

**CORRECTIONS AND UPDATES TO THE BOOK “VERTEX
ALGEBRAS AND ALGEBRAIC CURVES” BY EDWARD FRENKEL
AND DAVID BEN-ZVI**

To Remark 1.1.6: The maximal $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ -submodule of $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ supported on the diagonal is spanned by the series $f(w)\partial_w^i \delta(z-w)$, where $f(w) \in \mathbb{C}[[w^{\pm 1}]]$ and $i \in \mathbb{Z}_+$.

To § 3.2: The statement of Theorem 3.2.1 needs to be modified as follows. By locality, for any $C \in V$ the series $Y(A, z)Y(B, w)C$ is the expansion in $V((z))((w))$ of an element of the space $V[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$. We embed this space into $V((w))((z-w))$ by sending z to $w + (z-w)$ and z^{-1} to $(w + (z-w))^{-1}$ considered as a power series in positive powers of $(z-w)/w$. Observe also that for degree reasons for any $C \in V$ the power series $Y(Y(A, z-w)B, w)C$ in w and $z-w$ belongs to $V((w))((z-w))$.

3.2.1. Theorem. *Any vertex algebra V satisfies the following associativity property: for all $A, B, C \in V$ we have the equality in $V((w))((z-w))$,*

$$(1) \quad Y(A, z)Y(B, w)C = Y(Y(A, z-w)B, w)C,$$

where by the left hand side we understand its image in $V((w))((z-w))$ under the map described above.

When we write that $Y(A, z)Y(B, w) = Y(Y(A, z-w)B, w)$, the equality should be understood in this sense.

Proof of Theorem 3.2.1. The following is a simplified proof of Theorem 3.2.1. By Proposition 3.2.8 we have the equality in $\text{End } V[[z^{\pm 1}, w^{\pm 1}]]$,

$$Y(A, z)Y(B, w)C = e^{Tw}(e^{-Tw}Y(A, z)e^{wT})Y(C, -w)B.$$

Using Lemma 3.2.6 in the right hand side of this formula we obtain

$$(2) \quad Y(A, z)Y(B, w)C = e^{wT}Y(A, z-w)Y(C, -w)B,$$

where by $(z-w)^{-1}$ we understand its expansion in positive powers of w/z .

On the other hand, consider

$$Y(Y(A, z-w)B, w)C = \sum_{n \in \mathbb{Z}} (z-w)^{-n-1} Y(A_{(n)} \cdot B, w)C$$

as a formal power series in $(z-w)$ and w . By Proposition 3.2.8,

$$Y(A_{(n)} \cdot B, w)C = e^{wT}Y(C, -w)A_{(n)}B.$$

Hence

$$(3) \quad Y(Y(A, z-w)B, w)C = e^{wT}Y(C, -w)Y(A, z-w)B,$$

as formal power series in $(z - w)$ and w . By locality, the right hand side of (3) is the expansion in $V((w))((z - w))$ of an element of $V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$. Therefore the left hand side of (3) has the same properties.

By applying the locality axiom to the right hand sides of (2) and (3), we obtain that $Y(A, z)Y(B, w)C$ and $Y(Y(A, z - w)B, w)C$ are expansions of the same element of $V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$ in $V((z))((w))$ and $V((w))((z - w))$, respectively. Theorem 3.2.1 follows.

Remark. This proof shows that for large enough $N \in \mathbb{Z}_+$, $z^N Y(Y(A, z - w)B, w)C$ belongs to $V[[z, w]][w^{-1}, (z - w)^{-1}]$, and its image in $V[[z^{\pm 1}, w^{\pm 1}]]$, obtained by expanding $(z - w)^{-1}$ in positive powers of w/z , equals $z^N Y(A, z)Y(B, w)C$. This suggests another interpretation of formula (1): for any $A, B, C \in V$ and large enough $N \in \mathbb{Z}_+$ we have the identity

$$z^N Y(A, z)Y(B, w)C = z^N Y(Y(A, z - w)B, w)C$$

in $V[[z^{\pm 1}, w^{\pm 1}]]$.

To Definition 4.1.1: The second axiom should be modified as follows. For all $A, B \in V$ and $C \in M$, the series $Y_M(A, z)Y_M(B, w)C$ is the expansion of an element of $M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$ in $M((z))((w))$, and

$$Y_M(A, z)Y_M(B, w)C = Y_M(Y(A, z - w)B, w)C$$

in the sense of Theorem 3.2.1.

To § 15.6.6: $\mathcal{E}_{i+1}/\mathcal{E}_i \simeq \mathcal{E}_1 \otimes \Omega^{-i}$.