau \partial t}$, where the derivatives are computed by Eqs. (2.7) and (2.8).

The proof is the same as that of Proposition 1.7 (Lemma 2.4 now plays the role of Lemma 1.6).

2.2. This subsection is devoted to conservation laws for the Lax equation. If $P = \sum_{i=-\infty}^{m} b_i D^i$, $b_i \in B$, then we set $\operatorname{res} P = b_{-1}$.

LEMMA 2.6. Let $P, Q \in B((D^{-1}))$. Then $\operatorname{res}[P, Q]$ is a total derivative of some differential polynomial in the coefficients of P and Q.

Proof. It suffices to consider the case where $P = aD^m$, $Q = bD^e$. If $m + l < -1$, then $\operatorname{res}(PQ) = \operatorname{res}(QP) = 0$. If $m + l \geqslant -1$, then $\operatorname{res}(PQ)=\frac{m(m-1)\dots(1-l)(-l)}{(m+l+1)!}ab^{(m+l+1)}$, $\operatorname{res}(QP)=\frac{l(l-1)\dots(1-m)(-m)}{(m+l+1)!}\times ba^{(m+l+1)}$. Therefore $\operatorname{resp}[P, Q] = g'$, where

$$g=\frac{m(m-1)\dots(1-l)(-l)}{(m+l+1)!}\sum_{i=0}^{m+l}(-1)^i a^{(i)} b^{(m+l-i)}. \blacksquare$$

From Lemmas 2.4 and 2.6 we obtain the following result.

Proposition 2.7. For any $r\in\mathbf{N}$, $\operatorname{res} L^{r/k}$ is a density of a conservation law for the Lax equation. $\blacksquare$

Of course, nontrivial conservation laws correspond only to numbers r not a multiple of k.

We now discuss another means of constructing conservation laws.

LEMMA 2.8. Let P be an element of $B((D^{-1}))$ of the form $P=D+\sum_{l=0}^{\infty}g_lD^{-l}$, $g_l\in B$. Then any element $M\in B((D^{-1}))$ can be represented uniquely in the form $\sum_{l=n}^{\infty}h_lP^{-l}$, $n\in\mathbf{Z}$, $h_l\in B$. Here h_i are differential polynomials in the coefficients of M and P. $\blacksquare$

THEOREM 2.9. Let $P=D+\sum_{l=0}^{\infty}g_lD^{-l}$, $g_l\in B$. We represent D in the form $D=P+\sum_{l=0}^{\infty}h_lP^{-l}$. Then a) $h_0 = -g_0$; b) if $r > 0$, then $h_r + \operatorname{res} P^r/r$ is a total derivative of some differential polynomial in the coefficients of P.

COROLLARY. Let $L=D^k+\sum_{i=0}^{k-1}u_iD^i$. We represent D in the form

$$D=L^{1/k}+\sum_{i=0}^{\infty}f_iL^{-i/k},\quad f_i\in B. \tag{2.9}$$

Then a) $f_0 = -u_{k-1}/k$; b) if $r > 0$, then $f_r + \operatorname{res} L^{r/k}/r$ is a total derivative of a differential polynomial in $u_0,\dots,u_{k-1}$. $\blacksquare$

From the corollary, in particular, it follows that all f_r are densities of conservation laws for the Lax equation. This fact can be proved directly without difficulty (see, for example, [69, 71, 16, p. 48]).

Before proving Theorem 2.9, we present a means of computing the densities f_r by means of an equation of Ricatti type. We consider a formal solution of the equation $L\psi = \zeta^k\psi$ of the form

$$\psi(x,\zeta)=e^{\zeta x}\sum_{i=0}^{\infty}\varphi_i(x)\zeta^{-i},\quad \psi_0\neq 0. \tag{2.10}$$

This solution is uniquely determined up to multiplication by series of the form $\sum_{i=0}^{\infty}c_i\zeta^{-i}$, $c_i\in\mathbf{C}$, $c_0 \neq 0$.

Proposition 2.10.

$$\psi'(x,\zeta)\varphi^{-1}(x,\zeta)=\zeta+\sum_{i=0}^{\infty}f_i(x)\zeta^{-i}. \tag{2.11}$$

Proof. It follows from (2.9) that $\psi'=L^{1/k}\psi+\sum_{i=0}^{\infty}f_iL^{-i/k}\psi$. It remains to show that $L^{1/k}\psi = \zeta\psi$. Since $\zeta^{-1}L^{1/k}\psi$ is an eigenfunction of the operator L of the form (2.10), it follows that $L^{1/k}\psi = a(\zeta)\psi$, where $a(\zeta)=c\zeta+\sum_{i=0}^{\infty}a_i\zeta^{-i}$. It is easy to see that $c = 1$. Since $L\psi = \zeta^k\psi$, it follows that $a(\zeta)^k = \zeta^k$. Therefore, $a(\zeta) = \zeta$. $\blacksquare$

Thus, to find the densities f_i it suffices to find $\psi'\psi^{-1}$, i.e., a formal solution of an equation of Ricatti type. This is much more convenient than expanding D in a series in fractional powers of L. Therefore, as the definition of f_i, as a rule, we take formula (2.11) rather than (2.9).

Proof of Theorem 2.9 (I. V. Cherednik [43], Flaschka [51]). Assertion a) is obvious. To prove b) we use the formula

$$\operatorname{res} P^m = D\cdot(P^m)_- - (D\cdot P^m)_-. \tag{2.12}$$

Let $(P^m)_- = \sum_{l=1}^{\infty} \varphi_{lm} P^{-l}$; then

$$D\cdot(P^m)_- = D\cdot\sum_{l=1}^{\infty}\varphi_{lm}P^{-l} = \sum_{l=1}^{\infty}\varphi'_{lm}P^{-l} + \sum_{l=1}^{\infty}\varphi_{lm}D\cdot P^{-l} =$$

$$= \sum_{l=1}^{\infty}\varphi'_{lm}\ P^{-l} + \sum_{l=1}^{\infty}\varphi_{lm}P^{1-l} + \sum_{l=1}^{\infty}\sum_{j=0}^{\infty}\varphi_{lm}h_jP^{-l-j} = \sum_{l=1}^{\infty}\varphi'_{lm}P^{-l} + (P^m)_-\cdot P + \sum_{j=0}^{\infty}h_j(P^m)_-P^{-j};$$

$$(D\cdot P^m)_- = (P^{m+1})_- + \sum_{j=0}^{\infty}h_j(P^{m-j})_-.$$

Therefore, multiplying (2.12) by P^{-m} and summing on m, we obtain

$$\sum_{m=1}^{\infty}(\operatorname{res} P^m)\cdot P^{-m} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\varphi'_{lm}P^{-l-m} + \sum_{m=1}^{\infty}(P^m)_-\cdot P^{1-m} - \sum_{m=1}^{\infty}(P^{m+1})_-\cdot P^{-m} + \sum_{m=1}^{\infty}\sum_{j=0}^{\infty}h_j\cdot(P^m)_-\cdot P^{-j-m} -$$

$$\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}h_j\cdot(P^{m-j})_-P^{-m} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\varphi'_{lm}P^{-l-m} + P_- - \sum_{j=0}^{\infty}h_j\cdot\sum_{m=1}^{j}(P^{m-j})_-P^{-m}.$$

Since $P_- = -\sum_{j=0}^{\infty}h_jP^{-j}$, $(P^{m-j})_- = P^{m-j}$ for $m < j$, $(P^{m-j})_- = 0$ for $m = j$, it follows that

$$\sum_{m=1}^{\infty}(\operatorname{res} P^m)\cdot P^{-m} = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\varphi'_{lm}P^{-l-m} - \sum_{j=1}^{\infty}jh_jP^{-j}. \tag{2.13}$$

Equating coefficients of P^{-r} in (2.13), we obtain the desired relation $\operatorname{res} P^r + rh_r = \sum_{l+m=r}\varphi'_{lm}$. ■

2.3. We proceed now to a discussion of the Hamiltonian formalism for Lax equations. For reasons put forth in part 1.4, we shall discuss only the periodic case. The manifold M on which it is necessary to introduce a Hamiltonian structure consists of all differential operators L of the form $D^k + \sum_{i=0}^{k-1}u_iD^i$, $u_i\in B_0$. As in part 1.4, the Poisson bracket will be defined on the set of functionals of the form

$$\varphi(L) = \int_{x\in\mathbf{R}/\mathbf{Z}} f(x, u_0(x), \ldots, u_{k-1}(x), \ldots, u_0^{(n)}(x), \ldots, u_{k-1}^{(n)})\,dx, \tag{2.14}$$

where f is a polynomial in $u_0, \ldots, u_{k-1}, \ldots, u_0^{(n)}, \ldots, u_{k-1}^{(n)}$ with coefficients in B_0. It is convenient to first define the Poisson bracket for affine linear functionals of the form (2.14), i.e., for functionals of the form

$$\varphi(L) = c + \sum_{i=0}^{k-1}\int_{x\in\mathbf{R}/\mathbf{Z}} a_i(x)u_i(x)\,dx,\quad c\in\mathbf{C},\quad a_i\in B_0. \tag{2.15}$$

Since the differential of a functional of the form (2.14) at any point of M has the form (2.15), knowing the Poisson bracket of functionals of the form (2.15), it is possible to uniquely recover the Poisson bracket of any functional of the form (2.14).

Following [44], we now introduce a convenient form of writing functionals of the form (2.15). We define the functional $\operatorname{Tr}: B_0((D^{-1}))\to\mathbf{C}$ by the formula $\operatorname{Tr} P = \int_{x\in\mathbf{R}/\mathbf{Z}}(\operatorname{res} P)\,dx$. The notation "Tr" is justified by the formula $\operatorname{Tr}[P, Q] = 0$ which follows from Lemma 2.6. To each pseudodifferential symbol X of the form $\sum_{l=1}^{\infty} b_l D^{-l}$, $b_l \in B_0$ (such symbols will henceforth be called integral symbols) we assign the functional $l_X: M \to \mathbf{C}$ given by the formula $l_X(L) = \mathrm{Tr} \times (XL)$. It is clear that l_X has the form (2.15), and any functional of the form (2.15) can be written as l_X by setting $X = \sum_{l=0}^{k-1} D^{-l-1} a_l + cD^{-k-1}$.

THEOREM 2.11. 1) On the set of functionals of the form (2.15) there exist Poisson brackets $\{ , \}_1$ and $\{ , \}_2$ (called, respectively, the first and second Gel'fand–Dikii brackets) such that for any integral symbols X, Y and any $L \in M$

$$\{l_X, l_Y\}_1 (L) = \mathrm{Tr}\,(L\,[Y, X]), \tag{2.16}$$

$$\{l_X, l_Y\}_2 (L) = \mathrm{Tr}\,((LY)_+ LX - XL\,(YL)_+). \tag{2.17}$$

These brackets are coordinated. 2) The Lax equations are Hamiltonian relative to both Gel'fand–Dikii brackets. More precisely, the equation $dL/dt = [(L^{r/k})_+, L]$ is the Hamiltonian equation corresponding to the Hamiltonian H_r and the second Gel'fand–Dikii brackets and also to the Hamiltonian H_{r+k} and the first Gel'fand–Dikii bracket, where $H_r: M \to \mathbf{C}$ is defined by the formula $H_r(L) = k\,\mathrm{Tr}\, L^{r/k}/r$.

The proof will be carried out in Sec. 3, where we discuss the relation between the Gel'fand–Dikii brackets and the brackets given by formulas (1.13), (1.14). There is another proof in [5, 7].

Remark 1. It follows from Proposition 2.7 that the Hamiltonians H_r, $r \in \mathbf{N}$ are conservation laws for Eq. (2.1). From this and Theorem 2.11 it follows that the functionals H_r commute with one another relative to both Gel'fand–Dikii brackets.

Remark 2. The meaning of the first Gel'fand–Dikii bracket was clarified in [27, 44]. In these works it was noted that if we consider the set G of all integral symbols as a Lie algebra and identify G^* with the set of differential operators assigning to the operator L the functional $X \to -\mathrm{Tr}\,(Lx)$, then the Hamiltonian structure of A. A. Kirillov [37] on G^* is defined by formula (2.16). We note that formula (2.16) itself first appeared in [27, 44].

Remark 3. We shall indicate the connection between the Hamiltonians H_r and the coefficients of the formal monodromy series. By the formal monodromy series we mean the following. We consider a formal solution of the equation $L\psi = \zeta^k \psi$ having the form (2.10). Then $\psi(x + 1, \zeta)e^{-\zeta}$ is a formal solution of the same form, so that $e^{-\zeta}\psi(x + 1, \zeta) = M(\zeta)\psi(x, \zeta)$. The series $M(\zeta)$ we call the formal monodromy series. From Proposition 2.10 and the corollary of Theorem 2.9 we have the formula

$$M(\zeta) = \exp\left(\frac{1}{k}\left(\int_{R/Z} u_{k-1}(x)\,dx + \sum_{r=1}^{\infty} H_r \zeta^{-r}\right)\right).$$

3. THE SCALAR LAX EQUATION AS REDUCTION IN THE ZAKHAROV–SHABAT SCHEME

3.1. We consider an operator $\mathfrak{L}$ of the form

$$\mathfrak{L} = \frac{d}{dx} + q + \Lambda. \tag{3.1}$$

Here q(x) is a function with values in the set $\mathfrak{b}$ of upper triangular matrices of order k and $\Lambda = I + \lambda e$, where

$$I = \sum_{i=1}^{k-1} e_{i+1,i}, \quad e = e_{1,k}. \tag{3.2}$$

We recall that $e_{i,j}$ denotes the matrix having a one at the (i, j)-th site and zeros elsewhere. We denote by $\mathfrak{n}$ (respectively, N) the set of matrices of $\mathfrak{b}$ with zeros (respectively, ones) on the main diagonal. It is easy to see that if $\mathfrak{L}$ is an operator of the form (3.1), $S \in C^\infty(\mathbf{R}, N)$, then the operator

$$\tilde{\mathfrak{L}} = S^{-1}\mathfrak{L}S \tag{3.3}$$

also has the form (3.1). We call the transformation (3.3) a gauge transformation, and we call the operators $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ gauge equivalent.

Proposition 3. 1.[5] Any operator $\mathfrak{L}$ of the form (3.1) can be uniquely represented in the form $S\mathfrak{L}^{\mathrm{can}}S^{-1}$, where $S\in C^\infty(\mathbf{R}, N)$, $\mathfrak{L}^{\mathrm{can}}=\frac{d}{dx}+q^{\mathrm{can}}(x)+\Lambda$,

$$q^{\mathrm{can}}=v_1(x)e_{1,k}+v_2(x)e_{2,k}+\ldots+v_k(x)e_{k,k}. \tag{3.4}$$

Here S and q^{can} are differential polynomials in q.

Proof. We represent q in the form $\sum_{i=0}^{k-1}q_i$, $q_i(x)\in\mathfrak{b}_i$, where $\mathfrak{b}_i$ is the set of matrices $(a_{\alpha\beta})$ such that $a_{\alpha\beta}=0$ for $\beta-\alpha\neq i$. In exactly the same way, $q^{\mathrm{can}}=\sum_{i=0}^{k-1}q_i^{\mathrm{can}}$, $S=E+\sum_{i=1}^{k-1}S_i$. Since $[e, S]=0$, the relation $\mathfrak{L}=S\mathfrak{L}^{\mathrm{can}}S^{-1}$ can be written in the form $S'+[I, S]+qS-Sq^{\mathrm{can}}=0$. If $q_0^{\mathrm{can}},\ldots,q_{i-1}^{\mathrm{can}}$, $S_1,\ldots,S_i$ are already known, then q_i^{can} and S_{i+1} can be found from the relation $[I, S_{i+1}]-q_i^{\mathrm{can}}=\sum_{j=1}^{i}S_jq_{i-j}^{\mathrm{can}}-q_i-\sum_{j=1}^{i}q_{i-j}S_j-S_i'$ which is uniquely solvable. ■

COROLLARY. If the operators $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ are gauge equivalent, then 1) the matrix S in relation (3.3) is uniquely determined; 2) $\mathfrak{L}^{\mathrm{can}}=\tilde{\mathfrak{L}}^{\mathrm{can}}$. ■

Remark. The canonical form (3.4) is not the only possible one. If for any $i\in\{0, 1,\ldots,k-1\}$ we choose a vector subspace $V_i\subset\mathfrak{b}_i$ such that $\mathfrak{b}_i=[I, \mathfrak{b}_{i+1}]\oplus V_i$ and set $V=\bigoplus_{i=0}^{k-1}V_i$, then any operator of the form (3.1) can be uniquely represented in the form $S\left(\frac{d}{dx}+\tilde{q}+\Lambda\right)S^{-1}$, where $S\in C^\infty(\mathbf{R}, N)$, $\tilde{q}\in C^\infty(\mathbf{R}, V)$. Actually, $[I, \mathfrak{b}_{i+1}]$ consists of all matrices $(a_{\alpha\beta})\in\mathfrak{b}_i$ such that $\sum_{\alpha,\beta}a_{\alpha\beta}=0$; therefore, $\dim V_i=1$.

We denote by R the ring of differential polynomials in the elements of q which are invariant relative to gauge transformations.

Proposition 3.2. There exist elements $u_1,\ldots, u_k\in R$ such that any element of R can be represented uniquely in the form $p(u_1,\ldots,u_k)$, where p is a differential polynomial.

We call a collection of elements $u_1,\ldots,u_k$ possessing this property a system of generators in R.

Proof. We shall show that for $u_i(q)$ it is possible to take the differential polynomials $v_i(q)$ of formula (3.4). Indeed, let $f\in R$; then from Proposition 3.1 it follows that f can be represented as a differential polynomial in $v_1,\ldots,v_k$ and the elements of the matrix S, while from the gauge invariance of f it follows that this polynomial does not depend on S.

Definition. We say that the equation $\partial q/\partial t=p$, where p is a differential polynomial in q, preserves gauge equivalence if the derivative by the equation of any element of R again belongs to R.

Roughly speaking, an equation preserves gauge equivalence if from the fact that two of its solutions are gauge equivalent at $t=0$ it follows that they are gauge equivalent for any t (there is a particular gauge transformation for each t).

Suppose that there is given an equation $\partial q/\partial t=p$ preserving gauge equivalence. In R we choose a system of generators $u_1,\ldots,u_k$. Then

$$\frac{\partial u_i}{\partial t}=f_i(u_1,\ldots,u_k),\quad i=1,\ldots,k, \tag{3.5}$$

where f_i are uniquely determined differential polynomials. We consider (3.5) as a system of equations for the functions $u_i(x, t)$. If in R we choose another system of generators $\tilde{u}_1,\ldots,\tilde{u}_k$, then the system of equations $\partial\tilde{u}_i/\partial t=\tilde{f}_i(\tilde{u}_1,\ldots,\tilde{u}_k)$ corresponding to it is obtained from (3.5) by an invertible differential-polynomial change of unknowns. We have thus obtained an entire class of equivalent systems. Any one of them we call an equation for the class of gauge equivalence.

3.2. Following the scheme of Sec. 1, in this subsection we consider some differential equations for an operator $\mathfrak{L}$ of the form (3.1). It will be shown that these equations preserve gauge equivalence.

[5]Proposition 3.1 is essentially contained in [65, 25].

Proposition 3.3. Let the operator $\mathfrak{L}$ have the form (3.1). Then there exists a formal series T of the form $\sum_{i=0}^{\infty} T_i\lambda^{-i}$, where $T_0 \in C^\infty(\mathbf{R}, N)$, such that the operator $\mathfrak{L}_0 \overset{\text{def}}{=} T\mathfrak{L}T^{-1}$ has the form

$$\mathfrak{L}_0 = \frac{d}{dx} + \Lambda + \sum_{i=0}^{\infty} f_i\Lambda^{-i}, \quad f_i \in B. \tag{3.6}$$

The series T is uniquely determined up to multiplication on the left by series of the form $E + \sum_{i=1}^{\infty} t_i\Lambda^{-i}$, $t_i \in B$. T can be chosen in exactly one way so that the first column of T is equal to $(1, 0, \dots, 0)^t$. The T_i are hereby differential polynomials in q with zero free terms.

We first formulate an elementary lemma.

LEMMA 3.4. 1) Each element of $\mathrm{Mat}(k, \mathbf{C}((\lambda^{-1})))$ can be uniquely represented in the form $\sum_{i=-\infty}^{m} h_i\Lambda^i$, where $h_i \in \mathrm{Diag}$. 2) If $h = \mathrm{diag}(a_1, \dots, a_k)$, then $\Lambda h = h^\sigma \Lambda$, where $h^\sigma = \mathrm{diag}(a_k, a_1, a_2, \dots, a_{k-1})$. 3) A matrix $T \in \mathrm{Mat}(k, \mathbf{C}((\lambda^{-1})))$ has the form $\sum_{i=0}^{\infty} T_i\lambda^{-i}$, where $T_0 \in N$, if and only if T can be represented in the form $E + \sum_{i=1}^{\infty} h_i\Lambda^{-i}$, $h_i \in \mathrm{Diag}$. 4) An operator $\mathfrak{L}$ of the form (3.1) can be written as $\frac{d}{dx} + \Lambda + \sum_{i=0}^{k-1} d_i\Lambda^{-i}$, where $d_i \in C^\infty(\mathbf{R}, \mathrm{Diag})$. ■

Proof of Proposition 3.3. We write the desired series T in the form $E + \sum_{i=1}^{\infty} h_i\Lambda^{-i}$, $h_i \in C^\infty(\mathbf{R}, \mathrm{Diag})$. Equating coefficients of Λ^{-n} in the equality $\mathfrak{L}_0 T = T\mathfrak{L}$ we find that $h_{n+1} - h_{n+1}^\sigma - f_n E$ can be expressed in terms of $h_1, \dots, h_n, f_1, \dots, f_{n-1}$. The existence of the series T and the operator $\mathfrak{L}_0$ will be proved therefore if we show that any diagonal matrix can be represented in the form $h - h^\sigma - fE$. For this it suffices to verify that any diagonal matrix with zero trace can be represented in the form $h - h^\sigma$, and this is obvious. The fact that T is uniquely determined up to multiplication by series of the form $E + \sum_{i=1}^{\infty} t_i\Lambda^{-i}$, $t_i \in B$ can be verified in the same way as the corresponding assertion in Proposition 1.2. To prove the remainder of the proposition it suffices to note that the first column of the matrix $T = E + \sum_{i=1}^{\infty} h_i\Lambda^{-i}$ is equal to $(1, 0, \dots, 0)^t$ if and only if for any i the equality $(h_i)_{n,n} = 0$ holds where $n - 1$ is the remainder on dividing $-i$ by k. ■

For any operator $\mathfrak{L}$ of the form (3.1) we set $Z_{\mathfrak{L}} = \{M \in \mathrm{Mat}(k, B((\lambda^{-1}))) \mid [\mathfrak{L}, M] = 0\}$.

LEMMA 3.5. $Z_{\mathfrak{L}} = T^{-1}\mathbf{C}((\Lambda^{-1}))T$, where T is the same as in Proposition 3.3.

Proof. It suffices to show that if $\mathfrak{L}_0$ is an operator of the form (3.6), then $Z_{\mathfrak{L}_0} = \mathbf{C}((\Lambda^{-1}))$. Let $[M, \mathfrak{L}_0] = 0$, $M = \sum_{i=-\infty}^{n} h_i\Lambda^i$, $h_i \in C^\infty(\mathbf{R}, \mathrm{Diag})$. Equating the coefficient of Λ^{n+1} in the expansion of $[M, \mathfrak{L}_0]$ in powers of Λ to zero, we find that h_n is a scalar matrix. Equating to zero the trace of the coefficient of Λ^n, we see that $h_n' = 0$. We apply an analogous argument to the series $M - h_n\Lambda^n$, etc. ■

If $M = \sum_{i=-\infty}^{n} h_i\Lambda^i$, $h_i \in \mathrm{Diag}$, then we set $M^+ \overset{\text{def}}{=} \sum_{i \geq 0} h_i\Lambda^i$, $M^- \overset{\text{def}}{=} \sum_{i<0} h_i\Lambda^i$. Regarding the symbols M_+ and M_- see the list of notation.

Remark. It is not hard to verify that the difference $M^+ - M_+$ belongs to $\mathfrak{n}$ and does not depend on λ.

LEMMA 3.6. Let $M \in Z_{\mathfrak{L}}$. Then $[M^+, \mathfrak{L}]$ and $[M_+, \mathfrak{L}]$ are upper triangular matrices not depending on λ.

Proof. We consider the equalities $[M^+, \mathfrak{L}] = -[M^-, \mathfrak{L}]$, $[M_+, \mathfrak{L}] = -[M_-, \mathfrak{L}]$. The left sides of these equalities are polynomials in λ, while the right sides can be expanded in series in nonpositive powers of Λ. From this we easily deduce the assertion of the lemma. ■

For any $u\in\mathbf{C}((\Lambda^{-1}))$ we set $\varphi(u)=T^{-1}uT$, where T is the same as in Proposition 3.3. From this proposition it follows that φ is well defined and that the coefficients of the series $\varphi(u)$ are differential polynomials in q.

It follows from Lemma 3.6 that the equation

$$\frac{d\mathfrak{L}}{dt}=[\mathcal{A},\mathfrak{L}],\quad \mathcal{A}\overset{\text{def}}{=}\sum_{i=0}^{m}c_i(\varphi(\Lambda^i))^+,\quad c_i\in\mathbf{C} \tag{3.7}$$

is a self-consistent equation for the matrix q. Together with equation (3.7) we consider the equation

$$\frac{d\mathfrak{L}}{dt}=[\mathcal{B},\mathfrak{L}],\quad \mathcal{B}\overset{\text{def}}{=}\sum_{i=0}^{m}c_i(\varphi(\Lambda^i))_+,\quad c_i\in\mathbf{C}, \tag{3.8}$$

which is also self-consistent.

Proposition 3.7. Equations (3.7) and (3.8) preserve gauge equivalence and lead to the same equation for the class of gauge equivalence.

Proof. We shall show that if $f\in R$, then the derivatives of f by Eqs. (3.7) and (3.8) coincide. We denote the difference of these derivatives by g. It is clear that g is a derivative of f by the equation $\frac{d\mathfrak{L}}{dt}=[\mathcal{A}-\mathcal{B},\mathfrak{L}]$, i.e., $g(\mathfrak{L})=\frac{d}{dt}f(\mathfrak{L}(t))|_{t=0}$, where $\mathfrak{L}(t)$ is any function such that $\mathfrak{L}(0)=\mathfrak{L},\ \frac{d\mathfrak{L}(t)}{dt}\Big|_{t=0}=[\mathcal{A}-\mathcal{B},\mathfrak{L}]$. From the remark preceding Lemma 3.6 it follows that $\mathcal{A}-\mathcal{B}$ is a function of x with values in $\mathfrak{n}$, which does not depend on λ. Therefore, the function $S(x,t)=E+t(\mathcal{A}-\mathcal{B})$ does not depend on λ and takes values in N. For $\mathfrak{L}(t)$ it is possible to take $S(x,t)\mathfrak{L}S^{-1}(x,t)$. Then due to gauge invariance of f, $f(\mathfrak{L}(t))=f(\mathfrak{L})$ does not depend on t, and hence $g(\mathfrak{L})=0$. It remains to show that Eq. (3.8) preserves gauge equivalence. For this it suffices to note that if $\mathfrak{L}(t)$ satisfies (3.8) and $S(x)\in N$, then $\tilde{\mathfrak{L}}(t)=S^{-1}\mathfrak{L}(t)S$ also satisfies (3.8). ■

Remark 1. Since $\Lambda^k=\lambda E$, we have $\varphi(\Lambda^{kj})=\lambda^j E,\ j\in Z$. Hence, it may be assumed in formulas (3.7) and (3.8) that $c_i=0$ if i is divisible by k.

Remark 2. It follows from (3.7) that $\operatorname{tr}q(x,t)$ does not depend on t. Indeed, $\frac{\partial}{\partial t}\operatorname{tr}q(x,t)=-\frac{\partial}{\partial x}\operatorname{tr}\mathcal{A}$, and from the definition of $\mathcal{A}$ it is evident that $\operatorname{tr}\mathcal{A}$ is a constant. Thus, without violating the self-consistency of Eq. (3.7), it is possible to set $\operatorname{tr}q=0$. The same applies to Eq. (3.8).

Below we shall be interested not so much in Eqs. (3.7) and (3.8) as in the equation for the class of gauge equivalence corresponding to them. This equation can be written in the form

$$\frac{\partial q^{\text{can}}}{\partial t}=F\left(q^{\text{can}},\frac{\partial q^{\text{can}}}{\partial x},\dots\right). \tag{3.9}$$

We shall find the Lax representation for Eq. (3.9). If on the basis of the operator $\mathfrak{L}^{\text{can}}$ we find an operator $\mathcal{A}$ by formula (3.7), then the equation $\frac{d\mathfrak{L}^{\text{can}}}{dt}=[\mathcal{A},\mathfrak{L}^{\text{can}}]$ will not be self-consistent. In order to correct the operator $\mathcal{A}$ we use the following lemma which is proved in the same way as Proposition 3.1.

LEMMA 3.8. For any $\eta,\ q\in C^\infty(\mathbf{R},\mathfrak{b})$ there exists precisely one matrix $\theta\in C^\infty(\mathbf{R},\mathfrak{n})$ such that all columns of the matrix $[d/dx+I+q,\theta]-\eta$ except, possibly, the last are equal to zero. Here θ is a differential polynomial in q and η.

Using Lemma 3.8, we find a matrix $\theta\in C^\infty(\mathbf{R},\mathfrak{n})$ such that all columns of the difference $\left[\frac{d}{dx}+I+q,\theta\right]-[\mathcal{A},\mathfrak{L}]$ except the last are equal to zero (we recall that according to Lemma 3.6 $[\mathcal{A},\mathfrak{L}]$ is an upper triangular matrix not depending on λ). Since $[\mathfrak{L},\theta]=\left[\frac{d}{dx}+I+q,\theta\right]$, the equation

$$\frac{d\mathfrak{L}}{dt}=[\mathscr{A}+\theta,\ \mathfrak{L}] \tag{3.10}$$

admits the reduction $\mathfrak{L}=\mathfrak{L}^{can}$. In analogy to Proposition 3.7 it can be shown that Eqs. (3.7) and (3.10) lead to the same equation for the class of gauge equivalence. Therefore, the equation $\frac{d\mathfrak{L}^{can}}{dt}=[\mathscr{A}+\theta,\ \mathfrak{L}^{can}]$ is a Lax representation for Eq. (3.9).

It will be shown further on that (3.9), where q^{can} has the form (3.4), is nothing other than the scalar Lax equation for the operator $L=D^k-\sum_{i=0}^{k-1}v_{i+1}D^i$. For another choice of q^{can} (see the remark following Proposition 3.1) Eq. (3.9) is connected with the Lax equation by an invertible differential-polynomial substitution. It can happen that for an appropriate choice of q^{can} Eq. (3.9) is in a certain sense more simpatica than the corresponding Lax equation (has lower order, for example).

Example 1. The Lax equation with $L = D^3 + uD + v$, $A = L_+^{2/3}$ is a system of third order (see Example 2, part 2.1). However, if we set $q^{can} = (\alpha e_{12} + \beta e_{23})U + \gamma e_{13}V$, then for suitable constants α, β, γ the system (3.9) has the form

$$U_t=-U_{xx}+V_x,$$
$$V_t=V_{xx}+UU_x.$$

Example 2. The Lax equation with $L = D^4 + uD^2 + vD + w$, $A = L_+^{1/2}$ is a system of fourth order (see [35]). However, if we set $q^{can} = \alpha(e_{12} + e_{34})W + (\beta e_{13} + \gamma e_{24})V + \delta e_{14}U$, then for suitable α, β, γ, δ the system (3.9) has the form

$$U_t=-U_{xx}+2V_xW+VW_x,$$
$$V_t=V_{xx}+U_x$$
$$W_t=V_x.$$

3.3. Suppose an operator $\mathfrak{L}$ of the form (3.1) is given. On $B((\lambda^{-1}))^k$ we introduce the structure of a B[D]-module as follows: if $P=\sum_{i=0}^{n}b_iD^i\in B[D]$, $\eta\in B((\lambda^{-1}))^k$, then $P\cdot\eta\overset{\text{def}}{=}\sum_{i=0}^{n}b_i\mathfrak{L}^i(\eta)$. The axioms for a module are satisfied, since $[\mathfrak{L},b]=b'$ for $b\in B$. It is clear that $B[\lambda]^k$ is a B[D]-submodule in $B((\lambda^{-1}))^k$. We emphasize that $D\cdot\eta=\mathfrak{L}(\eta)\neq\eta'$! We set $\psi=(1,0,\ldots,0)^t\in B[\lambda]^k$.

LEMMA 3.9. Each element of $B[\lambda]^k$ can be uniquely represented in the form $P\cdot\psi$, where $P\in B[D]$.

Proof. It is easy to see that any element $\eta\in B[\lambda]^k$ can be uniquely represented in the form $\eta=\sum_{i=0}^{n}b_i\Lambda^i\psi$, $b_i\in B$, $b_n\neq 0$. The number n is henceforth called the order of η, and b_n is called the leading coefficient of η. To prove the lemma it suffices to note that $\mathfrak{L}(\eta)$ has order one higher than η and has the same leading coefficient as η.

Remark. It is evident from the proof of the lemma that the order and leading coefficient of $P\cdot\psi$ are equal to the order and leading coefficient of P.

From the lemma and the remark it follows that $\lambda\psi = L\cdot\psi$, where L is a uniquely determined differential operator of the form

$$L=D^k+\sum_{i=0}^{k-1}u_iD^i. \tag{3.11}$$

Thus, to each operator $\mathfrak{L}$ of the form (3.1) we have assigned an operator L of the form (3.11).

Proposition 3.10. 1) To gauge equivalent operators $\mathfrak{L}$ there correspond the same operators L. 2) The mapping obtained from the set of classes of gauge equivalence of operators $\mathfrak{L}$ into the set of operators L is bijective.

Proof. 1) Let $\tilde{\mathfrak{L}} = S^{-1}\mathfrak{L}S$, where $S(x) \in N$. It must be shown that if $\lambda\psi = \left(\mathfrak{L}^k + \sum_{i=0}^{k-1} u_i \mathfrak{L}^i\right)\psi$, then $\lambda\psi = \left(\tilde{\mathfrak{L}}^k + \sum_{i=0}^{k-1} u_i \tilde{\mathfrak{L}}^i\right)\psi$. For this it suffices to multiply both sides of the first equality on the left by S^{-1} and note that $\psi = S\psi$. 2) In view of Proposition 3.1, it suffices to prove that each operator L corresponds to exactly one operator $\mathfrak{L} = \frac{d}{dx} + q + \Lambda$, where q has the form (3.4). Indeed, it is easy to see that the coefficients $u_0, \ldots, u_{k-1}$ of the operator L can be expressed in terms of the elements $v_1, \ldots, v_k$ of the matrix q by the formula $u_i = -v_{i+1}$. ■

COROLLARY. The coefficients of the operator L considered as differential polynomials in q form a system of generators in R.

The remainder of this part will be devoted to the exposition of another point of view regarding the nature of the correspondence described above between the operators (3.1) and (3.11).

Let $\mathfrak{B}$ be a noncommutative ring with identity. We consider an arbitrary matrix $F \in \mathrm{Mat}(k, \mathfrak{B})$ of the form

$$F = \begin{pmatrix} \alpha & \beta \\ A & \gamma \end{pmatrix}, \tag{3.12}$$

where A is an invertible matrix in $\mathrm{Mat}(k-1, \mathfrak{B})$. We denote by $\mathfrak{N}$ the set of upper triangular matrices in $\mathrm{Mat}(k, \mathfrak{B})$ having ones on the main diagonal.

LEMMA 3.11. 1) There exist $S_1, S_2 \in \mathfrak{N}$ such that $\Phi \overset{\text{def}}{=} S_1 F S_2$ has the form

$$\Phi = \begin{pmatrix} 0 & d \\ \tilde{A} & 0 \end{pmatrix}, \tag{3.13}$$

where $\tilde{A} \in \mathrm{Mat}(k-1, \mathfrak{B})$. 2) The element d does not depend on the choice of S_1 and S_2.

Proof. To prove the first part of the lemma we use the fact that multiplication of a matrix on the left (respectively, right) by a matrix of the form $E + \alpha e_{ij}$, $i < j$ leads to an elementary row (respectively, column) transformation. In order to reduce F to the form (3.13), it suffices to subtract from the first row a linear combination of the remaining rows and from the last column a linear combination of the remaining columns. The coefficients of each of these linear combinations are found by solving a system of linear equations with matrix A. To prove the second part of the lemma it suffices to verify that if $S_1\Phi_1 = \Phi_2 S_2$, where Φ_i are matrices of the form (3.13), $S_i \in \mathfrak{N}$, then $d_1 = d_2$. Indeed, in the right upper corner the matrix $S_1\Phi_1$ has d_1, while $\Phi_2 S_2$ has d_2. ■

We denote the element d by $\Delta(F)$. It is easy to show that $\Delta F = \beta - \alpha A^{-1}\gamma$.

Let W be a left module over $\mathfrak{B}$. It is trivial to prove the following assertion.

LEMMA 3.12. If $F \cdot (u_1, \ldots, u_k)^t = (v, 0, \ldots, 0)^t$, where $u_i \in W$, $v \in W$, then $\Delta(F) \cdot u_k = v$. ■

We assume that there is given an antiautomorphism $x \to x^*$ of the ring $\mathfrak{B}$ such that $(x^*)^* = x$. For any matrix $A = (a_{i,j})$ in $\mathrm{Mat}(k, \mathfrak{B})$ we set $A^T = (a^T_{i,j})$, where $a^T_{i,j} \overset{\text{def}}{=} a^*_{k-j+1,k-i+1}$. It is not hard to verify that $(A_1A_2)^T = A_2^T A_1^T$ for any matrices A_1 and A_2.

LEMMA 3.13. $\Delta(F^T) = (\Delta(F))^*$.

Proof. Applying the operation T to both sides of the equality $S_1FS_2 = \Phi$, where Φ is a matrix of the form (3.13), we obtain the assertion of the lemma. ■

We return to our situation. Let $\mathfrak{B} = B[D, \lambda]$, $W = \mathfrak{B}((\lambda^{-1}))^k$, and let * be the operation of forming the formal adjoint, i.e., $D^* \overset{\text{def}}{=} -D$, $f^* = f$ for $f \in B[\lambda]$. The structure of a $B[D, \lambda]$-module is introduced by means of an operator $\mathfrak{L}$ of the form (3.1) (see the beginning of part 3.3). On the other hand, it is possible to consider $\mathfrak{L}$ as a matrix with coefficients in $\mathfrak{B}$. In order to avoid confusion, we denote this matrix by l. We set $l_0 = l - \lambda e$. It is clear that l_0 does not depend on λ and has the form (3.12). We denote the elements of the standard basis of W as a module over $B((\lambda^{-1}))$ by $\bar{e}_1, \ldots, \bar{e}_k$. We recall that $\psi \overset{\text{def}}{=} \bar{e}_1$.

Proposition 3.14. Let L be an operator of the form (3.11) such that $L\cdot\psi = \lambda\psi$. Then $L = -(\Delta(l_0))^*$.

Proof. It is not hard to verify that the equality $D\cdot\bar{e}_i = \mathfrak{L}(\bar{e}_i), i=1,\ldots,k$ can be written $l^T\cdot(\bar{e}_k,\ldots,\bar{e}_1)^t = 0$, i.e., $l_0^T\cdot(\bar{e}_k,\ldots,\bar{e}_1)^t = (-\lambda\bar{e}_1, 0,\ldots,0)^t$. Using Lemmas 3.12 and 3.13, we find that $-(\Delta(l_0))^*\cdot\bar{e}_1 = \lambda\bar{e}_1$. It remains to use Lemma 3.9. ■

3.4. In this subsection we show that the equation for the class of gauge equivalence corresponding to Eq. (3.7) coincides with the scalar Lax equation.

On $B((\lambda^{-1}))^k$ we introduce the structure of a $B((D^{-1}))$-module. For this it is necessary to assign a meaning to the expression $D^{-1}\cdot\eta$, where $\eta\in B((\lambda^{-1}))^k$.

LEMMA 3.15. The operator $\mathfrak{L}: B((\lambda^{-1}))^k \to B((\lambda^{-1}))^k$ is invertible.

Proof. It is easy to see that any element in $B((\lambda^{-1}))^k$ can be uniquely represented in the form $\sum_{i=-\infty}^{n} b_i\Lambda^i\psi$, $b_i\in B$. As in the proof of Lemma 3.9, this makes it possible to introduce the concepts of order and leading coefficient of an element of $B((\lambda^{-1}))^k$. We set $A=\mathfrak{L}-\Lambda= d/dx + q$. The operator inverse to $\mathfrak{L}$ is given by the formula $\mathfrak{L}^{-1}\eta=(\Lambda(E+\Lambda^{-1}A))^{-1}\eta=\Lambda^{-1}\eta-\Lambda^{-1}A\Lambda^{-1}\eta+\Lambda^{-1}A\Lambda^{-1}A\Lambda^{-1}\eta-\cdots$, which is meaningful, since the operator A does not increase the order. ■

It is clear that the operator $\mathfrak{L}^{-1}: B((\lambda^{-1}))^k\to B((\lambda^{-1}))^k$ reduces the order. Therefore, expressions of the form $\sum_{i=-\infty}^{n} b_i\mathfrak{L}^i\eta$, where $b_i\in B$, $\eta\in B((\lambda^{-1}))^k$ are meaningful. We thus obtain on $B((\lambda^{-1}))^k$ the structure of a $B((D^{-1}))$-module. It can be proved in analogy to Lemma 3.9 that each element $\eta\in B((\lambda^{-1}))^k$ can be uniquely represented in the form $P\cdot\psi$, where $P\in B((D^{-1}))$. Here the order and leading coefficient of P are the same as for η.

Proposition 3.16. $\varphi(\Lambda^n)(\psi)=L^{n/k}\cdot\psi$.

Proof. We represent $\varphi(\Lambda)(\psi)$ in the form $P\cdot\psi$, $P\in B((D^{-1}))$. Since $\varphi(\Lambda)$ has the form $\Lambda+\sum_{i=0}^{\infty} h_i(x)\Lambda^{-i}, h_i(x)\in \mathrm{Diag}$, the order and leading coefficient of $\varphi(\Lambda)(\psi)$, and hence also of P, are equal to one. Since $[\varphi(\Lambda), \mathfrak{L}]=0$, it follows that $\varphi(\Lambda)$ commutes with the action of any element of $B((D^{-1}))$. Therefore, $\varphi(\Lambda^n)(\psi)=P^n\cdot\psi$ for any n. In particular, for n = k we obtain $P^k\cdot\psi=\lambda\psi=L\cdot\psi$, since $P^k = L$. Thus, $P = L^{1/k}$, and hence $\varphi(\Lambda^n)(\psi)=L^{n/k}\cdot\psi$. ■

LEMMA 3.17. Let $\mathcal{M}\in \mathrm{Mat}(k, B((\lambda^{-1})))$, $M\in B((D^{-1}))$, $\mathcal{M}\psi=M\cdot\psi$. Then $\mathcal{M}^+\psi=M_+\cdot\psi$.

Proof. Since $\mathcal{M}^+\psi\in B[\lambda]^k$, it follows that $\mathcal{M}^+\psi=P\cdot\psi$, where $P\in B[D]$. The element $(M - P)\cdot\psi=\mathcal{M}^-\psi$ has negative order, since $\mathrm{ord}\,(M - P) < 0$. Therefore, $P = M_+$. ■

We choose the coefficients of L as a system of generators in R.

Proposition 3.18. The equation for the class of gauge equivalence corresponding to Eq. (3.7) with this choice of the system of generators coincides with the scalar Lax equation (2.1), where $A=-\sum_{i=0}^{m} c_i L_+^{i/k}$.

Proof. It must be shown that if $\mathfrak{L}$ satisfies (3.7), then the corresponding operator L satisfies the Lax equation. For the time being let B denote the ring of smooth functions of x and t. We denote by $B[D, D_t]$ the ring of differential operators of the form $\sum_{i,j} a_{ij}D^iD_t^j$, $a_{ij}\in B$. On $B[\lambda]^k$ we introduce the structure of a $B[D, D_t]$-module so that the operator of multiplication by D_t is equal to $\frac{d}{dt}-\mathcal{A}$ (this is possible, since $\left[\mathfrak{L}, \frac{d}{dt}-\mathcal{A}\right]=0$ and $\left[\frac{d}{dt}-\mathcal{A}, b\right]=\frac{\partial b}{\partial t}$ for $b\in B$). We recall that $\mathcal{A}=\mathcal{M}^+$, where $\mathcal{M}=\sum_{i=0}^{m} c_i\varphi(\Lambda^i)$. Since $\psi = (1, 0,\ldots,0)^t$, it follows that $D_t\cdot\psi=-\mathcal{A}\psi=-\mathcal{M}^+\psi$ which by Proposition 3.16 and Lemma 3.17 is equal to $A\cdot\psi$. From the equalities $L\cdot\psi = \lambda\psi$ and $(D_t - A)\cdot\psi = 0$ it follows that $[D_t - A, L]\cdot\psi = 0$. Since $[D_t - A, L]$ belongs to $B[D]$, from this it follows that $[D_t - A, L] = 0$. ■

3.5. Proposition 3.19. The functions f_i defined by formula (3.6) are densities of conservation laws for Eq. (3.7). The arbitrariness in the choice of T (see Proposition 3.3) leads only to the change of f_i by a total derivative.

The proof is the same as that of Proposition 1.5. ∎

We shall now show that the conservation laws found coincide with the conservation laws for the scalar Lax equation obtained in Sec. 2.

Proposition 3.20. Let $\mathfrak{L}$ be an operator of the form (3.1), let L be the corresponding operator of the form (3.11), and suppose that the functions f_i are defined by formula (3.6). Then $f_0=-\frac{1}{k}u_{k-1}$, and for $i>0$ $f_i+\frac{1}{i}\operatorname{res}L^{i/k}$ is a total derivative.

Proof. In view of the corollary of Theorem 2.9, it suffices to show that

$$D=L^{1/k}+\sum_{i=0}^{\infty}f_iL^{-i/k}, \tag{3.14}$$

where the functions f_i are defined by formula (3.6) in which $\mathfrak{L}_0$ is normalized by the condition $T\psi = \psi$. Conjugating both sides of (3.6) with T, we obtain $\mathfrak{L}=T^{-1}\frac{d}{dx}T+\varphi(\Lambda)+\sum_{i=0}^{\infty}f_i\varphi\times(\Lambda^{-i})$. From Proposition 3.16 and the equality $T\psi = \psi$ it follows that $D\cdot\psi\overset{\mathrm{def}}{=}\mathfrak{L}\psi=\varphi(\Lambda)\psi+\sum_{i=0}^{\infty}f_i\varphi(\Lambda^{-i})\psi=\left(L^{1/k}+\sum_{i=0}^{\infty}f_iL^{-i/k}\right)\cdot\psi$, whence we obtain (3.14). ∎

3.6. We now proceed to the discussion of the Hamiltonian formalism for Eqs. (3.7). The manifold $\mathcal{M}$ on which it is necessary to introduce a Hamiltonian structure is the set of classes of gauge equivalence of operators of the form (3.1), where the elements of the matrix q belong to B_0. We note that if in relation (3.3) the operators $\mathfrak{J}$ and $\tilde{\mathfrak{J}}$ have periodic coefficients, then it follows from the uniqueness of the matrix S that its elements are also periodic. It is convenient to represent functionals on $\mathcal{M}$ as gauge invariant functionals on the set of operators of the form (3.1). The Poisson bracket will be defined on the set $\mathcal{F}$ of gauge invariant functionals of the form (1.11).

For any $u, v\in\operatorname{Mat}(k, B_0)$ we set $(u, v)=\int\operatorname{tr}(u(x)v(x))dx$. If $l\in\mathcal{F}$, $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$, then $\operatorname{grad}_q l$ denotes any element of Mat (k, B_0) such that

$$\frac{d}{d\varepsilon}l(q+\varepsilon h)|_{\varepsilon=0}=(\operatorname{grad}_q l, h), \tag{3.15}$$

for any $h\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$. We emphasize that $\operatorname{grad}_q l$ is not uniquely determined by relation (3.15) but only up to the addition of functions with values in $\mathfrak{n}$.

We define the first and second Hamiltonian structures on $\mathcal{M}$ by formulas analogous to (1.13) and (1.14):

$$\{\varphi, \psi\}_1(q)=-(\operatorname{grad}_q\varphi, [\operatorname{grad}_q\psi, e]), \tag{3.16}$$

$$\{\varphi, \psi\}_2(q)=\left(\operatorname{grad}_q\varphi, \left[\operatorname{grad}_q\psi, \frac{d}{dx}+I+q\right]\right), \tag{3.17}$$

where $\varphi, \psi\in\mathcal{F}$, $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$, and I and e are defined by formula (3.2). It must be verified that a) the definition is correct, i.e., $\{\varphi, \psi\}_1$ and $\{\varphi, \psi\}_2$ do not depend on the arbitrariness in the choice of the gradient; b) gauge invariance of φ and ψ implies invariance of $\{\varphi, \psi\}_1$ and $\{\varphi, \psi\}_2$; c) the brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are skew-symmetric and satisfy the Jacobi identity. Verification of a) reduces to the proof of the equalities

$$(\operatorname{grad}_q\varphi, [\theta, e])=0, \tag{3.18}$$

$$(\theta, [\operatorname{grad}_q\psi, e])=0, \tag{3.19}$$

$$\left(\operatorname{grad}_q\varphi, \left[\theta, \frac{d}{dx}+I+q\right]\right)=0, \tag{3.20}$$

$$\left(\theta, \left[\operatorname{grad}_q\psi, \frac{d}{dx}+I+q\right]\right)=0, \tag{3.21}$$

for any function $\theta\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{n})$. Formulas (1.15) show that Eqs. (3.19) and (3.21) follow from (3.18) and (3.20). Equality (3.18) is obvious, since $[\theta, e] = 0$. Equality (3.20) follows from the gauge invariance of φ. Indeed, we define $q(\varepsilon)$ from the equality $(E+\varepsilon\theta)\mathfrak{L}(E+\varepsilon\theta)^{-1}=\frac{d}{dx}+I+\lambda e+q(\varepsilon)$. Then $\frac{d}{d\varepsilon}\varphi(q(\varepsilon))|_{\varepsilon=0}=\left(\operatorname{grad}_q\varphi, \left[\theta, \frac{d}{dx}+I+q\right]\right)=0$.

To prove b) it suffices to note that if q and $\tilde{q}$ are connected by the relation $\frac{d}{dx}+\tilde{q}+\Lambda=S^{-1}\left(\frac{d}{dx}+q+\Lambda\right)S$, where $S\in C^\infty(\mathbf{R}/\mathbf{Z}, N)$, then $\operatorname{grad}_{\tilde{q}}\varphi=S^{-1}(\operatorname{grad}_q\varphi)S$.

The skew-symmetry of the brackets (3.16) and (3.17) follows from formula (1.15). We shall show that the bracket (3.17) satisfies the Jacobi identity. For any functional $l:\mathrm{Mat}\times(k, B_0)\to\mathbf{C}$ of the form (1.11) we denote by $\tilde{l}$ the functional on $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$ given by the formula $\tilde{l}(q) = l(I + q)$. It is clear that if $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$, then for $\operatorname{grad}_q \tilde{l}$ it is possible to take $\operatorname{grad}_{I+q} l$. Therefore, for any functionals φ, ψ of the form (1.11) such that $\tilde{\varphi}$ and $\tilde{\psi}$ are gauge invariant we have $\{\tilde{\varphi}, \psi\}_2=\{\tilde{\varphi}, \tilde{\psi}\}_2$, where $\{\varphi, \psi\}_2$ is defined by formula (1.14). Hence, the Jacobi identity for the bracket (3.17) follows from the Jacobi identity for (1.14). The Jacobi identity for the bracket (3.16) is verified in exactly the same way [we note only that the bracket (1.13) satisfies the Jacobi identity for any matrix a including $a = -e$].

Remark. From the coordination of the Hamiltonian structures (1.13), (1.14) there follows the coordination of the structures (3.16), (3.17).

For any $n\in\mathbf{N}$ we define the functional $\mathcal{H}_n:\mathcal{M}\to\mathbf{C}$ by the formula $\mathcal{H}_n(\mathfrak{L})=k\int_{\mathbf{R}/\mathbf{Z}} f_n(x)\,dx$, where f_n is defined by formula (3.6). It is clear that $\mathcal{H}_n\in\mathcal{F}$. We note that in the present situation a Hamiltonian $\mathcal{H}\in\mathcal{F}$ defines the evolution not of the operator $\mathfrak{L}$ itself but only of its class of gauge equivalence. Therefore, the Hamiltonian equation is a system of evolution equations for the generators of R.

Proposition 3.21. The equation for the class of gauge equivalence corresponding to Eq. (3.8), where $\mathcal{B}=\varphi(\Lambda^n)_+$, is the Hamiltonian equation corresponding to the Hamiltonian $\mathcal{H}_n$ and the second Hamiltonian structure. This same equation is the Hamiltonian equation corresponding to the Hamiltonian $\mathcal{H}_{n+k}$ and the first Hamiltonian structure.

The proof is analogous to the proof of Proposition 1.9. ■

3.7. If by the means indicated in part 3.3 we identify $\mathcal{M}$ and the manifold M of part 2.3, then $\mathcal{F}$ obviously goes over into a class of functionals on M of the form (2.14).

THEOREM 3.22. The first and second Hamiltonian structures on $\mathcal{M}$ go over, respectively, into the first and second Gel'fand–Dikii structures on M.

In the proof of the theorem we use the following functional $\mathrm{Tr}:\mathrm{Mat}(n, B_0((D^{-1})))\to\mathbf{C}$ [which is a generalization of the functional $\mathrm{Tr}:B_0((D^{-1}))\to\mathbf{C}$ considered in part 2.3]: if $A\in\mathrm{Mat}(n, B_0((D^{-1})))$, then $\mathrm{Tr}\,A\overset{\mathrm{def}}{=}\int_{\mathbf{R}/\mathbf{Z}}(\operatorname{tr}\operatorname{res}A)\,dx$. It is easy to see that if X and Y are matrices over $B_0((D^{-1}))$ of dimension $m\times n$ and $n\times m$, respectively, then $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$.

Proof. Let $X=\sum_{i=1}^{\infty}D^{-i}a_i$, $Y=\sum_{i=1}^{\infty}D^{-i}b_i$ be integral symbols, let l_X and l_Y be the functionals on M they define (see part 2.3), and let $\tilde{l}_X$ and $\tilde{l}_Y$ be the functionals on $\mathcal{M}$, corresponding to l_X and l_Y. Let $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$, $\mathfrak{L}=\frac{d}{dx}+q+\Lambda$, and let $L=D^k+\sum_{i=0}^{k-1}u_iD^i$ be the operator corresponding to $\mathfrak{L}$. It must be shown that $\{\tilde{l}_X, \tilde{l}_Y\}_1(q)=\{l_X, l_Y\}_1(L)$, $\{\tilde{l}_X, \tilde{l}_Y\}_2(q)=\{l_X, l_Y\}_2(L)$, i.e.,

$$(\operatorname{grad}_q\tilde{l}_X, [\operatorname{grad}_q\tilde{l}_Y, e])=\mathrm{Tr}([Y, X)\cdot L), \tag{3.22}$$

$$\left(\operatorname{grad}_q\tilde{l}_X, \left[\operatorname{grad}_q\tilde{l}_Y, \frac{d}{dx}+I+q\right]\right)=\mathrm{Tr}((LY)_+LX-XL(YL)_+). \tag{3.23}$$

It may hereby be assumed that (see the second part of the proof of Proposition 3.10)

$$q=-(u_0e_{1,k}+u_1e_{2,k}+\ldots+u_{k-1}e_{k,k}). \tag{3.24}$$

We shall first prove (3.23). We shall find $\mathrm{grad}_q \tilde{l}_X$. Let $h \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$. We denote by $L(\varepsilon)$ the element of $B_0[D]$ corresponding to the operator $\mathfrak{L}+\varepsilon h$. We recall (see part 3.3) that $L(\varepsilon) = -(\Delta(P(\varepsilon)))^*$, where $P(\varepsilon) = I + q + \varepsilon h + \mathrm{diag}(D,\ldots,D)$. Thus, $\tilde{l}_X(q+\varepsilon h) = \mathrm{Tr}(X\cdot L(\varepsilon)) = \mathrm{Tr}(\Delta(P(\varepsilon))\cdot X^*)$ [we have used the identity $\mathrm{Tr}\, Z^* = -\mathrm{Tr}\, Z$, $Z \in B_0((D^{-1}))$]. We write $P(\varepsilon)$ in the form $\begin{pmatrix}\alpha(\varepsilon) & \beta(\varepsilon)\\ A(\varepsilon) & \gamma(\varepsilon)\end{pmatrix}$, where $A(\varepsilon)$ is a square matrix of order $k-1$. Then $\Delta(P(\varepsilon)) = \beta(\varepsilon) - \alpha(\varepsilon)A^{-1}(\varepsilon)\gamma(\varepsilon)$ (see part 3.3). Writing now h in the form $\begin{pmatrix}\xi & \zeta\\ C & \eta\end{pmatrix}$, where C is a matrix of order $k-1$, and setting $\alpha = \alpha(0)$, $\beta = \beta(0)$, $\gamma = \gamma(0)$, $A = A(0)$, we obtain $\frac{d}{d\varepsilon}\Delta(P(\varepsilon))|_{\varepsilon=0} = \zeta - \xi A^{-1}\gamma - \alpha A^{-1}\eta + \alpha A^{-1}CA^{-1}\gamma$. Therefore, $\frac{d}{d\varepsilon}\tilde{l}_X(q+\varepsilon h)|_{\varepsilon=0} = \mathrm{Tr}(\zeta X^*) - \mathrm{Tr}(\xi\cdot A^{-1}\gamma X^*) - \mathrm{Tr}(\eta\cdot X^*\alpha A^{-1}) + \mathrm{Tr}(C\cdot A^{-1}\gamma X^*\alpha A^{-1})$ whence $\mathrm{grad}_q \tilde{l}_X = \mathrm{res}\begin{pmatrix}-A^{-1}\gamma X^*, & A^{-1}\gamma X^*\alpha A^{-1}\\ X^*, & -X^*\alpha A^{-1}\end{pmatrix}$. In exactly the same way, $\mathrm{grad}_q \tilde{l}_Y = \mathrm{res}\begin{pmatrix}-A^{-1}\gamma Y^*, & A^{-1}\gamma Y^*\alpha A^{-1}\\ Y^*, & -Y^*\alpha A^{-1}\end{pmatrix}$, whence $\left[\mathrm{grad}_q \tilde{l}_Y, \frac{d}{dx}+I+q\right] = \mathrm{res}\left[\begin{pmatrix}-A^{-1}\gamma Y^*, & A^{-1}\gamma Y^*\alpha A^{-1}\\ Y^*, & -Y^*\alpha A^{-1}\end{pmatrix}, \begin{pmatrix}\alpha & \beta\\ A & \gamma\end{pmatrix}\right] = \mathrm{res}\begin{pmatrix}0, & A^{-1}\gamma Y^* L^*\\ 0, & -Y^*L^*\end{pmatrix} - \mathrm{res}\begin{pmatrix}-L^*Y, & L^*Y^*\alpha A^{-1}\\ 0, & 0\end{pmatrix}$. Using now the fact that q has the form (3.24) we have $L = D^k + \sum_{i=0}^{k-1} u_i D^i$, $\alpha A^{-1} = (D, -D^2, \ldots, (-1)^k D^{k-1})$, $A^{-1}\gamma = -((LD^{-1})_+^*, \ldots, (LD^{1-k})_+^*)^t$. Therefore,

$$\mathrm{tr}\left(\mathrm{grad}_q \tilde{l}_X\cdot\left[\mathrm{grad}_q \tilde{l}_Y, \frac{d}{dx}+I+q\right]\right) = -\sum_{i=1}^{k}\mathrm{res}\,(X^*(D^{i-1})^*)\times$$

$$\times\,\mathrm{res}\,((LD^{-i})_+^*\, Y^*L^*)_+ + \sum_{i=1}^{k}\mathrm{res}\,(L^*Y^*(D^{i-1})^*)\cdot\mathrm{res}\,((LD^{-i})_+^*\, X^*) =$$

$$= \sum_{i=1}^{k}\mathrm{res}\,(D^{i-1}YL)\cdot\mathrm{res}\,(X(LD^{-i})_+) - \sum_{i=1}^{k}\mathrm{res}\,(D^{i-1}X)\cdot\mathrm{res}\,(LY(LD^{-i})_+).$$

For any $Z \in B_0((D^{-1}))$ we have $\sum_{i=1}^{k}(LD^{-i})_+\cdot\mathrm{res}\,(D^{i-1}Z) = (LZ_-)_+$ (to see this it suffices to represent Z in the form $\sum_j D^j f_j$). Hence, $\mathrm{tr}\left(\mathrm{grad}_q \tilde{l}_X\cdot\left[\mathrm{grad}_q \tilde{l}_Y, \frac{d}{dx}+I+q\right]\right) = \mathrm{res}\,(X(L(YL)_-)_+) - \mathrm{res}\,(LY(LX)_+) = \mathrm{res}\,(XL(YL)_-) - \mathrm{res}\,((LY)_-LX)$. Thus, the left side of (3.23) is equal to $\mathrm{Tr}\,(XL(YL)_- - (LY)_-LX) = \mathrm{Tr}\,((LY)_+LX - XL(YL)_+)$, as was required to prove.

We shall now deduce (3.22) from (3.23), following [60]. We note that (3.23) remains in force if q is replaced by $q - e$ and L is replaced by $L + 1$. Since $X_+ = Y_+ = 0$, it follows that

$$\left(\mathrm{grad}_{q-e}\tilde{l}_X, \left[\mathrm{grad}_{q-e}\tilde{l}_Y, \frac{d}{dx}+I+q-e\right]\right) =$$

$$= \mathrm{Tr}\,((LY)_+LX - XL(YL)_+) + \mathrm{Tr}\,((LY)_+X - X(YL)_+) = \mathrm{Tr}\,((LY)_+LX - XL(YL)_+) + \mathrm{Tr}\,(L\cdot[Y, X]). \quad (3.25)$$

Since $\tilde{l}_X(L + 1) - \tilde{l}_X(L)$ does not depend on L, it follows that $\tilde{l}_X(q - e) - \tilde{l}_X(q)$ does not depend on q, and hence $\mathrm{grad}_{q-e}\,\tilde{l}_X = \mathrm{grad}_q\,\tilde{l}_X$. Therefore, subtracting (3.23) from (3.25), we obtain (3.22). ■

We note that in the proof of Theorem 3.22 we did not use the assertion that the Gel'fand–Dikii brackets satisfy the Jacobi identity. Moreover, in proving Propositions 3.20, 3.21, and Theorem 3.22, we proved Theorem 2.11 at the same time.

Theorem 3.22 and the results of [39] make it possible to give a group-theoretic interpretation of the second Hamiltonian structure of Gel'fand–Dikii. This interpretation will be presented in part 6.5.

3.8. In this subsection we consider so-called modified Lax equations.

LEMMA 3.23. Equation (3.7) admits the reduction

$$\mathfrak{L} = \frac{d}{dx} + q(x) + \Lambda, \quad q = \mathrm{diag}\,(q_1, \ldots, q_k). \quad (3.26)$$

Proof. Let $\mathfrak{L}$ be an operator of the form (3.26), and let $M \in Z_{\mathfrak{L}}$. It is necessary to prove that $[M^+, \mathfrak{L}]$ is a diagonal matrix. It is clear that in the expansion of $[M^+, \mathfrak{L}]$ in powers of Λ (see Lemma 3.4) there are no negative powers of Λ. On the other hand, it is evident from the equality $[M^+, \mathfrak{L}] = -[M^-, \mathfrak{L}]$ that in this expansion there are no positive powers of Λ. ■

Remark. It can be proved in exactly the same way that for any $r \in \{0, 1, \ldots, k-1\}$ (3.7) admits the reduction $q = (q_{ij})$, where $q_{ij} = 0$ for $j - i > r$.

Equation (3.7) with $q(x) \in \mathrm{Diag}$ we shall call the modified Lax equation. This equation may be considered a reduction of Eq. (1.1). In order to see this, it is necessary to set $\lambda = \zeta^k$ and pass from the operator $\mathfrak{L}$ to the operator $\bar{\mathfrak{L}} = \Phi \mathfrak{L} \Phi^{-1} = \frac{d}{dx} + \zeta a + q(x)$, where $\Phi = \mathrm{diag}(1, \zeta, \ldots, \zeta^{k-1})$, $a = I + e$. The modified Lax equation can then be written in the form $\frac{d\bar{\mathfrak{L}}}{dt} = [\bar{\mathcal{A}}, \bar{\mathfrak{L}}]$, where $\bar{\mathcal{A}}$ in the notation of part 1.1 is given by the formula $\bar{\mathcal{A}} = \sum_{i=0}^{m} c_i \varphi(a^i \zeta^i)_+$.

By the method indicated in part 3.3 we assign to each operator $\mathfrak{L}$ of the form (3.26) an operator L of the form (3.11). We call this mapping the Miura mapping. It is clear that the Miura mapping takes solutions of the modified Lax equation into solutions of the corresponding Lax equation.

Proposition 3.24. The Miura transformation is given by the formula

$$L = (D - q_k) \ldots (D - q_2)(D - q_1). \tag{3.27}$$

Proof. We denote the standard basis in $B[\lambda]^k$ by $e_1, \ldots, e_k$. We recall that $\psi \overset{\text{def}}{=} e_1$. It must be shown that the operator L defined by formula (3.27) satisfies the equality $L \cdot \psi = \lambda \psi$, i.e., $(\mathfrak{L} - q_k) \ldots (\mathfrak{L} - q_2)(\mathfrak{L} - q_1) e_1 = \lambda e_1$. Indeed, $(\mathfrak{L} - q_i) e_i = e_{i+1}$ for $i < k$, $(\mathfrak{L} - q_k) e_k = \lambda e_1$. ■

We note that the Miura transformation is not injective: knowing L, in order to find $q_1, \ldots, q_k$ from relation (3.27) it is necessary to solve a system of ordinary differential equations. This means that different operators of the form (3.26) may be gauge equivalent.

Example. Let $L = D^2 + u = (D + q)(D - q)$. If $A = L_+^{3/2}$, then the Lax equation (2.1) is the Korteweg–de Vries equation $u_t = (u''' + 6uu')/4$. The corresponding modified Lax equation $\frac{d\mathfrak{L}}{dt} = -[\varphi(\Lambda^3)^+, \mathfrak{L}]$, where $\mathfrak{L} = \frac{d}{dx} + \begin{pmatrix} q & \lambda \\ 1 & -q \end{pmatrix}$, is the modified Korteweg–de Vries equation $q_t = (q''' - 6q^2q')/4$. The relation $u = -q' - q^2$ connecting the solutions of these two equations was found by Miura.

We proceed to a discussion of the Hamiltonian formalism for the modified Lax equation. The manifold $\bar{\mathcal{M}}$, on which it is necessary to introduce a Hamiltonian structure consists of all operators $\mathfrak{L}$ of the form (3.26) where $q_i \in B_0$. The gradient of a functional $l: \bar{\mathcal{M}} \to \mathbf{C}$ at a point $q \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathrm{Diag})$ is a function $\mathrm{grad}_q l \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathrm{Diag})$ such that relation (3.15) is satisfied for any $h \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathrm{Diag})$. The gradient is uniquely determined by this condition. The Poisson bracket on $\bar{\mathcal{M}}$ is given by the formula

$$\{\varphi, \psi\}(q) = \left(\frac{d}{dx} \mathrm{grad}_q \varphi, \mathrm{grad}_q \psi\right). \tag{3.28}$$

We denote by $\bar{\mathcal{H}}_n$ the restriction to $\bar{\mathcal{M}}$ of the functional $\mathcal{H}_n: \mathcal{M} \to \mathbf{C}$ of part 3.6.

Proposition 3.25. The modified Lax equation $\frac{d\mathfrak{L}}{dt} = [\varphi(\Lambda^n)^+, \mathfrak{L}]$ is the Hamiltonian equation corresponding to the Hamiltonian $\bar{\mathcal{H}}_n$.

Proof. It is easy to see that the Hamiltonian equation corresponding to $\bar{\mathcal{H}}_n$ has the form $\frac{d\mathfrak{L}}{dt} = -(\mathrm{grad}\,\bar{\mathcal{H}}_n)'$. Just as in the proof of Proposition 1.9, it can be verified that $\mathrm{grad}\,\bar{\mathcal{H}}_n = \varphi(\Lambda^n)_0$, where $\varphi(\Lambda^n)^0$ is the free term in the expansion of $\varphi(\Lambda^n)$ in powers of Λ. It remains to show that $[\varphi(\Lambda^n)^+, \mathfrak{L}] = -\varphi(\Lambda^n)'$. Indeed, $[\varphi(\Lambda^n)^+, \mathfrak{L}] = \left[\varphi(\Lambda^n)_0, \frac{d}{dx} + q\right] = -\varphi(\Lambda^n)_0'$, since $\varphi(\Lambda^n)_0$ and q are diagonal matrices. ■

According to Proposition 3.19, the functionals $\bar{\mathcal{H}}_n$ are conservation laws for the modified Lax equations; from Proposition 3.25 we therefore obtained the following result.

COROLLARY. $\{\mathcal{H}_m^-, \bar{\mathcal{H}}_n\}=0$. ■

We introduce the important concept of a Hamiltonian mapping. A manifold with a Hamiltonian structure given on it we call a Hamiltonian manifold. Let M_1 and M_2 be Hamiltonian manifolds. We denote by F_i the class of functionals $M_i \to \mathbf{C}$ on which the Poisson bracket is defined. If l is a functional on M_2, then for any mapping $f:M_1 \to M_2$ $f^*(l)$ denotes the functional on M_1 given by the formula $f^*(l)(\eta)=l(f(\eta))$, $\eta \in M_1$.

Definition. A mapping $f:M_1 \to M_2$ is called a Hamiltonian mapping if $f^*(F_2)\subset F_1$ and for any $\varphi, \psi \in F_2$, $f^*\{\varphi, \psi\} = \{f^*(\varphi, f^*(\psi)\}$.

Proposition 3.26. The Miura transformation $\mu:\bar{\mathcal{M}}\to M$, where M is the manifold of part 2.3 equipped with the second Hamiltonian structure, is a Hamiltonian mapping.

Proof. It follows from Theorem 3.22 that in the assertion to be proved it is possible to replace M by $\mathcal{M}$. Thus, we must verify that if φ and ψ are functionals on $\mathcal{M}$, $\bar{\varphi}$, and $\bar{\psi}$ and their restrictions to $\bar{\mathcal{M}}$, then $\{\bar{\varphi}, \bar{\psi}\}=\overline{\{\varphi, \psi\}}_2$. In other words, we must prove the equality $((\mathrm{grad}_q\bar{\varphi})', \mathrm{grad}_q\bar{\psi})=\left(\mathrm{grad}_q\varphi, \left[\mathrm{grad}_q\psi, \frac{d}{dx}+I+q\right]\right)$, $q(x)\in\mathrm{Diag}$. We normalize $\mathrm{grad}_q\varphi$ and $\mathrm{grad}_q\psi$ so that they are lower triangular matrices; then $(\mathrm{grad}_q\varphi, [\mathrm{grad}_q\psi, I+q])=0$. It remains to note that $\mathrm{grad}_q\bar{\varphi}=(\mathrm{grad}_q\varphi)_{\mathrm{diag}}$, $\mathrm{grad}_q\bar{\psi}=(\mathrm{grad}_q\psi)_{\mathrm{diag}}$ and to use formula (1.15). ■

The Hamiltonian property of the Miura mapping was first proved in [60]. The simplicity of the proof of this assertion presented above as compared with the proof in [60] is achieved to considerable extent due to the use of the nontrivial Theorem 3.2.

4. THE METHOD OF ZAKHAROV—SHABAT FOR LIE ALGEBRAS

4.1. Let $\mathfrak{G}$ be a finite-dimensional Lie algebra, and let a be an element of $\mathfrak{G}$. We set $\mathfrak{H}\stackrel{\mathrm{def}}{=}\mathrm{Ker}\,\mathrm{ad}\,a$, $\mathfrak{H}^\perp\stackrel{\mathrm{def}}{=}\mathrm{Im}\,\mathrm{ad}\,a$. We assume that a) the Lie algebra $\mathfrak{H}$ is commutative; b) $\mathfrak{G}=\mathfrak{H}\oplus\mathfrak{H}^\perp$. For any $\varphi\in\mathfrak{G}$ we denote by $\varphi_{\mathfrak{H}}$ the projection of φ onto $\mathfrak{H}$. We note that if on $\mathfrak{G}$ there is given a nondegenerate, symmetric, invariant bilinear form then $\mathfrak{H}^\perp$ coincides with the orthogonal complement of $\mathfrak{H}$. In the case where $\mathfrak{G}=\mathrm{Mat}(k, \mathbf{C})$ for a it is possible to take any matrix with distinct eigenvalues. If $\mathfrak{G}$ is a simple Lie algebra, then for a it is possible to take an arbitrary regular element.

We consider the relation (1.1), where $L=\frac{d}{dx}+q-\lambda a$, $A=\sum_{i=0}^{m} A_i\lambda^i$, q and A_i are functions of x, t with values in $\mathfrak{G}$; a is an element of $\mathfrak{G}$ satisfying conditions a), b) formulated above. Our purpose is to carry over the main results of the first section to this case. As in Sec. 1, the key feature is the reduction of the operator

$$L=\frac{d}{dx}+q-\lambda a, \quad q\in C^\infty(\mathbf{R}, \mathfrak{G}) \tag{4.1}$$

to canonical form (see Proposition 1.2). However, now if no representation of $\mathfrak{G}$ has been chosen we cannot conjugate L with a formal series in λ^{-1} simply because the operation of multiplication is not defined in $\mathfrak{G}$. The analogue of conjugation in our situation is application of the operator $e^{\mathrm{ad}u}$. This operator acts as follows: if $u=\sum_{i=1}^{\infty} u_i\lambda^{-i}$, $u_i\in\mathfrak{G}$, $\psi\in\mathfrak{G}((\lambda^{-1}))$, then $e^{\mathrm{ad}u}(\psi)\stackrel{\mathrm{def}}{=}\psi+[u, \psi]+\frac{1}{2!}[u, [u, \psi]]+\dots$. We note that in the case where $\mathfrak{G}=\mathrm{Mat}(k, \mathbf{C})$, $e^{\mathrm{ad}u}(\psi)=e^u\psi e^{-u}$.

It is not hard to verify that the mapping $e^{\mathrm{ad}u}:\mathfrak{G}((\lambda^{-1}))\to\mathfrak{G}((\lambda^{-1}))$ is a Lie-algebra automorphism. From the Campbell—Hausdorff formula [40] it follows that automorphisms of this type form a group.

We consider the Lie algebra $\tilde{\mathfrak{G}}$ of formal series of the form $\sum_{i=-\infty}^{n}\left(c_i\frac{d}{dx}+p_i\right)\lambda^i$. $c_i\in\mathbf{C}$, $p_i\in C^\infty(\mathbf{R}, \mathfrak{G})$. We note that the operator L belongs to $\tilde{\mathfrak{G}}$. It is clear that the mapping $e^{\mathrm{ad}U}$, where $U=\sum_{i=1}^{\infty}u_i\lambda^{-i}$, $u_i\in C^\infty(\mathbf{R}, \mathfrak{G})$, is an automorphism of $\tilde{\mathfrak{G}}$. We denote the group of all such automorphisms by G.

Proposition 4.1. There exists a formal series $U=\sum_{i=1}^{\infty} u_i\lambda^{-i}$, $u_i\in C^\infty(\mathbf{R}, \mathfrak{G})$ such that $L_0 \overset{\text{def}}{=} e^{\mathrm{ad}U}(L)$ has the form

$$L_0=\frac{d}{dx}-\lambda a+\sum_{i=0}^{\infty} h_i\lambda^{-i}, \quad h_i\in C^\infty(\mathbf{R}, \mathfrak{H}). \tag{4.2}$$

The automorphism $e^{\mathrm{ad}U}$ is defined uniquely up to multiplication on the left by automorphisms of the form $e^{\mathrm{ad}U_0}$, where $U_0=\sum_{i=1}^{\infty} v_i\lambda^{-i}$, $v_i\in C^\infty(\mathbf{R}, \mathfrak{H})$. It is possible to choose the series U in exactly one way so that its coefficients u_i belong to $C^\infty(\mathbf{R}, \mathfrak{H}^\perp)$. Here the u_i are differential polynomials in q.[6]

Proof. Equating coefficients of λ^{-n} on both sides of the equality $e^{\mathrm{ad}\sum_{i=1}^{\infty} u_i\lambda^{-i}}(L)=\frac{d}{dx}-\lambda a+\sum_{i=0}^{\infty} h_i\lambda^{-i}$, we find that $[u_{n+1}, a] + h_n$ can be expressed in terms of $u_1,\ldots,u_n$, $h_0,\ldots,h_{n-1}$. Since the restriction of $\mathrm{ad}\, a$ to $\mathfrak{H}^\perp$ is an isomorphism, any element in $\mathfrak{G}$ can be uniquely represented in the form $[X, a] + Y$, where $X\in\mathfrak{H}^\perp$, $Y\in\mathfrak{H}$. Therefore, knowing $u_1,\ldots,u_n$, $h_0,\ldots,h_{n-1}$, it is possible to uniquely determine $u_{n+1}\in C^\infty(\mathbf{R}, \mathfrak{H}^\perp)$ and $h_n\in C^\infty(\mathbf{R}, \mathfrak{H})$.

Let $g_1, g_2\in G$ be such that the operators $L_1\overset{\text{def}}{=}g_1(L)$ and $L_2\overset{\text{def}}{=}g_2(L)$ have the form (4.2). Since G is a group, $g_2g_1^{-1}=\exp\left(\mathrm{ad}\sum_{i=1}^{\infty} v_i\lambda^{-i}\right)$, where $v_i\in C^\infty(\mathbf{R}, \mathfrak{G})$. We must verify that $v_i\in C^\infty(\mathbf{R}, \mathfrak{H})$. Equating coefficients of λ^{-n} in the equality $\exp\left(\mathrm{ad}\sum_{i=1}^{\infty} v_i\lambda^{-i}\right)(L_1)=L_2$, we see that if $v_1,\ldots, v_n\in C^\infty(\mathbf{R}, \mathfrak{H})$, then $[v_{n+1}, a]\in C^\infty(\mathbf{R}, \mathfrak{H})$, and hence $v_{n+1}\in C^\infty(\mathbf{R}, \mathfrak{H})$. ■

We set $Z_L=\{M\in C^\infty(\mathbf{R}, \mathfrak{G}((\lambda^{-1})))\,|\,[M, L]=0\}$. In exactly the same way as in Sec. 1, it can be proved that $Z_L=e^{-\mathrm{ad}U}(\mathfrak{H}((\lambda^{-1})))$, where U is the series of Proposition 4.1. For any $b\in\mathfrak{H}((\lambda^{-1}))$ we denote by $\varphi(b)$ the series $e^{-\mathrm{ad}U}(b)$. It is easy to see (see Lemma 1.1) that if $A=\varphi(b)_+$, where $b\in\mathfrak{H}((\lambda^{-1}))$, then the commutator [A, L] does not depend on λ. Hence, relation (1.1), where

$$A=\sum_{i=0}^{m}\varphi(b_i\lambda^i)_+, \quad b_i\in\mathfrak{H} \tag{4.3}$$

is an evolution equation for q. For brevity we call this equation Eq. (1.1).

4.2. The following assertions are proved in analogy to Propositions 1.4, 1.5, 1.7.

Proposition 4.2. If we set $A=\varphi(b\lambda^n)_+$, where $b\in\mathfrak{H}$, then Eq. (1.1) has the form $\partial q/\partial t = Pq^{(n)} + f(q, q',\ldots,q^{(n-1)})$, where P is a linear operator on $\mathfrak{G}$ annihilating $\mathfrak{H}$ and such that its restriction to $\mathfrak{H}^\perp$ is equal to $\mathrm{ad}\, b(\mathrm{ad}\, a)^{-n}$. If we assume that $q^{(i)}$ has degree of homogeneity i + 1, then f is a homogeneous polynomial of degree of homogeneity n + 1. ■

Proposition 4.3. The functions $h_i(x)$ defined by formula (4.2) are densities of conservation laws for Eq. (1.1). Here $h_0=q_{\mathfrak{H}}$; if i > 0, then up to total derivatives the linear part of h_i is equal to zero, while the quadratic part has the form $\frac{1}{2}\left[(\mathrm{ad}\, a)^{-i}\frac{d^{i-1}}{dx^{i-1}}(q_{\mathfrak{H}}-q), q\right]_{\mathfrak{H}}$. ■

Proposition 4.4. We consider the equations

$$\frac{\partial L}{\partial t}=[\varphi(u)_+, L], \quad u\in\mathfrak{H}((\lambda^{-1})),$$

$$\frac{\partial L}{\partial \tau}=[\varphi(\tilde{u})_+, L], \quad \tilde{u}\in\mathfrak{H}((\lambda^{-1})),$$

then $\partial^2L/\partial t\partial\tau = \partial^2L/\partial\tau\partial t$, where the derivatives are computed by these equations. ■

[6]The words "u is a differential polynomial in q," where u and q are functions with values in vector spaces V and W, mean here and henceforth that for some (and hence any) choice of bases in V and W the coordinates of u are differential polynomials in the coordinates of q.

4.3. In this and the following subsection we shall discuss the Hamiltonian formalism for Eqs. (1.1).

As the manifold M on which a Hamiltonian structure is introduced we take $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$, while for the class of functionals F we take the set of all functionals from M to C of the form (1.11).

We assume that on $\mathfrak{G}$ there is given a nondegenerate, symmetric, invariant bilinear form (,). We recall the invariance of a form means that for any u, v, $w \in \mathfrak{G}$ the equality (u, ad v(w)) = −(ad v(u), w) holds. For any functions $g, h \in M$ we set $(g, h) = \int_{\mathbf{R}/\mathbf{Z}} (g(x), h(x))\,dx$. It is obvious that (g, h') = −(h, g').

We define the Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ on M by formulas (1.13) and (1.14). In exactly the same way as in part 1.4, it can be verified that any linear combination of these brackets is a Poisson bracket.

Let $u = \sum_{i=0}^{m} b_i \lambda^i$, $b_i \in \mathfrak{H}$. We define the functional $H_u: M \to \mathbf{C}$ by the formula $H_u(q) = \sum_{i=0}^{m} (b_i, h_i)$, where h_i are the same as in Proposition 4.1. By Proposition 4.3, the functionals H_u are conservation laws for Eq. (1.1).

Proposition 4.5. 1) Equation (1.1), where A is defined by formula (4.3), is the Hamiltonian equation corresponding to the Hamiltonian H_u and the second Hamiltonian structure. 2) This same equation is the Hamiltonian equation corresponding to the Hamiltonian $H_{\lambda u}$ and the first Hamiltonian structure.

The proof of this assertion differs in only minor details from the proof of Proposition 1.9. ■

COROLLARY. For any $u, \tilde{u} \in \mathfrak{H}((\lambda^{-1}))$, $\{H_u, H_{\tilde{u}}\}_1 = \{H_u, H_{\tilde{u}}\}_2 = 0$. ■

4.4. In [37-39] Reiman and Semenov-Tyan-Shanskii constructed an imbedding of the manifold M in the dual space to a remarkable infinite-dimensional Lie algebra $\hat{\mathfrak{G}}$ such that the second Hamiltonian structure on M is induced by the Kirillov structure on $\hat{\mathfrak{G}}^*$. In this subsection we present the construction of Reiman and Semenov-Tyan-Shanskii. It will be used in part 6.5 in constructing a group-theoretic interpretation of the second Hamiltonian structure of Gel'fand–Dikii.

We first make a remark of general character. Let X be a Hamiltonian manifold, and let $\mathfrak{A}$ be a subalgebra of the Lie algebra of functionals on X. Then the mapping $i: X \to \mathfrak{A}^*$ assigning to each point $x \in X$ the functional $l_x: \mathfrak{A} \to \mathbf{C}$ given by the formula $l_x(f) = f(x)$ is a Hamiltonian mapping (see the end of Sec. 3) if the Hamiltonian structure of A. A. Kirillov is considered on $\mathfrak{A}^*$. If, moreover, functions in $\mathfrak{A}$ separate points of X, then i is an imbedding. In this case X may be considered a submanifold of $\mathfrak{A}^*$, and the Hamiltonian structure on X is induced by the Kirillov structure on $\mathfrak{A}^*$. Of course, an imbedding i of the type described above can be useful only if the algebra $\mathfrak{A}$ is not too large.

As X we now take the manifold $M = C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$, equipped with the second Hamiltonian structure. For any $u \in M$, $c \in \mathbf{C}$ we define the functional $\varphi_{u,c}: M \to \mathbf{C}$ by the formula $\varphi_{u,c}(q) = (q, u) + c$. We have $\{\varphi_{u_1,c_1}, \varphi_{u_2,c_2}\} = \varphi_{u,c}$, where

$$u = [u_1, u_2], \quad c = (u_1', u_2). \tag{4.4}$$

Thus, the set $\hat{\mathfrak{G}} \overset{\text{def}}{=} \{\varphi_{u,c} \mid u \in M, c \in \mathbf{C}\}$ is a Lie algebra. Applying the construction described above to the pair $(M, \hat{\mathfrak{G}})$, we obtain the desired imbedding $i: M \to \hat{\mathfrak{G}}^*$. It is clear that $i(M) \subset \{l \in \hat{\mathfrak{G}}^* \mid l(\varphi_{0,1}) = 1\}$. This inclusion is not an equality: if $l \in \hat{\mathfrak{G}}^*$, $l(\varphi_{0,1}) = 1$, then l has the form $l(\varphi_{u,c}) = (q, u) + c$, where q is a generalized function $\mathbf{R}/\mathbf{Z} \to \mathfrak{G}$. Following [38], we understand by $\hat{\mathfrak{G}}^*$ below the set of "smooth" linear functionals on $\hat{\mathfrak{G}}$ (i.e., functionals of the form $\varphi_{u,c} \to$ (q, u) + αc, where $\alpha \in \mathbf{C}$, $q \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$). With this interpretation of $\hat{\mathfrak{G}}^*$ we have i(M) = $\{l \in \hat{\mathfrak{G}}^* \mid l(\varphi_{0,1}) = 1\}$.

We shall discuss the structure of $\hat{\mathfrak{G}}$ as a Lie algebra. It follows from (4.4) that the one-dimensional subspace in $\hat{\mathfrak{G}}$, consisting of functionals of the form $\varphi_{0,c}$, $c \in \mathbf{C}$ is contained in the center of $\hat{\mathfrak{G}}$. Identifying this subspace with $\mathbf{C}$, we see that the algebra $\hat{\mathfrak{G}}/\mathbf{C}$ is isomorphic to $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$. The algebra $\hat{\mathfrak{G}}$ is hereby not isomorphic to the direct sum of the algebras $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$ and $\mathbf{C}$. Thus, $\hat{\mathfrak{G}}$ is a nontrivial central extension of $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$ by $\mathbf{C}$. It is known that if the algebra $\mathfrak{G}$ is simple, then such an extension is unique up to isomorphism. Therefore, in the case where $\mathfrak{G}$ is a simple Lie algebra the manifold M has the following abstract description: let $\hat{\mathfrak{G}}$ be a nontrivial central extension of $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$ by $\mathbf{C}$; then M is the hyperplane in $\hat{\mathfrak{G}}^*$ consisting of functionals $l\in\hat{\mathfrak{G}}^*$ taking the value 1 on the element $1\in\mathbf{C}\subset\hat{\mathfrak{G}}$, and the Hamiltonian structure on M is induced by the Kirillov structure on $\hat{\mathfrak{G}}^*$.

In conclusion, we note that the imbedding $M\to\hat{\mathfrak{G}}^*$ described above makes it possible to interpret the Hamiltonian H_u (see Proposition 4.5) in terms of a so-called Adler scheme (see [37-39]).

5. SOME FACTS CONCERNING SEMISIMPLE LIE ALGEBRAS AND KATS—MOODY ALGEBRAS

As always in this work, in the present section we consider Lie algebras only over $\mathbf{C}$.

5.1. Definition. A Lie algebra is called simple if it is finite-dimensional, non-Abelian, and contains no nontrivial ideals. A semisimple Lie algebra is the direct product of a finite number of simple algebras.

We shall recall the structure and classification of semisimple Lie algebras.

Definition. A system of Weyl generators of a Lie algebra $\mathfrak{G}$ is a system of generators X_i, Y_i, H_i, $1\leqslant i\leqslant r$ of it such that a) $X_i\neq 0$, $Y_i\neq 0$, $H_i\neq 0$ for all i; b) for any i, j the relations

$$[H_i, H_j]=0, \tag{5.1}$$
$$[X_i, Y_j]=\delta_{ij}H_i, \tag{5.2}$$
$$[H_i, X_j]=N_{ij}X_j, \tag{5.3}$$
$$[H_i, Y_j]=-N_{ij}Y_j, \tag{5.4}$$

hold where (N_{ij}) is a nondegenerate matrix such that $N_{ii} = 2$ for any i.

Remark. If $N_{ii}\neq 0$, then, multiplying X_i and H_i by $2/N_{ii}$, it can be arranged that $N_{ii} = 2$.

Proposition 5.1. 1) In order that in a finite-dimensional Lie algebra $\mathfrak{G}$ there exist a system of Weyl generators it is necessary and sufficient that $\mathfrak{G}$ be semisimple. 2) Suppose that the Lie algebra $\mathfrak{G}$ is semisimple and $\{X_i, Y_i, H_i\}$ and $\{\tilde{X}_j, \tilde{Y}_j, \tilde{H}_j\}$, $i\in\{1,\ldots,r\}$, $j\in\{1,\ldots,\tilde{r}\}$ are Weyl generators of it. Then a) $\tilde{r} = r$; b) there exist an inner automorphism $\varphi:\mathfrak{G}\to\mathfrak{G}$ and a permutation $\sigma\in S_r$ such that $\tilde{X}_i=\varphi(X_{\sigma(i)})$, $\tilde{Y}_i=\varphi(Y_{\sigma(i)})$, $\tilde{H}_i=\varphi(H_{\sigma(i)})$ for any i. Here φ and σ are unique.

We recall that an automorphism of $\mathfrak{G}$ is inner if it can be represented in the form of a product of a finite number of automorphisms of the form $\exp(\mathrm{ad}\,a)$, $a\in\mathfrak{G}$. If $\mathfrak{G}$ is realized as a subalgebra of Mat (n, $\mathbf{C}$) and G is a connected Lie subgroup in GL (n, $\mathbf{C}$) with Lie algebra $\mathfrak{G}$, then inner automorphisms of $\mathfrak{G}$ are automorphisms of the form $x\to TxT^{-1}$, $T\in G$.

To prove Proposition 5.1 it suffices to use Theorem 1 and Lemma 16 of the work [21] and also Proposition 1 of Sec. 4, Theorem 1 of Sec. 4, and Proposition 5 of Sec. 5 of Chap. 8 of the book [4].

Let $\mathfrak{G}$ be a semisimple Lie algebra. In it we choose a system of Weyl generators X_i, Y_i, H_i, $1\leqslant i\leqslant r$. The number r is called the rank of $\mathfrak{G}$ [this definition is correct by assertion 2) of Proposition 5.1]. The matrix (N_{ij}) [see formulas (5.3) and (5.4)] is called the Cartan matrix of the algebra $\mathfrak{G}$. Assertion 2) of Proposition 5.1 shows that up to simultaneous permutation of rows and columns this matrix does not depend on the choice of the system of Weyl generators.

Proposition 5.2. 1) $N_{ij}\in\mathbf{Z}$. 2) If $i\neq j$, then $N_{ij}\leqslant 0$, $N_{ij}N_{ji}\leqslant 3$. 3) $N_{ij}=0\Leftrightarrow N_{ji}=0$.

A proof is given, for example, in part 3 of the book [40].

Proposition 5.3. If $i\neq j$, then

$$(\mathrm{ad}\,X_i)^{1-N_{ij}}\cdot X_j=(\mathrm{ad}\,Y_i)^{1-N_{ij}}Y_j=0. \tag{5.5}$$

2) Equalities (5.1)-(5.5) form a complete system of relations between the elements X_i, Y_i, H_i.

A proof is given, for example, in [4], Chap. 8, Sec. 4, Proposition 4.

COROLLARY. A semisimple Lie algebra is uniquely determined by its Cartan matrix.

Instead of writing out the Cartan matrix, it is customary to present the corresponding Dynkin scheme — a graph with vertices $c_1,\ldots,c_r$ such that 1) the number of segments connecting c_i and c_j, $i \neq j$ is equal to $N_{ij}N_{ji}$; 2) these segments are equipped with an arrow pointing to c_i if and only if $N_{ij} < N_{ji}$. From Proposition 5.2 and the equality $N_{ii} = 2$ it follows that the Cartan matrix can be uniquely recovered from the Dynkin scheme.

Proposition 5.4. 1) The Dynkin schemes of simple Lie algebras are those and only those graphs presented in Table 1. 2) Let $\mathfrak{G}_1,\ldots,\mathfrak{G}_k$ be simple Lie algebras. Then the Dynkin schemes of the algebra $\mathfrak{G}_1\times\ldots\times\mathfrak{G}_k$ is the disjoint union of the Dynkin schemes of the algebras $\mathfrak{G}_1,\ldots,\mathfrak{G}_k$.

Assertion 1) is proved in Sec. 4, Chap. 6 of the book [3], while assertion 2) follows from the definitions (as a system of Weyl generators of the algebra $\mathfrak{G}_1\times\ldots\times\mathfrak{G}_k$ it is possible to take the union of the systems of Weyl generators of the algebras $\mathfrak{G}_1,\ldots,\mathfrak{G}_k$).

Remark. When we speak, say, of a Lie algebra of type E_6 we have in mind the simple Lie algebra whose Dynkin scheme has type E_6. The index 6 designates the number of vertices of the Dynkin scheme, and hence the rank of the algebra is equal to 6.

Examples of simple Lie algebras are the algebras $\mathfrak{sl}(n)$ for $n\geqslant 2$, $\mathfrak{o}(2n+1)$ for $n\geqslant 1$, $\mathfrak{sp}(2n)$ for $n \geqslant 1$, and $\mathfrak{o}(2n)$ for $n \geqslant 3$ which are usually called the classical Lie algebras (see Appendix 1). Their Dynkin schemes are A_{n-1}, B_n, C_n, and D_n, respectively. Since $A_1 = B_1 = C_1$, $B_2 = C_2$, $A_3 = D_3$, it follows that $\mathfrak{sl}(2)\simeq\mathfrak{o}(3)\simeq\mathfrak{sp}(2)$, $\mathfrak{o}(5)\simeq\mathfrak{sp}(4)$, $\mathfrak{sl}(4)\simeq\mathfrak{o}(6)$. We note further that $\mathfrak{o}(4)$ is a semisimple Lie algebra isomorphic to $\mathfrak{sl}(2)\times\mathfrak{sl}(2)$ (see Appendix 1).

We recall the structure of the group of automorphisms of a semisimple Lie algebra $\mathfrak{G}$. We denote by $\mathrm{Aut}^0\mathfrak{G}$ the set of inner automorphisms of $\mathfrak{G}$. $\mathrm{Aut}^0\mathfrak{G}$ is a normal subgroup of $\mathrm{Aut}\,\mathfrak{G}$. Let Γ be the Dynkin scheme of $\mathfrak{G}$, $\tau\in\mathrm{Aut}\,\Gamma$, and let $\tau(c_i)=c_{\sigma(i)}$. Then there exists exactly one automorphism $\varphi_\tau:\mathfrak{G}\to\mathfrak{G}$ such that $\varphi_\tau(X_i)=X_{\sigma(i)}$, $\varphi_\tau(Y_i)=Y_{\sigma(i)}$, $\varphi_\tau(H_i)=H_{\sigma(i)}$ (the existence of φ_τ follows from Proposition 5.3). We call φ_τ the automorphism $\mathfrak{G}$ induced by τ. It follows from assertion 2) of Proposition 5.1 that any automorphism of $\mathfrak{G}$ can uniquely be represented in the form $f\circ\varphi_\tau$, where $f\in\mathrm{Aut}^0\mathfrak{G}$, $\tau\in\mathrm{Aut}\,\Gamma$. Therefore, $\mathrm{Aut}\,\mathfrak{G}/\mathrm{Aut}^0\mathfrak{G}\simeq\mathrm{Aut}\,\Gamma$. If $\psi\in\mathrm{Aut}\,\mathfrak{G}$ and $\psi=f\circ\varphi_\tau$, where $f\in\mathrm{Aut}^0\mathfrak{G}$, $\tau\in\mathrm{Aut}\,\Gamma$, then we call τ the automorphism of Γ determined by the automorphism ψ.

In conclusion we note that with each system of Weyl generators of a semisimple Lie algebra $\mathfrak{G}$ there are connected three subalgebras in $\mathfrak{G}$, which will play an important role in Sec. 6: the Cartan subalgebra $\mathfrak{X}$, generated by all elements H_i, the Borel subalgebra $\mathfrak{H}$, generated by all elements H_i and Y_i, and also the subalgebra $\mathfrak{n}$, generated by all elements Y_i. It is easy to see that a) $\mathfrak{H}$ is commutative; b) the elements H_i form a basis in $\mathfrak{H}$; c) $\mathfrak{H}=\mathfrak{H}\oplus\mathfrak{H}$. If for the classical Lie algebras we choose the Weyl generators as done in Appendix 1, then $\mathfrak{G}$ is the set of diagonal matrices in $\mathfrak{G}$, $\mathfrak{H}$ is the set of upper triangular matrices in $\mathfrak{G}$, and $\mathfrak{n}$ is the set of matrices in $\mathfrak{H}$ with zeros on the main diagonal.

5.2. Let $\mathfrak{A}$ be a Lie algebra; then on $\mathfrak{A}[\zeta,\zeta^{-1}]$ there is a natural structure of a Lie algebra. If $\varphi:\mathfrak{A}\to\mathfrak{A}$ is an automorphism of finite order n, then we set $L(\mathfrak{A},\varphi)=\{f\in\mathfrak{A}[\zeta,\zeta^{-1}]\mid f(\zeta e^{\frac{2\pi i}{n}})=\varphi(f(\zeta))\}$. $L(\mathfrak{A},\varphi)$ is a Lie subalgebra in $\mathfrak{A}[\zeta,\zeta^{-1}]$.

Definition. A Kats–Moody algebra is a Lie algebra of the form $L(\mathfrak{A},\varphi)$, where $\mathfrak{A}$ is a simple Lie algebra and $\varphi:\mathfrak{A}\to\mathfrak{A}$ is an automorphism of finite order.

The next result follows from results of the work [23].

Proposition 5.5. 1) If $L(\mathfrak{A}_1,\varphi_1)\simeq L(\mathfrak{A}_2,\varphi_2)$, where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple Lie algebras, then $\mathfrak{A}_1\simeq\mathfrak{A}_2$. 2) Let φ_1 and φ_2 be automorphisms of finite order of a simple Lie algebra $\mathfrak{A}$. In order that $L(\mathfrak{A},\varphi_1)\simeq L(\mathfrak{A},\varphi_2)$ it is necessary and sufficient that the automorphisms of the Dynkin scheme $\mathfrak{A}$, determined by φ_1 and φ_2, be conjugate.

We note that the group of automorphisms of the Dynkin scheme of a simple Lie algebra (see Table 1) either has order 1 or 2 or is isomorphic to S_3 (the last possibility is realized only in the case of D_4). Therefore, the conjugacy class of an automorphism of the Dynkin scheme is uniquely determined by its order. If $\mathfrak{A}$ has, say, type E_6 and $\varphi\in\mathrm{Aut}\,\mathfrak{A}$ determines an automorphism of the Dynkin scheme of order 2, then it is said that the algebra $L(\mathfrak{A},\varphi)$

TABLE 2

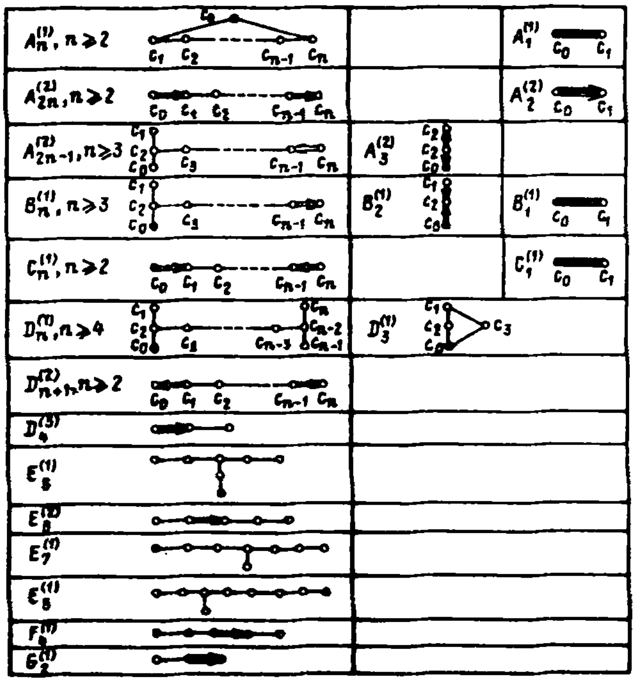
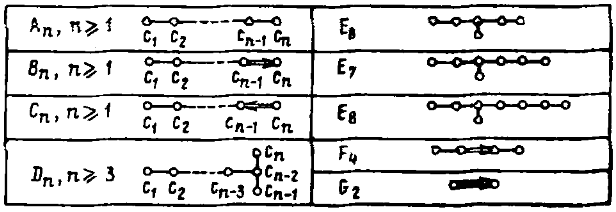


TABLE 1



has type $E_6^{(2)}$. The Kats–Moody algebras of types $B_n^{(1)}$, $C_n^{(1)}$, $A_n^{(k)}$, $D_n^{(k)}$ (k = 1, 2) are called classical. Their explicit realizations are presented in Appendix 2.

Proposition 5.5 shows that all Kats–Moody algebras are exhausted by algebras of the form $L(\mathfrak{A}, \varphi)$, where $\varphi: \mathfrak{A} \to \mathfrak{A}$ is induced by an automorphism of the Dynkin scheme of the algebra $\mathfrak{A}$. For our purposes, however, it is more convenient to choose as representers of the cosets of the group $\mathrm{Aut}\,\mathfrak{A}$ by the subgroup $\mathrm{Aut}^0\mathfrak{A}$ not the automorphisms induced by automorphisms of the Dynkin scheme but the so-called Coxeter automorphisms.

Definition. An automorphism C of a simple Lie algebra $\mathfrak{A}$ is called a Coxeter automorphism if 1) the algebra $\mathfrak{A}^C \overset{\mathrm{def}}{=} \{x \in \mathfrak{A} \mid Cx = x\}$ is Abelian; 2) among all automorphisms $\varphi \in C \cdot \mathrm{Aut}^0\mathfrak{A}$ such that the algebra $\mathfrak{A}^\varphi$ is Abelian C has least order.

It follows from the results of [23] that for any automorphism τ of the Dynkin scheme $\mathfrak{A}$ in the corresponding coset K_τ of the group $\mathrm{Aut}\,\mathfrak{A}$ by the subgroup $\mathrm{Aut}^0\mathfrak{A}$ there exists a Coxeter automorphism C, and C is unique up to conjugation with inner automorphisms. The order h of the automorphism C is called the Coxeter number of the algebra $L(\mathfrak{A}, C)$. The Coxeter numbers of Kats–Moody algebras are presented in Table 3 (see part 5.6) taken from [57]. From the results of [23] it is possible to extract the following means of constructing C if h is known. In $\mathfrak{A}$ we consider a canonical system of generators $\{X_j, Y_j, H_j\}$, $1 \leqslant j \leqslant r$. Suppose that τ takes the j-th vertex of the Dynkin scheme into a vertex with index $\sigma(j)$. Then the action of C on the generators can be given by the formulas $C(X_j) = e^{2\pi i/h} X_{\sigma(j)}$, $C(Y_j) = e^{-2\pi i/h} Y_{\sigma(j)}$, $C(H_j) = H_{\sigma(j)}$.

Let C be a Coxeter automorphism of a simple Lie algebra $\mathfrak{A}$, and let $\omega = e^{2\pi i/h}$, where h is the Coxeter number. We set $G = L(\mathfrak{A}, C)$. We have $G = \bigoplus_{j \in Z} \mathfrak{A}_j \zeta^j$, where $\mathfrak{A}_j \overset{\mathrm{def}}{=} \{x \in \mathfrak{A} \mid Cx = \omega^j x\}$. We set $G^j = \mathfrak{A}_j \zeta^j$. It is clear that $G = \bigoplus_{j \in Z} G^j$, $[G^j, G^k] \subset G^{j+k}$, so that G is a graded Lie algebra. According to the definition of the Coxeter automorphism, the algebra $G^0 = \mathfrak{A}_0$ is Abelian. The number $r \overset{\mathrm{def}}{=} \dim G^0$ we call the rank of the Kats–Moody algebra G.

Proposition 5.6. 1) There exist elements $e_0, \ldots, e_r \in G^1$, $f_0, \ldots, f_r \in G^{-1}$, $h_0, \ldots, h_r \in G^0$ such that a) $e_0, \ldots, e_r$ form a basis in G^1, $f_0, \ldots, f_r$ form a basis in G^{-1}, $h_0, \ldots, h_r$ generate G^0; b) the following relations hold:

$$[h_i, h_j] = 0, \tag{5.6}$$

$$[e_i, f_j] = \delta_{ij} h_i, \tag{5.7}$$

$$[h_i, e_j] = A_{ij} e_j, \tag{5.8}$$
$$[h_i, f_j] = -A_{ij} f_j, \tag{5.9}$$

where $A_{ii} = 2$ for all i. 2) The elements e_i, f_i, h_i are uniquely determined up to enumeration and transformations of the form $\tilde{e}_i = c \exp(\mathrm{ad}\, a) e_i$, $\tilde{f}_i = c^{-1} \exp(\mathrm{ad}\, a) f_i$, $\tilde{h}_i = h_i$, where $c \in \mathbf{C}$, $a \in G^0$.

A proof is given at the beginning of part 4 of the work [23]. We note that if concrete $\mathfrak{A}$ and C are given, then it is not hard to find the explicit form of the elements e_i, f_i, h_i (e_i and f_i are eigenvectors of the operators $\mathrm{ad}\, a$, $a \in G^0$).

The matrix (A_{ij}) is called the Cartan matrix of the algebra G. It is proved in part 5 of the work [23] that this matrix possesses the following properties.

Proposition 5.7. 1) $A_{ij} \in \mathbf{Z}$. 2) If $i \neq j$, then $A_{ij} \leqslant 0$, $A_{ij}A_{ji} \leqslant 4$. 3) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$. 4) The space of solutions of the system of equations

$$\sum_{i=0}^{r} A_{ij} x_i = 0, \quad j = 0, \ldots, r, \tag{5.10}$$

is one-dimensional.

From the results of part 4 of the work [23] and Proposition 13 of the work [21] we obtain the following results.

Proposition 5.8. 1) The elements e_i, f_i, h_i, $0 \leqslant i \leqslant r$ generate G. 2) If $i \neq j$, then

$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0. \tag{5.11}$$

3) Let $(\alpha_0, \ldots, \alpha_r)$ be a nonzero solution of the system (5.10). Then

$$\sum_{i=0}^{r} \alpha_i h_i = 0. \tag{5.12}$$

4) Equalities (5.6)-(5.9), (5.11), (5.12) form a complete system of relations between the elements e_i, f_i, h_i.

We call the system of generators $\{e_i, f_i, h_i\}$ of the algebra G canonical.

Remark 1. Assertion 1) of Proposition 5.8 means that G is generated as a space by multiple commutators of the elements e_j, f_j, h_j. Using the Jacobi identity and relations (5.6)-(5.9), from this it is not hard to deduce that G is generated as a vector space by elements of the form $[e_{i_1}, \ldots, e_{i_n}]$, $[f_{i_1}, \ldots, f_{i_n}]$ ($n = 1, 2, \ldots$) and h_i. It is clear that for any $n \in \mathbf{N}$ the space G^n is generated by elements of the form $[e_{i_1}, \ldots, e_{i_n}]$ and G^{-n} by elements of the form $[f_{i_1}, \ldots, f_{i_n}]$.

Remark 2. Frequently not the algebra G but the algebra $\tilde{G}$ with generators e_i, f_i, h_i and defining relations (5.6)-(5.9), (5.11) is called the Kats–Moody algebra. The algebra $\tilde{G}$ has a one-dimensional center generated by the element $\sum_{i=0}^{r} \alpha_i h_i$, and its factor by the center is isomorphic to G. We note that the most interesting applications of Kats–Moody algebras (see [22, 31, 47, 48, 56, 66, 68]) are connected with $\tilde{G}$ rather than G.

The concept of the Dynkin scheme for a Kats–Moody algebra is introduced in the same way as for semisimple Lie algebras. It follows from Proposition 5.7 that the Cartan matrix can be uniquely recovered from the Dynkin scheme of a Kats–Moody algebra and thus so can the algebra itself. The Dynkin schemes of Kats–Moody algebras are presented in Table 2 borrowed from [23].

Let G be a Kats–Moody algebra with canonical generators e_i, f_i, h_i, $0 \leqslant i \leqslant r$.

Proposition 5.9. Let $S \subset \{0, 1, \ldots, r\}$ be a proper subset. Then the subalgebra generated by the elements e_i, f_i, h_i, $i \in S$, is semisimple, and these elements themselves are Weyl generators.

Proof. Comparing Tables 1 and 2, it is not hard to see that if from the Dynkin scheme of a Kats–Moody algebra part of the vertices and adjacent segments are removed, then a Dynkin scheme of a semissimple Lie algebra is obtained. ∎

COROLLARY. In relation (5.12) $\alpha_i \neq 0$ for all i.

Proof. The complementary minors of the diagonal elements of the matrix (A_{ij}) are nonzero, since they are the determinants of the Cartan matrix of semisimple Lie algebras. ■

The gradation in G that we have so far considered is called canonical. Moreover, to each vertex of the Dynkin scheme of G there corresponds a gradation $G=\bigoplus_{j\in\mathbf{Z}} G_j$ called a standard gradation. The standard gradation corresponding to a vertex c_m is characterized by the following property: $e_m \in G_1$, $f_m \in G_{-1}$, and the remaining canonical generators belong to G_0.

Proposition 5.10. Let $G=\bigoplus_{j\in\mathbf{Z}} G_j$ be the standard gradation corresponding to the vertex c_m of the Dynkin scheme. Then 1) if $i > 0$, then $G_i \subset \sum_{j>0} G^j$. 2) If $i < 0$, then $G_i \subset \sum_{j<0} G^j$. 3) G_0 is a semisimple Lie algebra with Weyl generators e_i, f_i, h_i, $i \neq m$. 4) The algebra $G_0 \cap \sum_{j>0} G^j$ is generated by the elements e_i, $i \neq m$. 5) The algebra $G_0 \cap \sum_{j<0} G^j$ is generated by the elements f_i, $i \neq m$.

Proof. It is clear that for any $n \in \mathbf{N}$ the space G_n (respectively, G_{-n}) is generated by elements of the form $[e_{j_1},\ldots,e_{j_k}]$ (respectively, $[f_{j_1},\ldots,f_{j_k}]$), where m occurs among the numbers $j_1,\ldots,j_k$ precisely n times. G_0 is generated by elements of the form $[e_{j_1},\ldots,e_{j_k}]$, $[f_{j_1},\ldots,f_{j_k}]$, where $j_1 \neq m,\ldots,j_k \neq m$ and by the elements h_j. From this we obtain assertions 1), 2), 4), and 5). To prove 3) it is necessary to use Proposition 5.9 and further note that because of Proposition 5.9 h_m can be expressed in terms of the elements h_j, $j \neq m$ from relation (5.12). ■

Remark. It follows from assertion 3) of Proposition 5.10 that the Dynkin scheme of G_0 is obtained from the Dynkin scheme of G by deleting the vertex c_m and the edge contiguous to it.

5.3. If σ is an automorphism of finite order of a simple Lie algebra $\mathfrak{A}$ such that $C^{-1}\cdot\sigma$ is an inner automorphism, then, according to Proposition 5.5, there exists an isomorphism $L(\mathfrak{A}, C) \xrightarrow{\sim} L(\mathfrak{A}, \sigma)$. We shall indicate the explicit form of this isomorphism in the case where $\sigma = C\cdot e^{\mathrm{ad}\,x}$, $x \in G^0$. We recall that

$$L(\mathfrak{A}, \sigma)=\left\{f\in\mathfrak{A}[\lambda, \lambda^{-1}] \mid f\left(\lambda e^{\frac{2\pi i}{n}}\right)=\sigma(f(\lambda))\right\}, \tag{5.13}$$

where n is the order of σ. We make the substitution $\lambda = e^{\frac{2\pi i u}{n}}$. We denote by T the algebra of functions $g:\mathbf{C}\to\mathfrak{A}$ of the form $g(u)=\sum_{j=1}^{k} e^{a_j u}\cdot v_j$, where $a_j \in \mathbf{C}$, $v_j \in \mathfrak{A}$. Then

$$L(\mathfrak{A}, \sigma)=\{g\in T \mid g(u+1)=\sigma(g(u))\}. \tag{5.14}$$

In exactly the same way $L(\mathfrak{A}, C) \overset{\text{def}}{=} \{f\in\mathfrak{A}[\zeta, \zeta^{-1}] \mid f\left(\zeta e^{\frac{2\pi i}{h}}\right) = C(f(\zeta))\} = \{g\in T \mid g(u+1)=C(g(u))\}$, where $\zeta = e^{\frac{2\pi i u}{n}}$. It is not hard to verify that the mapping $\tau: T \to T$ given by the formula

$$\tau(g)=\tilde{g},\quad \tilde{g}(u)\overset{\text{def}}{=}e^{u\,\mathrm{ad}\,x}g(u), \tag{5.15}$$

isomorphically maps $L(\mathfrak{A}, C)$ onto $L(\mathfrak{A}, \sigma)$.

Proposition 5.11. For each vertex c_m of the Dynkin scheme of G there exists exactly one element $x\in G^0$, possessing the following property: if we set $\sigma_m = C\cdot e^{\mathrm{ad}\,x}$ and define τ by formula (5.15), then τ takes the standard gradation of the algebra $G=L(\mathfrak{A}, C)$, corresponding to the vertex c_m into a gradation of the algebra $L(\mathfrak{A}, \sigma_m)$ in powers of λ.

Proof. Since the elements h_j, $j \neq m$ form a basis in G^0, the desired element x can be represented in the form $x=\sum_{j\neq m} x_j h_j$, $x_j\in\mathbf{C}$. We recall that $e_k=\tilde{e}_k\zeta=\tilde{e}_k\exp\left(\frac{2\pi i u}{h}\right)$, $f_k=\tilde{f}_k\zeta^{-1}=\tilde{f}_k\exp\left(-\frac{2\pi i u}{h}\right)$, where $\tilde{e}_k$, $\tilde{f}_k\in\mathfrak{A}$. The element x possesses the property that $\exp(u\cdot\mathrm{ad}\,x)e_k$ does not depend on u for any $k \neq m$. From this we obtain the system

$$\sum_{j\neq m} A_{jk}x_j = -\frac{2\pi i}{h}, \cdot\ 0\leqslant k\leqslant r,\quad k\neq m, \tag{5.16}$$

for determining x_j. Since this system has exactly one solution (see the proof of the corollary to Proposition 5.9), the uniqueness of x has been proved. We shall now prove that the element $x=\sum_{j\neq m} x_j h_j$, defined from the system (5.16) is the desired one. We set $\sigma_m = C\cdot e^{\mathrm{ad}x}$ and define $L(\mathfrak{A}, \sigma_m)$ by formula (5.14) [it is not possible to define $L(\mathfrak{A}, \sigma_m)$ by formula (5.13) until it has been proved that σ_m has finite order]. By the construction of x we have $\tau(e_j) = \tilde{e}_j$ for $j \neq m$. Moreover, it is easy to see that $\tau(f_j) = \tilde{f}_j$ for $j \neq m$, $\tau(h_j) = h_j$ for all j, and there exists $c\in\mathbf{C}$ such that $\tau(e_m) = \exp(cu)\tilde{e}_m$, $\tau(f_m) = \exp(-cu)\tilde{f}_m$. We shall show that c has the form $\frac{2\pi i}{n}$, $n\in\mathbf{N}$. For any $y\in G^0$ the element $z = y\cdot\zeta^h$ belongs to G^h. Since z lies in the subalgebra generated by the elements e_j, it follows that $\tau(z)$ lies in the subalgebra generated by the elements $\tau(e_j)$ and hence has the form $\sum_{l=0}^{k}\exp(lcu)\cdot v_l$, $v_l\in\mathfrak{A}$. On the other hand, $\tau(z)=z=y\cdot\exp(2\pi i l u)$. Therefore, c has the form $\frac{2\pi i}{n}$, $n\in\mathbf{N}$. It is not hard to verify that $\sigma_m \times (\tilde{e}_m) = e^c\cdot\tilde{e}_m$, $\sigma_m(\tilde{e}_j) = \tilde{e}_j$ for $j \neq m$, $\sigma_m(\tilde{f}_m) = e^{-c}\tilde{f}_m$, $\sigma_m(\tilde{f}_j) = \tilde{f}_j$ for $j \neq m$, $\sigma_m(h_j) = h_j$ for all j. From this it follows that σ_m has order n. Since $\tau(e_m)=\exp\left(\frac{2\pi i u}{n}\right)\tilde{e}_m=\lambda\tilde{e}_m$, $\tau(f_m)=\lambda^{-1}\tilde{f}_m$, and the images of the remaining elements e_j, f_j, h_j under the mapping τ do not depend on λ, it follows that τ takes the standard gradation $L(\mathfrak{A}, C)$, corresponding to the vertex c_m into the gradation of $L(\mathfrak{A}, \sigma_m)$ in powers of λ. ■

COROLLARY. $\dim G_j < \infty$.

The automorphism $\sigma_m = C\cdot e^{\mathrm{ad}x}$, where x is the same as in Proposition 5.11, we shall call the standard automorphism corresponding to the vertex c_m, while the realization of G in the form $L(\mathfrak{A}, \sigma_m)$ we call the standard realization.

We make several remarks concerning a practical way to find the standard realization G. If $\mathfrak{A}$ is realized as a subalgebra in Mat (k, $\mathbf{C}$), then $\exp(u\cdot\mathrm{ad}\,x)L(\mathfrak{A}, C)=e^{ux}L(\mathfrak{A}, C)e^{-ux}$. If, moreover, $G^0\subset\mathrm{Diag}$, then e_{ux} is a diagonal matrix of the form $\mathrm{diag}(\zeta^{n_1},\ldots,\zeta^{n_k})$ (we recall that $\zeta=e^{\frac{2\pi i u}{h}}$). Thus, $L(\mathfrak{A}, \sigma_m) = U(\zeta)L(\mathfrak{A}, C)U^{-1}(\zeta)$, where $U(\zeta)=\mathrm{diag}(\zeta^{n_1},\ldots,\zeta^{n_k})$, and the numbers $n_1,\ldots,n_k$ can be found uniquely from the following conditions: 1) the matrix $U(\zeta)e_iU^{-1}(\zeta)$ does not depend on ζ for $i \neq m$; 2) $U(\zeta)\in H$, where $H=\{e^y\,|\,y\in G^0\}$. The automorphism σ_m is given by the formula $\sigma_m(X)=U\left(e^{\frac{2\pi i}{h}}\right)\cdot C(X)\cdot U\left(e^{\frac{2\pi i}{h}}\right)^{-1}$. Finally, λ and ζ, figuring in the definitions of $L(\mathfrak{A}, \sigma_m)$ and $L(\mathfrak{A}, C)$, are connected by the relation $\lambda = \zeta^{h/n}$, where n is the order of σ_m.

5.4. The height of the Kats–Moody algebra $L(\mathfrak{A}, \varphi)$ is the order of the automorphism of the Dynkin scheme of $\mathfrak{A}$, defined by φ. This subsection is devoted to Kats–Moody algebras of height 1.

Let $\mathfrak{A}$ be a simple Lie algebra with Weyl generators X_i, Y_i, H_i, $i = 1,\ldots,r$. It is easy to see that on $\mathfrak{A}$ there exists exactly one gradation $\mathfrak{A}=\bigoplus_{j\in\mathbf{Z}}\mathfrak{A}^j$ such that $X_i\in\mathfrak{A}^1$, $Y_i\in\mathfrak{A}^{-1}$, $H_i\in\mathfrak{A}^0$ for all $i\in\{1,\ldots,r\}$. We note that $\mathfrak{A}^0$ coincides with the Cartan subalgebra $\mathfrak{H}$ (see the end of part 5.1). We set $S=\{j\in\mathbf{Z}\,|\,\mathfrak{A}^j\neq 0\}$. The largest and least elements of S we denote by k and l, respectively.

Proposition 5.12. 1) $l=-k$; 2) $\dim\mathfrak{A}^k=\dim\mathfrak{A}^{-k}=1$; 3) there exist nonzero elements $X_0\in\mathfrak{A}^{-k}$, $Y_0\in\mathfrak{A}^k$, $H_0\in\mathfrak{H}$ such that $[X_0, Y_0] = H_0$, $[H_0, X_0] = 2X_0$, $[H_0, Y_0] = -2Y_0$; 4) the center of the algebra $\mathfrak{n}$, generated by the elements $Y_1,\ldots,Y_r$ is equal to $\mathfrak{A}^{-k}$.

This proposition follows from the theorem on the existence of a maximal root (see [3], Chap. 6, Sec. 1, Proposition 25).

The number $h = k + 1$ is called the Coxeter number of the algebra $\mathfrak{A}$.

We set $C\overset{\mathrm{def}}{=}\exp\left(\frac{2\pi i}{h}\cdot\mathrm{ad}\,a\right)$, where the element $a\in\mathfrak{H}$, is such that $[a, X_j] = X_j$, $[a, Y_j] = -Y_j$, $1 \leqslant j \leqslant r$ (the existence and uniqueness of a follow from the nondegeneracy of the Cartan matrix of $\mathfrak{A}$). We define elements e_i, f_i, $h_i \in \mathfrak{A}[\zeta, \zeta^{-1}]$ by the formulas $e_i = X_i\zeta$, $f_i = Y_i\zeta^{-1}$, $h_i = H_i$, $0 \leqslant i \leqslant r$. It is easy to see that $e_i, f_i, h_i \in L(\mathfrak{A}, C)$ for all $i \in \{0, \ldots, r\}$.

Proposition 5.13. 1) The automorphism C is a Coxeter automorphism (in particular, the Coxeter number of Kats–Moody algebra of height 1 is equal to the Coxeter number of the corresponding simple Lie algebra). 2) The elements e_i, f_i, h_i, $0 \leqslant i \leqslant r$ are canonical generators of $L(\mathfrak{A}, C)$.

This proposition is essentially proved in [21] in the proof of Lemma 22.

Suppose that the canonical generators of a Kats–Moody algebra G of height 1 are chosen as in Proposition 5.13. Then the vertex of the Dynkin scheme of G having index 0 is called special. It is clear that if the special vertex and the edge contiguous to it are eliminated from the Dynkin scheme of G, then the Dynkin scheme of the corresponding simple Lie algebra is obtained. In Table 2 the special vertices of the Dynkin scheme of the Kats–Moody algebras of height 1 are represented by black circles.

Proposition 5.14. The standard automorphism of $\mathfrak{A}$ (see part 5.3) corresponding to a special vertex is the identity automorphism.

Proof. We recall (see the proof of Proposition 5.11) that the standard automorphism corresponding to a special vertex is equal to $C \cdot e^{\mathrm{ad}x}$, where $x \in \mathfrak{H}$ is found from the following condition: $e^{u\,\mathrm{ad}\,x} \cdot e^{\frac{2\pi i u}{h}} X_k$ does not depend on u for all $k \neq 0$. Since $[a, X_k] = X_k$ for all $k \neq 0$, it follows that $x = -\frac{2\pi i}{h} a$. ■

Thus, the standard realization of a Kats–Moody algebra of height 1 corresponding to a special vertex is $\mathfrak{A}[\lambda, \lambda^{-1}]$. In correspondence with part 5.3, the isomorphism $\tau: L(\mathfrak{A}, C) \to \mathfrak{A}[\lambda, \lambda^{-1}]$ is given by the formula $\tau(f) = \tilde{f}$, where $\tilde{f}(\lambda) = \exp(-\ln\zeta \cdot \mathrm{ad}\, a) \cdot f(\zeta)$, $\zeta = \lambda^{1/h}$. The canonical generators of $\mathfrak{A}[\lambda, \lambda^{-1}]$ are λX_0, $\lambda^{-1} Y_0$, H_0, X_i, Y_i, H_i, $1 \leqslant i \leqslant r$, while the canonical gradation of $\mathfrak{A}[\lambda, \lambda^{-1}]$ has the form $\mathfrak{A}[\lambda, \lambda^{-1}] = \bigoplus_{i \in \mathbb{Z}} G^i$, where $G^i = \bigoplus_{j+kh=i} \mathfrak{A}^j \lambda^k$.

5.5. Let $\mathfrak{A}$ be a semisimple Lie algebra of rank r; let X_i, where $i = 1, \ldots, r$, be the elements of the system of Weyl generators of $\mathfrak{A}$; let $\mathfrak{A} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{A}^j$ be the gradation of $\mathfrak{A}$ introduced in part 5.4. We set $I = \sum_{i=1}^{r} X_i$. It is clear that the operator ad I maps $\mathfrak{A}^j$ into $\mathfrak{A}^{j+1}$.

Proposition 5.15. If $j \leqslant 0$, then the operator $\mathrm{ad}\, I: \mathfrak{A}^j \to \mathfrak{A}^{j+1}$ is injective. If $j \geqslant -1$, then the operator $\mathrm{ad}\, I: \mathfrak{A}^j \to \mathfrak{A}^{j+1}$ is surjective.

A proof is given in [58].

Definition. The number j is called the exponent of the algebra $\mathfrak{A}$ if the operator ad I: $\mathfrak{A}^{-j-1} \to \mathfrak{A}^{-j}$ is not surjective. The difference $\dim \mathfrak{A}^{-j} - \dim \mathfrak{A}^{-j-1}$ is called the multiplicity of the exponent j.

It is shown in [58] that the definition of the exponent presented above coincides with the definition of Bourbaki [3]. It follows from Proposition 5.15 that if the algebra $\mathfrak{A}$ is simple, then all exponents belong to the segment [1, h − 1], where h is the Coxeter number of $\mathfrak{A}$. We note that the number of all exponents (counting multiplicity) is equal to the rank of $\mathfrak{A}$: indeed, $\sum_{j>0} (\dim \mathfrak{A}^{-j} - \dim \mathfrak{A}^{-j-1}) = \dim \mathfrak{A}^0 = r$.

5.6. Let G be a Kats–Moody algebra with canonical generators e_i, f_i, h_i, $0 \leqslant i \leqslant r$. We set $\Lambda = \sum_{i=0}^{r} e_i$, $\mathfrak{Z} = \mathrm{Ker}\,\mathrm{ad}\,\Lambda = \{u \in G \mid [\Lambda, u] = 0\}$. The next result is important for our subsequent purposes.

Proposition 5.16. 1) The algebra $\mathfrak{Z}$ is commutative. 2) $G = \mathfrak{Z} \oplus \mathrm{Im}\,\mathrm{ad}\,\Lambda$.

A proof is given in [57] (Proposition 3.8).

As B. A. Magadeev has informed us, the following converse assertion also holds: if G is a contragradient Lie algebra in the sense of [21] and $G = \mathrm{Ker}\,\mathrm{ad}\,\Lambda \oplus \mathrm{Im}\,\mathrm{ad}\,\Lambda$, then G is the direct product of a finite number of Kats–Moody algebras.

TABLE 3

Type of Kats–Moody algebra	Coxeter number	Exponents	Type of Kats–Moody algebra	Coxeter number	Exponents
$A_n^{(1)}$	$n+1$	$1,2,\dots,n$	$D_4^{(3)}$	12	1,5,7,11
$A_{2n}^{(2)}$	$4n+2$	$1,3,5,\dots,2n-1,2n+3,\dots,4n+1$	$E_6^{(1)}$	12	1,4,5,7,8,11
$A_{2n-1}^{(2)}$	$4n-2$	$1,3,5,\dots,4n-3$	$E_6^{(2)}$	18	1,5,7,11,13,17
$B_n^{(1)}$	$2n$	$1,3,5,\dots,2n-1$	$E_7^{(1)}$	18	1,5,7,9,11,13,17
$C_n^{(1)}$	$2n$	$1,3,5,\dots,2n-1$	$E_8^{(1)}$	30	1,7,11,13,17,19,23,29
$D_n^{(1)}$	$2n-2$	$1,3,5,\dots,2n-3,n-1$	$F_4^{(1)}$	12	1,5,7,11
$D_{n+1}^{(2)}$	$2n+2$	$1,3,5,\dots,2n+1$	$G_2^{(1)}$	6	1,5

Let $G=\bigoplus_{j\in\mathbf{Z}} G^j$ be the canonical gradation. It is clear that $\mathfrak{Z}=\bigoplus_{j\in\mathbf{Z}}\mathfrak{Z}^j$, where $\mathfrak{Z}^j=\mathfrak{Z}\cap G^j$. It is easy to see that $\mathfrak{Z}^0=0$, and for any j, $\dim\mathfrak{Z}^{j+h}=\dim\mathfrak{Z}^j$, where h is the Coxeter number of G.

Definition. The number $j\in[1, h-1]$ is called the exponent of the algebra G if $\mathfrak{Z}^j\neq 0$. The dimension of $\mathfrak{Z}^j$ is called the multiplicity of the exponent j.

The exponents of Kats–Moody algebras are presented in Table 3. We note that only algebras of type $D_{2n}^{(1)}$ have an exponent of multiplicity greater than 1 (in this case $2n-1$ is an exponent of multiplicity 2). It is known (see [57], Proposition 3.7) that the exponents of a Kats–Moody algebra of height 1 coincide with the exponents of the corresponding simple Lie algebra. Therefore, the exponents of simple Lie algebras can also be found from Table 3.

We note (see Table 3) that j is an exponent of G if and only if $h-j$ is an exponent, and the multiplicities of these exponents coincide. In other words, $\dim\mathfrak{Z}^i=\dim\mathfrak{Z}^{-i}$ for any $i\in\mathbf{Z}$. This equality can also be derived from Proposition 5.20 below.

Proposition 5.17. If $u\in\mathfrak{Z}$ and $[u, G^0]=0$, then $u=0$.

Proof. By hypothesis $[u, \Lambda]=0$, $[u, G^0]=0$. It is not hard to see that the elements Λ and h_i, $0\leqslant i\leqslant r$ generate the algebra $\bigoplus_{k>0}G^k$. Therefore, $[u, G^k]=0$ for $k\geqslant 0$. We recall that $G=L(\mathfrak{A}, C)=\bigoplus_{k\in\mathbf{Z}}\mathfrak{A}_k\zeta^k$, where $\mathfrak{A}_k=\{x\in\mathfrak{A}\mid Cx=e^{\frac{2\pi i k}{h}}x\}$. Here $G^k=\mathfrak{A}_k\zeta^k$. Thus, $[u(\zeta), \mathfrak{A}_k]=0$ for $k\geqslant 0$. Since $\mathfrak{A}=\bigoplus_{k=0}^{h-1}\mathfrak{A}_k$, it follows that $[u(\zeta), \mathfrak{A}]=0$. It remains to note that the center of $\mathfrak{A}$ is equal to zero, since the algebra $\mathfrak{A}$ is simple. ■

5.7. This subsection is devoted to invariant bilinear forms on semisimple Lie algebras and Kats–Moody algebras.

We recall that the Killing form on a finite-dimensional Lie algebra $\mathfrak{A}$ is defined by the formula $(x, y)_K=\operatorname{tr}(\operatorname{ad}x\cdot\operatorname{ad}y)$. It is easy to see that this form is invariant and symmetric.

Proposition 5.18. 1) A finite-dimensional Lie algebra is semisimple if and only if the Killing form on it is nondegenerate. 2) Let $\mathfrak{A}=\mathfrak{A}_1\times\ldots\times\mathfrak{A}_n$, where $\mathfrak{A}_1,\dots,\mathfrak{A}_n$ are simple Lie algebras. Then any invariant bilinear form on $\mathfrak{A}$ has the form $(x, y)=\sum_{i=1}^n c_i(x_i, y_i)_K$, $c_i\in\mathbf{C}$, where x_i and y_i are the components of x and y in $\mathfrak{A}_i$. In particular, any invariant bilinear form on $\mathfrak{A}$ is symmetric.

Assertion 1) is proved, for example, in [2] (Chap. 1, Sec. 6, Theorem 1). Concerning the proof of 2) see Exercise 18 in Sec. 6, Chap. 1 of the book [2].

Proposition 5.19. Let $\mathfrak{A}$ be a semisimple Lie algebra on which a nondegenerate invariant bilinear form has been fixed. Let $\mathfrak{H}$, $\mathfrak{b}$, $\mathfrak{n}$ be the same as at the end of part 5.1, and let $\mathfrak{A}^i$ be the same as in part 5.4. Then 1) if $x\in\mathfrak{A}^i$, $y\in\mathfrak{A}^k$, $j+k\neq 0$, then $(x, y)=0$; 2) on $\mathfrak{H}$ the bilinear form is nondegenerate; 3) the orthogonal complement of $\mathfrak{b}$ is equal to $\mathfrak{n}$.

Proof. It is easy to show that there exists an element $h\in\mathfrak{H}$ such that $[h, X_i]=X_i$, $[h, Y_i]=-Y_i$ for all $i\in\{1,\dots,r\}$. Then $[h, x]=jx$ if $x\in\mathfrak{A}^j$. If $x\in\mathfrak{A}^j$, $y\in\mathfrak{A}^k$, then $(j+k)(x, y)=([h, x], y)+(x, [h, y])=0$. Assertion 1) has been proved. It implies assertions

2) and 3)), since $\mathfrak{H}=\mathfrak{A}^0$ $\mathfrak{b}=\bigoplus_{j<0}\mathfrak{A}^j$, $\mathfrak{n}=\bigoplus_{j<0}\mathfrak{A}^j$. ■

Let $G=\bigoplus_j G_j$ be a graded Lie algebra. It is said that a bilinear form (,) on G is coordinated with the gradation if (x, y) = 0 for $x\in G_k$, $y\in G_l$, $k+l\neq 0$.

Proposition 5.20. On a Kats–Moody algebra there exists a nondegenerate, symmetric, bilinear form which is unique up to a factor and is coordinated with the canonical gradation. This form is also coordinated with the standard gradation.

Proof. Let $G=L(\mathfrak{A}, C)$, where $\mathfrak{A}$ is a simple Lie algebra and C is the Coxeter automorphism. On $\mathfrak{A}$ we choose a nonzero, invariant, bilinear form. On G we define a bilinear form B by the formula $B(u,v)=\sum_i (u_i, v_{-i})$, where $u=\sum_i u_i\zeta^i$, $v=\sum_i v_i\zeta^i$, $u_i\, v_i\in\mathfrak{A}$. It is easy to see that the form B is the desired one. Noting that B(u, v) is the free term of (u(ζ), v(ζ)) and using (5.15), we find that in the standard realization of the algebra C (see part 5.3) the form B has the form $B\left(\sum_i u_i\lambda^i, \sum_i v_i\lambda^i\right)=\sum_i (u_i, v_{-i})$. Therefore, B is coordinated with the standard gradation. The uniqueness of the form follows from the lack of nontrivial, homogeneous ideals in G which is proved in [23] (see Exercise 18 in Sec. 6, Chap. 1 of the book [2]).

6. ANALOGOUS OF THE KdV EQUATION FOR KATS–MOODY ALGEBRAS

6.1. Let G be a Kats–Moody algebra, let e_i, f_i, h_i be its canonical generators (i = 0,...,r), and let $G=\bigoplus_j G^j$ be the canonical gradation (we recall that $e_i\in G^1$, $f_i\in G^{-1}$, $h_i\in G^0$).

We fix a vertex c_m of the Dynkin scheme of G. Let $G=\bigoplus_j G_j$ be the standard gradation corresponding to this vertex. We set $\mathfrak{C}\overset{\text{def}}{=}G_0$, $\mathfrak{H}\overset{\text{def}}{=}G^0$, $\mathfrak{b}\overset{\text{def}}{=}G_0\cap\sum_{l\leqslant 0}G^l$, $\mathfrak{n}\overset{\text{def}}{=}G_0\cap\sum_{l<0}G^l$. We recall (see Proposition 5.10) that $\mathfrak{C}$ is a semisimple Lie algebra, $\mathfrak{H}$ and $\mathfrak{b}$ are its Cartan and Borel subalgebras, and e_i, f_i, h_i, where i = 0,...,m − 1, m + 1,...,r, are the Weyl generators of the algebra $\mathfrak{C}$. Let $\mathfrak{C}=\bigoplus_i\mathfrak{C}^i$ be the gradation of $\mathfrak{C}$, corresponding to this choice of generators. Then $\mathfrak{H}=\mathfrak{C}^0$, $\mathfrak{b}=\bigoplus_{i\leqslant 0}\mathfrak{C}^i$. For any i ⩾ 0 we set $\mathfrak{b}_i=\mathfrak{C}^{-i}$. Thus, $\mathfrak{b}=\bigoplus_{i\geqslant 0}\mathfrak{b}_i$, $\mathfrak{H}=\mathfrak{b}_0$, $\mathfrak{n}=\bigoplus_{i>0}\mathfrak{b}_i$. We note that $[e_m, \mathfrak{n}]=0$. Indeed, the algebra $\mathfrak{n}$ is generated by elements f_i, i ≠ m which commute with e_m.

We consider the operator

$$\mathfrak{L}=\frac{d}{dx}+q+\Lambda, \tag{6.1}$$

where $q\in C^\infty(\mathbf{R}, \mathfrak{b})$, $\Lambda=\sum_{i=0}^r e_i$. We note that the operator (3.1) considered in Sec. 3 under the additional condition tr q = 0 corresponds to the case where $G=\mathfrak{sl}(k, \mathbf{C}[\lambda,\lambda^{-1}])$ and c_m is a special vertex of the Dynkin scheme. All assertions of the present section constitute a generalization of the assertions of Sec. 3 to the case of arbitrary Kats–Moody algebras.

If $\mathfrak{L}$ is an operator of the form (6.1) and $S\in C^\infty(\mathbf{R}, \mathfrak{N})$, then the operator

$$\tilde{\mathfrak{L}}=e^{\operatorname{ad}S}(\mathfrak{L}) \tag{6.2}$$

also has the form (6.1). This follows from the fact that $[\mathfrak{n}, \mathfrak{b}]\subset\mathfrak{n}$, $[\mathfrak{n}, e_m]=0$, $[\mathfrak{n}, e_i]\subset\mathfrak{b}$ for i ≠ m. We call the transformation (6.2) a gauge transformation, and we call the operators $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ gauge equivalent.

It follows from Proposition 5.15 that for any i > 0 the operator ad I, where $I\overset{\text{def}}{=}\Lambda-e_m$, acts from $\mathfrak{b}_i$ to $\mathfrak{b}_{i-1}$ injectively. For any exponent j of the algebra $\mathfrak{D}$ (see part 5.5) we choose a vector subspace $V_j\subset\mathfrak{b}_j$ so that $\mathfrak{b}_j=[I, \mathfrak{b}_{j+1}]\oplus V_j$. We set $V=\bigoplus_j V_j$. Since $\mathfrak{b}=V\oplus[I, \mathfrak{n}]$ and the mapping $\operatorname{ad} I: \mathfrak{n}\to\mathfrak{b}$ is injective, it follows that $\dim V=\dim\mathfrak{b}-\dim\mathfrak{n}=\dim\mathfrak{H}=r$.

Proposition 6.1. Any operator $\mathfrak{L}$ of the form (6.1) can be uniquely represented in the form $\mathfrak{L}=e^{\operatorname{ad}S}(\mathfrak{L}^{\text{can}})$, where $S\in C^\infty(\mathbf{R}, \mathfrak{n})$, $\mathfrak{L}^{\text{can}}=\frac{d}{dx}+q^{\text{can}}+\Lambda$, $q^{\text{can}}\in C^\infty(\mathbf{R}, V)$. Here S and q^{can} are differential polynomials in q.

Proof. Since $[S, e_m] = 0$, we must actually represent $\frac{d}{dx}+q+I$ in the form $e^{\mathrm{ad}S}\left(\frac{d}{dx}+q^{can}+I\right)$. Let $q^{can}=\sum_{i>0} q_i^{can}$, $S=\sum_{i>0} S_i$, where S_i, $q_i^{can} \in C^\infty(\mathbf{R}, \mathfrak{b}_i)$. Equating components lying in $\mathfrak{B}_i$ in the relation $\frac{d}{dx}+q+I=e^{\mathrm{ad}S}\left(\frac{d}{dx}+q^{can}+I\right)$, we find that $q_i^{can} + [S_{i+1}, I]$ can be expressed in terms of $q_0^{can},\ldots,q_{i-1}^{can}$, $S_1,\ldots,S_i$. Therefore, if $q_i^{can},\ldots,q_{i-1}^{can}$, $S_1,\ldots,S_i$ have already been found, then q_i^{can} and S_{i+1} can be found uniquely because $\mathrm{ad}\, I$ is injective. ■

Thus, the choice of the space V provides a "coordinate system" in the set of classes of gauge equivalence.

6.2. In this subsection we write out equations for the class of gauge equivalence (see part 3.1) which are naturally called generalized KdV equations.

If W is a subspace of G, $W=\oplus_i W^i$, $W^i \subset G^i$, then we set $W^+ \overset{\mathrm{def}}{=} \oplus_{i\geq 0} W^i$, $W^- \overset{\mathrm{def}}{=} \prod_{i<0} W^i$, $\hat{W} \overset{\mathrm{def}}{=} W^+ \oplus W^-$. The elements of $\hat{W}$ are series of the form $\sum_{i=-\infty}^{n} w_i$, $w_i \in W^i$. For example, if $G=\mathfrak{A}[\lambda, \lambda^{-1}]$, then $\tilde{G} = \mathfrak{A}((\lambda^{-1}))$.

Let $\mathfrak{Z}$, $\mathfrak{Z}^i$ be the same as in part 5.6. We set $\mathfrak{Z}^\perp = \mathrm{Im}\,\mathrm{ad}\,\Lambda$. It is easy to see that $\mathfrak{Z}^\perp$ is the orthogonal complement of $\mathfrak{Z}$ relative to the scalar product on G (see Proposition 5.20). It is clear that $\mathfrak{Z}^\perp = \oplus_i (\mathfrak{Z}^i)^\perp$, where $(\mathfrak{Z}^i)^\perp = \mathfrak{Z}^\perp \cap G^i$.

Proposition 6.2. For any operator $\mathfrak{L}$ of the form (6.1) there exists an element $U = \sum_{i=1}^{\infty} U_i$, $U_i \in C^\infty(\mathbf{R}, G^{-i})$ such that the operator $\mathfrak{L}_0 \overset{\mathrm{def}}{=} e^{\mathrm{ad}U}(\mathfrak{L})$ has the form

$$\mathfrak{L}_0=\frac{d}{dx}+\Lambda+H, \quad H \in C^\infty(\mathbf{R}, \mathfrak{Z}^-). \tag{6.3}$$

If U and $\tilde{U}$ are two such elements, then $e^{\mathrm{ad}U} \cdot e^{-\mathrm{ad}\tilde{U}} = e^{\mathrm{ad}T}$, where $T \in C^\infty(\mathbf{R}, \mathfrak{Z}^-)$. U can be chosen in precisely one way so that $U \in C^\infty(\mathbf{R}, (\mathfrak{Z}^\perp)^-)$. The U_i are hereby differential polynomials in q.

We note that $C^\infty(\mathbf{R}, \mathfrak{Z}^-) \overset{\mathrm{def}}{=} \prod_{i<0} C^\infty(\mathbf{R}, \mathfrak{Z}^i)$. Notation of the type $C^\infty(\mathbf{R}, \mathfrak{Z}^+)$ and $C^\infty(\mathbf{R}, \hat{\mathfrak{Z}})$ has an analogous meaning.

Proof. Let $H=\sum_{i=0}^{\infty} H_i$, $H_i \in C^\infty(\mathbf{R}, \mathfrak{Z}^{-i})$. Equating in the relation $\mathfrak{L}_0 = e^{\mathrm{ad}U}(\mathfrak{L})$ the components lying in G^{-i}, we find that $H_i + [U_{i+1}, \Lambda]$ can be expressed in terms of $U_1,\ldots,U_i$, $H_0,\ldots,H_{i-1}$. It follows from Proposition 5.16 that any element $g \in G^{-i}$ can be uniquely represented in the form $a + [b, \Lambda]$, where $a \in \mathfrak{Z}^{-i}$, $b \in (\mathfrak{Z}^{-i-1})^\perp$. Therefore, knowing $U_1,\ldots,U_i$, $H_0,\ldots,H_{i-1}$, it is possible to determine $U_{i+1} \in C^\infty(\mathbf{R}, (\mathfrak{Z}^{-i-1})^\perp)$, $H_i \in C^\infty(\mathbf{R}, \mathfrak{Z}^{-i})$ uniquely, etc. Since $\mathfrak{Z}^0=0$ (see part 5.6), it follows that $H \in C^\infty(\mathbf{R}, \mathfrak{Z}^-)$.

Suppose that $\mathfrak{L}_0=e^{\mathrm{ad}U}(\mathfrak{L})$ and $\tilde{\mathfrak{L}}_0=e^{\mathrm{ad}\tilde{U}}(\mathfrak{L})$ have the form $e^{\mathrm{ad}U} \cdot e^{-\mathrm{ad}\tilde{U}} = e^{\mathrm{ad}T}$, where $T=\sum_{i=1}^{\infty} T_i$, $T_i \in C^\infty(\mathbf{R}, G^{-i})$. We must verify that $T_i \in C^\infty(\mathbf{R}, \mathfrak{Z}^{-i})$. Equating in the relation $\mathfrak{L}_0=e^{\mathrm{ad}T}(\tilde{\mathfrak{L}}_0)$ components lying in G^{-i}, we see that if $T_1,\ldots,T_i \in C^\infty(\mathbf{R}, \mathfrak{Z})$, then $[\Lambda, T_{i+1}] \in C^\infty(\mathbf{R}, \mathfrak{Z})$ and hence $T_{i+1} \in C^\infty(\mathbf{R}, \mathfrak{Z}^{-i-1})$. ■

For any operator $\mathfrak{L}$ of the form (6.1) we set $Z_{\mathfrak{L}} \overset{\mathrm{def}}{=} \{M \in C^\infty(\mathbf{R}, \hat{G}) \mid [M, \mathfrak{L}]=0\}$.

LEMMA 6.3. $Z_{\mathfrak{L}}=e^{-\mathrm{ad}U}(\hat{\mathfrak{Z}})$, where U is the same as in Proposition 6.2.

Proof. It suffices to show that if $\mathfrak{L}_0$ is an operator of the form (6.3), then $Z_{\mathfrak{L}_0}=\hat{\mathfrak{Z}}$. Let $[M, \mathfrak{L}_0]=0$, $M=\sum_{i=-\infty}^{n} M_i$, $M_i \in G^i$. Equating to zero the component of G^{n+1} in $[M, \mathfrak{L}_0]$, we find that $M_n \in C^\infty(\mathbf{R}, \mathfrak{Z}^n)$. Equating now to zero the component of G^n, we obtain $\frac{d}{dx}M_n=[M_{n-1}, \Lambda]$. Since the left side of this equality belongs to $C^\infty(\mathbf{R}, \mathfrak{Z}^n)$, while the right side lies in $C^\infty(\mathbf{R}, (\mathfrak{Z}^n)^\perp)$, we have $\frac{d}{dx}M_n=0$, i.e., $M_n \in \mathfrak{Z}^n$. We then apply analogous considerations to $M-M_n \in Z_{\mathfrak{L}_0}$ etc.

Let $g=\sum_{l=-\infty}^{m} g^l,\ g^l \in G^l$. Since $G^l = \underset{j\in Z}{\oplus}(G^l \cap G_j)$, and the spaces G_j are finite-dimensional, g can be represented uniquely in the form $\sum_{j=-\infty}^{n} g_j$, where $g_j \in G_j$. We set $g_+ \overset{\text{def}}{=} \sum_{j>0} g_j,\ g^+ \overset{\text{def}}{=} \sum_{l>0} g^l$, $g_- \overset{\text{def}}{=} g - g_+,\ g^- \overset{\text{def}}{=} g - g^+$. It is easy to see that $g_+ - g^+ \in \mathfrak{n}$.

LEMMA 6.4. Let $M \in Z_{\mathfrak{L}}$. Then $[M_+, \mathfrak{L}]$ and $[M^+, \mathfrak{L}]$ belong to $C^\infty(\mathbf{R}, \mathfrak{b})$.

Proof. We shall prove that $[M_+, \mathfrak{L}] \in C^\infty(\mathbf{R}, \mathfrak{b})$. The fact that $[M^+, \mathfrak{L}] \in C^\infty(\mathbf{R}, \mathfrak{b})$ is proved similarly. We must verify that the element $[M_+, \mathfrak{L}]$ is nonpositive in the canonical gradation and belongs to $C^\infty(\mathbf{R}, G_0)$. We have $[M_+, \mathfrak{L}] = -[M_-, \mathfrak{L}]$. The right side of this equality does not contain positive components relative to the canonical gradation, since in this gradation M_- is negative (see Proposition 5.10) and $\mathfrak{L} - \frac{d}{dx}$ contains no components of degree greater than 1. In the standard gradation the right side is nonpositive, while the left is nonnegative. ∎

Let U be the element of Proposition 6.2. For any $u \in \hat{\mathfrak{Z}}$ we set $\varphi(u) \overset{\text{def}}{=} e^{-\operatorname{ad} U}(u)$. From the commutativity of $\mathfrak{Z}$ it follows that $\varphi(u)$ does not depend on the arbitrariness in the choice of U. From Lemma 6.4 it follows that the relation

$$\frac{d\mathfrak{L}}{dt} = [\varphi(u)^+, \mathfrak{L}] \tag{6.4}$$

is a self-consistent equation for $q \in C^\infty(\mathbf{R}^2, \mathfrak{b})$. Together with Eq. (6.4) we consider the equation

$$\frac{d\mathfrak{L}}{dt} = [\varphi(u)_+, \mathfrak{L}], \tag{6.5}$$

which is also self-consistent.

In analogy to what was done in Sec. 3, it can be proved that Eqs. (6.4) and (6.5) preserve gauge equivalence and lead to the same equation for the class of gauge equivalence which we call the generalized KdV equation corresponding to the algebra G and the vertex c_m. If an operator $\mathfrak{L}$ of the form (6.1) satisfies Eq. (6.4) [or Eq. (6.5)] and $\mathfrak{L}^{\text{can}} = \frac{d}{dx} + q^{\text{can}} + \Lambda$ is the operator of Proposition 6.1, then q^{can} satisfies an equation of the form $\frac{\partial q^{\text{can}}}{\partial t} = F\left(q^{\text{can}}, \frac{\partial q^{\text{can}}}{\partial x}, \ldots\right)$, which is a coordinate realization of the generalized KdV equation. An (L, A)-pair for this equation relative to q^{can} can be constructed as in part 3.2.

It is clear that Eq. (6.4), and hence also the generalized KdV equation, does not change if an element of $\mathfrak{Z}^-$ is added to u. Therefore, it may be assumed with no loss of generality that $u \in \mathfrak{Z}^+$.

Remark 1. We write the generalized KdV equation corresponding to an element $u \in \mathfrak{Z}^n$, in the form of the system of equation

$$\frac{\partial q_i^{\text{can}}}{\partial t} = F_i\left(q^{\text{can}}, \frac{\partial q^{\text{can}}}{\partial x}, \ldots\right), \tag{6.6}$$

where i runs through the set of exponents of the algebra $\mathfrak{G}$, and q^{can} and q_i^{can} are the same as in the proof of Proposition 6.1. It can be shown that the polynomial F_i is homogeneous of degree i + 1 + n if it is assumed that $\partial^j q_i^{\text{can}}/\partial x^j$ has degree of homogeneity i + j + 1. From this it follows that the order of the system (6.6) does not exceed n + s, where s is the difference between the largest and least exponents of the algebra $\mathfrak{G}$. Generally speaking, the order of the system (6.6) depends on the choice of q^{can}.

Remark 2. If two vertices of the Dynkin scheme of G go over into one another under an automorphism of this scheme, then the series of generalized KdV equations corresponding to them coincide.

6.3. The following assertions are proved in the same ways as the analogous assertions in Sec. 1.

Proposition 6.5. We consider the equations $\frac{\partial \mathfrak{L}}{\partial t} = [\varphi(u)_+, \mathfrak{L}]$ and $\frac{\partial \mathfrak{L}}{\partial \tau} = [\varphi(\tilde{u})_+, \mathfrak{L}]$, where $u, \tilde{u} \in \mathfrak{Z}^+$. Then the mixed derivatives $\frac{\partial^2 \mathfrak{L}}{\partial t \partial \tau}$ and $\frac{\partial^2 \mathfrak{L}}{\partial \tau \partial t}$, computed by means of these equations coincide.

An analogous assertion holds also for the equations $\frac{\partial \mathfrak{L}}{\partial t}=[\tilde{\varphi}(u)^+, \mathfrak{L}]$ and $\frac{\partial \mathfrak{L}}{\partial \tau}=[\varphi(\tilde{u})^+, \mathfrak{L}]$. ■

Proposition 6.6. Let H be the same as in Proposition 6.2, $H=\sum_{i=1}^{\infty} H_i$, $H_i \in C^\infty(\mathbf{R}, \mathfrak{Z}^{-i})$. Then H_i are densities of conservation laws for Eqs. (6.4) and (6.5). Up to total derivatives these densities do not depend on the arbitrariness in the choice of U (see Proposition 6.2). ■

Remark 1. Since $\mathfrak{n} \subset G^-$, the densities H_i are gauge-invariant up to total derivatives (i.e., if the operator $\mathfrak{L}=\frac{d}{dx}+q+\Lambda$ is replaced by $e^{\operatorname{ad} S}(\mathfrak{L})$, $S(x) \in \mathfrak{n}$, then a total derivative of a differential polynomial in q and S is added in H_i). From this it follows easily that it is possible to add a total derivative to H_i in such a way that a gauge-invariant differential polynomial in q is obtained as a result.

Remark 2. According to part 5.6, $H_i \neq 0$ only if the remainder on dividing i by the Coxeter number of G is an exponent of this algebra; the number of scalar conservation laws corresponding to this exponent is equal to its multiplicity. Later (see Proposition 6.12) it will be proved that the densities of the conservation laws obtained are linearly independent modulo total derivatives.

Remark 3. It is not hard to show that if the element U (see Proposition 6.2) is normalized by the condition $U \in C^\infty(\mathbf{R}, (\mathfrak{Z}^\perp)^-)$, then the H_i is a homogeneous (in the sense indicated at the end of part 6.2) differential polynomial in q of degree of homogeneity i + 1.

6.4. In this subsection we consider generalizations of the modified KdV equation.

LEMMA 6.7. The relation (6.4) admits the reduction

$$\mathfrak{L}=\frac{d}{dx}+q+\Lambda, \quad q \in C^\infty(\mathbf{R}, \mathfrak{H}). \tag{6.7}$$

Proof. Let $\mathfrak{L}$ be an operator of the form (6.7), and let $M \in Z_{\mathfrak{L}}$. It must be shown that $[M^+, \mathfrak{L}] \in C^\infty(\mathbf{R}, \mathfrak{H})=C^\infty(\mathbf{R}, G^0)$. We have $[M^+, \mathfrak{L}[=-[M^-, \mathfrak{L}]$. The left side of this equality does not contain components that are negative relative to the canonical gradation, while the right contains none that are positive. ■

For any $u \in \mathfrak{Z}^+$ Eq. (6.4), where $\mathfrak{L}$ has the form (6.7), we call the generalized modified KdV equation corresponding to the algebra G (it does not depend on the choice of the vertex c_m, since it is defined in terms of the canonical gradation).

The mapping μ assigning to each operator $\mathfrak{L}$ of the form (6.7) its class of gauge equivalence we call a generalized Miura mapping. It is clear that μ takes solutions of the generalized mKdV into solutions of the corresponding generalized KdV.

It is obvious (see Proposition 6.6) that H_i are densities of conservation laws for generalized mKdV. It is not hard to verify that with a suitable normalization of U (see Remark 3 of the preceding subsection) H_i is a homogeneous differential polynomial in $q \in C^\infty(\mathbf{R}, \mathfrak{H})$ of degree of homogeneity i + 1 if it is assumed that $\deg q^{(j)} = j + 1$.

Proposition 6.8. Let $u \in \mathfrak{Z}^n$, $n>0$. Then the corresponding generalized mKdV has the form $\frac{dq}{dt}=\frac{d}{dx} f(q,\ldots,q^{(n-1)})$. If it is assumed that $q^{(j)}$ has degree of homogeneity j + 1, then f is a homogeneous polynomial of degree n. Its part linear in q is equal to $(-\operatorname{ad}\Lambda)^{-n}[u, q^{(n-1)}]$.

We mention that $(-\operatorname{ad}\Lambda)^{-n}$ here and below is considered as an operator on $\mathfrak{Z}^\perp$ (it follows from Proposition 5.16 that the operator $\operatorname{ad}\Lambda$ acts bijectively on $\mathfrak{Z}^\perp$). The correctness of the expression $(-\operatorname{ad}\Lambda)^{-n}[u, q^{(n-1)}]$ can be verified as follows. Since $\mathfrak{Z}^0=0$ (see part 5.6), it follows that $\mathfrak{H} \subset \mathfrak{Z}^\perp$. On the other hand, $[\operatorname{ad} u, \operatorname{ad}\Lambda] = 0$, and hence $[u, \mathfrak{Z}^\perp] \subset \mathfrak{Z}^\perp$. Thus, $[u, \mathfrak{H}] \subset \mathfrak{Z}^\perp$, so that the expression $(-\operatorname{ad}\Lambda)^{-n}[u, q^{(n-1)}]$ makes sense.

Proof. Let $\varphi(u)=\sum_{i=-\infty}^{n} A_i$, $A_i \in C^\infty(\mathbf{R}, G^i)$. Equating in (6.4) the components in G^0 and using the commutativity of G^0, we obtain $\frac{dq}{dt}=-\frac{d}{dx}A_0$. It remains to prove that A_0 is a homogeneous differential polynomial in q of degree n and to find its linear part. This is done in the same way as in the proof of Proposition 1.4. ■

Proposition 6.9. The generalized mKdV and KdV corresponding to a nonzero element $u \in \mathfrak{Z}^+$, are nontrivial.

By definition, nontriviality of the generalized mKdV means that its right side as a differential polynomial in q is not equal to zero. Nontriviality of the generalized KdV means that if this equation is considered as an equation for q^{can}, then its right side is nonzero.

Proof. In the modified case it suffices to use Propositions 6.8 and 5.17. We shall now show that triviality of the generalized KdV implies triviality of the modified equation. We represent the operator $\mathfrak{L}$ in Eq. (6.4) in the form $\mathfrak{L}=e^{\mathrm{ad}S}(\mathfrak{L}^{can})$ (see Proposition 6.1). Suppose that $\frac{d\mathfrak{L}^{can}}{dt}=0$. Then $\frac{d\mathfrak{L}}{dt}=[R, \mathfrak{L}]=\left[R, \frac{d}{dx}+I+q\right]$, where R is a differential polynomial in q with values in $\mathfrak{n}$. Suppose now that $q\in C^\infty(\mathbf{R}, \mathfrak{H})$. Then $\frac{d\mathfrak{L}}{dt}\in C^\infty(\mathbf{R}, \mathfrak{H})$ (see Lemma 6.7). Therefore, $P\left(\left[R, \frac{d}{dx}+I+q\right]\right)=0$, where P is the projector $\mathfrak{b}\to\mathfrak{n}$ such that $\operatorname{Ker}P=\mathfrak{H}$. We rewrite the equality obtained in the form

$$\frac{dR}{dx}=P[(R, I])+[R, q]. \tag{6.8}$$

We suppose that as a differential polynomial in q [where $q(x)\in\mathfrak{H}$] R has order k, i.e., R depends on $q^{(k)}$ but not on $q^{(k+1)}$, $q^{(k+2)}$, etc. Then the left side of (6.8) has order k + 1, while the right side has order no more than k. Therefore, R does not depend on q. From this and (6.8) it follows without difficulty that R = 0, i.e., the generalized mKdV is trivial contrary to what has been proved. ■

We note that together with the generalized modified and unmodified KdV equations it would be possible to consider a "partially modified" equation corresponding to an arbitrary proper subset S of the set of vertices of the Dynkin scheme. For this in the definition of the unmodified equation it is necessary to replace $\mathfrak{n}$ and $\mathfrak{b}$ by algebras $\mathfrak{n}_s$ and $\mathfrak{b}_s$, where $\mathfrak{n}_s$ is generated by elements f_i corresponding to the vertices in S and $\mathfrak{b}_s\overset{\text{def}}{=}\mathfrak{n}_s\oplus\mathfrak{H}$. The modified (respectively, unmodified) equation is obtained if S = ∅ (respectively, S consists of all vertices except one).

6.5. The remainder of the section is devoted to the Hamiltonian formalism for generalized KdV equations (including modified equations). In this subsection we define the corresponding Hamiltonian manifolds and in the following subsection we prove that the generalized mKdV and KdV are Hamiltonian. In the case of the unmodified equation the manifold on which it is necessary to introduce a Hamiltonian structure is the set of equivalence classes of operators $\mathfrak{L}$ of the form $\frac{d}{dx}+q+\Lambda$, $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$, but it can equally well be taken to be the set of equivalence classes of operators $\mathfrak{L}$ of the form $\frac{d}{dx}+q+I$ (see the beginning of the proof of Proposition 6.1). In the modified case the manifold on which it is necessary to introduce a Hamiltonian structure is $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$. Thus, both manifolds are defined in terms of the semisimple algebra $\mathfrak{G}$, and not the Kats–Moody algebra G. Therefore, in the present subsection we shall assume that $\mathfrak{G}$ is an arbitrary semisimple Lie algebra (not related to any Kats–Moody algebra).

Thus, let $\mathfrak{G}$ be a semisimple Lie algebra with Weyl generators X_i, Y_i, H_i, $1 \leqslant i \leqslant r$. We denote by $\mathfrak{n}$ and $\mathfrak{H}$, as always, the subalgebras generated by the elements $Y_1,\ldots,Y_r$ and $H_1,\ldots,H_r$, respectively. We set $\mathfrak{b}=\mathfrak{H}\oplus\mathfrak{n}$, $I=\sum_{i=1}^{r}X_i$. We consider operators $\mathscr{L}$ of the form $\frac{d}{dx}+q+I$, $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$. We call two such operators $\mathscr{L}$ and $\tilde{\mathscr{L}}$ gauge equivalent if $\tilde{\mathfrak{L}}=e^{\mathrm{ad}S}(\mathfrak{L})$, where $S\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{n})$. It is clear that Proposition 6.1 remains in force if $\mathfrak{L}$ is replaced by $\mathscr{L}$. We denote by $\mathscr{M}(\mathfrak{G})$ the set of classes of gauge equivalence of the operators $\mathscr{L}$. We call the mapping $\mu: C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})\to\mathscr{M}(\mathfrak{G})$, assigning to a function $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$ the equivalence class of the operator d/dx + q + I the Miura transformation.

On $\mathfrak{G}$ we fix a nondegenerate, invariant, bilinear form (according to Proposition 5.18, this form is always symmetric). We extend it to a bilinear form on $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$ by the formula

$$(u, v)\overset{\text{def}}{=}\int_{\mathbf{R}/\mathbf{Z}}(u(x), v(x))\,dx, \tag{6.9}$$

where $u, v\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$. The gradient of a functional $l: C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})\to\mathbf{C}$ at a point $q\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$ is a function $\operatorname{grad}_q l\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$ such that the relation (3.15) is satisfied for any $h\in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$. The gradient is uniquely determined by this condition, since the scalar product on $\mathfrak{H}$

is nondegenerate. We define a Poisson bracket on $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$ by formula (3.28).

As in Sec. 3, on $\mathscr{M}(\mathfrak{G})$ we define two Hamiltonian structures. Here the first structure will depend on the choice of the element e in the center of $\mathfrak{N}$. If l is a functional on $\mathscr{M}(\mathfrak{G})$ [i.e., a gauge-invariant functional on $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$], then $\mathrm{grad}_q l$ denotes any element in $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$ satisfying (3.15) for all $h \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b})$. From assertion 3) of Proposition 5.19 it follows that $\mathrm{grad}_q l$ is defined up to the addition of elements of $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{n})$. We define the first and second Hamiltonian structures on $\mathscr{M}(\mathfrak{G})$ by formulas (3.16) and (3.17). Just as in part 3.6, it can be verified that $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ are well defined and are coordinated Poisson brackets. Moreover, just as in part 3.8, it can be proved that the mapping $\mu : C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{h}) \to \mathscr{M}(\mathfrak{G})$ is Hamiltonian if the second Hamiltonian is considered on $\mathscr{M}(\mathfrak{G})$.

The manifold denoted by $\mathscr{M}$, in part 3.6 will henceforth be denoted by $\mathscr{M}(\mathfrak{gl}(k))$ [we are obliged to specify this, since the algebra $\mathfrak{gl}(k)$ is not semisimple]. In Sec. 3 $\mathscr{M}(\mathfrak{gl}(k))$ was identified with the set of differential operators of the form $D^k+\sum_{l=1}^{k-1} u_l D^l$, so that the first and second Hamiltonian structures on $\mathscr{M}(\mathfrak{gl}(k))$ go over into the corresponding Gel'fand–Dikii structures. Analogous realizations of $\mathscr{M}(\mathfrak{G})$ for all classical simple algebras $\mathfrak{G}$ will be constructed in Sec. 8.

Remark 1. By definition, $\mathscr{M}(\mathfrak{G})$ depends on the choice of Weyl generators of the algebra $\mathfrak{G}$, but from assertion 2) of Proposition 5.1 it follows that this dependence is actually inconsequential.

Remark 2. Both Hamiltonian structures on $\mathscr{M}(\mathfrak{G})$ depend on the choice in $\mathfrak{G}$ of an invariant scalar product, while the first structure additionally depends on the choice of an element e in the center of $\mathfrak{n}$. If the algebra $\mathfrak{G}$ is simple, then e and the invariant scalar product are uniquely determined up to multiplication by a number (see Propositions 5.12 and 5.18). Therefore, the arbitrariness in their choice leads only to multiplication of both brackets by constants.

Remark 3. Let $\mathfrak{G}=\mathfrak{G}_1\times\mathfrak{G}_2$. We assume that the scalar products on $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are taken to be the restrictions of the scalar product on $\mathfrak{G}$. Then the manifold $\mathscr{M}(\mathfrak{G})$, equipped with the second Hamiltonian structure is the direct product of the Hamiltonian manifolds $\mathscr{M}(\mathfrak{G}_1)$ and $\mathscr{M}(\mathfrak{G}_2)$ (on which the second structure is also introduced). The same holds for the first structure if the elements $e\in\mathfrak{G}$, $e_1\in\mathfrak{G}_1$, $e_2\in\mathfrak{G}_2$ are chosen in a coordinated way. Thus, it suffices to study $\mathscr{M}(\mathfrak{G})$ in the case where $\mathfrak{G}$ is a simple algebra. The words "a Hamiltonian manifold M is the direct product of Hamiltonian manifolds M_1 and M_2" mean, by definition, that a) M as a set is equal to $M_1 \times M_2$; b) the projections $\pi_i: M \to M_i$, i = 1, 2 are Hamiltonian mappings; c) for any functionals $\varphi_1 : M_1 \to \mathbf{C}$, $\varphi_2 : M_2 \to \mathbf{C}$ the equality $\{\pi_1{}^*\varphi_1, \pi_2{}^*\varphi_2\}=0$ is satisfied, where $\pi_i{}^*\varphi_i$ is the functional on M defined by the formula $(\pi_i{}^*\varphi_i)(x)=\varphi_i(\pi_i(x))$.

The remainder of this subsection is devoted to an interpretation of the second Hamiltonian structure on $\mathscr{M}(\mathfrak{G})$ (and, in particular, the Gel'fand–Dikii structure) in terms of Hamiltonian reduction. This interpretation will not be used below but is of interest in itself.

We denote by M the set of operators of the form d/dx + q, $q \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$, equipped with the second Hamiltonian structure (see part 4.3). We denote by N the connected and simply connected Lie group with Lie algebra $\mathfrak{n}$ [if $\mathfrak{G}$ is realized as a subalgebra of $\mathfrak{gl}(k)$, then N = $\{e^x \mid x \in \mathfrak{n}\}$]. We set $\tilde{N} \overset{\text{def}}{=} C^\infty(\mathbf{R}/\mathbf{Z}, N)$, $\tilde{\mathfrak{n}} = C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{n})$. The group $\tilde{N}$ acts on M by conjugation. It is easy to verify that this action is a Poisson action (see [1], p. 337) and the corresponding moment mapping P: $M \to \tilde{\mathfrak{n}}^*$ (see [1], p. 338) assigns to the operator d/dx + q the functional $l_q : \tilde{\mathfrak{n}} \to \mathbf{C}$, given by the formula $l_q(f) = (f, q)$. According to the general scheme of Hamiltonian reduction (see [37], p. 11; [1], pp. 339, 340), if $l \in \tilde{\mathfrak{n}}^*$, and $\mathcal{O}_l$ is the orbit of l under the coadjoint action of $\tilde{N}$, then under some additional assumptions there is a natural Hamiltonian structure on $P^{-1}(\mathcal{O}_l)/\tilde{N}$. It is said that $P^{-1}(\mathcal{O}_l)/\tilde{N}$ is obtained by reduction of M by the action of $\tilde{N}$ on l [sometimes $P^{-1}(\mathcal{O}_l)/\tilde{N}$ is called reduced phase space]. It is not hard to verify that the manifold $\mathscr{M}(\mathfrak{G})$, equipped with the second Hamiltonian structure is obtained by reduction of M by the action of $\tilde{N}$ on the functional $l_I \in \tilde{\mathfrak{n}}^*$, given by the formula $l_I(f) = (f, I)$.

If we now use the interpretation of M presented in part 4.4, then we obtain the following abstract description of $\mathscr{M}(\mathfrak{G})$. Suppose first that $\mathfrak{A}_1$ is an arbitrary Lie algebra, $\mathfrak{A}_2$

is a subalgebra of it, and A_2 is a connected Lie group with algebra $\mathfrak{A}_2$, $l \in \mathfrak{A}_2^*$. We then denote by $M(\mathfrak{A}_1, \mathfrak{A}_2, l)$ the Hamiltonian manifold obtained from the reduction of $\mathfrak{A}_1^*$ by the coadjoint action of A_2 on l (we consider the Kirillov Hamiltonian structure on $\mathfrak{A}_1^*$). It is not hard to see that the manifold $\mathcal{M}(\mathfrak{G})$, equipped with the second Hamiltonian structure has the form $M(\mathfrak{A}_1, \mathfrak{A}_2, l)$: as $\mathfrak{A}_1$ it is necessary to take the algebra $\hat{\mathfrak{G}}$, considered in part 4.4, as $\mathfrak{A}_2$ the preimage of $\tilde{\mathfrak{n}}$ under the natural mapping $\hat{\mathfrak{G}} \to C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{G})$ (we note that the algebra $\mathfrak{A}_2$ is canonically isomorphic to $\tilde{\mathfrak{n}} \times \mathbf{C}$), and define the functional $l: \tilde{\mathfrak{n}} \times \mathbf{C} \to \mathbf{C}$ by the formula $l(f, \alpha) = (f, I) + \alpha$. All that has been said above is valid, in particular, for $\mathfrak{G} = \mathfrak{gl}(k)$. Thus, we have obtained a group-theoretic interpretation of the second Hamiltonian structure of Gel'fand–Dikii.

6.6. We now return to the Kats–Moody algebra G. Let $\mathfrak{G}$, $\mathfrak{b}$, etc. be the same as in part 6.1. On G we fix a nondegenerate, symmetric, invariant, bilinear form coordinated with the canonical gradation (see Proposition 5.20). Since this form is coordinated with the standard gradations, its restriction to G_0 (i.e., to $\mathfrak{G}$) is nondegenerate. We shall define the Hamiltonian structures on $C^\infty(\mathbf{R}/\mathfrak{Z}, \mathfrak{H})$ and $\mathcal{M}(\mathfrak{G})$ (see part 6.5) using just this bilinear form on $\mathfrak{G}$.

The bilinear form on $\hat{G}$ obviously extends to $\hat{G}$ and then to $C^\infty(\mathbf{R}/\mathbf{Z}, \hat{G})$ [see formula (6.9)]. For any $u \in \hat{\mathfrak{Z}}$ we define the functional $\mathcal{H}_u: C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{b}) \to \mathbf{C}$ by the formula $\mathcal{H}_u(q) = (H(q), u)$, where H(q) is defined by formula (6.3). If $u \in \mathfrak{Z}^-$, then $\mathcal{H}_u = 0$, and hence it may be assumed with no loss of generality that $u \in \mathfrak{Z}^+$. We note that $\mathcal{H}_u$ does not depend on the arbitrariness in the definition of H (see Proposition 6.6). From Remark 1 following Proposition 6.6 it follows that $\mathcal{H}_u$ is gauge-invariant and hence can be considered a functional on $\mathcal{M}(\mathfrak{G})$.

Proposition 6.10. The generalized KdV corresponding to an element $u \in \hat{\mathfrak{Z}}$, is the Hamiltonian equation corresponding to the Hamiltonian $\mathcal{H}_u$ and the second Hamiltonian structure.

Proof. Let $\varphi(u)$ be the same as in formula (6.5), $\varphi(u) = \sum_{i=-\infty}^{n} A_i$, $A_i \in C^\infty(\mathbf{R}/\mathbf{Z}, G_i)$. Exactly as in the proof of Proposition 1.9, it can be verified that for $\operatorname{grad} \mathcal{H}_u$ it is possible to take A_0. It remains to show that $\left[A_0, \frac{d}{dx} + q + I\right] = \left[\varphi(u)_+, \frac{d}{dx} + q + \Lambda\right]$. This equality follows from Lemma 6.4 and the fact that the projection of q + Λ onto G_0 is equal to q + I. ■

Since the functionals $\mathcal{H}_u$ are conservation laws for generalized KdV (see Proposition 6.6), the next result follows from Proposition 6.10.

COROLLARY. $\{\mathcal{H}_u, \mathcal{H}_{\tilde{u}}\}_2 = 0$ for any $u, \tilde{u} \in \hat{\mathfrak{Z}}$.

We denote by $\bar{\mathcal{H}}_u$ the restriction of $\mathcal{H}_u$ to $C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H})$. From the equality $\{\mathcal{H}_u, \mathcal{H}_{\tilde{u}}\}_2 = 0$ and the fact that the mapping $\mu: C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H}) \to \mathcal{M}(\mathfrak{D})$ is Hamiltonian it follows that $\{\bar{\mathcal{H}}_u, \bar{\mathcal{H}}_{\tilde{u}}\} = 0$.

Proposition 6.11. The generalized mKdV corresponding to an element $u \in \hat{\mathfrak{Z}}$, is the Hamiltonian equation corresponding to the Hamiltonian $\bar{\mathcal{H}}_u$.

Proof. Let $\varphi(u) = \sum_{i=-\infty}^{k} A^i$, where $A^i \in C^\infty(\mathbf{R}/\mathbf{Z}, G^i)$. In the proof of Proposition 6.8 it was shown that the generalized mKdV has the form $dq/dt = -(A^0)'$. On the other hand, just as in the proof of Proposition 1.9, it can be verified that $\operatorname{grad} \bar{\mathcal{H}}_u = A^0$. Thus, the equation in question has the form $\frac{dq}{dt} = -(\operatorname{grad} \bar{\mathcal{H}}_u)'$, and this is a Hamiltonian equation. ■

The next result follows from Propositions 6.9-6.11.

Proposition 6.12. If $u \in \mathfrak{Z}^+$, $u \neq 0$, then $\mathcal{H}_u \neq 0$, $\bar{\mathcal{H}}_u \neq 0$. ■

The equations considered in Sec. 3 are Hamiltonian relative to not only the second but also the first Hamiltonian structure. For generalized KdV this is not true, generally speaking. A counterexample is provided by generalized KdV corresponding to the algebra $A_2^{(2)}$ (see Sec. 9). We shall show, however, that the assertion regarding the Hamiltonian character relative to the first structure is valid for generalized KdV corresponding to a Kats–Moody algebra of height 1 and a special vertex of its Dynkin scheme (see part 5.4).

We recall that the standard realization of a Kats–Moody algebra G of height 1 corresponding to a special vertex has the form $\mathfrak{A}(\lambda, \lambda^{-1}]$, where $\mathfrak{A}$ is a simple Lie algebra. In this realization $G_k = \mathfrak{A}\lambda^k$ and, in particular, $\mathfrak{G} = \mathfrak{A}$. Let X_i, Y_i, H_i, where $1 \leqslant i \leqslant r$, be the Weyl generators of the algebra $\mathfrak{A}$, and let X_0 be the same as in Proposition 5.12. From the explicit form of the canonical generators of $\mathfrak{A}[\lambda, \lambda^{-1}]$ (see part 5.4) it follows that $\Lambda = I + \lambda X_0$, where $I=\sum_{i=1}^{r} X_i$. For the element e contained in the definition of the first Hamiltonian structure we take X_0 (we recall that according to Proposition 5.12 the center of $\mathfrak{n}$ is equal to CX_0). The following result is proved just as analogous assertions in Secs. 1 and 3.

Proposition 6.13. In the situation described above the generalized KdV corresponding to an element $u \in \hat{\mathfrak{Z}}$, is the Hamiltonian equation corresponding to the Hamiltonian $\mathcal{H}_{\lambda u}$ and the first Hamiltonian structure. Moreover, $\{\mathcal{H}_u, \mathcal{H}_{\tilde{u}}\}_1 = 0$ for any u, $\tilde{u} \in \hat{\mathfrak{Z}}$.

7. SCALAR (L, A)-PAIRS FOR GENERALIZED KdV EQUATIONS

7.1. In Sec. 3 the scalar Lax equation (2.1) was interpreted as the equation for the class of gauge equivalence for Eq. (3.8). In Sec. 6 for an arbitrary pair (G, c_m), where G is a Kats–Moody algebra and c_m is a vertex of the Dynkin scheme of G, an analogue of Eq. (3.8) [Eq. (6.5)] was constructed. The equation for the class of gauge equivalence corresponding to this analogue was called a generalized KdV equation. A generalization of the generalized KdV corresponding to $(\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}]), c_0)$, where c_0 is a special vertex of the Dynkin scheme, is the scalar Lax equation (2.1) in which the additional condition $u_{k-1} = 0$ is imposed on the operator L (regarding the possibility of such a reduction see the remarks following Propositions 2.3 and 3.7). It is possible to not consider vertices of the Dynkin scheme of $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$, distinct from c_0, since they go over into c_0 under automorphisms of this scheme (see Table 2, type $A_n^{(1)}$).

In this section for generalized KdV corresponding to classical Kats–Moody algebras distinct from $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$, we will find realizations analogous to the scalar Lax equations. It turns out that a generalized KdV corresponding to a classical Kats–Moody algebra G distinct from $\mathfrak{sl}(k, \mathbf{C}[\lambda, \lambda^{-1}])$, and a vertex c_m of its Dynkin scheme can with some conventions (see parts 7.3 and 7.4) be written in the form

$$\begin{cases} \frac{dL_1}{dt} = A_2L_1 - L_1A_1, \\ \frac{dL_2}{dt} = A_1L_2 - L_2A_2, \end{cases} \tag{7.1}$$

where L_1 and L_2 are scalar pseudodifferential symbols, $A_1=\sum_i b_i (L_2L_1)_+^{\frac{2i+1}{k}}$, $A_2=\sum_i b_i (L_1L_2)_+^{\frac{2i+1}{k}}$, $b_i \in \mathbf{C}$, $k \overset{\text{def}}{=} \operatorname{ord} L_1 + \operatorname{ord} L_2$. Here the operators L_i are of three types (we denote these types by P_n, Q_n, and R_n):

$$1)\ P_n \overset{\text{def}}{=} D^{2n+1} + \sum_{i=0}^{n-1} (u_i(x) D^{2i+1} + D^{2i+1} u_i(x)),$$

$$2)\ Q_n \overset{\text{def}}{=} D^{2n} + \sum_{i=0}^{n-1} (u_i(x) D^{2i} + D^{2i} u_i(x)),$$

$$3)\ R_n \overset{\text{def}}{=} D^{2n-1} + \sum_{i=1}^{n-1} (u_i(x) D^{2i-1} + D^{2i-1} u_i(x)) + u_0(x) D^{-1} u_0(x).$$

In order to determine the types of operators L_1 and L_2 corresponding to a given pair (G, c_m) it is convenient to use the language of Dynkin schemes. The Dynkin scheme for G after removing c_m decomposes into Dynkin schemes of two simple Lie algebras of types B_n, C_n, and D_n (see Tables 1 and 2). To one of these algebras there corresponds the operator L_1 and to the other the operator L_2. To the algebra B_n there hereby corresponds an operator of type P_n, while to the algebras C_n and D_n there correspond operators of types Q_n and R_n.

A detailed proof of the assertion formulated above will be presented for the cases $(A_{2n}^{(2)}, c_m)$ and $(D_n^{(1)}, c_m)$, where c_m is a vertex of general position. In the remaining cases the proof is analogous.

We note that if L_1 and L_2 satisfy (7.1), then the operator $L \overset{\text{def}}{=} L_2L_1$ satisfies the equation $\frac{dL}{dt} = [A_1, L]$, where $A_1 = \sum_i b_i L_+^{\frac{2i+1}{k}}$, $k \overset{\text{def}}{=} \operatorname{ord} L$. Therefore, the system (7.1), where L_1 and L_2 have type P_n or Q_n, is in a certain sense a reduction of the scalar Lax equation. Special cases of such reductions were considered in [41].

7.2. Suppose that the algebra G of type $A_{2n}^{(2)}$ is realized in the form $L(\mathfrak{sl}(2n+1), \sigma_m)$, $0 < m < n$, where σ_m is the standard automorphism corresponding to the vertex c_m (see Appendix 2). Then the operator $\mathfrak{L}$ corresponding to the pair (G, c_m) [see formula (6.1)] has the form

$$\mathfrak{L}=\frac{d}{dx}+\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}+\Lambda, \qquad (7.2)$$

where the blocks $a_1(x)$ and $a_2(x)$ are upper triangular and belong, respectively, to $\mathfrak{o}(2n-2m+1)$ and $\mathfrak{sp}(2m)$; $\Lambda=\sum_{i=0}^{n}\tilde{e}_i$ (the form of $\tilde{e}_i$ is indicated in Appendix 2). We denote by W_1 the set of columns of the form $(u_1, u_2,\ldots,u_{2n+1})^t$, where $u_i(\lambda)\in B((\lambda^{-1}))$, $i=1,\ldots,2n+1$, whereby $u_i(\lambda) = u_i(-\lambda)$ for $i \leqslant 2n - 2m + 1$, $u_i(\lambda) = -u_i(-\lambda)$ for $i > 2n - 2m + 1$. We set $W_2\overset{\text{def}}{=}\lambda W_1$. If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in G$, where $\alpha\in \mathrm{Mat}(2n-2m+1, \mathbf{C}[\lambda, \lambda^{-1}])$, $\delta\in\mathrm{Mat}(2m, \mathbf{C}[\lambda, \lambda^{-1}])$, then the blocks α and δ contain only even powers of λ, while the blocks β and γ contain only odd powers. Therefore, $X(W_i)\subset W_i$ for any $X\in G$. Moreover, $\mathfrak{L}(W_i)\subset W_i$. By means of the operator $\mathfrak{L}$ we introduce the structure of a $B((D^{-1}))$-module on W_i in the same way as this was done in part 3.4. The following result is proved by direct verification.

<u>LEMMA 7.1.</u> Any vector in W_i whose components contain λ only in nonnegative powers can be uniquely represented in the form $A\cdot\psi_i$, where $A\in B[D]$, $\psi_1=(1, 0, \ldots, 0)^t$, $\psi_2=(\underbrace{0,\ldots,0}_{2n-2m+1}, 1, 0,\ldots, 0)^t$. ■

It follows from Lemma 7.1 that there exist uniquely determined operators L_1, $L_2\in B[D]$ such that $L_1\cdot\psi_1 = \lambda\psi_2$, $L_2\cdot\psi_2 = \lambda\psi_1$. It is easy to see that the orders of these operators are equal, respectively, to $2n - 2m + 1$ and $2m$, while the leading coefficients are equal to one. It is not hard to verify that L_1 and L_2 do not change under gauge transformations of the operator $\mathfrak{L}$.

<u>LEMMA 7.2.</u> $L_1^* = -L_1$, $L_2^* = L_2$.

<u>Proof.</u> It is proved in analogy to Proposition 3.14 that $L_i = -(\Delta(P_i))^*$, where $P_i = a_i(x)+\begin{pmatrix} D & 0\ldots 0 \\ 1 & D\ldots 0 \\ \cdot & \cdot\ \cdot\ \cdot \\ 0 & \ldots 1\ D \end{pmatrix}$. According to Lemma 3.13, $(\Delta(P_i))^* = \Delta(P_i^T)$. On the other hand, $P_i^T = -\mathrm{diag}(1, -1,\ldots,(-1)^{i+1}) P_i \,\mathrm{diag}(1, -1,\ldots,(-1)^{i+1})$, whence $\Delta(P_1^T)=-\Delta(P_1)$, $\Delta(P_2^T)=\Delta(P_2)$. ■

We have thus constructed a mapping from the set of classes of gauge equivalence of operators $\mathfrak{L}$ of the form (7.2) to the set of pairs (L_1, L_2), where L_1 and L_2 are operators of type P_{n-m} and Q_m, respectively (see part 7.1). In Sec. 8 it will be shown that this mapping is bijective (see Propositions 8.2 and 8.4).

It is not hard to verify that elements of the form Λ^{2i+1}, where $2i+1\in\mathbf{Z}$ is not divisible by $2n + 1$, generate the centralizer $\mathfrak{Z}$ of the element Λ as a vector space over $\mathbf{C}$ (see Appendix 2).

<u>Proposition 7.3.</u> Suppose that an operator $\mathfrak{L}$ of the form (7.2) satisfies Eq. (6.5), where $u=\sum_{i=0}^{k}b_i\Lambda^{2i+1}$, $b_i\in\mathbf{C}$. Then the operators L_1 and L_2 satisfy the system (7.1), where $A_1 = -\sum_{i=0}^{k}b_i(L_2L_1)_+^{\frac{2i+1}{2n+1}}$, $A_2=-\sum_{i=0}^{k}b_i(L_1L_2)_+^{\frac{2i+1}{2n+1}}$.

<u>Proof.</u> Throughout the proof B will denote a ring of functions of x and t. On W_i, $i = 1, 2$ we introduce the structure of a $B[D, D_t]$-module so that the operator of multiplication by D_t is equal to $\frac{d}{dt}-\varphi(u)_+$. This is possible since 1) $\left[\frac{d}{dt}-\varphi(u)_+, \mathfrak{L}\right]=0$, 2) $\varphi(u)_+\in G$, and the elements of G take W_i into itself. We shall find operators $A_i\in B[D]$, $i=1, 2$, such that $D_t\cdot\psi_i = A_i\cdot\psi_i$. Since $D_t\cdot\psi_i=-\varphi(u)_+\psi_i$, according to Lemma 7.1 the A_i exist and are unique. We have $\left(D_t-\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right)\cdot\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}=0$, $\begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix}\cdot\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}=\lambda\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Hence

$$\left[D_t - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} 0 & L_2 \\ L_1 & 0 \end{pmatrix}\right] \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0, & \frac{dL_2}{dt} - A_1L_2 + L_2A_2 \\ \frac{dL_1}{dt} - A_2L_1 + L_1A_1, & 0 \end{pmatrix} \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

and hence the operators L_i satisfy the system (7.2). In order to express A_1 in terms of L_1 and L_2 we note that $L_2L_1 \cdot \psi_1 = \lambda^2\psi_1$. From this, as in the proof of Proposition 3.16, it follows that $\varphi(\Lambda)\psi_1 = (L_2L_1)^{\frac{1}{2n+1}} \cdot \psi_1$. Further, following the proof of Lemma 3.17, we find that $A_1 = -\sum_{l=0}^{k} b_l (L_2L_1)_+^{\frac{2l+1}{2n+1}}$. The formula for A_2 is derived similarly. ■

For algebras of types $C_n^{(1)}$ and $D_n^{(2)}$ the realization of the generalized KdV in the form of a system (7.1) is found in exactly the same way. In the case of the algebras $B_n^{(1)}$, $A_{2n-1}^{(2)}$, and $D_n^{(1)}$ the arguments are somewhat more involved, since $\det \Lambda = 0$. In the next subsection we treat the case $D_n^{(1)}$ which is the most complicated, since the zero eigenvalue of the matrix Λ is multiple (multiplicity 2).

7.3. We denote by G the standard realization of the algebra of type $D_n^{(1)}$ corresponding to a vertex c_m, $1 < m < n - 1$ (see Appendix 2). We note that if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, $\alpha \in \mathrm{Mat}(2n-2m, \mathbf{C}[\lambda, \lambda^{-1}])$, $\delta \in \mathrm{Mat}(2m, \mathbf{C}[\lambda, \lambda^{-1}])$, then $\alpha(\lambda) = \alpha(-\lambda)$, $\delta(\lambda) = \delta(-\lambda)$, $\beta(\lambda) = -\beta(-\lambda)$, $\gamma(\lambda) = -\gamma(-\lambda)$. We henceforth write matrices of order 2n in the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where the orders of α and δ are equal, respectively, to $2n - 2m$ and $2m$. The operator $\mathfrak{L}$, corresponding to the pair (G, c_m) has the form $\mathfrak{L} = \frac{d}{dx} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \Lambda$, where $\Lambda = \sum_{l=0}^{n} \tilde{e}_l$ (the form of $\tilde{e}_i$ is indicated in Appendix 2), the matrices $a_i(x)$ are upper triangular, and $a_1 \in \mathfrak{b}^0(2n-2m)$, $a_2 \in \mathfrak{b}^0(2m)$.

We shall formulate some properties of the matrix Λ that we need below.

LEMMA 7.4. 1) The eigenvalues of Λ are roots of degree $2n - 2$ in λ^2 and zero (having multiplicity two). 2) $B((\lambda^{-1}))^{2n} = \mathrm{Ker}\,\Lambda \oplus \mathrm{Im}\,\Lambda$. 3) The matrix $P \overset{\text{def}}{=} \lambda^{-2}\Lambda^{2n-2}$ is the projector onto $\mathrm{Im}\,\Lambda$. P does not depend on λ, and the first and $(2n - 2m + 1)$-th columns of P have the form $\psi_1 \overset{\text{def}}{=} (1, 0, \ldots, 0)^t$ and $\psi_2 \overset{\text{def}}{=} (\underbrace{0, \ldots, 0}_{2n-2m}, 1, 0, \ldots, 0)^t$, respectively. 4) The centralizer of Λ in G is generated as a vector space over $\mathbf{C}$ by the elements Λ^{2l+1} and $\lambda^{2l+1}F$, $l \in \mathbf{Z}$. Here Λ^k for $k < 0$ is defined by the formula $\Lambda^k \overset{\text{def}}{=} \lambda^{-2r}\Lambda^{k+(2n-2)r}$, $r \gg 0$;

$$F \overset{\text{def}}{=} \frac{1}{2}(e_{n-m,2n-m} + e_{n-m+1,2n-m+1} + (-1)^n e_{2n-m,n-m} +$$
$$+ (-1)^n e_{2n-m+1,n-m+1}) - e_{n-m,2n-m+1} + (-1)^{n+1} e_{2n-m,n-m+1} -$$
$$- \frac{1}{4}(e_{n-m+1,2n-m} + (-1)^n e_{2n-m+1,n-m}).$$

5) For $k < 0$, Λ^k does not contain positive powers of λ; the first and $(2n - 2m + 1)$-th columns of Λ^k contain only strictly negative powers of λ. ■

We denote by W_1 the set of columns of the form $(u_1, \ldots, u_{2n})^t$ such that $u_i \in B((\lambda^{-1}))$ for $i = 1, \ldots, 2n$, $u_i(\lambda) = u_i(-\lambda)$ for $i \leqslant 2n - 2m$, $u_i(\lambda) = -u_i(-\lambda)$ for $i > 2n - 2m$. We set $W_2 \overset{\text{def}}{=} \lambda W_1$. It is clear that $B((\lambda^{-1}))^{2n} = W_1 \oplus W_2$, $\mathfrak{L}(W_i) \subset W_i$, $X(W_i) \subset W_i$ for all $X \in G$.

According to Proposition 6.2, there exists a series T such that the operator $\mathfrak{L}_0 = T\mathfrak{L}T^{-1}$ has the form

$$\mathfrak{L}_0 = \frac{d}{dx} + \Lambda + \sum_{l=0}^{\infty} f_l \Lambda^{-(2l+1)} + \sum_{l=0}^{\infty} g_l \lambda^{-(2l+1)} F,$$

where f_l, $g_l \in B$. We shall need the following properties of T.

LEMMA 7.5. 1) $T(W_i) = W_i$. 2) $T(\psi_i) - \psi_i$ is a series in strictly negative powers of λ.

Proof. Since $T = e^U$, where $U \in G$, it follows that $T(W_i) = W_i$. Since $U \in G^-$, it follows that T does not contain positive powers of λ while the free term of T has the form $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, where α_i are upper triangular matrices with ones on the main diagonal. From this we obtain assertion 2). ■

We set $W_i' = T^{-1}(W_i \cap \operatorname{Im}\Lambda)$, $W_i'' = T^{-1}(W_i \cap \operatorname{Ker}\Lambda)$. Since $B((\lambda^{-1}))^{2n} = \operatorname{Ker}\Lambda \oplus \operatorname{Im}\Lambda = W_1 \oplus W_2$ and $\Lambda(W_i) \subset W_i$, it follows that $W_i = (W_i \cap \operatorname{Ker}\Lambda) \oplus (W_i \cap \operatorname{Im}\Lambda)$, and hence $W_i = W_i' \oplus W_i''$. It is clear that $\mathfrak{L}(W_i') \subset W_i'$, $\mathfrak{L}(W_i'') \subset W_i''$. Moreover, it is not hard to show that the operator $\mathfrak{L}$ is invertible on W_i'. On W_i' we introduce by means of the operator $\mathfrak{L}$ the structure of a $B((D^{-1}))$-module in the same way as this was done in part 3.4.

LEMMA 7.6. Any element of W_i' can be represented uniquely in the form $A \cdot \tilde{\psi}_i$, where $A \in B((D^{-1}))$, and $\tilde{\psi}_i$ is the projection of ψ_i onto W_i'.

Proof. It suffices to verify that any element of $W_i \cap \operatorname{Im}\Lambda$ can be represented uniquely in the form $\sum_{j=-\infty}^{k} f_j \mathfrak{L}_0^j P T \psi_i$, $f_j \in B$, where P is the projector onto $\operatorname{Im}\Lambda$ such that $\operatorname{Ker} P = \operatorname{Ker}\Lambda$. This can easily be seen directly. It is only necessary to note that $PT\psi_i = \psi_i + R_i$, where R_i is a series in strictly negative powers of λ (this follows from assertion 2) of Lemma 7.5 and assertion 3) of Lemma 7.4). ■

According to Lemma 7.6, there exist pseudodifferential symbols L_1 and L_2 such that $L_1 \cdot \tilde{\psi}_1 = \lambda \tilde{\psi}_2$, $L_2 \cdot \tilde{\psi}_2 = \lambda \tilde{\psi}_1$.

LEMMA 7.7. L_1 and L_2 have the forms R_{n-m} and R_m, respectively (see part 7.1).

Proof. In analogy to the way this was done in the proof of Lemma 7.2, it can be shown that $L_i^* = -L_i$, and $L_i = \Delta(P_i)$, where

$$P_i = a_i(x) + \begin{pmatrix}
D & 0 & \cdots & & & & 0\\
1 & D & \cdots & & & & 0\\
0 & 1 & \ddots & \cdots & & & 0\\
\vdots & \ddots & \ddots & D & & & \\
 & & & 1 & D & & \\
 & 1/2 & 0 & 1 & \ddots & D & \\
 & & 1/2 & \ddots & \ddots & \ddots & 0\\
 & & & \ddots & 1 & & D\\
0 & & & 0 & & 1 & D
\end{pmatrix}$$

It remains to find the order and leading coefficient of L_i and also prove that $(L_i)_-$ has the form $fD^{-1}f$, $f \in B$. Let, say, $i = 1$. We recall that P_1 has dimension $2n - 2m$. It is convenient to go over from the matrix P_1 to the matrix $\tilde{P}_1 \stackrel{\text{def}}{=} SP_1S^{-1}$, where $S \stackrel{\text{def}}{=} E - \frac{1}{2}e_{n-m,n-m} - 2e_{n-m+1,n-m+1} + e_{n-m,n-m+1} + \frac{1}{2}e_{n-m+1,n-m}$ and then in $\tilde{P}_1$ permute the columns with indices $n - m$ and $n - m + 1$. We denote the matrix obtained as a result of this by $\overline{P}_1$. It is clear that $\Delta(P_1) = \Delta(\overline{P}_1)$. $\overline{P}_1$ has the form $\overline{P}_1 = q(x) + (E - e_{n-m,n-m} - e_{n-m+1,n-m+1} + e_{n-m,n-m+1} + e_{n-m+1,n-m})D + \sum_{i=1}^{2n-2m-1} e_{i+1,i} - e_{n-m+1,n-m}$, where the matrix $q(x)$ is upper triangular. From this it is not hard to deduce that $(L_1)_+$ has the form $D^{2n-2m+1} + \sum_{l=0}^{2n-2m} u_l(x) D^l$, and $(L_1)_-$ has the following structure. Let $\overline{P}_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{ij} are blocks of order $n - m$. We write $\Delta(A_{11})$ in the form $\sum_l a_l D^l$ and $\Delta(A_{22})$ in the form $\sum_l D^l b_l$. Then $(L_1)_- = -a_0 D^{-1} b_0$. It remains to note that $A_{22} = \operatorname{diag}(1, 1, -1, 1, -1, \ldots, (-1)^{n-m} \times A_{11}^T \operatorname{diag}(1, -1, 1, -1, \ldots, (-1)^{n-m+1}$, whence $\Delta(A_{22})) = (-1)^{n-m+1}(\Delta(A_{11}))^*$ and hence $b_0 = (-1)^{n-m+1} a_0$. Thus, $(L_1)_- = (-1)^{n-m} a_0 D^{-1} a_0$. ■

We set $f_1 = i^{n-m} a_0$, where a_0 is the same as in the proof of the lemma. The analogue of f_1 constructed on the basis of the matrix P_2 we denote by f_2. It is easy to see that L_i and f_i do not change under gauge transformations of $\mathfrak{L}$. Thus, we have constructed a mapping from the set of classes of gauge equivalence of operators (L_1, L_2, f_1, f_2), where $f_i \in B$ and L_i are skew-symmetric pseudodifferential symbols with leading coefficient 1 such that

$(L_i)^- = f_i D^{-1} f_i$, ord $L_1 = 2n-2m-1$, ord $L_2 = 2m-1$. Later (see Proposition 8.5) it will be shown that this mapping is bijective. We note that the mapping from the set of classes of gauge equivalence of operators $\mathfrak{L}$ into the set of pairs (L_1, L_2) is not bijective, since, knowing L_i, it is possible to recover f_i only up to sign.

In part 7.1 an assertion was formulated that the generalized KdV corresponding to a classical Kats–Moody algebra distinct from $A_n^{(1)}$ reduces to the system (7.1), where A_1 and A_2 are expressed in terms of fractional powers of L_1L_2 and L_2L_1. In the case $D_n^{(1)}$ this assertion needs refinement: the generalized KdV in question must correspond to an element $u \in \mathfrak{Z}$, having the form $u = \sum_i b_i \Lambda^{2i+1}$, $b_i \in \mathbf{C}$ (we note that in all cases except $D_n^{(1)}$ any element of $\mathfrak{Z}$ has this form; see Appendix 2).

Proposition 7.8. Suppose the operator $\mathfrak{L}$ satisfies Eq. (6.5), where $u = \sum_{j=0}^{k} b_j \Lambda^{2j+1}$, $b_j \in \mathbf{C}$. Then L_1 and L_2 satisfy the system (7.1), where $A_i = (M_i)_+$, $M_1 = -\sum_{j=0}^{k} b_j (L_2L_1)^{\frac{2j+1}{2n-2}}$, $M_2 = -\sum_{j=0}^{k} b_j \times (L_1L_2)^{\frac{2j+1}{2n-2}}$.

The proof is entirely analogous to the proof of Proposition 7.3. However, because the $\tilde{\psi}_i$ are not entirely explicitly defined the proof of the formula $P_t \cdot \tilde{\psi}_i = A_i \cdot \tilde{\psi}_i$ is nontrivial. It is based on the following lemmas.

LEMMA 7.9. $D_t \cdot W_i' \subset W_i'$, $D_t \cdot W_i'' \subset W_i''$.

Proof. We denote by ∇_t the operator on W acting according to the formula $\nabla_t(w) = D_t \cdot w$ [i.e., $\nabla_t = \frac{d}{dt} - \varphi(u)_+$]. Since $[\nabla_t, \mathfrak{L}] = 0$, it follows that $[T\nabla_t T^{-1}, \mathfrak{L}_0] = 0$. From this it is easily deduced that $[T\nabla_t T^{-1}, \Lambda] = 0$, and hence the kernel and image of Λ are invariant under $T\nabla_t T^{-1}$. Using assertion 1) of Lemma 7.5, we find that $T\nabla_t T^{-1}(W_i) \subset W_i$ whence the lemma follows. ■

LEMMA 7.10. $(M \cdot \tilde{\psi}_i)_+ = M_+ \cdot \psi_i$ for any $M \in B((D^{-1}))$.

Proof. $(M \cdot \tilde{\psi}_i)_+ - M_+ \cdot \psi_i = (M \cdot \tilde{\psi}_i - M_+ \cdot \psi_i)_+ = (M_- \cdot \tilde{\psi}_i - M_+ \cdot \bar{\psi}_i)_+$, where $\bar{\psi}_i \overset{\text{def}}{=} \psi_i - \tilde{\psi}_i$. It remains to verify that $(\mathfrak{L}^j \tilde{\psi}_i)_+ = 0$ for $j < 0$, $(\mathfrak{L}^j \bar{\psi}_i)_+ = 0$ for $j \geq 0$. It suffices to prove that $(\mathfrak{L}_0^j T\tilde{\psi}_i)_+ = 0$ for $j < 0$, $(\mathfrak{L}_0^j T\bar{\psi}_i)_+ = 0$ for $j \geqslant 0$. We have $T\tilde{\psi}_i = PT\psi_i$, $T\bar{\psi}_i = (E-P)T\psi_i$, where P is the same as in Lemma 7.4. Therefore from assertion 2) of Lemma 7.5 and assertion 3) of Lemma 7.4 it follows that $(T\bar{\psi}_i)_+ = 0$, $(T\tilde{\psi}_i)_+ = \psi_i$. Since $T\bar{\psi}_i \in \operatorname{Ker}\Lambda$, the equality $(T\bar{\psi}_i)_+ = 0$ implies that $(\mathfrak{L}_0^j T\bar{\psi}_i)_+ = 0$ for $j \geqslant 0$. The equality $(\mathfrak{L}_0^j T\tilde{\psi}_i)_+ = 0$, where $j < 0$, follows from assertion 5) of Lemma 7.4. ■

The formula $D_t \cdot \tilde{\psi}_i = A_i \cdot \tilde{\psi}_i$ is induced from these lemmas as follows. It follows from Lemma 7.9 that $D_t \cdot \tilde{\psi}_i = \pi_i(D_t \cdot \psi_i)$, where $\pi_i : W_i \to W_i'$ is the projector with kernel W_i''. By definition, D_t, $D_t \cdot \psi_i = -\mathscr{A}_+ \psi_i = -(\mathscr{A}\psi_i)_+$, where $\mathscr{A} = \sum_{j=0}^{k} b_j T^{-1} \Lambda^{2j+1} T$. Since $\mathscr{A}(W_i'') = 0$, it follows that $\mathscr{A}\psi_i = \mathscr{A}\tilde{\psi}_i$. Since $\mathscr{A}\tilde{\psi}_i = -M \cdot \tilde{\psi}_i$ (see the proof of Proposition 7.3), we have $D_t \cdot \tilde{\psi}_i = \pi_i((M_i \cdot \tilde{\psi}_i)_+)$. Applying now Lemma 7.10, we find that $D_t \cdot \tilde{\psi}_i = \pi_i(A_i \cdot \psi_i) = A_i \cdot \tilde{\psi}_i$.

If u has the form $\sum_j b_i \Lambda^{2j+1}$, then, as before, the corresponding generalized KdV reduces to the system (7.1), but unfortunately in this case we do not know an explicit formula for A_1 and A_2. In the simplest nontrivial case $A_1 = f_2 D^{-1} f_1$, $A_2 = f_1 D^{-1} f_2$.

7.4. In part 7.1 a rule was formulated for determining the types of the operators L_1 and L_2 corresponding to a given pair (G, c_m), where G is a classical Kats–Moody algebra and c_m is a vertex of its Dynkin scheme. However, in the case of algebras G of low rank and also in the case where c_m is an extreme vertex this rule cannot be understood literally. We therefore present a table (Table 4) in which for each pair (G, c_m) the types of the corresponding operators L_1 and L_2 are indicated. In this table it is understood that $P_0 \overset{\text{def}}{=} D$, $Q_0 \overset{\text{def}}{=} 1$, $R_0 \overset{\text{def}}{=} D^{-1}$. The vertices of the Dynkin scheme of G are numbered as in Table 2.

It was noted in part 7.1 that if L_1 and L_2 satisfy the system (7.1), then the operator $L \overset{\text{def}}{=} L_2L_1$ satisfies the equation $dL/dt = [A_1, L]$. It is clear that if one of the operators

TABLE 4

$\mathfrak{G}$	c_m	L_1	L_2
$A_{2n}^{(2)}$ $n\geqslant 1$	$0\leqslant m\leqslant n$	P_{n-m}	Q_m
$A_{2n-1}^{(2)}$ $n\geqslant 1$	$2\leqslant m\leqslant n$	Q_{n-m}	R_m
" —— "	$m=0,1$	Q_n	R_0
$B_n^{(1)}$ $n\geqslant 1$	$2\leqslant m\leqslant n$	P_{n-m}	R_m
" —— "	$m=0,1$	P_n	R_0
$C_n^{(1)}$ $n\geqslant 1$	$0\leqslant m\leqslant n$	Q_{n-m}	Q_m
$D_n^{(1)}$ $n\geqslant 3$	$2\leqslant m\leqslant n-2$	R_{n-m}	R_m
" —— "	$m=0,1$	R_n	R_0
" —— "	$m=n-1,n$	R_0	R_n
$D_{n+1}^{(2)}$ $n\geqslant 1$	$0\leqslant m\leqslant n$	P_{n-m}	P_m

L_i belongs to type P_0, Q_0, or R_0 (this condition is almost equivalent to c_m being an extreme vertex) the system (7.1) is equivalent to this equation.

In conclusion we consider conservation laws for generalized KdV. In analogy to Proposition 3.20, it can be proved that in the case of classical Kats–Moody algebras distinct from $D_n^{(1)}$ the densities of the conservation laws H_i considered in Proposition 6.6 up to constant multiples and addition of total derivatives are equal to res $(L_1L_2)^{i/k}$, where k = ord (L_1L_2). In the case of $D_n^{(1)}$ the situation is as follows. We write the element $H\in C^\infty(\mathbf{R}, \mathfrak{Z}^-)$, considered in Proposition 6.6, in the form $H=\sum_{l=0}^{\infty} h_l\Lambda^{-(2l+1)}+\sum_{l=0}^{\infty} g_l\lambda^{-(2l+1)}F$, where h_l, $g_l\in B$ (see Lemma 7.4). Then up to a constant multiple and addition of total derivatives h_i is equal to $\operatorname{res}(L_1L_2)^{\frac{2l+1}{k}}$. Unfortunately, we do not know a general formula expressing g_i in terms of L_1, L_2, f_1, f_2 (the f_i are the same as in part 7.3). We note only that up to a multiple and total derivatives g_0 is equal to f_1f_2.

8. HAMILTONIAN MANIFOLDS $M(\mathfrak{G})$

In part 6.5 for any semisimple Lie algebra $\mathfrak{G}$ we defined a manifold $\mathcal{M}(\mathfrak{G})$ equipped with two Hamiltonian structures. Moreover, $\mathcal{M}(\mathfrak{gl}(k))$ is by definition the manifold denoted in part 3.6 by $\mathcal{M}$. We denote by $M(\mathfrak{gl}(k))$ the set of operators of the form $D^k+\sum_{i=0}^{k-1} u_iD^i$, $u_i\in B_0$. In part 3.3 we constructed a bijection $F:\mathcal{M}(\mathfrak{gl}(k))\to M(\mathfrak{gl}(k))$, and in part 3.7 it was shown that the first and second Hamiltonian structures on $\mathcal{M}(\mathfrak{gl}(k))$ go over under the mapping F into the corresponding Gel'fand–Dikii structures on $M(\mathfrak{gl}(k))$. In the present section for each classical simple Lie algebra $\mathfrak{G}$ we define a manifold $M(\mathfrak{G})$ consisting of scalar differential [and in the case $\mathfrak{G}=\mathfrak{o}(2n)$ pseudodifferential] operators of special form[7] and a bijective mapping $F:\mathcal{M}(\mathfrak{G})\to M(\mathfrak{G})$. Moreover, the Hamiltonian structures on $M(\mathfrak{G})$ corresponding to the first and second Hamiltonian structures on $\mathcal{M}(\mathfrak{G})$ will be found.

We recall (see part 6.5) that the definition of the manifold $\mathcal{M}(\mathfrak{G})$ and the Hamiltonian structures on it involves the Weyl generators of $\mathfrak{G}$, an invariant scalar product on $\mathfrak{G}$, and an element e of the center of the algebra $\mathfrak{n}$. For all classical algebras $\mathfrak{G}$ we use the Weyl generators presented in Appendix 1 and the scalar product (X, Y) = tr (XY). We shall indicate the element e each time.

8.1. We begin with the simplest case $\mathfrak{G}=\mathfrak{sl}(k)$. It is easy to see that $\mathcal{M}(\mathfrak{sl}(k))\subset\mathcal{M}(\mathfrak{gl}(k))$. We set $M(\mathfrak{sl}(k))=\left\{D^k+\sum_{i=0}^{k-2} u_iD^i \mid u_i\in B_0\right\}$. It is clear that the bijection $\mathcal{M}(\mathfrak{gl}(k))\to M(\mathfrak{gl}(k))$ constructed in part 3.3 maps $\mathcal{M}(\mathfrak{sl}(k))$ onto $M(\mathfrak{sl}(k))$.

We denote by $\{\cdot,\ \cdot\}_1$ and $\{\cdot,\ \cdot\}_2$ the Poisson brackets on $M(\mathfrak{sl}(k))$, corresponding to the first and second Hamiltonian structures on $\mathcal{M}(\mathfrak{sl}(k))$ (as the element e in the definition of the first Hamiltonian structure we take the matrix $e_{1,k}$ as in Sec. 3). We shall find the explicit form of these brackets. For this it suffices for any integral symbols X, $Y\in B_0((D^{-1}))$ to find $\{l_X, l_Y\}_1$ and $\{l_X, l_Y\}_2$, where $l_X:M(\mathfrak{sl}(k))\to\mathbf{C}$ is defined by the formula $l_X(L)=\operatorname{Tr}(XL)$. It is not hard to see that if φ and ψ are functionals on $\mathcal{M}(\mathfrak{gl}(k))$ and $\bar\varphi$ and $\bar\psi$ are their restrictions to $\mathcal{M}(\mathfrak{sl}(k))$, then $\{\bar\varphi, \bar\psi\}_1$ and $\{\bar\varphi, \bar\psi\}_2$ are equal to the restrictions to $\mathcal{M}(\mathfrak{sl}(k))$ of

[7]Mikhailov [65] pointed out the connection between the types of scalar differential operators and the types of the classical simple Lie algebras.

the functionals $\{\varphi, \psi\}_1$ and $\{\varphi, \psi\}_2$. It therefore follows from Theorem 3.22 that $\{l_X, l_Y\}_1$ and $\{l_X, l_Y\}_2$ are defined by formulas (2.16), (2.17).

8.2. Now let $\mathfrak{G} = \mathfrak{o}(2n+1) = \{A \in \mathfrak{gl}(2n+1) \mid \sigma(A) = A\}$, where σ is the automorphism of $\mathfrak{gl}(2n+1)$, given by the formula $\sigma(A) = -\mathrm{diag}(1, -1, \ldots, 1) \cdot A^T \cdot \mathrm{diag}(1, -1, \ldots, 1)$. The Borel subalgebra in $\mathfrak{G}$ will be denoted by $\mathfrak{b}_{\mathfrak{G}}$, while $\mathfrak{b}$ denotes the set of all upper triangular matrices (we recall that $\mathfrak{b}_{\mathfrak{G}} = \mathfrak{b} \cap \mathfrak{G}$). The meaning of the notation $\mathfrak{n}_{\mathfrak{G}}$ and $\mathfrak{n}$ is similar. Since $\mathfrak{b}_{\mathfrak{G}} \subset \mathfrak{b}$, $\mathfrak{n}_{\mathfrak{G}} \subset \mathfrak{n}$, while the elements I for the algebras $\mathfrak{G}$ and $\mathfrak{gl}(2n+1)$ coincide (namely, $I = \sum_{i=1}^{2n} e_{i+1,i}$), there is the natural mapping $\varphi: \mathcal{M}(\mathfrak{G}) \to \mathcal{M}(\mathfrak{gl}(2n+1))$. Since $\sigma(\mathfrak{b}) = \mathfrak{b}$, $\sigma(\mathfrak{n}) = \mathfrak{n}$, $\sigma(I) = I$, it follows that σ induces a mapping $\mathcal{M}(\mathfrak{gl}(2n+1)) \to \mathcal{M}(\mathfrak{gl}(2n+1))$, which we also denote by σ.

<u>LEMMA 8.1.</u> φ maps $\mathcal{M}(\mathfrak{G})$ in one-to-one fashion onto the set of elements of $\mathcal{M}(\mathfrak{gl}(2n+1))$, invariant under σ.

<u>Proof.</u> We denote by $\mathfrak{b}_i$ the set of matrices $(a_{\alpha\beta})$ such that $a_{\alpha\beta} = 0$ for $\beta - \alpha \neq i$. For any $i \geqslant 0$ we choose a vector subspace $V_i \subset \mathfrak{b}_i$ so that $\mathfrak{b}_i = [I, \mathfrak{b}_{i+1}] \oplus V_i$ and $\sigma(V_i) = V_i$ (for example, for V_i it is possible to take the one-dimensional subspace generated by the matrix $\sum_{j=1}^{2n+1-i} e_{j,j+i}$). We set $V \overset{\text{def}}{=} \underset{i}{\oplus} V_i$. It is clear that $\mathfrak{b}_{\mathfrak{G}} = [I, \mathfrak{n}_{\mathfrak{G}}] \oplus V^{\sigma}$, where $V^{\sigma} \overset{\text{def}}{=} \{A \in V \mid \sigma(A) = A\}$. According to Proposition 6.1, it is possible to replace $\mathcal{M}(\mathfrak{gl}(2n+1))$ by $C^\infty(\mathbf{R}/\mathbf{Z}, V)$ and $\mathcal{M}(\mathfrak{G})$ by $C^\infty(R/Z, V^\sigma)$ after which the assertion of the lemma becomes obvious. ■

We henceforth identify $\mathcal{M}(\mathfrak{o}(2n+1))$ with its image in $\mathcal{M}(\mathfrak{gl}(2n+1))$. We recall that the bijective mapping constructed in Sec. 3 $F: \mathcal{M}(\mathfrak{gl}(2n+1)) \to M(\mathfrak{gl}(2n+1))$ assigns to a class of gauge equivalence of the operator $d/dx + I + q$ the operator $L = -(\Delta(P))^*$, where $P = I + q + \mathrm{diag}(D, D, \ldots, D)$ (see part 3.3). If q is replaced by $\sigma(q)$, then P is replaced by $-\mathrm{diag}(1, -1, \ldots, 1) \cdot P^T \cdot \mathrm{diag}(1, -1, \ldots, 1)$, and hence (see Lemma 3.13) L is replaced by $-L^*$. Thus, the mapping $F\sigma F^{-1}: M(\mathfrak{gl}(2n+1)) \to M(\mathfrak{gl}(2n+1))$ takes L into $-L^*$. From Lemma 8.1 we therefore obtain the following result.

<u>Proposition 8.2.</u> F maps $\mathcal{M}(\mathfrak{o}(2n+1))$ in one-to-one fashion onto the set $M(\mathfrak{o}(2n+1)) \overset{\text{def}}{=} \{L \in M(\mathfrak{gl}(2n+1)) \mid L^* = -L\}$.

<u>Remark 1.</u> It is clear that the mapping $F: \mathcal{M}(\mathfrak{o}(2n+1)) \to M(\mathfrak{o}(2n+1))$ and the mapping inverse to it are given by differential polynomials.

<u>Remark 2.</u> Let $\mathfrak{H}$ be the Cartan subalgebra of $\mathfrak{o}(2n+1)$. Composition of the Miura transformation $\mu: C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{H}) \to \mathcal{M}(\mathfrak{o}(2n+1))$ and the mapping $F: \mathcal{M}(\mathfrak{o}(2n+1)) \to M(\mathfrak{o}(2n+1))$ takes the matrix $\mathrm{diag}(f_1, \ldots, f_n, 0, -f_n, \ldots, -f_1)$ into the operator $L = (D + f_1) \ldots (D + f_n) D (D - f_n) \ldots (D - f_1)$.

We denote by $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ the Poisson brackets on $M(\mathfrak{o}(2n+1))$, corresponding to the first and second Hamiltonian structures on $\mathcal{M}(\mathfrak{o}(2n+1))$ [for e we take $(e_{1,2n} + e_{2,2n+1})/2$]. We shall find the explicit form of these brackets. For this it suffices for any integral symbols $X, Y \in B_0((D^{-1}))$ to find $\{l_X, l_Y\}_1$ and $\{l_X, l_Y\}_2$, where $l_X: M(\mathfrak{o}(2n+1)) \to \mathbf{C}$ is defined by the formula $l_X(L) = \mathrm{Tr}(XL)$. Moreover, it may be assumed with no loss of generality that $X^* = X$, $Y^* = Y$ [indeed, the formula $\mathrm{Tr}\, Z^* = -\mathrm{Tr}\, Z$ valid for any $Z \in B_0((D^{-1}))$, implies that if $L^* = -L$, then $\mathrm{Tr}(XL) = \mathrm{Tr}\left(\frac{X + X^*}{2} \cdot L\right)$].

<u>Proposition 8.3.</u> Let $L \in M(\mathfrak{o}(2n+1))$, $X, Y \in B_0((D^{-1}))$, $\mathrm{ord}\, X < 0$, $\mathrm{ord}\, Y < 0$, $X^* = X$, $Y^* = Y$. Then

$$\{l_X, l_Y\}_1(L) = \mathrm{Tr}(L \cdot (YDX - XDY)), \tag{8.1}$$

$$\{l_X, l_Y\}_2(L) = \mathrm{Tr}((LY)_+ LX - XL(YL)_+). \tag{8.2}$$

We note that since $L^* = -L$, $X^* = X$, $Y^* = Y$, it follows that $\mathrm{Tr}(L \cdot (YDX - XDY)) = 2\,\mathrm{Tr} \times (LYDX)$, $\mathrm{Tr}((LY)_+ LX - XL(YL)_+) = 2\,\mathrm{Tr}((LY)_+ LX)$.

<u>Proof.</u> From formula (3.17) it follows that if the functionals $l_1, l_2: \mathcal{M}(\mathfrak{gl}(2n+1)) \to \mathbf{C}$ are invariant under σ and $\bar{l}_1$ and $\bar{l}_2$ are their restrictions to $\mathcal{M}(\mathfrak{o}(2n+1))$, then $\{\bar{l}_1, \bar{l}_2\}_2$ is equal to the restriction of $\{l_1, l_2\}_2$ to $\mathcal{M}(\mathfrak{o}(2n+1))$ [to see this it suffices to note that if $\sigma(q) = q$, then for a suitable normalization of the gradient $\sigma(\mathrm{grad}_q l_i) = \mathrm{grad}_q l_i$, whence $\mathrm{grad}_q l_i \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathfrak{o}(2n+1))$ and hence $\mathrm{grad}_q l_i = \mathrm{grad}_q \bar{l}_i$]. Therefore, if φ_1 and φ_2 are functionals on $M(\mathfrak{gl}(2n+1))$ such that $\varphi_i(L) = \varphi_i(-L^*)$ and $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are their restrictions to $M(\mathfrak{o}(2n+1))$, then

$\{\bar{\varphi}_1, \bar{\varphi}_2\}_2$ is equal to the restriction of $\{\varphi_1, \varphi_2\}_2$ to $M(\mathfrak{o}(2n+1))$. Setting now $\varphi_1(L)=\mathrm{Tr}(XL)$, $\varphi_2(L)=\mathrm{Tr}(YL)$, we obtain (8.2). Since the element e for the algebras $\mathfrak{o}(2n+1)$ and $\mathfrak{gl}(2n+1)$ do not coincide, equality (8.1) cannot be proved in a similar way. It is not hard, however, to deduce (8.1) from (8.2) by arguing in the same way as at the end of the proof of Theorem 3.22. Here we use the following simple assertion: if to the operator d/dx + I + q(x), where q(x) is an upper triangular matrix in $\mathfrak{o}(2n+1)$, there corresponds $L\in M(\mathfrak{o}(2n+1))$, then to the operator d/dx + I + q(x) − e there corresponds L + D. ■

8.3. We consider the case $\mathfrak{G}=\mathfrak{sp}(2n)=\{A\in\mathfrak{gl}(2n)\mid\sigma(A)=A\}$, where σ is the automorphism of $\mathfrak{gl}(2n)$, given by the formula $\sigma(A)=-\mathrm{diag}(1,-1,\ldots,1,-1)\cdot A^T\cdot\mathrm{diag}(1,-1,\ldots,1,-1)$. Lemma 8.1 also holds in this case. In particular, $\mathscr{M}(\mathfrak{sp}(2n))$ can be considered a subset of $\mathscr{M}(\mathfrak{gl}(2n))$. The following assertion is proved in the same way as Proposition 8.2, 8.3 [and even somewhat more simply since the elements e in the definition of the first Hamiltonian structure for the algebras $\mathfrak{sp}(2n)$ and $\mathfrak{gl}(2n)$ coincide].

Proposition 8.4. 1) The mapping $F:\mathscr{M}(\mathfrak{gl}(2n)\to M(\mathfrak{gl}(2n))$ maps $\mathscr{M}(\mathfrak{sp}(2n))$ in one-to-one fashion onto the set $M(\mathfrak{sp}(2n)\overset{\mathrm{def}}{=}\{L\in M(\mathfrak{gl}(2n))\mid L^*=L\}$. The mapping $F:\mathscr{M}(\mathfrak{sp}(2n))\to M(\mathfrak{sp}(2n))$ and the mapping inverse to it are given by differential polynomials.

2) Let $\mathfrak{H}$ be the Cartan subalgebra of $\mathfrak{sp}(2n)$. Composition of the Miura transformation $\mu:C^\infty(\mathbf{R}/\mathbf{Z},\mathfrak{H})\to\mathscr{M}(\mathfrak{sp}(2n))$ and the mapping $F:\mathscr{M}(\mathfrak{sp}(2n))\to M(\mathfrak{sp}(2n))$ takes the matrix $\mathrm{diag}(f_1,\ldots,f_n,-f_n,\ldots,-f_1)$ into the operator $L=(D+f_1)\ldots(D+f_n)(D-f_n)\cdots(D-f_1)$.

3) We denote by $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ the Poisson brackets on $M\mathfrak{sp}(2n))$, corresponding to the first and second Hamiltonian structures on $\mathscr{M}(\mathfrak{sp}(2n))$ (for e we take $e_{1,2n}$). For any integral symbol $X\in B_0(D^{-1}))$ we define $l_X:M(\mathfrak{sp}(2n))\to\mathbf{C}$ by the formula $l_X(L)=\mathrm{Tr}(XL)$. Then for any $L\in M(\mathfrak{sp}(2n))$ and any skew-symmetric integral symbols $X, Y\in B_0(D^{-1}))$ the following equalities hold:

$$\{l_X, l_Y\}_1(L)=\mathrm{Tr}(L\cdot[Y,X])=2\cdot\mathrm{Tr}(LYX),$$
$$\{l_X, l_Y\}_2(L)=\mathrm{Tr}((LY)_+LX-XL(YL)_+)=2\,\mathrm{Tr}((LY)_+LX).$$

Remark. The analogue found in [27, 44] of the interpretation of the first Gel'fand–Dikii bracket as the Kirillov bracket also holds for the first Hamiltonian structures on $M(\mathfrak{sp}(2n))$ and $M(\mathfrak{o}(2n+1))$. In the case $\mathfrak{sp}(2n)$ it is necessary to consider the Kirillov bracket for the Lie algebra of skew-symmetric integral symbols. In the case $\mathfrak{o}(2n+1)$ it is necessary to consider the Lie algebra of symmetric integral symbols with the unusual commutator $[X, Y]\overset{\mathrm{def}}{=}XDY-YDX$.

8.4. We consider, finally, the most difficult case $\mathfrak{G}=\mathfrak{o}(2n)$. We denote by $M(\mathfrak{o}(2n))$ the set of pairs (L, f), where $f\in B_0$, $L\in B_0((D^{-1}))$, $L^*=-L$, $L_-=fD^{-1}f$, $\mathrm{ord}\,L=2n-1$, and the leading coefficient of L is equal to 1. In part 7.3 (see the proof of Lemma 7.7) we essentially constructed a mapping $F:\mathscr{M}(\mathfrak{o}(2n))\to M(\mathfrak{o}(2n))$.

Proposition 8.5. 1) F is bijective. The mappings F and F^{-1} are given by differential polynomials.

2) Let $\mathfrak{H}$ be the Cartan subalgebra of $\mathfrak{o}(2n)$. Composition of the Miura transformation $\mu:C^\infty(\mathbf{R}/\mathbf{Z},\mathfrak{H})\to\mathscr{M}(\mathfrak{o}(2n))$ and the mapping $F:\mathscr{M}(\mathfrak{o}(2n))\to M(\mathfrak{o}(2n))$ takes the matrix $\mathrm{diag}(f_1,\ldots,f_n,-f_n,\ldots,-f_1)$ into the pair (L, f), where $L=(D+f_1)\cdots(D+f_n)D^{-1}(D-f_n)\cdots(D-f_1)$, $f=(-1)^{n-1}i^nP(1)$, where $P=(D+f_1)\cdots(D+f_n)$.

3) We denote by $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ the Poisson brackets on $M(\mathfrak{o}(2n))$, corresponding to the first and second Hamiltonian structures on $\mathscr{M}(\mathfrak{o}(2n))$ [for e we take $(e_{1,2n-1}+e_{2,2n})/2$]. For any integral symbol $X\in B_0((D^{-1}))$, such that $X^*=X$ we define $l_X:M(\mathfrak{o}(2n))\to\mathbf{C}$ by the formula $l_X(L,f)=\mathrm{Tr}(XL)$. Moreover, for any $s\in B_0$ we set $\lambda_s(L,f)\overset{\mathrm{def}}{=}\int_{x\in\mathbf{R}/\mathbf{Z}}s(x)f(x)\,dx$. Then

$$\{l_X, l_Y\}_1(L,f)=2\,\mathrm{Tr}(LYDX),$$
$$\{l_X,\lambda_s\}_1=0,\quad \{\lambda_s,\lambda_t\}_1=\frac{1}{2}\int_{x\in\mathbf{R}/\mathbf{Z}}s(x)t'(x)\,dx;$$
$$\{l_X, l_Y\}_2(L,f)=2\,\mathrm{Tr}((LY)_+LX),\quad \{\lambda_s, l_X\}_2(L,f)=\mathrm{Tr}(LXfD^{-1}s),$$
$$\{\lambda_s,\lambda_t\}_2(L,f)=\frac{1}{2}\mathrm{Tr}(LtD^{-1}s).$$

We shall only outline the proof. Assertion 2), of course, is proved by direct computation. The difficulty in the proof of the remaining assertions as compared with the analogous assertions for $\mathfrak{o}(2n+1)$ (see part 8.2) is connected with the fact that the matrices I for $\mathfrak{o}(2n)$ and $\mathfrak{gl}(2n)$ do not coincide, and hence there is no natural mapping $\mathcal{M}(\mathfrak{o}(2n))\to\mathcal{M}(\mathfrak{gl}(2n))$. In order to overcome this difficulty, we define a new manifold $\mathcal{M}'$, in which $\mathcal{M}(\mathfrak{o}(2n))$ is imbedded in a natural way. For this we consider operators $\mathfrak{L}$ of the form

$$\mathfrak{L}=\frac{d}{dx}+I+q,\ q\in C^\infty(\mathbf{R}/\mathbf{Z},\mathfrak{B}), \tag{8.3}$$

where I is the sum of the Weyl generators X_i of the algebra $\mathfrak{o}(2n)$ (see Appendix 1) and $\mathfrak{B}$ is a set of matrices $(a_{\alpha\beta})$ of order 2n such that $a_{\alpha\beta}=0$ for $\alpha>\beta$ and $(\alpha,\beta)\neq(n+1,n)$. We denote by $\mathfrak{N}$ the set of matrices $(a_{\alpha\beta})$ of order 2n such that $a_{\alpha\beta}=0$ for $\alpha\geqslant\beta$ and, moreover, $a_{n,n+1}=0$. It is easy to verify that if $\mathfrak{L}$ has the form (8.3) and $S\in C^\infty(\mathbf{R}/\mathbf{Z},\mathfrak{N})$, then $e^S\mathfrak{L}e^{-S}$ also has the form (8.3). Such transformations we call gauge transformations. The set of classes of gauge equivalence of operators $\mathfrak{L}$ of the form (8.3) we denote by $\mathcal{M}'$. We recall that $\mathfrak{o}(2n)=\{A\in\mathfrak{gl}(2n)\mid\sigma(A)=A\}$, where σ is the automorphism of $\mathfrak{gl}(2n)$, given by the formula $\sigma(A)=-\mathrm{diag}(1,-1,\ldots,(-1)^{n-1},(-1)^{n-1},(-1)^n,\ldots,1)\,A^T\cdot\mathrm{diag}(1,-1,\ldots,(-1)^{n-1},\ (-1)^{n-1},\ (-1)^n,\ldots,1)$. Since $\sigma(I)=I$, $\sigma(\mathfrak{B})=\mathfrak{B}$, $\sigma(\mathfrak{N})=\mathfrak{N}$, it follows that σ induces a mapping $\mathcal{M}'\to\mathcal{M}'$, which we also denote by σ. In analogy to Lemma 8.1 it can be shown that the natural mapping from $\mathcal{M}(\mathfrak{o}(2n))$ into the set of elements $\mathcal{M}'$, fixed under σ is bijective. We denote by M' the set of quadruples of the form (L, f, g, h), where $f,g,h\in B_0$, and L is a pseudodifferential symbol of order $2n-1$ with leading coefficient 1 such that $L_-=f(D+h)^{-1}g$. We define the mapping $F':\mathcal{M}'\to M'$, assigning to the class of the operator $d/dx+I+q(x)$ the following quadruple (L,f,g,h): a) $L\overset{\text{def}}{=}\Delta(P)$, where $P=\mathrm{diag}(D,D,\ldots,D)+q+I$, b) $h=\frac{1}{2}(q_{n,n}+q_{n+1,n+1})-q_{n+1,n}-\frac{1}{4}q_{n,n+1}$, where $q_{i,j}$ are the elements of the matrix q; c) let A_{ij} be the same as in the proof of Lemma 7.7; we write $\Delta(A_{11})$ in the form $\sum_i a_j(D+h)^j$, and $\Delta(A_{22})$ in the form $\sum_j(D+h)^jb_j$; then $f\overset{\text{def}}{=}i^na_0$, $g\overset{\text{def}}{=}(-1)^{n+1}i^nb_0$. It is not hard to verify that F' is well defined and is bijective, whereby the elements of the matrix q corresponding to the quadruple (L, f, g, h) can be chosen in the form of differential polynomials in f, g, h and the coefficients of L; the mapping $F'\cdot\sigma\cdot(F')^{-1}:M'\to M'$ acts by the formula $(L,f,g,h)\to(-L^*,g,f,-h)$. From this we obtain assertion 1) of Proposition 8.5.

On $\mathcal{M}'$ we consider the Poisson bracket given by formula (3.17) and carry it over to M' by means of the mapping F'. For any integral symbol $X\in B_0((D^{-1}))$ we define $l_X:M'\to\mathbf{C}$ by the formula $l_X(L,f,g,h)=\mathrm{Tr}(XL)$. Moreover, for any $s\in B_0$ we define functionals φ_s and ψ_s by the formulas $\varphi_s(L,f,g,h)=\int\limits_{x\in\mathbf{R}/\mathbf{Z}}f(x)s(x)\,dx$, $\psi_s(L,f,g,h)=\int\limits_{x\in\mathbf{R}/\mathbf{Z}}g(x)s(x)\,dx$. Computations analogous to those done in the proof of Theorem 3.22 but more involved show that $\{\varphi_s,\varphi_t\}=\{\psi_s,\psi_t\}=0$, $\{\varphi_s,\psi_t\}(L,f,g,h)=\mathrm{Tr}(Lt(D+h)^{-1}s)$, $\{\varphi_s,l_X\}(L,f,g,h)=\mathrm{Tr}(LXf(D+h)^{-1}s)$, $\{\psi_s,l_X\}(L,f,g,h)=-\mathrm{Tr}(XLg(D+h)^{-1}s)$, $\{l_X,l_Y\}(L,f,g,h)=\mathrm{Tr}((LY)_+LX-L(YL)_+X)$. The derivation from these formulas of assertion 3) of Proposition 8.5 is analogous to the proof of Proposition 8.3.

8.5. We note that since $\mathfrak{o}(3)\simeq\mathfrak{sl}(2)$, $\mathfrak{o}(4)\simeq\mathfrak{sl}(2)\times\mathfrak{sl}(2)$, $\mathfrak{o}(5)\simeq\mathfrak{sp}(4)$, $\mathfrak{o}(6)\simeq\mathfrak{sl}(4)$ (see part 5.1), there are the canonical bijections $f_1:M(\mathfrak{sl}(2))\tilde{\to}M(\mathfrak{o}(3))$, $f_2:M(\mathfrak{sl}(2))\times M(\mathfrak{sl}(2))\tilde{\to}M(\mathfrak{o}(4))$, $f_3:M(\mathfrak{sp}(4))\tilde{\to}M(\mathfrak{o}(5))$, $f_4:M(\mathfrak{sl}(4))\tilde{\to}M(\mathfrak{o}(6))$. It is not hard to obtain the following explicit formulas: $f_1(D^2+u)=D^3+2(uD+Du)$; $f_2(D^2+u,D^2+v)=(D^3+(u+v)D+D(u+v)+(u-v)D^{-1}(u-v),u-v)$; $f_3(D^4+uD^2+D^2u+v)=D^5+2(uD^3+D^3u)+2(sD+Ds)$, where $s=u^2-u''-v$; $f_4(D^4+uD^2+D^2u+v+wD+Dw)=(D^5+2(uD^3+D^3u)+2(sD+Ds)-4wD^{-1}w,2iw)$, $s\overset{\text{def}}{=}u^2-u''-v$.

We call a mapping of Hamiltonian manifolds $f:M_1\to M_2$ almost Hamiltonian if for any functionals $\varphi,\psi:M_2\to\mathbf{C}$ $\{f^*\varphi,f^*\psi\}=\lambda\cdot f^*\{\varphi,\psi\}$, where λ is a number not depending on φ and ψ (we recall that f is Hamiltonian if $\lambda=1$). It is clear that the mappings f_i, $1\leqslant i\leqslant 4$ are almost Hamiltonian relative to the first and second Hamiltonian structures. They are not Hamiltonian, since the isomorphisms $\mathfrak{sl}(2)\tilde{\to}\mathfrak{o}(3)$, $\mathfrak{sl}(2)\times\mathfrak{sl}(2)\tilde{\to}\mathfrak{o}(4)$, etc. preserve the scalar product only up to a constant multiple (in addition, under these isomorphisms the elements e in the definition of the first Hamiltonian structure go over into one another again up to a multiple).

8.6. If φ_τ is the automorphism of the semisimple Lie algebra τ induced by an automorphism $\mathfrak{g}$, of its Dynkin scheme (see part 5.1), then $\varphi_\tau(I)=I$, $\varphi_\tau(\mathfrak{b})=\mathfrak{b}$, $\varphi_\tau(\mathfrak{n})=\mathfrak{n}$, and hence φ_τ induces a mapping $S_\tau:\mathcal{M}(\mathfrak{g})\to\mathcal{M}(\mathfrak{g})$. It can be shown that if $\mathfrak{g}$ is a simple algebra, then φ_τ preserves the scalar product, and hence the mapping S_τ is Hamiltonian relative to the second Hamiltonian structure (relative to the first structure it is almost Hamiltonian). Among the Dynkin schemes of the classical Lie algebras A_n and D_n have nontrivial automorphisms (see Table 1). It is not hard to verify that to the nontrivial automorphism of A_n there corresponds the mapping $M(\mathfrak{sl}(n+1))\to M(\mathfrak{sl}(n+1))$, acting by the formula $L\to(-1)^{n+1}L^*$, while to the automorphism of the scheme of D_n changing the places of c_n and c_{n-1} there corresponds a mapping $M(\mathfrak{o}(2n))\to M(\mathfrak{o}(2n))$, taking (L, f) into (L, –f). In the case of an algebra of type D_4 the group of automorphisms of the Dynkin scheme is isomorphic to S_3. We shall not present the formulas giving the action of S_3 on $M(\mathfrak{o}(8))$. We note only that the set of points of $M(\mathfrak{o}(8))$, fixed under the action of S_3 is none other than $M(\mathfrak{A})$, where $\mathfrak{A}=\{X\in\mathfrak{o}(8)|\forall\sigma\in S_3,\ \sigma(X)=X\}$. It is not hard to verify that $\mathfrak{A}$ is a simple Lie algebra of type G_2. Computations show that $M(\mathfrak{A})$ is the set of all operators of $B_0[D]$ of the form $D^7+uD^5+D^5u+\left(\frac{u^2}{2}-2u''\right)D^3+D^3\left(\frac{u^2}{2}-2u''\right)+vD+Dv$.

9. EXAMPLES OF GENERALIZED KdV and mKdV EQUATIONS

In this section we present some examples of generalized KdV and mKdV corresponding to classical Kats–Moody algebras distinct from $\mathfrak{sl}(k,\mathbf{C}[\lambda,\lambda^{-1}]$. The generalized KdV are hereby considered as equation for $\mathfrak{L}^{can}$ (see part 6.2). The form of $\mathfrak{L}^{can}$ is chosen so that the simplest equation of the corresponding series has minimal possible order.

Table 5 is devoted to generalized mKdV for which the type of the algebra G and the Hamiltonian H is shown that leads to the simplest equation of the series [the generalized mKdV corresponding to the Hamiltonian $H(u_1,\ldots,u_k)$ has the form $\frac{\partial u_i}{\partial t}=\frac{\partial}{\partial x}\frac{\delta H}{\delta u_i}, i=1,\ldots,k$].

We further indicate the Hamiltonian for mKdV corresponding to $D_4^{(1)}$ and having the form $\frac{d\mathfrak{L}}{dt}=[\varphi(F)^+,\mathfrak{L}]$, where F is the same as in Appendix 2. H in this case is equal to $-4(u_1'u_4'+u_1'u_3u_4-u_4'u_1u_2-u_1u_2u_3u_4)$.

Examples of generalized KdV are presented in Tables 6 and 7. Table 6 is devoted to equations corresponding to pairs (G, c_m) such that after removal of the vertex c_m of the Dynkin scheme G decomposes into unconnected points. In this case the algebra $\mathfrak{G}$ (see part 6.1) is isomorphic to the direct product of several copies of $\mathfrak{sl}(2)$. Therefore, for suitable choice of the "coordinate system" $u_1,\ldots,u_k$ the generalized KdV corresponding to the Hamiltonian H and the second Hamiltonian structure has the form

$$\frac{\partial u_i}{\partial t}=\lambda_i(D^3+2(u_iD+Du_i))\frac{\delta H}{\delta u_i},\quad i=1,\ldots,k, \tag{9.1}$$

where $\lambda_i\in\mathbf{C}$. We mention that the operator $D^3+2(u_iD+Du_i)$ corresponds to the second Hamiltonian structure on $M(\mathfrak{sl}(2))$, and the occurrence of the factors λ_i is connected with the fact that the isomorphism $\mathfrak{G}\rightleftarrows\mathfrak{sl}(2)\times\cdots\times\mathfrak{sl}(2)$ does not preserve scalar products. The simplest example of an equation admitting the form (9.1) is the KdV equations for which $k = 1$, $\lambda_1 = 1$, $H = u_1^2/2$. In Table 6 for each pair (G, c_m) of this type the factors λ_i and Hamiltonian H are presented that correspond to the simplest equation of the series.

The simplest generalized KdV corresponding to algebras of types $A_2^{(2)}$, $A_3^{(2)}$, $B_2^{(1)}$ are presented in Table 7. We recall that each generalized KdV possesses the Lax representation $\frac{d\mathfrak{L}^{can}}{dt}=[\mathcal{A}^{can},\mathfrak{L}^{can}]$, where $\mathfrak{L}^{can}=\frac{d}{dx}+\Lambda+q^{can}$ (see part 6.2). For each equation Table 7 shows the form of the matrix q^{can}. The matrix Λ is equal to the sum of all canonical generators e_i. The explicit form of the e_i is indicated in Appendix 2.

Remark 1. The equations of Table 7 corresponding to the pairs $(A_2^{(2)}, c_0)$ $(A_2^{(2)}, c_1)$, $(A_3^{(2)}, c_2)$, $(B_2^{(1)}, c_2)$ after linear changes of unknowns and scale transformations can be written in the Hamiltonian form (9.1) (see Table 6). The remaining two equations, which are of a rather simple form, have a very complicated Hamiltonian form.

Remark 2. Tables 6 and 7 contain all the simplest generalized KdV corresponding to algebras G of ranks 1 and 2 except the equations corresponding to $(A_4^{(2)}, c_0)$ and $(A_4^{(2)}, c_1)$. In

TABLE 5

G	H
$A_2^{(2)}$	$3(u_1'')^2-5(u_1')^3+15(u_1')^2u_1^2+u_1^6$
$A_3^{(2)}$	$(u_1')^2+u_1^2u_2^2+u_2'u_1^2$
$A_4^{(2)}$	$(u_1')^2+4(u_2')^2+6u_1'u_2'+3u_1'u_2^2-6u_2'u_1^2+u_1^4+u_2^4-3u_1^2u_2^2$
$B_2^{(1)}$	$8(u_1')^2-4(u_2')^2+12u_2'u_1^2-u_1^4-u_2^4+6u_1^2u_2^2$

TABLE 6

G	c_m	λ_i	H
$A_2^{(2)}$	c_0	$\lambda_1=1$	$3(u_1')^2-16u_1^3$
$A_2^{(2)}$	c_1	$\lambda_1=1$	$3(u_1')^2-u_1^3$
$A_3^{(2)}$	c_2	$\lambda_1=\lambda_2=1$	$u_1^2+u_2^2-2u_1u_2$
$A_4^{(2)}$	c_1	$\lambda_1=1, \lambda_2=4$	$16u_1^2+u_2^2-12u_1u_2$
$A_5^{(2)}$	c_2	$\lambda_1=\lambda_2=1$, $\lambda_3=2$	$u_1^2+u_2^2-u_3^2-18u_1u_2+6u_1u_3+6u_2u_3$
$B_2^{(1)}$	c_2	$\lambda_1=\lambda_2=1$	$u_1^2+u_2^2-6u_1u_2$
$B_3^{(1)}$	c_2	$\lambda_1=\lambda_2=2$, $\lambda_3=1$	$u_1u_2-u_1u_3-u_2u_3+u_3^2$
$D_4^{(1)}$	c_2	$\lambda_1=\lambda_2=$ $=\lambda_3=\lambda_4=1$	$\alpha_1(-u_1u_2+u_1u_3+u_2u_4-u_3u_4)+$ $+\alpha_2(u_1u_3-u_1u_4-u_2u_3+u_2u_4)$, $\alpha_i\in\mathbf{C}$

TABLE 7

G	c_m	Simplest generalized KdV	q^{can}
$A_2^{(2)}$	c_1	$u_t=u_{xxxxx}+10uu_{xxx}+25u_xu_{xx}+20u^2u_x$	$-u(e_{1,2}+e_{2,3})\zeta^{-1}$
$A_2^{(2)}$	c_0	$u_t=u_{xxxxx}+5uu_{xxx}+5u_xu_{xx}+5u^2u_x$	$-ue_{3,1}\zeta^{-1}$
$A_3^{(2)}$	c_0	$\begin{cases} u_t=-v_x \\ v_t=v_{xxx}+uv_x+u_xv \end{cases}$	$-ue_{2,3}\zeta^{-1}+\frac{1}{3}ve_{1,4}\zeta^{-3}$
$A_3^{(2)}$	c_2	$\begin{cases} u_t=vv_x \\ v_t=v_{xxx}+2uv_x+u_xv \end{cases}$	$-\frac{1}{2}u(e_{1,2}+e_{3,4}+2e_{3,1}+2e_{4,2})\zeta^{-1}+$ $+\frac{\sqrt{3}}{6}v(e_{1,2}+e_{3,4}-2e_{3,1}-2e_{4,2})\zeta^{-1}$
$B_2^{(1)}$	c_0	$\begin{cases} u_t=u_{xxx}+uu_x+v_x \\ v_t=-2v_{xxx}-uv_x \end{cases}$	$-\frac{1}{3}u(e_{2,3}+e_{3,4})\zeta^{-1}-$ $-\frac{1}{18}v(e_{1,4}+e_{2,5})\zeta^{-3}$
$B_2^{(1)}$	c_2	$\begin{cases} u_t=u_{xxx}+uu_x-vv_x \\ v_t=-2v_{xxx}-uv_x \end{cases}$	$-\frac{1}{6}u(e_{1,2}+e_{4,5}+2e_{4,1}+2e_{5,2})\zeta^{-1}+$ $+\frac{\sqrt{2}}{12}v(e_{1,2}+e_{4,5}-2e_{4,1}-2e_{5,2})\zeta^{-1}$

these two cases it is possible to choose $\mathfrak{L}^{can}$ so that the simplest equations of the series have third order. However, due to the complexity of the formulas, we note here only that both these equations have the form

$$u_t=u_{xxx}+uu_x+v_x,$$
$$v_t=\alpha_1v_{xxx}+\alpha_2u_xu_{xx}+\alpha_3u^2u_x+\alpha_4(uv_x-vu_x),$$

where the coefficients α_i appropriate to each equation belong to $\mathbf{Q}[\sqrt{5}]$.

10. TWO-DIMENSIONALIZED TODA LATTICES

10.1. In this subsection we recall the definition of the two-dimensionalized Toda lattice corresponding to a Kats–Moody algebra, and for it we present the Zakharov–Shabat representation found in [61]. In the next subsection, following [65, 72, 12], we consider local conservation laws and the connection of the Toda lattices with generalized mKdV.

Let G be a Kats–Moody algebra with canonical generators e_i, f_i, h_i, $0 \leqslant i \leqslant r$. On G we fix a nondegenerate, invariant, symmetric, bilinear form coordinated with the canonical gradation $G=\bigoplus_j G^j$ (see Proposition 5.20). We set $\mathfrak{H}\overset{\text{def}}{=}G^0$. We recall that the elements h_i generate $\mathfrak{H}$ as a vector space, and there is exactly one linear relation (5.12) between them.

For any i, $0 \leqslant i \leqslant r$ we denote by α_i the linear functional on $\mathfrak{H}$, such that $[h, e_i] = \alpha_i(h)e_i$ for all $h\in\mathfrak{H}$ (the α_i are called simple roots of the algebra G). It is clear that $[h, f_i] = -\alpha_i(h)f_i$. We note that $\alpha_j(h_i) = A_{ij}$, where (A_{ij}) is the Cartan matrix.

We call the two-dimensionalized Toda lattice corresponding to G the equation

$$\frac{\partial^2\psi}{\partial x\partial\tau}=\sum_{i=0}^{r}e^{\alpha_i(\psi)}\cdot h_i, \quad \psi(x, \tau)\in\mathfrak{H}. \tag{10.1}$$

This name is also sometimes applied to the system of equations

$$\frac{\partial^2 u_i}{\partial x\partial\tau}=\exp\sum_{j=0}^{r}A_{ji}u_j, \quad 0\leqslant i\leqslant r, \quad u_i(x, \tau)\in\mathbf{C}. \tag{10.2}$$

We shall discuss the connection between (10.1) and (10.2). There is a mapping from the set of solutions of (10.2) into the set of solutions of (10.1) given by the formula

$\psi=\sum_{i=0}^{r} u_i h_i$. Due to relation (5.12) this mapping is not bijective. However, for any solution ψ or Eq. (10.1) it is not hard to find all solutions $(u_0,\ldots,u_r)$ of the system (10.2) such that $\sum_{i=0}^{r} u_i h_i=\psi$. Namely, for u_0 it is possible to take any solution of the equation $\frac{\partial^2 u_0}{\partial x \partial \tau}= e^{\alpha_0(\psi)}$, and after u_0 has been chosen the remaining u_i are found uniquely from the relation $\sum_{i=0}^{r} u_i h_i=\psi$ [it is not hard to see that the u_i found in this manner satisfy (10.2)]. Thus, Eq. (10.1) is almost equivalent to (10.2).

We further present the Lagrangian form of Eq. (10.1):

$$\frac{\partial^2\psi}{\partial x\partial\tau}=\operatorname{grad} U(\psi),\quad U(\psi)=\sum_{i=0}^{r} a_i e^{\alpha_i(\psi)},\quad a_i\overset{\mathrm{def}}{=}\frac{(h_i,h_i)}{2}. \tag{10.3}$$

The equivalence of (10.1) and (10.3) follows from the following well-known lemma.

LEMMA 10.1. For any $y\in\mathfrak{H}$, $\alpha_i(y)=\frac{2(y,h_i)}{(h_i,h_i)}$.

Proof. $(y, h_i) = (y, [e_i, f_i]) = ([y, e_i], f_i) = \alpha_i(y)(e_i, f_i)$. Setting $y = h_i$ and using the equality $\alpha_i(h_i) = A_{ii} = 2$, we obtain $(h_i, h_i) = 2(e_i, f_i)$, whence the lemma follows. ■

Remark. The equation $\frac{\partial^2\psi}{\partial x\partial\tau}=\operatorname{grad} U(\psi)$, $U(\psi)=\sum_{i=0}^{r} c_i e^{\alpha_i(\psi)}$ for any $c_i\in\mathbf{C}$ reduces to (10.3) by means of a transformation of the form $\tilde{\psi}=\psi+\psi_0$, $\tilde{\tau}=\alpha\tau$, where $\psi_0\in\mathfrak{H}$, $\alpha\in\mathbf{C}$.

Finally, we present the Zakharov–Shabat form of Eq. (10.1):

$$[\mathfrak{L},\overline{\mathfrak{L}}]=0,\quad \mathfrak{L}=e^{-\psi}\cdot\frac{\partial}{\partial x}\cdot e^{\psi}+\Lambda=\frac{\partial}{\partial x}+\psi'+\Lambda,$$
$$\overline{\mathfrak{L}}=\frac{\partial}{\partial\tau}+e^{-\psi}\cdot\overline{\Lambda}e^{\psi}, \tag{10.4}$$

where $\Lambda\overset{\mathrm{def}}{=}\sum_{i=0}^{r} e_i$, $\overline{\Lambda}\overset{\mathrm{def}}{=}\sum_{i=0}^{r} f_i$, and, strictly speaking, $e^{-\psi}\overline{\Lambda}e^{\psi}$ must be understood to be $e^{-\operatorname{ad}\psi}(\overline{\Lambda})$.

The equivalence of (10.1) and (10.4) follows immediately from the relations $[e_i, f_j] = \delta_{ij}h_j$, $[h, f_i] = -\alpha_i(h)f_i$.

Remark 1. Equation (10.4) admits the more symmetric form $\left[e^{-\psi/2}\frac{\partial}{\partial x}e^{\psi/2}+e^{\psi/2}\Lambda e^{-\psi/2},\ e^{\psi/2}\times\frac{\partial}{\partial\tau}e^{-\psi/2}+e^{-\psi/2}\overline{\Lambda}e^{\psi/2}\right]=0$.

Remark 2. If we interpret Λ, $\overline{\Lambda}$, and $\psi(x, \tau)$ as elements of a "genuine" Kats–Moody algebra $\tilde{G}$ (see Remark 2 following Proposition 5.8), then relation (10.4) is equivalent to the system (10.2).

To conclude this subsection we clarify the origin of the term "two-dimensionalized Toda lattice." The Toda lattice is properly the system of equations $\ddot{\psi}_i=\frac{\partial U}{\partial\psi_i}$, $i\in\mathbf{Z}$, where $U = -\sum_i e^{\psi_i-\psi_{i+1}}$ (it was investigated in [33, 52, 53]). From this infinite system there are two ways to obtain finite systems: a) require that $\psi_{i+n} = \psi_i$ for some n (the periodic Toda lattice); b) require that $\psi_0 = \psi_{n+1} = 0$ for some n (the corresponding system of equations for $\psi_1,\ldots,\psi_n$ is called the nonclosed Toda lattice). The two-dimensionalization consists in replacing $\partial^2/\partial t^2$ by the operator $\partial^2/\partial x\partial\tau$. The two-dimensionalized periodic Toda lattice is essentially equivalent to Eq. (10.1) for an algebra G of type $A_n^{(1)}$, while the two-dimensionalized nonclosed Toda lattice is equivalent to the system (10.2) for the case where (A_{ij}) is the Cartan matrix of an algebra of type $A_{n-1}^{(1)}$. The history of two-dimensionalization of the periodic Toda lattice and passage from $A_n^{(1)}$ to arbitrary Kats–Moody algebras can be traced in [30, 36, 45, 61, 64, 46, 55]. Concerning the nonclosed Toda lattice and its generalizations see [28, 29, 37, 44, 61].

10.2. The relation between two-dimensionalized Toda lattices and generalized mKdV corresponding to the same Kats-Moody algebra G is based on the fact that on operator $\mathfrak{L}$ of the form (6.7) after the substitution $q=\psi_x$ is converted into an operator $\mathfrak{L}$ of the form (10.4).

Proposition 10.2. Let H_i be the densities of conservation laws for generalized mKdV considered in Proposition 6.6. Then H_i, considered as differential polynomials in ψ, are densities of conservation laws for (10.1).

The proof is essentially the same as that of Proposition 1.5.

We recall (see Sec. 6) that H_i has degree of homogeneity i + 1 if it is assumed that $\deg \frac{\partial^j\psi}{\partial x^j}=j$. Moreover, $H_i \neq 0$ if and only if the remainder on division of i by the Coxeter number of G is an exponent.

Definition. It is said that the equation

$$\frac{\partial\psi}{\partial t}=f(\psi, \psi_x, \psi_\tau, \psi_{xx}, \psi_{x\tau}, \psi_{\tau\tau},\ldots) \tag{10.5}$$

is a symmetry for the equation

$$F(\psi, \psi_x, \psi_\tau, \psi_{xx}, \psi_{x\tau}, \psi_{\tau\tau},\ldots)=0, \tag{10.6}$$

if the derivative with respect to t of the left side of (10.6) computed by Eq. (10.5) vanishes on substituting for ψ any solution of Eq. (10.6).

Remark. If the set of solutions of Eq. (10.6) is thought of as a submanifold $M\subset W$, where W is the manifold of all functions $\psi(x, \tau)$, and Eq. (10.5) is considered a vector field v on W, then the definition presented above can be reformulated very briefly: v must be tangent to M.

We recall (see Proposition 6.8) that a generalized mKdV has the form $\frac{\partial q}{\partial t}=\frac{\partial}{\partial x}f(q, q_x, q_{xx},\ldots)$. After the substitution $q = \psi_x$ it acquires the form $\frac{\partial\psi}{\partial t}=f(\psi_x, \psi_{xx}, \psi_{xxx},\ldots)$. This equation for ψ by abuse of language we call, as before, a generalized mKdV.

Proposition 10.3. The generalized mKdV corresponding to the algebra G are symmetries for Eq. (10.1).

This proposition can be derived from Proposition 10.2 by means of a Hamiltonian formalism (see [72]). We present a direct proof.

Proof. We recall (see part 6.4) that a generalized mKdV has the form $\frac{d\mathfrak{L}}{dt}=[\varphi(u)^+, \mathfrak{L}]$, where $u\in\mathfrak{Z}^+$. It must be shown that if $[\mathfrak{L}, \bar{\mathfrak{L}}]=0$, then the derivative $\frac{d}{dt}[\mathfrak{L}, \bar{\mathfrak{L}}]$, computed by the generalized mKdV is equal to zero. For this it suffices to verify that $\frac{d\bar{\mathfrak{L}}}{dt}=[\varphi(u)^+, \bar{\mathfrak{L}}]$.

LEMMA 10.4. $[\bar{\mathfrak{L}}, \varphi(u)]=0$.

Proof. Let U and $\mathfrak{L}_0$ be the same as in Proposition 6.2. We set $\tilde{\mathfrak{L}}=e^{-\operatorname{ad}U}(\bar{\mathfrak{L}})$. We recall that $\mathfrak{L}=e^{\operatorname{ad}U}(\mathfrak{L}_0)$, $\varphi(u)\overset{\text{def}}{=}e^{-\operatorname{ad}U}(u)$. Since $[\mathfrak{L}, \bar{\mathfrak{L}}]=0$, it follows that $[\mathfrak{L}_0, \tilde{\mathfrak{L}}]=0$. From this it is easy to deduce that $\tilde{\mathfrak{L}}$ has the form $\tilde{\mathfrak{L}}=\frac{\partial}{\partial\tau}+v(x, \tau)$, $v(x,\tau)\in\mathfrak{Z}$. Therefore, $[\tilde{\mathfrak{L}}, u]=0$, and hence $[\bar{\mathfrak{L}}, \varphi(u)]=0$. ■

From the lemma it follows that $[\varphi(u)^+, \bar{\mathfrak{L}}]=-[\varphi(u)^-, \bar{\mathfrak{L}}]$. The degree of the left side (in the sense of the canonical gradation) is not less than −1, while the degree of the right side is not more than −1. Therefore, $[\varphi(u)^+, \bar{\mathfrak{L}}]\in C^\infty(\mathbf{R}^2, G^{-1})$, and if $\varphi(u)=\sum_i A_i$, where $A_i\in C^\infty(\mathbf{R}^2, G^i)$, then $[\varphi(u)^+, \bar{\mathfrak{L}}]=[A_0, e^{-\psi}\bar{\Lambda}e^{\psi}]$. On the other hand, $\frac{d\bar{\mathfrak{L}}}{dt}=-\left[\frac{d\psi}{dt}, e^{-\psi}\bar{\Lambda}e^{\psi}\right]$. In the course of the proof of Proposition 6.8 it was shown that generalized mKdV as an equation for q has the form $dq/dt = -\partial A_0/\partial x$. Therefore, as an equation for ψ it has the form $d\psi/\partial t = -A_0$. Thus, $\frac{d\bar{\mathfrak{L}}}{dt}=[\varphi(u)^+, \bar{\mathfrak{L}}]$, as was required to prove. ■

The conservation laws considered in Proposition 10.2 are only half of the known local conservation laws for Eq. (10.1). In order to obtain the second half it is necessary, roughly speaking, to interchange x and τ. The same pertains to the symmetries considered in Proposition 10.3.

APPENDIX 1

CLASSICAL SIMPLE LIE ALGEBRAS

In this appendix for each classical simple Lie algebra $\mathfrak{G}$ we list the system of Weyl generators X_i, Y_i, H_i, $1 \leqslant i \leqslant n$ (the numbering of the generators corresponds to the numbering of the vertices in Table 1). In all cases the realization and the system of Weyl generators are chosen that the Borel (Cartan) subalgebra of $\mathfrak{G}$ is equal, respectively, to the intersection of $\mathfrak{G}$ with the set of all upper triangular (diagonal) matrices.

We recall that X^T denotes the matrix obtained from X by transposition relative to the secondary diagonal.

Type A_n, $n \geqslant 1$

$\mathfrak{G}=\mathfrak{sl}(n+1)\overset{\text{def}}{=}\{A\in \mathrm{Mat}(n+1, \mathbf{C}) \mid \operatorname{tr} A=0\}$. System of Weyl generators: $X_i=e_{i+1,i}$, $Y_i=e_{i,i+1}$, $H_i=e_{i+1,i+1}-e_{i,i}$, $1\leqslant i\leqslant n$.

Type B_n, $n \geqslant 1$

$\mathfrak{G}=\mathfrak{o}(2n+1)\overset{\text{def}}{=}\{A\in \mathrm{Mat}(2n+1, \mathbf{C}) \mid A=-SA^TS^{-1}, S\overset{\text{def}}{=}\operatorname{diag}(1, -1, \ldots, -1, 1)$. System of Weyl generators: $X_i=e_{i+1,i}+e_{2n+2-i,2n+1-i}$, $Y_i=e_{i,i+1}+e_{2n+1-i,2n+2-i}$, $H_i=-e_{i,i}+e_{i+1,i+1}-e_{2n+1-i,2n+1-i}+e_{2n+2-i,2n+2-i}$, $i=1,2,\ldots,n-1$; $X_n=e_{n+1,n}+e_{n+2,n+1}$, $Y_n=2(e_{n,n+1}+e_{n+1,n+2})$, $H_n=2(e_{n+2,n+2}-e_{n,n})$.

Type C_n, $n \geqslant 1$

$\mathfrak{G}=\mathfrak{sp}(2n)\overset{\text{def}}{=}\{A\in \mathrm{Mat}(2n, \mathbf{C}) \mid A=-SA^TS^{-1}$, $S\overset{\text{def}}{=}\operatorname{diag}(1, -1, \ldots, 1, -1)$. System of Weyl generators: $X_i=e_{i+1,i}+e_{2n+1-i,2n-i}$, $Y_i=e_{i,i+1}+e_{2n-i,2n+1-i}$, $H_i=-e_{i,i}+e_{i+1,i+1}-e_{2n-i,2n-i}+e_{2n+1-i,2n+1-i}$, $i=1,2,\ldots,n-1$; $X_n=e_{n+1,n}$, $Y_n=e_{n,n+1}$, $H_n=-e_{n,n}+e_{n+1,n+1}$.

Type D_n, $n \geqslant 3$

$\mathfrak{G}=\mathfrak{o}(2n)\overset{\text{def}}{=}\{A\in \mathrm{Mat}(2n, \mathbf{C}) \mid A=-SA^TS^{-1}$, $S\overset{\text{def}}{=}\operatorname{diag}(1, -1, \ldots, (-1)^{n-1}, (-1)^{n-1}, (-1)^n, \ldots, 1)$. System of Weyl generators: $X_i=e_{i+1,i}+e_{2n+1-i,2n-i}$, $Y_i=e_{i,i+1}+e_{2n-i,2n+1-i}$, $H_i=-e_{i,i}+e_{i+1,i+1}-e_{2n-i,2n-i}+e_{2n+1-i,2n+1-i}$, $i=1, \ldots, n-1$; $X_n=\frac{1}{2}(e_{n+1,n-1}+e_{n+2,n})$, $Y_n=2(e_{n-1,n+1}+e_{n,n+2})$, $H_n=-e_{n-1,n-1}-e_{n,n}+e_{n+1,n+1}+e_{n+2,n+2}$. For n = 2 all formulas remain in force, but the Dynkin scheme is not connected, so that the algebra $\mathfrak{o}(4)$ is semisimple rather than simple. Since the Dynkin scheme of $\mathfrak{o}(4)$ consists of two vertices not connected by edges it follows that $\mathfrak{o}(4)\simeq\mathfrak{sl}(2)\times\mathfrak{sl}(2)$.

APPENDIX 2

CLASSICAL KATS–MOODY ALGEBRAS

In this appendix for each classical Kats–Moody algebra we present 1) a realization of G in the form $L(\mathfrak{A}, C)$, where C is the Coxeter automorphism; 2) a basis of the vector space $\mathfrak{Z}\overset{\text{def}}{=}\{x\in G \mid [\Lambda, x]=0\}$ and the eigenvalues of the matrix Λ in the realization; 3) the standard realization of G.

We shall consider part 3) in more detail. Generally speaking, for each vertex c_m of the Dynkin scheme of G we ought to present the corresponding standard automorphism $\sigma_m:\mathfrak{A}\to\mathfrak{A}$. In constructing the standard realization corresponding to c_m we actually replace $\mathfrak{A}$ by an algebra $\mathfrak{A}_m$ of the form $R_m\mathfrak{A}R_m^{-1}$, where R_m is a permutation matrix, so that σ_m is an automorphism of $\mathfrak{A}_m$ rather than $\mathfrak{A}$. R_m is chosen so that the elements of the semisimple algebra $\mathfrak{G}\overset{\text{def}}{=}\{x\in \mathfrak{A}_m \mid \sigma_m(x)=x\}$ have block-diagonal form. Further, we do not consider all vertices of the Dynkin scheme (for our purposes it suffices that each vertex be carried by some automorphism of the Dynkin scheme into one of the vertices considered). For all vertices c_m considered we indicate $\mathfrak{A}_m$, σ_m, $\mathfrak{G}$, and also the canonical generators $\tilde{e}_i\in L(\mathfrak{A}_m, \sigma_m)$. Due to the special choice of the matrices R_m the algebras $\mathfrak{H}$ and $\mathfrak{b}$ (see part 6.1) in all cases are equal, respectively, to $\mathfrak{G}\cap\mathrm{Diag}$ and $\mathfrak{G}\cap\mathfrak{t}$, where $\mathfrak{t}$ is the set of upper triangular matrices.

In this appendix $\mathfrak{o}(k)$ and $\mathfrak{sp}(2n)$ are the same as in Appendix 1. The number of the canonical generators of the Kats–Moody algebras corresponds to the number of the vertices of the Dynkin schemes in Table 2.

Type $A_n^{(1)}$, $n \geqslant 1$

$\mathfrak{A}=\mathfrak{sl}(n+1)$, $C(X)=SXS^{-1}$, $S\overset{\text{def}}{=}\operatorname{diag}(1, \omega, \ldots, \omega^n)$, where $\omega=e^{\frac{2\pi i}{h}}$, $h=n+1$.

System of canonical generators: $e_0=e_{1,n+1}\zeta$, $f_0=e_{n+1,1}\zeta^{-1}$, $h_0=e_{1,1}-e_{n+1,n+1}$; $e_i=e_{i+1,i}\zeta$, $f_i=e_{i,i+1}\zeta^{-1}$, $h_i=e_{i+1,i+1}-e_{i,i}$, $i=1,\ldots,n$.

Eigenvalues of Λ equal to $\zeta\omega^i$, $i = 0, 1,\ldots,n$. Basis in $\mathfrak{Z}$ is formed by Λ^i, where $i\in\mathbf{Z}$, i is not divisible by $n + 1$.

Standard realization corresponding to c_0: $\mathfrak{A}_0=\mathfrak{A}$, $\sigma_0(X)=X$. In this realization $\tilde{e}_0 = e_{1,n+1}\lambda$, $\tilde{e}_i = e_{i+1,i}$ for $i = 1,\ldots,n$.

Type $A_{2n}^{(2)}$, $n \geqslant 1$

$\mathfrak{A}=\mathfrak{sl}(2n+1)$, $C(X)=-SX^TS^{-1}$, $S\overset{\text{def}}{=}\operatorname{diag}(1, -\omega, \omega^2,\ldots, -\omega^{2n-1}, \omega^{2n})$, $\omega=e^{\frac{2\pi i}{h}}$, $h=4n+2$.

System of canonical generators: $e_0=e_{1,2n+1}\zeta$, $f_0=e_{2n+1,1}\zeta^{-1}$, $h_0=e_{1,1}-e_{2n+1,2n+1}$; $e_i=(e_{i+1,i}+e_{2n+2-i,2n+1-i})\zeta$, $f_i=(e_{i,i+1}+e_{2n+1-i,2n+2-i})\zeta^{-1}$, $h_i=-e_{i,i}+e_{i+1,i+1}-e_{2n+1-i,2n+1-i}+e_{2n+2-i,2n+2-i}$, $i = 1,\ldots, n-1$; $e_n=(e_{n+1,n}+e_{n+2,n+1})\zeta$, $f_n=2(e_{n,n+1}+e_{n+1,n+2})\zeta^{-1}$, $h_n=2(-e_{n,n}+e_{n+2,n+2})$.

Eigenvalues of Λ equal to $\zeta\omega^{2i}$, $i = 0, 1,\ldots,2n$. Basis in $\mathfrak{Z}$ formed by Λ^{2i+1}, where $i\in\mathbf{Z}$, $2i+1$ is not divisible by $2n + 1$.

Standard realization corresponding to c_0: $\mathfrak{A}_0=\mathfrak{A}$, $\sigma_0(X)=-QX^TQ^{-1}$, where $Q = \operatorname{diag}(1, -1,\ldots,-1, 1)$. Order of σ_0 equal to two, $\mathfrak{G}=\mathfrak{o}(2n+1)$. In this realization $\tilde{e}_0=e_{1,2n+1}\lambda$; $\tilde{e}_i=e_{i+1,i}+e_{2n+2-i,2n+1-i}$, $i=1,\ldots,n$.

Standard realization corresponding to c_m, $m=1,\ldots,n$: $\mathfrak{A}_m=\mathfrak{A}$, $\sigma_m(X)=-QX^TQ^{-1}$, where $Q=\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$, $\alpha=\operatorname{diag}(i, -i,\ldots,-i,i)\in\mathfrak{gl}(2n-2m+1)$, $\beta=\operatorname{diag}(1,-1,\ldots,1,-1)\in\mathfrak{gl}(2m)$. Order of σ_m equal to four, $\mathfrak{G}=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}, a\in\mathfrak{o}(2n-2m+1), b\in\mathfrak{sp}(2m)\right\}$.[8] In this realization $\tilde{e}_0=e_{2n-m+2,2n-m+1}$; $\tilde{e}_j=e_{2n-m+2-j,2n-m+1-j}+e_{2n-m+2+j,2n-m+1+j}$, $j=1,\ldots,m-1$; $\tilde{e}_m=(e_{1,2n+1}+e_{2n-2m+2,2n-2m+1})\lambda$; $\tilde{e}_{m+j}=e_{j+1,j}+e_{2n-2m+2-j,2n-2m+1-j}$, $j=1,\ldots,n-m$.

Type $A_{2n-1}^{(2)}$, $n \geqslant 2$

$\mathfrak{A}=\mathfrak{sl}(2n)$, $C(X)=-SX^TS^{-1}$, $S\overset{\text{def}}{=}\operatorname{diag}(1, -\omega, \omega^2,\ldots, \omega^{2n-2}, -\omega^{2n-1})$, $\omega=e^{\frac{2\pi i}{h}}$, $h=4n-2$.

System of canonical generators: $e_0=\frac{1}{2}(e_{1,2n-1}+e_{2,2n})\zeta$, $f_0=2(e_{2n-1,1}+e_{2n,2})\zeta^{-1}$, $h_0=e_{1,1}+e_{2,2}-e_{2n-1,2n-1}-e_{2n,2n}$; $e_i=(e_{i+1,i}+e_{2n+1-i,2n-i})\zeta$, $f_i=(e_{i,i+1}+e_{2n-i,2n+1-i})\zeta^{-1}$, $h_i=-e_{i,i}+e_{i+1,i+1}-e_{2n-i,2n-i}+e_{2n+1-i,2n+1-i}$, $i=1,\ldots,n-1$; $e_n=e_{n+1,n}\zeta$, $f_n=e_{n,n+1}\zeta^{-1}$, $h_n=-e_{n,n}+e_{n+1,n+1}$.

The eigenvalues of Λ are equal to $0, \zeta, \omega^2\zeta,\ldots, \omega^{4n-4}\zeta$. A basis in the space $\mathfrak{Z}$ is formed by $\Lambda^{2i+2}-\frac{1}{2n}\operatorname{tr}(\Lambda^{2i+1})E$, $i\in\mathbf{Z}$, where for $i<0$ $\Lambda^{2i+1}\overset{\text{def}}{=}\zeta^{-kh}\Lambda^{2i+1+kh}$, $k\gg 0$ (we note that $\operatorname{tr}\Lambda^{2i+1}\neq 0$ only if $2i + 1$ is divisible by $2n - 1$).

The standard realization corresponding to c_0: $\mathfrak{A}_0=\mathfrak{A}$, $\sigma_0(X)=-QX^TQ^{-1}$, where $Q = \operatorname{diag}(1, -1,\ldots,1, -1)$. The order of σ_0 is equal to two, $\mathfrak{G}=\mathfrak{sp}(2n)$. In this realization $\tilde{e}_0=\frac{1}{2}\times(e_{1,2n-1}+e_{2,2n})\lambda$; $\tilde{e}_i=e_{i+1,i}+e_{2n+1-i,2n-i}$, $i=1,\ldots,n-1$; $\tilde{e}_n=e_{n+1,n}$.

Standard realization corresponding to c_n: $\mathfrak{A}_n=\mathfrak{A}$, $\sigma_n(X)=-QX^TQ^{-1}$, where $Q=\operatorname{diag}(1, -1,\ldots,(-1)^{n+1}, (-1)^{n+1}, (-1)^{n+2},\ldots, -1, 1)$. The order of σ_n is equal to two, $\mathfrak{G}=\mathfrak{o}(2n)$. In this realization $\tilde{e}_0=\frac{1}{2}(e_{n+1,n-1}+e_{n+2,n})$; $\tilde{e}_i=e_{n+1-i,n-i}+e_{n+1+i,n+i}$, $i=1,\ldots,n-1$; $\tilde{e}_n=e_{1,2n}\lambda$.

Standard realization corresponding to c_m, $m=2, 3,\ldots, n-1$: $\mathfrak{A}_m=\mathfrak{A}$, $\sigma_m(X)=-QX^TQ^{-1}$, where $Q=\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$, $\alpha=\operatorname{diag}(-i, i,\ldots, -i, i)\in\mathfrak{gl}(2n-2m)$, $\beta=\operatorname{diag}(1, -1,\ldots,(-1)^{m+1}, (-1)^{m+1}, (-1)^{m+2},\ldots, 1)\in\mathfrak{gl}(2m)$. The order of σ_m is equal to four, $\mathfrak{G}=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}, a\in\mathfrak{sp}(2n-2m), b\in\mathfrak{o}(2m)\right\}$. In

[8] The fact that $\mathfrak{G}\simeq \mathrm{o}(2n-2m+1)\times \mathrm{sp}(2m)$ follows from the general theory of Kats–Moody algebras (we recall that the Dynkin scheme of $\mathfrak{G}$ is obtained from the Dynkin scheme of G by removing the vertex c_m and the edges continuous to it).

this realization $\tilde{e}_0=\frac{1}{2}(e_{2n-m+1,\,2n-m-1}+e_{2n-m+2,\,2n-m})$; $\tilde{e}_j=e_{2n-m+1-j,\,2n-m-j}+e_{2n-m+1+j,\,2n-m+j}$, $j=1,\ldots,m-1$; $\tilde{e}_m=(e_{1,2n}+e_{2n-2m+1,\,2n-2m})\lambda$; $\tilde{e}_{m+j}=e_{j+1,\,j}+e_{2n-2m+1-j,\,2n-2m-j}$, $j=1,\ldots,n-m-1$; $\tilde{e}_n=e_{n-m+1,n-m}$.

Type $B_n^{(1)}$, $n \geqslant 2$

$$\mathfrak{A}=\mathfrak{o}(2n+1),\ C(X)=SXS^{-1},\ S\overset{\text{def}}{=}\operatorname{diag}(1,\ \omega,\ \omega^2,\ldots,\ \omega^{2n-1},\ 1),\ \omega=e^{\frac{2\pi i}{h}},\ h=2n.$$

System of canonical generators: $e_0=\frac{1}{2}(e_{1,\,2n}+e_{2,\,2n+1})\zeta$, $f_0=2(e_{2n,\,1}+e_{2n+1,\,2})\zeta^{-1}$, $h_0=e_{1,1}+e_{2,2}-e_{2n,\,2n}-e_{2n+1,\,2n+1}$; $e_i=f_0=(e_{i+1,\,i}+e_{2n+2-i,\,2n+1-i})\zeta$, $f_i=(e_{i,\,i+1}+e_{2n+1-i,\,2n+2-i})\zeta^{-1}$, $h_i=-e_{i,i}+e_{i+1,\,i+1}-e_{2n+1-i,\,2n+1-i}+e_{2n+2-i,\,2n+2-i}$, $i=1,\ldots,n-1$; $e_n=(e_{n+1,\,n}+e_{n+2,\,n+1})\zeta$, $f_n=2(e_{n,\,n+1}+e_{n+1,\,n+2})\zeta^{-1}$, $h_n=2(e_{n+2,\,n+2}-e_{n,n})$.

The eigenvalues of Λ are equal to $0,\ \zeta,\ \omega\zeta,\ldots,\ \omega^{2n-1}\zeta$. A basis in $\mathfrak{Z}$ is formed by Λ^{2i+1}, $i\in\mathbf{Z}$, where for $i<0$ $\Lambda^{2i+1}\overset{\text{def}}{=}\zeta^{-kh}\Lambda^{2i+1+kh}$, $k\gg 0$.

Standard realization corresponding to c_0: $\mathfrak{A}_0=\mathfrak{A}$, $\sigma_0(X)=X$, $\mathfrak{G}=\mathfrak{o}(2n+1)$. In this realization $\tilde{e}_0=\frac{1}{2}(e_{1,\,2n}+e_{2,\,2n+1})\lambda$; $\tilde{e}_i=e_{i+1,\,i}+e_{2n+2-i,\,2n+1-i}$, $i=1,\ldots,n$.

Standard realization corresponding to c_m, $m=2,\ldots,n$: $\mathfrak{A}_m\overset{\text{def}}{=}\{X\in\mathfrak{gl}(2n+1)\,|\,X=-RX^TR^{-1}\}$, where $R=\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$, $\alpha=\operatorname{diag}(1,\ -1,\ \ldots,\ -1,\ 1)\in\mathfrak{gl}(2n-2m+1)$, $\beta=\operatorname{diag}(-1,\ 1,\ \ldots,\ (-1)^m,\ (-1)^m,\ (-1)^{m+1},\ \ldots,\ -1)\in\mathfrak{gl}(2m)$; $\sigma_m(X)=QXQ^{-1}$, where $Q=\operatorname{diag}(\underbrace{-1,\ -1,\ \ldots,\ -1}_{2n-2m+1},\ 1,\ 1,\ \ldots,\ 1)$. The order of σ_m is equal to two, $\mathfrak{G}=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix},\ a\in\mathfrak{o}(2n-2m+1),\ b\in\mathfrak{o}(2m)\right\}$. In this realization $\tilde{e}_0=\frac{1}{2}(e_{2n-m+2,2n-m}+e_{2n-m+3,2n-m+1})$; $\tilde{e}_j=e_{2n-m+2+j,2n-m+1+j}+e_{2n-m+2-j,2n-m+1-j}$, $j=1,\ldots,m-1$; $\tilde{e}_m=(e_{1,2n+1}+e_{2n-2m+2,2n-2m+1})\lambda$; $\tilde{e}_{m+j}=e_{j+1,j}+e_{2n-2m+2-j,2n-2m+1-j}$, $j=1,\ldots,n-m$.

Type $C_n^{(1)}$, $n \geqslant 1$

$$\mathfrak{A}=\mathfrak{sp}(2n),\quad C(X)=SXS^{-1},\quad S\overset{\text{def}}{=}\operatorname{diag}(1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{2n-1}),\quad \omega=e^{\frac{2\pi i}{h}},\ h=2n.$$

System of canonical generators: $e_0=e_{1,2n}\zeta$, $f_0=e_{2n,1}\zeta^{-1}$, $h_0=e_{1,1}-e_{2n,2n}$; $e_i=(e_{i+1,i}+e_{2n+1-i,2n-i})\zeta$, $f_i=(e_{i,i+1}+e_{2n-i,2n+1-i})\zeta^{-1}$, $h_i=-e_{i,i}+e_{i+1,i+1}-e_{2n-i,2n-i}+e_{2n+1-i,2n+1-i}$, $i=1,\ldots,n-1$; $e_n=e_{n+1,n}\zeta$, $f_n=e_{n,n+1}\zeta^{-1}$, $h_n=-e_{n,n}+e_{n+1,n+1}$.

The eigenvalues of Λ are equal to $\zeta\omega^i$, $i=0,\ 1,\ldots,2n-1$. A basis in $\mathfrak{Z}$ is formed by Λ^{2i+1}, $i\in\mathbf{Z}$.

Standard realization corresponding to c_0: $\mathfrak{A}_0=\mathfrak{A}$, $\sigma_0(X)=X$, $\mathfrak{G}=\mathfrak{sp}(2n)$. In this realization $\tilde{e}_0=e_{1,2n}\lambda$; $\tilde{e}_i=e_{i+1,i}+e_{2n+1-i,2n-i}$, $i=1,\ldots,n-1$; $\tilde{e}_n=e_{n+1,n}$.

Standard realization corresponding to c_m, $m=1,\ldots,n-1$: $\mathfrak{A}_m\overset{\text{def}}{=}\{X\in\mathfrak{gl}(2n)\,|\,X=-RX^TR^{-1}\}$, where $R=\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$, $\alpha=\operatorname{diag}(1,\ -1,\ \ldots,\ 1,\ -1)\in\mathfrak{gl}(2n-2m)$, $\beta=\operatorname{diag}(1,\ -1,\ \ldots,\ 1,\ -1)\in\mathfrak{gl}(2m)$; $\sigma_m(X)=QXQ^{-1}$, where $Q=\operatorname{diag}(\underbrace{-1,\ -1,\ \ldots,\ -1}_{2n-2m},\ 1,\ 1,\ \ldots,\ 1)$. The order of σ_m is equal to two, $\mathfrak{G}=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix},\ a\in\mathfrak{sp}(2n-2m),\ b\in\mathfrak{sp}(2m)\right\}$. In this realization $\tilde{e}_0=e_{2n-m+1,2n-m}$; $\tilde{e}_j=e_{2n-m+1+j,2n-m+j}+e_{2n-m+1-j,2n-m-j}$, $j=1,\ldots,m-1$; $\tilde{e}_m=(e_{1,2n}+e_{2n-2m+1,2n-2m})\lambda$; $\tilde{e}_{m+j}=e_{j+1,j}+e_{2n-2m+1-j,2n-2m-j}$, $j=1,\ldots,n-m-1$; $\tilde{e}_n=e_{n-m+1,n-m}$.

Type $D_n^{(1)}$, $n \geqslant 3$

$$\mathfrak{A}=\mathfrak{o}(2n),\ C(X)=SXS^{-1},\ S\overset{\text{def}}{=}\operatorname{diag}(1,\ \omega,\ \ldots,\ \omega^{n-1},\ \omega^{n-1},\ \omega^n,\ \ldots,\ \omega^{2n-3},\ 1),\ \omega=e^{\frac{2\pi i}{h}},\ h=2n-2.$$

System of canonical generators: $e_0=\frac{1}{2}(e_{1,2n-1}+e_{2,2n})\zeta$, $f_0=2(e_{2n-1,1}+e_{2n,2})\zeta^{-1}$, $h_0=e_{1,1}+e_{2,2}-e_{2n-1,2n-1}-e_{2n,2n}$; $e_i=(e_{i+1,i}+e_{2n+1-i,2n-i})\zeta$, $f_i=(e_{i,i+1}+e_{2n-i,2n+1-i})\zeta^{-1}$, $h_i=-e_{i,i}+e_{i+1,i+1}-e_{2n-i,2n-i}+e_{2n+1-i,2n+1-i}$, $i=1,\ldots,n-1$; $e_n=\frac{1}{2}(e_{n+1,n-1}+e_{n+2,n})\zeta$, $f_n=2(e_{n-1,n+1}+e_{n,n+2})\zeta^{-1}$, $h_n=-e_{n-1,n-1}-e_{n,n}+e_{n+1,n+1}+e_{n+2,n+2}$.

The eigenvalues of Λ are equal to 0 (of multiplicity two), $\zeta, \omega\zeta, \ldots, \omega^{2n-3}\zeta$. A basis in $\mathfrak{Z}$ is formed by Λ^{2i+1}, $i\in Z$, where for $i < 0$, $\Lambda^{2i+1} \stackrel{\text{def}}{=} \zeta^{-kh}\Lambda^{2i+1+kh}$, $k \gg 0$ and $\zeta^{n-1+ih}F$, $i\in Z$. Here $F = \Phi+(-1)^n\Phi^T$, $\Phi=e_{1,n}-2e_{1,n+1}-2e_{2n,n}+4e_{2n,n+1}$.

Standard realization corresponding to $c_0\colon \mathfrak{A}_0=\mathfrak{A}$, $\sigma_0(X)=X$, $\mathfrak{G}=\mathfrak{o}(2n)$. In this realization $\tilde{e}_0=\frac{1}{2}(e_{1,2n-1}+e_{2,2n})\lambda$; $\tilde{e}_i=e_{i+1,i}+e_{2n+1-i,2n-i}$, $i=1,\ldots,n-1$; $e_n=\frac{1}{2}(e_{n+1,n-1}+e_{n+2,n})$.

Standard realization corresponding to c_m, $m=2,3,\ldots,n-2$: $\mathfrak{A}_m \stackrel{\text{def}}{=} \{X\in\mathfrak{gl}(2n)\,|\,X=-RX^TR^{-1}\}$, where $R=\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$, $\alpha=\operatorname{diag}(1,-1,\ldots,(-1)^{n-m-1},(-1)^{n-m-1},\ldots,1)\in\mathfrak{gl}(2n-2m)$, $\beta=\operatorname{diag}(-1,1,\ldots,(-1)^m,(-1)^m,\ldots,-1)\in\mathfrak{gl}(2m)$; $\sigma_m(X)=QXQ^{-1}$, where $Q=\operatorname{diag}(\underbrace{-1,-1,\ldots,-1}_{2n-2m},1,1,\ldots,1)$. The order of σ_m is equal to two, $\mathfrak{G}=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}, a\in\mathfrak{o}(2n-2m), b\in\mathfrak{o}(2m)\right\}$. In this realization $\tilde{e}_0=\frac{1}{2}\times(e_{2n-m+1,2n-m-1}+e_{2n-m+2,2n-m})$; $\tilde{e}_j=e_{2n-m+1+j,2n-m+j}+e_{2n-m+1-j,2n-m-j}$, $j=1,\ldots,m-1$; $\tilde{e}_m=(e_{1,2n}+e_{2n-2m+1,2n-2m})\lambda$; $\tilde{e}_{m+j}=e_{j+1,j}+e_{2n-2m+1-j,2n-2m-j}$, $j=1,\ldots,n-m-1$; $\tilde{e}_n=\frac{1}{2}(e_{n-m+1,n-m-1}+e_{n-m+2,n-m})$.

Type $D_{n+1}^{(2)}$, $n \geqslant 2$

$\mathfrak{A}=\{X\in\mathfrak{gl}(2n+2)\,|\,X=-RX^TR^{-1}\}$, $R\stackrel{\text{def}}{=}\operatorname{diag}(\underbrace{1,-1,\ldots,(-1)^{n+1}}_{n},0,0,(-1)^{n+1},\ldots,1)+(-1)^n e_{n+1,n+2}-e_{n+2,n+1}$, $C(X)=SXS^{-1}$, $S\stackrel{\text{def}}{=}\operatorname{diag}(\omega,\omega^2,\ldots,\omega^n,-1,1,\omega^{n+2},\omega^{n+3},\ldots,\omega^{2n+1})$, $\omega=e^{\frac{2\pi i}{h}}$, $h=2n+2$.

System of canonical generators: $e_0=(e_{1,n+2}+e_{n+2,2n+2})\zeta$, $f_0=2(e_{n+2,1}+e_{2n+2,n+2})\zeta^{-1}$, $h_0=2(e_{1,1}-e_{2n+2,2n+2})$; $e_i=(e_{i+1,i}+e_{2n+3-i,2n+2-i})\zeta$, $f_i=(e_{i,i+1}+e_{2n+2-i,2n+3-i})\zeta^{-1}$, $h_i=-e_{i,i}+e_{i+1,i+1}-e_{2n+2-i,2n+2-i}+e_{2n+3-i,2n+3-i}$, $i=1,\ldots,n-1$; $e_n=(e_{n+1,n}+e_{n+3,n+1})\zeta$, $f_n=2(e_{n,n+1}+e_{n+1,n+3})\zeta^{-1}$, $h_n=2(-e_{n,n}+e_{n+3,n+3})$.

The eigenvalues of Λ are equal to $\zeta\omega^i$, $i = 0,\ldots,2n+1$. A basis in $\mathfrak{Z}$ is formed by Λ^{2i+1}, $i\in Z$.

Standard realization corresponding to c_m, $m=0,1,\ldots,n$: $\mathfrak{A}_m\stackrel{\text{def}}{=}\{X\in\mathfrak{gl}(2n+2)\,|\,X=-RX^TR^{-1}\}$, where $R=\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$, $\alpha=\operatorname{diag}(1,-1,\ldots,1)\in\mathfrak{gl}(2n-2m+1)$, $\beta=\operatorname{diag}(-1,1,\ldots,-1)\in\mathfrak{gl}(2m+1)$; $\sigma_m(X)=QXQ^{-1}$, where $Q=\operatorname{diag}(-1,-1,\ldots,-1,1,1,\ldots,1)$. The order of σ_m is equal to two, $\mathfrak{G}=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}, a\in\mathfrak{o}(2n-2m+1), b\in\mathfrak{o}(2m+1)\right\}$. In this realization $\tilde{e}_j=e_{2n-m+2-j,2n-m+1-j}+e_{2n-m+3+j,2n-m+2+j}$, $j=0,1,\ldots,m-1$; $\tilde{e}_m=(e_{1,2n+2}+e_{2n-2m+2,2n-2m+1})\lambda$; $\tilde{e}_{m+j}=e_{j+1,j}+e_{2n-2m+2-j,2n-2m+1-j}$, $j=1,\ldots,n-m$.

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HIGHER REGULATORS AND VALUES OF L-FUNCTIONS

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In the work conjectures are formulated regarding the value of L-functions of motives and some computations are presented corroborating them.

INTRODUCTION

Let X be a complex algebraic manifold, and let $K_j(X)$, $H^j_{\mathcal{B}}(X, \mathbf{Q})$ be its algebraic K-groups and singulary cohomology, respectively. We consider the Chern character ch: $K_j(X)\otimes\mathbf{Q}\to\oplus H^{2i-j}_{\mathcal{B}}(X,\mathbf{Q})$. It is easy to see that there are the Hodge conditions on the image of ch: we have ch $(K_j(X))\subset\oplus(W_{2i}H^{2i-j}_{\mathcal{B}}(X, \mathbf{Q}))\cap(F^iH^{2i-j}_{\mathcal{B}}(X,\mathbf{C}))$, where $W_\cdot$, $F^\cdot$ are the filtration giving the mixed Hodge structure on $H^\cdot_{\mathcal{B}}(X)$. For example, if X is compact, then ch $(K_j(X)) = 0$ for $j > 0$. It turns out that the Hodge conditions can be used, and, untangling them, it is possible to obtain finer analytic invariants of the elements of $K_\cdot(X)$ than the usual cohomology classes. For the case of Chow groups they are well known: they are the Abel–Jacobi–Griffiths periods of an algebraic cycle. Apparently, these invariants are closely related to the values of L-functions; we formulate conjectures and some computations corroborating them.

In Sec. 1 our main tool appears: the groups $H^j_{\mathcal{D}}(x,\mathbf{Z}(i))$ of "topological cycles lying in the i-th term of the Hodge filtration." These groups are written in a long exact sequence

$$\dots\to H^{j-1}_{\mathcal{B}}(X,\mathbf{C})\to H^j_{\mathcal{D}}(X,\mathbf{Z}(i))\xrightarrow{\varepsilon_{\mathcal{B}}\oplus\varepsilon_{\mathcal{F}}} H^j_{\mathcal{B}}(X,\mathbf{Z})\oplus F^iH^j_{\mathcal{B}}(X,C)\to\dots.$$

On $H^\cdot_{\mathcal{D}}$ we construct a $\cup$-product such that $\varepsilon_{\mathcal{B}}$ becomes a ring morphism, and we show that $H^\cdot_{\mathcal{D}}$ form a cohomology theory satisfying Poincaré duality. Therefore, it is possible to apply the machinery of characteristic classes to $H^\cdot_{\mathcal{D}}$ [22] and obtain a morphism $\mathrm{ch}_{\mathcal{D}}: K_j(X)\otimes\mathbf{Q}\to\oplus H^{2i-j}_{\mathcal{D}}(X,\mathbf{Q}(i))$. The corresponding constructions are recalled in Sec. 2. Let $H^{2i-j}_{\mathcal{A}}(X, \mathbf{Q}(i))\subset K_j(X)\otimes\mathbf{Q}$ be the eigenspace of weight i relative to the Adams operator [2]; then $\mathrm{ch}_{\mathcal{D}}$ defines a regulator – a morphism $r_{\mathcal{D}}: H^j_{\mathcal{A}}(X,\mathbf{Q}(i))\to H^j_{\mathcal{D}}(X,\mathbf{Q}(i))$. [It is thought that for any schemes there exists a universal cohomology theory $H^j_{\mathcal{A}}(X, \mathbf{Z}(i))$, satisfying Poincaré duality and related to Quillen's K-theory in the same way as in topology the singular cohomology is related to K-theory; $H^\cdot_{\mathcal{A}}$ must be closely connected with the Milnor ring.] In the appendix we study the connection between deformations of $\mathrm{ch}_{\mathcal{D}}$ and Lie algebra cohomologies; as a consequence we see that if X is a point, then our regulators coincide with Borel regulators. There we present a formulation of a remarkable theory of Tsygan–Feigin regarding stable cohomologies of algebras of flows. Finally, Sec. 3 contains formulations of the basic conjectures connecting regulators with the values of L-functions at integral points distinct from the middle of the critical strip; the arithmetic intersection index defined in part 2.5 is responsible for the behavior in the middle of the critical strip. From these conjectures (more precisely, from the part of them that can be applied to any complex manifold) there follow rather unexpected assertions regarding the connection of Hodge structures with algebraic cycles. The remainder of the work contains computations corroborating the conjectures in Sec. 3. Thus, in Sec. 7 we prove these conjectures for the case of Dirichlet series;

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