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VERTEX ALGEBRAS AND ALGEBRAIC CURVES

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1. INTRODUCTION

Vertex operators appeared in the early days of string theory as local operators describing propagation of string states. Mathematical analogues of these operators were discovered in representation theory of affine Kac-Moody algebras in the works of Lepowsky–Wilson [LW] and I. Frenkel–Kac [FK]. In order to formalize the emerging structure and motivated in particular by the I. Frenkel–Lepowsky–Meurman construction of the Moonshine Module of the Monster group, Borcherds gave the definition of vertex algebra in [B1]. The foundations of the theory were subsequently laid down in [FLM, FHL]; in particular, it was shown in [FLM] that the Moonshine Module indeed possessed a vertex algebra structure. In the meantime, Belavin, Polyakov and Zamolodchikov [BPZ] initiated the study of two-dimensional conformal field theory (CFT). Vertex algebras can be seen in retrospect as the mathematical equivalent of the central objects of CFT called the chiral symmetry algebras. Moreover, the key property of associativity of vertex algebras is equivalent to the property of operator product expansion in CFT, which goes back to the pioneering works of Polyakov and Wilson. Thus, vertex algebras may be thought of as the mathematical language of two-dimensional conformal field theory.

Vertex algebras have turned out to be extremely useful in many areas of mathematics. They are by now ubiquitous in representation theory of infinite-dimensional Lie algebras. They have also found applications in such fields as algebraic geometry, theory of finite groups, modular functions and topology. Recently Beilinson and Drinfeld have introduced a remarkable geometric version of vertex algebras which they called chiral algebras [BD3]. Chiral algebras give rise to some novel concepts and techniques which are likely to have a profound impact on algebraic geometry.

In this talk we review the theory of vertex algebras with a particular emphasis on their algebro-geometric interpretation and applications. We start in §2 with the axiomatic definition of vertex algebra, which is somewhat different from, but equivalent to Borcherds' original definition (see [DL, FKRW, K2]). We then discuss some of their most important properties and give the first examples, which come from infinite-dimensional Lie algebras, such as Heisenberg, affine Kac-Moody and the Virasoro algebra. More unconventional examples of vertex algebras which are not generated by a Lie algebra, such as the $\mathcal{W}$–algebras, are reviewed in §3.

In §4 we explain how to make vertex algebras coordinate independent, thus effectively getting rid of the formal variable omnipresent in the standard algebraic approach. This is achieved by attaching to each conformal vertex algebra a vector bundle with a flat connection on a formal disc, equipped with an intrinsic operation. The formal variable is restored when we choose a coordinate on the disc; the independence of the operation from the choice of coordinate follows from the fact that the group of changes of coordinates is an "internal symmetry" of the vertex algebra. Once we obtain a coordinate independent object, we can give a rigorous definition of the space of conformal blocks associated to a conformal vertex algebra and an algebraic curve X (see §5). From the physics point of view, conformal blocks give rise to "chiral correlation functions" on powers of X with singularities along the diagonals.

As we vary the curve X and other data on X reflecting the internal symmetry of a vertex algebra (such as G–bundles), the corresponding spaces of coinvariants, which are the duals of the spaces of conformal blocks, combine into a sheaf on the relevant moduli space. This sheaf carries the structure of a (twisted) $\mathcal{D}$–module, as explained in §6. One can gain new insights into the structure of moduli spaces from the study of these $\mathcal{D}$–modules. For instance, one obtains a description of the formal deformation spaces of the complex structure or a G–bundle on a curve in terms of certain sheaves on the symmetric powers of the curve [BG, BD2, Gi]. Thus, vertex algebras appear as the local objects controlling deformations of curves with various extra structures. This raises the possibility that more exotic vertex algebras, such as $\mathcal{W}$–algebras, also correspond to some still unknown moduli spaces.

Finally, in §7 we discuss the relation between vertex algebras and the Beilinson–Drinfeld chiral algebras. We review briefly the description of chiral algebras as factorization algebras, i.e., sheaves on the Ran space of finite subsets of a curve, satisfying certain compatibilities. Using this description, Beilinson and Drinfeld have introduced the concept of chiral homology, which can be thought of as a derived functor of the functor of coinvariants.

The formalism of vertex and chiral algebras appears to be particularly suitable for the construction of the conjectural geometric Langlands correspondence between automorphic $\mathcal{D}$–modules on the moduli space of G–bundles on a smooth projective curve X over $\mathbb{C}$ and flat ${}^LG$–bundles on X, where G is a simple algebraic group and ${}^LG$ is the Langlands dual group (see [BD2]). The idea is to construct the automorphic $\mathcal{D}$–modules corresponding to flat ${}^LG$–bundles as the sheaves of coinvariants (or, more generally, chiral homology) of a suitable vertex algebra. We mention below two such constructions (both due to Beilinson and Drinfeld): in one of them the relevant vertex algebra is associated to an affine Kac-Moody algebra of critical level (see §6.5), and in the other it is the chiral Hecke algebra (see §7).

Another application of vertex algebras to algebraic geometry is the recent construction by Malikov, Schechtman and Vaintrob [MSV] of a sheaf of vertex superalgebras on an arbitrary smooth algebraic variety, called the chiral deRham complex, which is reviewed in §6.6.

Vertex algebras form a vast and rapidly growing subject, and it is impossible to cover all major results (or even give a comprehensive bibliography) in one survey. For example, because of lack of space, I have not discussed such important topics as the theory of conformal algebras [K2, K3] and their chiral counterpart, Lie* algebras [BD2]; quantum deformations of vertex algebras [B3, EK, FR]; and the connection between vertex algebras and integrable systems.

Most of the material presented below (note that §§4–6 contain previously unpublished results) is developed in the forthcoming book [FB].

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2. DEFINITION AND FIRST PROPERTIES OF VERTEX ALGEBRAS

2.1. Let R be a $\mathbb{C}$–algebra. An R–valued *formal power series* (or formal distribution) in variables $z_1, z_2, \ldots, z_n$ is an arbitrary infinite series

$$A(z_1, \ldots, z_n) = \sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} a_{i_1,\ldots,i_n} z_1^{i_1} \cdots z_n^{i_n},$$

where each $a_{i_1,\ldots,i_n} \in R$. These series form a vector space denoted by $R[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$. If $P(z_1, \ldots, z_n) \in R[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ and $Q(w_1, \ldots, w_m) \in R[[w_1^{\pm 1}, \ldots, w_m^{\pm 1}]]$, then their product is a well-defined element of $R[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, w_1^{\pm 1}, \ldots, w_m^{\pm 1}]]$. In general, a product of two formal power series in the same variables does not make sense, but the product of a formal power series by a polynomial (i.e., a series, such that $a_{i_1,\ldots,i_n} = 0$ for all but finitely many indices) is always well-defined.

Let V be a $\mathbb{Z}_+$–graded vector space $V = \oplus_{n=0}^{\infty} V_n$ with finite-dimensional homogeneous components. An endomorphism T of V is called homogeneous of degree n if $T(V_m) \subset V_{n+m}$. Denote by $\operatorname{End} V$ the vector space of linear endomorphisms of V, which are finite linear combinations of homogeneous endomorphisms. This is a $\mathbb{Z}$–graded algebra.

A *field* of conformal dimension $\Delta \in \mathbb{Z}_+$ is an $\operatorname{End} V$–valued formal power series in z,

$$\phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j-\Delta}$$

where each ϕ_j is a homogeneous linear endomorphism of V of degree $-j$. Two fields $\phi(z)$ and $\psi(z)$ are called mutually *local* if there exists $N \in \mathbb{Z}_+$, such that

$$(z-w)^N [\phi(z), \psi(w)] = 0 \tag{1}$$

(as an element of $\operatorname{End} V[[z^{\pm 1}, w^{\pm 1}]]$).

Now we can formulate the axioms of vertex algebra.

DEFINITION 2.1. — *A* vertex algebra *is a collection of data:*

- space of states, *a $\mathbb{Z}_+$–graded vector space $V = \oplus_{n=0}^{\infty} V_n$, with* $\dim(V_n) < \infty$*;*
- vacuum vector $|0\rangle \in V_0$*;*
- shift operator $T : V \to V$ *of degree one;*
- vertex operation, *a linear map $Y(\cdot, z) : V \to \operatorname{End} V[[z, z^{-1}]]$ taking each $A \in V_n$ to a field of conformal dimension n.*

These data are subject to the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{Id}_V$. *Furthermore, for any $A \in V$ we have $Y(A,z)|0\rangle \in A + zV[[z]]$ (i.e., $Y(A,z)|0\rangle$ has a well–defined value at $z = 0$, which is equal to A).*
- (translation axiom) *For any $A \in V$, $[T, Y(A,z)] = \partial_z Y(A,z)$ and $T|0\rangle = 0$.*
- (locality axiom) *All fields $Y(A,z)$ are mutually local with each other.*

REMARK 2.2. — The locality axiom was first introduced in [DL], where it was shown that it can be used as a replacement for the original axioms of [B1, FLM] (see also [Li]).

It is easy to adopt the above definition to the supercase (see, e.g., [K2]). Then V is a superspace, and the above structures and axioms should be modified appropriately; in particular, we need to replace the commutator by the supercommutator in the definition of locality. Then we obtain the definition of vertex superalgebra. We mostly consider below purely even vertex superalgebras, but general vertex superalgebras are very important; for instance, $N = 2$ superconformal vertex superalgebras appear in physical models relevant to mirror symmetry.

The above conditions on V can be relaxed: it suffices to require that for any $A \in V, v \in V$, we have: $A_n \cdot v = 0$ for n large enough. It is not necessary to require that $\dim V_n < \infty$ and even that V is graded (in that case however one needs to be careful when dealing with dual spaces). We impose the above stronger conditions in order to simplify the exposition.

It is straightforward to define homomorphisms between vertex algebras, vertex subalgebras, ideals and quotients. If V_1 and V_2 are two vertex algebras, then $V_1 \otimes V_2$ carries a natural vertex algebra structure.

2.2. **Example: commutative vertex algebras.**

Let V be a $\mathbb{Z}$–graded commutative algebra (with finite-dimensional homogeneous components) with a unit and a derivation T of degree 1. Then V carries a canonical structure of vertex algebra. Namely, we take the unit of V as the vacuum vector $|0\rangle$, and define the operation Y by the formula

$$Y(A,z)=\sum_{n\geq 0}\frac{1}{n!}m(T^nA)z^n,$$

where for $B\in V$, $m(B)$ denotes the operator of multiplication by B on V. It is straightforward to check that all axioms of vertex algebra are satisfied; in fact, the locality axiom is satisfied in a strong sense: for any $A,B\in V$, we have: $[Y(A,z),Y(B,w)]=0$ (so we have $N=0$ in formula (1)).

Conversely, let V be a vertex algebra, in which locality holds in the strong sense (we call such a vertex algebra *commutative*). Then locality and vacuum axioms imply that $Y(A,z)\in \operatorname{End} V[[z]]$ for all $A\in V$ (i.e., there are no terms with negative powers of z). Denote by Y_A the endomorphism of V, which is the constant term of $Y(A,z)$, and define a bilinear operation $\circ$ on V by setting $A\circ B=Y_A\cdot B$. By construction, $Y_AY_B=Y_BY_A$ for all $A,B\in V$. This implies commutativity and associativity of $\circ$ (see, e.g., [Li]). We obtain a commutative and associative product on V, which respects the $\mathbb{Z}$–gradation. Furthermore, the vacuum vector $|0\rangle$ is a unit, and the operator T is a derivation with respect to this product. Thus, we see that the notion of commutative vertex algebra is equivalent to that of $\mathbb{Z}$–graded commutative associative algebra with a unit and a derivation of degree 1.

REMARK 2.3. — The operator T may be viewed as the generator of infinitesimal translations on the formal additive group with coordinate z. Therefore a commutative vertex algebra is the same as a commutative algebra equipped with an action of the formal additive group. Thus one may think of general vertex algebras as meromorphic generalizations of commutative algebras with an action of the formal additive group. This point of view has been developed by Borcherds [B3], who showed that vertex algebras are "singular commutative rings" in a certain category. He has also considered generalizations of vertex algebras, replacing the formal additive group by other (formal) groups or Hopf algebras.

2.3. **Non-commutative example: Heisenberg vertex algebra.**

Consider the space $\mathbb{C}((t))$ of Laurent series in one variable as a commutative Lie algebra. We define the Heisenberg Lie algebra $\mathcal{H}$ as follows. As a vector space, it is the direct sum of the space of formal Laurent power series $\mathbb{C}((t))$ and a one-dimensional space $\mathbb{C}\mathbf{1}$, with the commutation relations

$$[f(t),g(t)]=-\operatorname{Res} fdg\mathbf{1},\qquad [\mathbf{1},f(t)]=0. \tag{2}$$

Here Res denotes the (-1)st Fourier coefficient of a Laurent series. Thus, $\mathcal{H}$ is a one-dimensional central extension of the commutative Lie algebra $\mathbb{C}((t))$. Note that the relations (2) are independent of the choice of local coordinate t. Thus we may define a

Heisenberg Lie algebra canonically as a central extension of the space of functions on a punctured disc without a specific choice of formal coordinate t.

The Heisenberg Lie algebra $\mathcal{H}$ is topologically generated by the generators $b_n = t^n, n \in \mathbb{Z}$, and the central element $\mathbf{1}$, and the relations between them read

$$[b_n, b_m] = n\delta_{n,-m}\mathbf{1}, \qquad [\mathbf{1}, b_n] = 0. \tag{3}$$

The subspace $\mathbb{C}[[t]] \oplus \mathbf{1}$ is a commutative Lie subalgebra of $\mathcal{H}$. Let π be the $\mathcal{H}$–module induced from the one-dimensional representation of $\mathbb{C}[[t]]\oplus\mathbf{1}$, on which $\mathbb{C}[[t]]$ acts by 0 and $\mathbf{1}$ acts by 1. Equivalently, we may describe π as the polynomial algebra $\mathbb{C}[b_{-1}, b_{-2}, \ldots]$ with $b_n, n < 0$, acting by multiplication, b_0 acting by 0, and $b_n, n > 0$, acting as $n\dfrac{\partial}{\partial b_{-n}}$. The operators b_n with $n < 0$ are known in this context as creation operators, since they "create the state b_n from the vacuum 1", while the operators b_n with $n \geq 0$ are the annihilation operators, repeated application of which will kill any vector in π. The module π is called the Fock representation of $\mathcal{H}$.

We wish to endow π with a structure of vertex algebra. This involves the following data (Definition 2.1):

- $\mathbb{Z}_+$ grading: $\deg b_{j_1} \ldots b_{j_k} = -\sum_i j_i$.
- Vacuum vector: $|0\rangle = 1$.
- The shift operator T defined by the rules: $T \cdot 1 = 0$ and $[T, b_i] = -ib_{i-1}$.

We now need to define the fields $Y(A, z)$. To the vacuum vector $|0\rangle = 1$, we are required to assign $Y(|0\rangle, z) = \mathrm{Id}$. The key definition is that of the field $b(z) = Y(b_{-1}, z)$, which generates π in an appropriate sense. Since $\deg(b_{-1}) = 1$, $b(z)$ needs to have conformal dimension one. We set

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}.$$

The Reconstruction Theorem stated below implies that once we have defined $Y(b_{-1}, z)$, there is (at most) a unique way to extend this definition to other vectors of π. This is not very surprising, since as a commutative algebra with derivation T, π is freely generated by b_{-1}. Explicitly, the fields corresponding to other elements of π are constructed by the formula

$$Y(b_{-n_1} b_{-n_2} \ldots b_{-n_k}, z) = \frac{1}{(n_1-1)! \ldots (n_k-1)!} {:}\partial_z^{n_1-1} b(z) \ldots \partial_z^{n_k-1} b(z){:}. \tag{4}$$

The columns in the right hand side of the formula stand for the *normally ordered product*, which is defined as follows. First, let $:b_{n_1} \ldots b_{n_k}:$ be the monomial obtained from $b_{n_1} \ldots b_{n_k}$ by moving all b_{n_i} with $n_i < 0$ to the left of all b_{n_j} with $n_j \geq 0$ (in other words, moving all "creation operators" $b_n, n < 0$, to the left and all "annihilation operators" $b_n, n \leq 0$, to the right). The important fact that makes this definition correct is that the operators b_n with $n < 0$ (resp., $n \geq 0$) commute with each other, hence it does not matter how we order the creation (resp., annihilation) operators among themselves (this property does not hold for more general vertex algebras, and in those cases one needs to use a different definition of normally ordered product, which is given below). Now define $:\partial_z^{m_1} b(z) \dots \partial_z^{m_k} b(z):$ as the power series in z obtained from the ordinary product $\partial_z^{m_1} b(z) \dots \partial_z^{m_k} b(z)$ by replacing each term $b_{n_1} \dots b_{n_k}$ with $:b_{n_1} \dots b_{n_k}:$.

PROPOSITION 2.4. — *The Fock representation π with the structure given above satisfies the axioms of vertex algebra.*

The most difficult axiom to check is locality. The Reconstruction Theorem stated below allows us to reduce it to verifying locality property for the "generating" field $b(z)$. This is straightforward from the commutation relations (3):

$$\begin{aligned}
[b(z), b(w)] &= \sum_{n,m \in \mathbb{Z}} [b_n, b_m] z^{-n-1} w^{-m-1} = \sum_{n \in \mathbb{Z}} [b_n, b_{-n}] z^{-n-1} w^{n-1} \\
&= \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1} = \partial_w \delta(z-w),
\end{aligned}$$

where we use the notation $\delta(z-w) = \sum_{m \in \mathbb{Z}} w^m z^{-m-1}$.

Now $\delta(z-w)$ has the property that $(z-w)\delta(z-w) = 0$. In fact, the annihilator of the operator of multiplication by $(z-w)$ in $R[[z^{\pm 1}, w^{\pm 1}]]$ equals $R[[w^{\pm 1}]] \cdot \delta(z-w)$ (note that the product of $\delta(z-w)$ with any formal power series in z or in w is well-defined). More generally, the annihilator of the operator of multiplication by $(z-w)^N$ in $R[[z^{\pm 1}, w^{\pm 1}]]$ equals $\oplus_{n=0}^{N-1} R[[w^{\pm 1}]] \cdot \partial_w^n \delta(z-w)$ (see [K1]). This implies in particular that $(z-w)^2 [b(z), b(w)] = 0$, and hence the field $b(z)$ is local with itself. Proposition 2.4 now follows from the Reconstruction Theorem below.

2.4. **Reconstruction Theorem.**

We state a general result, which provides a "generators–and–relations" approach to the construction of vertex algebras. Let V be a $\mathbb{Z}_+$–graded vector space, $|0\rangle \in V_0$ a non-zero vector, and T a degree 1 endomorphism of V. Let S be a countable ordered set and $\{a^\alpha\}_{\alpha \in S}$ be a collection of homogeneous vectors in V, with a^α of degree Δ_α. Suppose we are also given fields

$$a^\alpha(z) = \sum_{n \in \mathbb{Z}} a_n^\alpha z^{-n-\Delta_\alpha}$$

on V, such that the following hold:

- For all α, $a^\alpha(z)|0\rangle \in a^\alpha + zV[[z]]$;
- $T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all α;
- All fields $a^\alpha(z)$ are mutually local;
- V is spanned by lexicographically ordered monomials

$$a^{\alpha_1}_{-\Delta_{\alpha_1}-j_1} \dots a^{\alpha_m}_{-\Delta_{\alpha_m}-j_m} |0\rangle$$

(including $|0\rangle$), where $j_1 \geq j_2 \geq \ldots \geq j_m \geq 0$, and if $j_i = j_{i+1}$, then $\alpha_i \geq \alpha_{i+1}$ with respect to the order on the set S.

THEOREM 2.5 ([FKRW]). — *Under the above assumptions, the above data together with the assignment*

$$Y(a^{\alpha_1}_{-\Delta_{\alpha_1}-j_1} \ldots a^{\alpha_m}_{-\Delta_{\alpha_m}-j_m}|0\rangle, z) = \frac{1}{j_1! \ldots j_m!} {:}\partial_z^{j_1} a^{\alpha_1}(z) \ldots \partial_z^{j_m} a^{\alpha_m}(z){:}, \tag{5}$$

define a vertex algebra structure on V.

Here we use the following general definition of the normally ordered product of fields. Let $\phi(z), \psi(w)$ be two fields of respective conformal dimensions Δ_ϕ, Δ_ψ and Fourier coefficients ϕ_n, ψ_n. The normally ordered product of $\phi(z)$ and $\psi(z)$ is by definition the formal power series

$${:}\phi(z)\psi(z){:} = \sum_{n \in \mathbb{Z}} \left(\sum_{m \leq -\Delta_\phi} \phi_m \psi_{n-m} + \sum_{m > -\Delta_\phi} \psi_{n-m} \phi_m \right) z^{-n-\Delta_\phi-\Delta_\psi}.$$

This is a field of conformal dimension $\Delta_\phi + \Delta_\psi$. The normal ordering of more than two fields is defined recursively from right to left, so that by definition ${:}A(z)B(z)C(z){:} = {:}A(z)({:}B(z)C(z){:}){:}$. It is easy to see that in the case of the Heisenberg vertex algebra π this definition of normal ordering coincides with the one given in §2.3.

2.5. The meaning of locality.

The product $\phi(z)\psi(w)$ of two fields is a well–defined $\operatorname{End} V$–valued formal power series in $z^{\pm 1}$ and $w^{\pm 1}$. Given $v \in V$ and $\varphi \in V^*$, consider the matrix coefficient

$$\langle \varphi, \phi(z)\psi(w)v \rangle \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]].$$

Since $V_n = 0$ for $n < 0$, we find by degree considerations that it belongs to $\mathbb{C}((z))((w))$, the space of formal Laurent series in w, whose coefficients are formal Laurent series in z. Likewise, $\langle \varphi, \psi(w)\phi(z)v \rangle$ belongs to $\mathbb{C}((w))((z))$.

As we have seen in §2.2, the condition that the fields $\phi(z)$ and $\psi(w)$ literally commute is too strong, and it essentially keeps us in the realm of commutative algebra. However, there is a natural way to relax this condition, which leads to the more general notion of locality. Let $\mathbb{C}((z, w))$ be the field of fractions of the ring $\mathbb{C}[[z, w]]$; its elements may be viewed as meromorphic functions in two formal variables. We have natural embeddings

$$\mathbb{C}((z))((w)) \longleftarrow \mathbb{C}((z, w)) \longrightarrow \mathbb{C}((w))((z)). \tag{6}$$

which are simply the inclusions of $\mathbb{C}((z, w))$ into its completions in two different topologies, corresponding to the z and w axes. For example, the images of $1/(z - w) \in \mathbb{C}((z, w))$ in $\mathbb{C}((z))((w))$ and in $\mathbb{C}((w))((z))$ are equal to

$$\delta(z - w)_- = \sum_{n \geq 0} w^n z^{-n-1}, \qquad -\delta(z - w)_+ = -\frac{1}{z} \sum_{n < 0} w^n z^{-n-1},$$

respectively.

Now we can relax the condition of commutativity of two fields by requiring that for any $v \in V$ and $\varphi \in V^*$, the matrix elements $\langle \varphi, \phi(z)\psi(w)v\rangle$ and $\langle \varphi, \psi(w)\phi(z)v\rangle$ are the images of the same element $f_{v,\varphi}$ of $\mathbb{C}((z,w))$ in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$, respectively. In the case of vertex algebras, we additionally require that

$$f_{v,\varphi} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

for all $v \in V, \varphi \in V^*$, and that the order of pole of $f_{v,\varphi}$ is universally bounded, i.e., there exists $N \in \mathbb{Z}_+$, such that $(z-w)^N f_{v,\varphi} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}]$ for all v, φ. The last condition is equivalent to the condition of locality given by formula (1).

From the analytic point of view, locality of the fields $\phi(z)$ and $\phi(w)$ means the following. Suppose that φ is an element of the restricted dual space $\oplus_{n\geq 0} V_n^*$. Then $\langle \varphi, \phi(z)\psi(w)v\rangle$ converges in the domain $|z| > |w|$ whereas $\langle \varphi, \psi(w)\phi(z)v\rangle$ converges in the domain $|w| > |z|$, and both can be analytically continued to the same meromorphic function $f_{v,\varphi}(z,w) \in \mathbb{C}[z,w][z^{-1}, w^{-1}, (z-w)^{-1}]$. Then their commutator (considered as a distribution) is the difference between boundary values of meromorphic functions, and hence is a delta-like distribution supported on the diagonal. In the simplest case, this amounts to the formula $\delta(z-w) = \delta(z-w)_- + \delta(z-w)_+$, which is a version of the Sokhotsky–Plemelj formula well-known in complex analysis.

2.6. Associativity.

Now we state the "associativity" property of vertex algebras. Consider the V–valued formal power series

$$Y(Y(A,z-w)B,w)C = \sum_{n\in\mathbb{Z}} Y(A_n \cdot B, w)C(z-w)^{-n-\Delta_A}$$

in w and $z-w$. By degree reasons, this is an element of $V((w))((z-w))$. By locality, $Y(A,z)Y(B,w)C$ is the expansion in $V((z))((w))$ of an element of $V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$, which we can map to $V((w))((z-w))$ sending z^{-1} to $(w+(z-w))^{-1}$ considered as a power series in positive powers of $(z-w)/w$.

Proposition 2.6 ([FHL, K2]). — *Any vertex algebra V satisfies the following associativity property: for any $A, B, C \in V$ we have the equality in $V((w))((z-w))$*

$$Y(A,z)Y(B,w)C = Y(Y(A,z-w)B,w)C. \tag{7}$$

One can also show that for large enough $N \in \mathbb{Z}_+$, $z^N Y(Y(A,z-w)B,w)C$ belongs to $V[[z,w]][w^{-1}, (z-w)^{-1}]$, and its image in $V[[z^{\pm 1}, w^{\pm 1}]]$, obtained by expanding $(z-w)^{-1}$ in positive powers of w/z, equals $z^N Y(A,z)Y(B,w)C$.

Formula (7) is called the operator product expansion (OPE). From the physics point of view, it manifests the important property in quantum field theory that the product of two fields at nearby points can be expanded in terms of other fields and the small parameter

$z-w$. From the analytic point of view, this formula expresses the fact that the matrix elements of the left and right hand sides of the formula, well-defined in the appropriate domains, can be analytically continued to the same rational functions in z, w, with poles only at $z=0, w=0$, and $z=w$. To simplify formulas we will often drop the vector C in formula (7).

For example, in the case of Heisenberg algebra, we obtain:

$$(8) \quad b(z)b(w) = \frac{1}{(z-w)^2} + \sum_{n \geq 0} \frac{1}{n!} : \partial_w^n b(w) b(w) : (z-w)^n = \frac{1}{(z-w)^2} + : b(z)b(w) : .$$

Using the Cauchy formula, one can easily extract the commutation relations between the Fourier coefficients of the fields $Y(A,z)$ and $Y(B,w)$ from the singular (at $z=w$) terms of their OPE. The result is

$$(9) \qquad [A_m, B_k] = \sum_{n > -\Delta_A} \begin{pmatrix} m+\Delta_A-1 \\ n+\Delta_A-1 \end{pmatrix} (A_n \cdot B)_{m+k}.$$

2.7. **Correlation functions.**

Formula (7) and the locality property have the following "multi–point" generalization. Let V^* be the space of all linear functionals on V.

PROPOSITION 2.7 ([FHL]). — *Let $A_1, \ldots, A_n \in V$. For any $v \in V, \varphi \in V^*$, and any permutation σ on n elements, the formal power series in $z_1, \ldots, z_n$,*

$$(10) \qquad \varphi \left(Y(A_{\sigma(1)}, z_{\sigma(1)}) \ldots Y(A_{\sigma(n)}, z_{\sigma(n)}) |0\rangle \right)$$

is the expansion in $\mathbb{C}((z_{\sigma(1)}))\ldots((z_{\sigma(n)}))$ of an element $f^{\sigma}_{A_1,\ldots,A_n}(z_1,\ldots,z_n)$ of $\mathbb{C}[[z_1,\ldots,z_n]][(z_i-z_j)^{-1}]_{i\neq j}$, which satisfies the following properties: it does not depend on σ (so we suppress it in the notation);

$$f_{A_1,\ldots,A_n}(z_1,\ldots,z_n) = f_{Y(A_i,z_i-z_j)A_j,A_1,\ldots,\widehat{A_i},\ldots,\widehat{A_j},\ldots,A_n}(z_j,z_1,\ldots,\widehat{z_i},\ldots,\widehat{z_j},\ldots,z_n)$$

for all $i \neq j$; and $\partial_{z_i} f_{A_1,\ldots,A_n}(z_1,\ldots,z_n) = f_{A_1,\ldots,TA_i,\ldots,A_n}(z_1,\ldots,A_n)$.

Thus, to each $\varphi \in V^*$ we can attach a collection of matrix elements (10), the "n–point functions" on $\operatorname{Spec}\mathbb{C}[[z_1,\ldots,z_n]][(z_i-z_j)^{-1}]_{i\neq j}$. They satisfy a symmetry condition, a "bootstrap" condition (which describes the behavior of the n–point function near the diagonals in terms of $(n-1)$–point functions) and a "horizontality" condition. We obtain a linear map from V^* to the vector space $\mathfrak{F}_n$ of all collections $\{f_{A_1,\ldots,A_m}(z_1,\ldots,z_m), A_i \in V\}_{m=1}^n$ satisfying the above conditions. This map is actually an isomorphism for each $n \geq 1$. The inverse map $\mathfrak{F}_n \to V^*$ takes $\{f_{A_1,\ldots,A_m}(z_1,\ldots,z_m), A_i \in V\}_{m=1}^n \in \mathfrak{F}_n$ to the functional φ on V, defined by the formula $\varphi(A) = f_A(0)$. Thus, we obtain a "functional realization" of V^*. In the case when V is generated by fields such that the singular terms in their OPEs are linear combinations of the same fields and their derivatives, we can simplify this functional realization by considering only the n–point functions of the generating fields.

For example, in the case of the Heisenberg vertex algebra we consider for each $\varphi \in \pi^*$ the n–point functions

$$\omega_n(z_1, \dots, z_n) = \varphi(b(z_1) \dots b(z_n)|0\rangle). \tag{11}$$

By Proposition 2.7 and the OPE (8), these functions are symmetric and satisfy the bootstrap condition

$$\omega_n(z_1, \dots, z_n) = \left(\frac{\omega_{r-2}(z_1, \dots, \widehat{z_i}, \dots, \widehat{z_j}, \dots, z_n)}{(z_i - z_j)^2} + \text{regular} \right). \tag{12}$$

Let Ω_∞ be the vector space of infinite collections $(\omega_n)_{n \geq 0}$, where

$$\omega_n \in \mathbb{C}[[z_1, \dots, z_n]][(z_i - z_j)^{-1}]_{i \neq j}$$

satisfy the above conditions. Using the functions (11), we obtain a map $\pi^* \to \Omega_\infty$. One can show that this map is an isomorphism.

3. MORE EXAMPLES

3.1. Affine Kac-Moody algebras.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Consider the formal loop algebra $L\mathfrak{g} = \mathfrak{g}((t))$ with the obvious commutator. The affine algebra $\widehat{\mathfrak{g}}$ is defined as a central extension of $L\mathfrak{g}$. As a vector space, $\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$, and the commutation relations read: $[K, \cdot] = 0$ and

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) + (\mathrm{Res}_{t=0}\, f dg (A, B))K \tag{13}$$

Here $(\cdot, \cdot)$ is an invariant bilinear form on $\mathfrak{g}$ (such a form is unique up to a scalar multiple; we normalize it as in [K1] by the requirement that $(\alpha_{\max}, \alpha_{\max}) = 2$).

Consider the Lie subalgebra $\mathfrak{g}[[t]]$ of $L\mathfrak{g}$. If $f, g \in \mathbb{C}[[t]]$, then $\mathrm{Res}_{t=0}\, f dg = 0$. Hence $\mathfrak{g}[[t]]$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. Let $\mathbb{C}_k$ be the one–dimensional representation of $\mathfrak{g}[[t]] \oplus \mathbb{C}K$, on which $\mathfrak{g}[[t]]$ acts by 0 and K acts by the scalar $k \in \mathbb{C}$. We define the vacuum representation of $\widehat{\mathfrak{g}}$ of level k as the representation induced from $\mathbb{C}_k$: $V_k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} \mathbb{C}_k$.

Let $\{J^a\}_{a=1,\dots \dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$. Denote $J^a_n = J^a \otimes t^n \in L\mathfrak{g}$. Then $J^a_n, n \in \mathbb{Z}$, and K form a (topological) basis for $\widehat{\mathfrak{g}}$. By the Poincaré–Birkhoff–Witt theorem, $V_k(\mathfrak{g})$ has a basis of lexicographically ordered monomials of the form $J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_k$, where v_k is the image of $1 \in \mathbb{C}_k$ in $V_k(\mathfrak{g})$, and all $n_i < 0$. We are now in the situation of the Reconstruction Theorem, and hence we obtain a vertex algebra structure on $V_k(\mathfrak{g})$, such that

$$Y(J^a_{-1} v_k, z) = J^a(z) := \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}.$$

The fields corresponding to other monomials are obtained by formula (5). Explicit computation shows that the fields $J^a(z)$ (and hence all other fields) are mutually local.

3.2. Virasoro algebra.

The Virasoro algebra Vir is a central extension of the Lie algebra $\mathrm{Der}\,\mathbb{C}((t)) = \mathbb{C}((t))\frac{\partial}{\partial t}$. It has (topological) basis elements $L_n = -t^{n+1}, n \in \mathbb{Z}$, and C satisfying the relations that C is central and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}C$$

(a coordinate independent version is given in §5.2). These relations imply that the Lie algebra $\mathrm{Der}\,\mathbb{C}[[t]] = \mathbb{C}[[t]]\frac{\partial}{\partial t}$ generated by $L_n, n \geq 0$, is a Lie subalgebra of Vir. Consider the induced representation $\mathrm{Vir}_c = \mathrm{Ind}^{Vir}_{\mathrm{Der}\,\mathbb{C}[[t]] \oplus \mathbb{C}C} \mathbb{C}_c$, where $\mathbb{C}_c$ is a one–dimensional representation, on which C acts as c and $\mathrm{Der}\,\mathbb{C}[[t]]$ acts by zero. By the Poincaré–Birkhoff–Witt theorem, Vir_c has a basis consisting of monomials of the form $L_{j_1} \dots L_{j_m} v_c, j_1 \leq j_2 \leq j_m \leq -2$, where v_c is the image of $1 \in \mathbb{C}_c$ in the induced representation. By Reconstruction Theorem, we obtain a vertex algebra structure on Vir_c, such that

$$Y(L_{-2}v_c, z) = T(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

Note that the translation operator is equal to L_{-1}.

The Lie algebra $\mathrm{Der}\,\mathcal{O}$ generates infinitesimal changes of coordinates. As we will see in §4, it is important to have this Lie algebra (and even better, the whole Virasoro algebra) act on a given vertex algebra by "internal symmetries". This property is formalized by the following definition.

DEFINITION 3.1. — *A vertex algebra V is called* conformal, *of central charge $c \in \mathbb{C}$, if V contains a non-zero* conformal vector *$\omega \in V_2$, such that the corresponding vertex operator $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies: $L_{-1} = T$, $L_0|_{V_n} = n\,\mathrm{Id}$, and $L_2\omega = \frac{1}{2}c|0\rangle$.*

These conditions imply that the operators $L_n, n \in \mathbb{Z}$, satisfy the commutation relations of the Virasoro algebra with central charge c. We also obtain a non-trivial homomorphism $\mathrm{Vir}_c \to V$, sending $L_{-2}v_c$ to ω.

For example, the vector $\frac{1}{2}b_{-1}^2 + \lambda b_{-2}$ is a conformal vector in π for any $\lambda \in \mathbb{C}$. The corresponding central charge equals $1 - 12\lambda^2$. The Kac-Moody vertex algebra $V_k(\mathfrak{g})$ is conformal if $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The conformal vector is given by the Sugawara formula $\frac{1}{2(k+h^\vee)} \sum_a (J^a_{-1})^2 v_k$, where $\{J^a\}$ is an orthonormal basis of $\mathfrak{g}$. Thus, each $\widehat{\mathfrak{g}}$–module from the category $\mathcal{O}$ of level $k \neq -h^\vee$ is automatically a module over the Virasoro algebra.

3.3. Boson–fermion correspondence.

Let $\mathcal{C}$ be the Clifford algebra associated to the vector space $\mathbb{C}((t)) \oplus \mathbb{C}((t))dt$ equipped with the inner product induced by the residue pairing. It has topological generators $\psi_n = t^n, \psi_n^* = t^{n-1}dt, n \in \mathbb{Z}$, satisfying the anti-commutation relations

$$[\psi_n, \psi_m]_+ = [\psi_n^*, \psi_m^*]_+ = 0, \qquad [\psi_n, \psi_m^*]_+ = \delta_{n,-m}. \tag{14}$$

Denote by $\bigwedge$ the fermionic Fock representation of $\mathcal{C}$, generated by vector $|0\rangle$, such that $\psi_n|0\rangle = 0, n \geq 0, \psi_n^*|0\rangle = 0, n > 0$. This is a vertex superalgebra with

$$Y(\psi_{-1}|0\rangle, z) = \psi(z) := \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}, \qquad Y(\psi_0^*|0\rangle, z) = \psi^*(z) := \sum_{n\in\mathbb{Z}} \psi_n^* z^{-n}.$$

Boson–fermion correspondence establishes an isomorphism between $\bigwedge$ and a vertex superalgebra built out of the Heisenberg algebra $\mathcal{H}$ from §2.3. For $\lambda \in \mathbb{C}$, let π_λ be the $\mathcal{H}$–module generated by a vector $\mathbf{1}_\lambda$, such that $b_n \cdot \mathbf{1}_\lambda = \lambda\delta_{n,0}\mathbf{1}_\lambda, n \geq 0$. For $N \in \mathbb{Z}_{>0}$, set $V_{\sqrt{N}\mathbb{Z}} = \oplus_{m\in\mathbb{Z}}\pi_{m\sqrt{N}}$. This is a vertex algebra (resp., superalgebra) for any even N (resp., odd N), which contains $\pi_0 = \pi$ as a vertex subalgebra. The gradation is given by the formulas $\deg b_n = -n, \deg \mathbf{1}_\lambda = \lambda^2/2$ (note that it can be half-integral). The translation operator is $T = \frac{1}{2}\sum_{n\in\mathbb{Z}} b_n b_{-1-n}$, so $T \cdot \mathbf{1}_\lambda = \lambda b_{-1}\mathbf{1}_\lambda$. In order to define the vertex operation on $V_{\sqrt{N}\mathbb{Z}}$, it suffices to define the fields $Y(\mathbf{1}_{m\sqrt{N}}, z)$. They are determined by the identity

$$\partial_z Y(\mathbf{1}_\lambda, z) = Y(T\cdot \mathbf{1}_\lambda, z) = \lambda : b(z)Y(\mathbf{1}_\lambda, z) : \quad .$$

Explicitly,

$$Y(\mathbf{1}_\lambda, z) = S_\lambda z^{\lambda b_0} \exp\left(-\lambda \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\lambda \sum_{n>0} \frac{b_n}{n} z^{-n}\right),$$

where $S_\lambda : \pi_\mu \to \pi_{\mu+\lambda}$ is the shift operator, $S_\lambda \cdot \mathbf{1}_\mu = \mathbf{1}_{\mu+\lambda}, [S_\lambda, b_n] = 0, n \neq 0$.

The *boson–fermion correspondence* is an isomorphism of vertex superalgebras $\bigwedge \simeq V_{\mathbb{Z}}$, which maps $\psi(z)$ to $Y(\mathbf{1}_{-1}, z)$ and $\psi^*(z)$ to $Y(\mathbf{1}_1, z)$. For more details, see, e.g., [K2].

3.4. **Rational vertex algebras.**

Rational vertex algebras constitute an important class of vertex algebras, which are particularly relevant to conformal field theory [dFMS]. In order to define them, we first need to give the definition of a module over a vertex algebra.

Let V be a vertex algebra. A vector space M is called a *V–module* if it is equipped with the following data:

- gradation: $M = \oplus_{n\in\mathbb{Z}_+} M_n$;
- operation $Y_M : V \to \operatorname{End} M[[z, z^{-1}]]$, which assigns to each $A \in V_n$ a field $Y_M(A, z)$ of conformal dimension n on M;

subject to the following conditions:

- $Y_M(|0\rangle, z) = \mathrm{Id}_M$;
- For all $A, B \in V$ and $C \in M$, the series $Y_M(A, z)Y_M(B, w)C$ is the expansion of an element of $M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ in $M((z))((w))$, and $Y_M(A, z)Y_M(B, w)C$ $= Y_M(Y(A, z-w)B, w)C$, in the sense of Proposition 2.6.

A conformal vertex algebra V (see Definition 3.1) is called *rational* if every V–module is completely reducible. It is shown in [DLM] that this condition implies that V has *finitely many* inequivalent simple modules, and the graded components of each simple V–module M are finite-dimensional. Furthermore, the gradation operator on M coincides with the Virasoro operator L_0^M up to a shift. Hence we can attach to a simple V–module M its *character* $\mathrm{ch}\, M = \mathrm{Tr}_M\, q^{L_0^M - c/24}$, where c is the central charge of V. Set $q = e^{2\pi i \tau}$. Zhu [Z1] has shown that if V satisfies a certain finiteness condition, then the characters give rise to holomorphic functions in τ on the upper-half plane. Moreover, he proved the following remarkable

THEOREM 3.2. — *Let V be a rational vertex algebra satisfying Zhu's finiteness condition, and $\{M_1, \ldots, M_n\}$ be the set of all inequivalent simple V–modules. Then the vector space spanned by* $\mathrm{ch}\, M_i, i = 1, \ldots, n$, *is invariant under the action of* $SL_2(\mathbb{Z})$.

This result has the following heuristic explanation. To each vertex algebra we can attach the sheaves of coinvariants on the moduli spaces of curves (see §6). It is expected that the sheaves associated to a rational vertex algebra satisfying Zhu's condition are vector bundles with projectively flat connection. The characters of simple V–modules should form the basis of the space of horizontal sections of the corresponding bundle on the moduli of elliptic curves near $\tau = \infty$. The $SL_2(\mathbb{Z})$ action is just the monodromy action on these sections.

Here are some examples of rational vertex algebras.

(1) Let L be an even positive definite lattice in a real vector space W. One can attach to it a vertex algebra V_L in the same way as in §3.3 for $L = \sqrt{N}\mathbb{Z}$. Namely, we define the Heisenberg Lie algebra associated to $W_{\mathbb{C}}$ as the Kac-Moody type central extension of the commutative Lie algebra $W_{\mathbb{C}}((t))$. Its Fock representation π_W is a vertex algebra. The vertex structure on π_W can be extended to $V_L = \pi_W \otimes \mathbb{C}[L]$, where $\mathbb{C}[L]$ is the group algebra of L [B1, FLM]. Then V_L is rational, and its inequivalent simple modules are parameterized by L'/L where L' is the dual lattice [D1]. The corresponding characters are theta-functions. The vertex algebra V_L is the chiral symmetry algebra of the free bosonic conformal field theory compactified on the torus W/L.

Note that if we take as L an arbitrary integral lattice, then V_L is a vertex superalgebra.

(2) Let $\mathfrak{g}$ be a simple Lie algebra and $k \in \mathbb{Z}_+$. Let $L_k(\mathfrak{g})$ be the irreducible quotient of the $\widehat{\mathfrak{g}}$–module $V_k(\mathfrak{g})$. This is a rational vertex algebra, whose modules are integrable representations of $\widehat{\mathfrak{g}}$ of level k [FZ]. The corresponding conformal field theory is the Wess-Zumino-Witten model (see [dFMS]).

(3) Let $L_{c(p,q)}$ be the irreducible quotient of $\mathrm{Vir}_{c(p,q)}$ (as a module over the Virasoro algebra), where $c(p,q) = 1 - 6(p-q)^2/pq,\ p, q > 1, (p,q) = 1$. This is a rational vertex algebra [Wa], whose simple modules form the "minimal model" of conformal field theory defined by Belavin, Polyakov, and Zamolodchikov [BPZ] (see also [dFMS]).

(4) The Moonshine Module vertex algebra $V^\natural$ (see below).

Zhu [Z1] has attached to each vertex algebra V an associative algebra $A(V)$, such that simple V–modules are in one-to-one correspondence with simple $A(V)$–modules.

3.5. **Orbifolds and the Monster.**

Let G be a group of automorphisms of a vertex algebra V. Then the space V^G of G–invariants in V is a vertex subalgebra of V. When G is a finite group, *orbifold theory* allows one to construct V^G–modules from twisted V–modules. For a finite order automorphism g of V, one defines a g–twisted V–module by modifying the above axioms of V–module in such a way that $Y_M(A,z)$ has monodromy λ^{-1} around the origin, if $g \cdot A = \lambda v$ (see [DLM]). Then conjecturally all V^G–modules can be obtained as V^G–submodules of g–twisted V–modules for $g \in G$.

Now suppose that V is a *holomorphic* vertex algebra, i.e., rational with a unique simple module, namely itself. Let G be a cyclic group of automorphisms of V of order n generated by g. Then the following pattern is expected to hold (see [D2]): (1) for each $h \in G$ there is a unique simple h–twisted V–module V_h; (2) V_h has a natural G–action, so we can write $V_h = \oplus_{i=0}^{n-1} V_h(i)$, where $V_h(i) = \{v \in V_h | g \cdot v = e^{2\pi i k/n} v\}$; then each $V_h(i)$ is a simple V^G–module, and these are all simple V^G–modules; (3) The vector space $\widetilde{V} = \oplus_{h \in G} V_h(0)$ carries a canonical vertex algebra structure, which is holomorphic.

A spectacular example is the construction of the *Moonshine Module* vertex algebra $V^\natural$ by I. Frenkel, Lepowsky and Meurman [FLM]. In this case V is the vertex algebra V_Λ associated to the Leech lattice Λ (it is self-dual, hence V_Λ is holomorphic), and g is constructed from the involution -1 on Λ. Then $V^\natural = V_\Lambda(0) \oplus V_{\Lambda,g}(0)$ is a vertex algebra, whose group of automorphisms is the Monster group [FLM]. Moreover, $V^\natural$ is holomorphic [D2]. Conjecturally, $V^\natural$ is the unique holomorphic vertex algebra V with central charge 24, such that $V_1 = 0$. Its character is the modular function $j(\tau) - 744$. More generally, for each element x of the Monster group, consider the Thompson series $\mathrm{Tr}_{V^\natural}\, x q^{L_0 - 1}$. The Conway–Norton conjecture, in the formulation of [FLM], states that these are Hauptmoduls for genus zero subgroups of $SL_2(\mathbb{R})$. This was first proved in [FLM] in the case when x commutes with a certain involution, and later in the general case (by a different method) by R. Borcherds [B2].

3.6. **Coset construction.**

Let V be a vertex algebra, and W a vector subspace of V. Denote by $C(V,W)$ the vector subspace of V spanned by vectors v such that $Y(A,z) \cdot v \in V[[z]]$ for all $A \in W$. Then $C(V,W)$ is a vertex subalgebra of V, which is called the *coset vertex algebra* of the pair (V,W). In the case $W = V$, the vertex algebra $C(V,V)$ is commutative, and is called the *center* of V. An example of a coset vertex algebra is provided by the Goddard–Kent–Olive construction [GKO], which identifies $C(L_1(\mathfrak{sl}_2) \otimes L_k(\mathfrak{sl}_2), L_{k+1}(\mathfrak{sl}_2))$ with $L_{c(k+2,k+3)}$ (see §3.4). For other examples, see [dFMS].

3.7. BRST construction and $\mathcal{W}$–algebras.

For $A \in V$, let $y(A) = \operatorname{Res} Y(A,z)$. Then associativity implies: $[y(A), Y(B,z)] = Y(y(A)\cdot B, z)$, and so $y(A)$ is an infinitesimal automorphism of V. Now suppose that $V^\bullet$ is a vertex (super)algebra with an additional $\mathbb{Z}$–gradation and $A \in V^1$ is such that $y(A)^2 = 0$. Then $(V^\bullet, y(A))$ is a complex, and its cohomology is a graded vertex (super)algebra. Important examples of such complexes are provided by *topological vertex superalgebras* introduced by Lian and Zuckerman [LZ]. In this case the cohomology is a graded commutative vertex algebra, but it has an additional structure of Batalin-Vilkovisky algebra.

Another example is the BRST complex of quantum hamiltonian reduction. We illustrate it in the case of quantum Drinfeld-Sokolov reduction [FF3], which leads to the definition of $\mathcal{W}$–algebras. Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ, and $\mathfrak{n}$ its upper nilpotent subalgebra.

Let $\mathcal{C}$ be the Clifford algebra associated to the vector space $\mathfrak{n}((t)) \oplus \mathfrak{n}^*((t))dt$ equipped with the inner product induced by the residue pairing. It has topological generators $\psi_{\alpha,n} = e_\alpha \otimes t^n, \psi^*_{\alpha,n} = e^*_\alpha \otimes t^{n-1}dt, \alpha \in \Delta_+, n \in \mathbb{Z}$, satisfying the anti-commutation relations (14). Let $\bigwedge_{\mathfrak{n}}$ be its Fock representation, generated by vector $|0\rangle$, such that $\psi_{\alpha,n}|0\rangle = 0, n \geq 0, \psi^*_{\alpha,n}|0\rangle = 0, n > 0$. This is a vertex superalgebra which is the tensor product of several copies of the vertex superalgebra $\bigwedge$ from §3.3. Introduce an additional $\mathbb{Z}$–gradation on $\mathcal{C}$ and $\bigwedge_{\mathfrak{n}}$ by setting $\deg \psi^*_{\alpha,n} = -\deg \psi_{\alpha,n} = 1, \deg |0\rangle = 0$.

Now consider the vertex superalgebra $C^\bullet_k(\mathfrak{g}) = V_k(\mathfrak{g}) \otimes \bigwedge^\bullet_{\mathfrak{n}}$. It carries a canonical differential d_{st} of degree 1, the standard differential of semi-infinite cohomology of $\mathfrak{n}((t))$ with coefficients in V_k [Fe1, FGZ]; it is equal to the residue of a field from $C^\bullet_k(\mathfrak{g})$ (see [FF3]). Let $\chi = \sum_{i=1}^\ell \operatorname{Res} \psi^*_{\alpha_i}(z)$ be the Drinfeld-Sokolov character [DS] of $\mathfrak{n}((t))$. Then $d = d_{\mathrm{st}} + \chi$ is a differential on $C^\bullet_k(\mathfrak{g})$, and the cohomology $H^\bullet_k(\mathfrak{g})$ of this differential is a vertex superalgebra.

THEOREM 3.3. — *$H^0_k(\mathfrak{g})$ is a vertex algebra generated by elements W_i of degrees $d_i+1, i = 1,\ldots,\ell$, where d_i is the ith exponent of $\mathfrak{g}$ (in the sense of the Reconstruction Theorem), and $H^i_k(\mathfrak{g}) = 0, i \neq 0$.*

This theorem is proved in [FF3, FF4] for generic k and in [dBT] for all k. The vertex algebra $H^0_k(\mathfrak{g})$ is called the $\mathcal{W}$*–algebra* associated to $\mathfrak{g}$ and denoted by $\mathcal{W}_k(\mathfrak{g})$. We have: $\mathcal{W}_k(\mathfrak{sl}_2) = \mathrm{Vir}_{c(k)}$, where $c(k) = 1 - 6(k+1)^2/(k+2)$. For $\mathfrak{g} = \mathfrak{sl}_3$, the $\mathcal{W}$–algebra was first constructed by Zamolodchikov, and for $\mathfrak{g} = \mathfrak{sl}_N$ by Fateev and Lukyanov [FL]. Since $V_k(\mathfrak{g})$ is conformal for $k \neq -h^\vee$ (see §3.2), $\mathcal{W}_k(\mathfrak{g})$ is also conformal, with W_1 playing the role of conformal vector. On the other hand, $\mathcal{W}_{-h^\vee}(\mathfrak{g})$ is a commutative vertex algebra, which is isomorphic to the center of $V_{-h^\vee}(\mathfrak{g})$ [FF3] (see §6.5). The simple quotient of $\mathcal{W}_k(\mathfrak{g})$ for $k = -h^\vee + p/q$, where p, q are relatively prime integers greater than or equal to $h^\vee$, is believed to be a rational vertex algebra. Moreover, if $\mathfrak{g}$ is simply-laced and $q = p+1$, this vertex algebra is conjecturally isomorphic to the coset algebra of $(L_1(\mathfrak{g}) \otimes L_{p-2}(\mathfrak{g}), L_{p-1}(\mathfrak{g}))$, see [FKW].

For any $V_k(\mathfrak{g})$–module M, the cohomology of the complex $(M \otimes \bigwedge^\bullet_{\mathfrak{n}}, d)$ is a $\mathcal{W}_k(\mathfrak{g})$–module. This defines a functor, which was studied in [FKW].

The $\mathcal{W}$–algebras exhibit a remarkable duality: $\mathcal{W}_k(\mathfrak{g}) \simeq \mathcal{W}_{k'}({}^L\mathfrak{g})$, where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra to $\mathfrak{g}$ and $(k+h^\vee)r^\vee = (k' + {}^Lh^\vee)^{-1}$ (here $r^\vee$ denotes the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$), see [FF3]. In the limit $k \to -h^\vee$ it becomes the isomorphism of Theorem 6.3.

4. COORDINATE INDEPENDENT DESCRIPTION OF VERTEX ALGEBRA STRUCTURE

Up to this point, we have discussed vertex algebras in the language of formal power series. The vertex operation is a map $Y : V \to \operatorname{End} V[[z,z^{-1}]]$, or, equivalently, an element of the completed tensor product $V^*((z)) \widehat{\otimes} \operatorname{End} V$. Thus, we can view Y as an $\operatorname{End} V$–valued section of a vector bundle over the punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((z))$ with fiber V^*. The question is whether one can define this bundle in such a way that this section is canonical, i.e., independent of the choice of z. In order to do that, we need a precise description of the transformation properties of Y. Once we understand what type of geometric object Y is, we will be able to give a global geometric meaning to vertex operators on arbitrary curves.

4.1. **The group** $\operatorname{Aut} \mathcal{O}$.

Denote by $\mathcal{O}$ the complete topological $\mathbb{C}$–algebra $\mathbb{C}[[z]]$, and let $\operatorname{Aut}\mathcal{O}$ be the group of continuous automorphisms of $\mathcal{O}$. Such an automorphism is determined by its action on the generator $z \in \mathbb{C}[[z]]$. Thus as a set this group consists of elements $\rho(z) \in \mathbb{C}[[z]]$ of the form $\rho_1 z + \rho_2 z^2 + \ldots$, with $\rho_1 \in \mathbb{C}^\times$, endowed with the composition law $(\rho * \mu)(z) = \mu(\rho(z))$. It is easy to see that $\operatorname{Aut}\mathcal{O}$ is a proalgebraic group, $\varprojlim \operatorname{Aut}\mathbb{C}[[z]]/(z^n)$. It is the semi-direct product $\mathbb{G}_m \ltimes \operatorname{Aut}_+ \mathcal{O}$, where $\operatorname{Aut}_+\mathcal{O}$ is the subgroup that consists of the transformations of the form $z \to z + a_2 z^2 + \cdots$, and $\mathbb{G}_m$ is the group of rescalings $z \to az, a \neq 0$. Furthermore, $\operatorname{Aut}_+\mathcal{O}$ is a prounipotent group. The Lie algebra of $\operatorname{Aut}\mathcal{O}$ is $\operatorname{Der}_0\mathcal{O} = z\mathbb{C}[[z]]\partial_z$, which is a semi–direct product of $\mathbb{C}\partial_z$ and $\operatorname{Der}_+\mathcal{O} = z^2\mathbb{C}[[z]]\partial_z$. Any representation of $\operatorname{Der}_0\mathcal{O}$, on which $z\partial_z$ is diagonalizable with integral eigenvalues and $\operatorname{Der}_+\mathcal{O}$ acts locally nilpotently, can be exponentiated to a representation of $\operatorname{Aut}\mathcal{O}$.

Given a vertex algebra and a field $Y(A,z)$, it makes sense to consider a new field $Y(A,\rho(z))$. We now seek an action of $\operatorname{Aut}\mathcal{O}$ on V, $A \mapsto R(\rho)\cdot A$, such that $Y(A,\rho(z))$ is related to $Y(R(\rho)A,z)$ in some reasonable way. In the theory of vertex algebras, actions of $\operatorname{Aut}\mathcal{O}$ usually arise from the action of the Virasoro algebra. Namely, let V be a conformal vertex algebra (see Definition 3.1). Then the Fourier coefficients L_n of $Y(\omega,z)$ satisfy the commutation relations of the Virasoro algebra with central charge c. The operators $L_n, n \geq 0$, then define an action of the Lie algebra $\mathrm{Der}_+ \mathcal{O}$ (L_n corresponds to $-t^{n+1}\partial_t$). It follows from the axioms of vertex algebra that this action can be exponentiated to an action of $\mathrm{Aut}\, \mathcal{O}$. For $f(z) \in \mathrm{Aut}\, \mathcal{O}$, denote by $R(f) : V \to V$ the corresponding operator.

Given a vector field $v = \sum_{n \geq -1} v_n z^{n+1} \partial_z$, we assign to it the operator $\mathbf{v} = -\sum_{n \geq -1} v_n L_n$. From formula (9) we obtain:

$$[\mathbf{v}, Y(A, w)] = -\sum_{m \geq -1} \frac{1}{(m+1)!} (\partial_w^{m+1} v(w)) Y(L_m A, w). \tag{15}$$

By "exponentiating" this formula, we obtain the following identity, due to Y.-Z. Huang [Hu]:

$$Y(A, t) = R(\rho) Y(R(\rho_t)^{-1} A, \rho(t)) R(\rho)^{-1}, \tag{16}$$

for any $\rho \in \mathrm{Aut}\, \mathbb{C}[[z]]$ (here $\rho_t(z) = \rho(t+z) - \rho(t)$, considered as formal power series in z with coefficients in $\mathbb{C}[[t]]$).

4.2. **Vertex algebra bundle.**

Now we can give a coordinate-free description of the operation Y. Let X be a smooth complex curve. Given a point $x \in X$, denote by $\mathcal{O}_x$ the completion of the local ring of x, by $\mathfrak{m}_x$ its maximal ideal, and by $\mathcal{K}_x$ the field of fractions of $\mathcal{O}_x$. A formal coordinate t_x at x is by definition a topological generator of $\mathfrak{m}_x$. Consider the set of pairs (x, t_x), where $x \in X$ and t_x is a formal coordinate at x. This is the set of points of a scheme $\widehat{X}$ of infinite type, which is a principal $\mathrm{Aut}\, \mathcal{O}$–bundle over X. Its fiber at $x \in X$ is the $\mathrm{Aut}\, \mathcal{O}$–torsor $\mathcal{L}_x$ of all formal coordinates at x. Given a finite-dimensional $\mathrm{Aut}\, \mathcal{O}$–module V, let $\mathcal{V} = \widehat{X} \underset{\mathrm{Aut}\, \mathcal{O}}{\times} V$ be the vector bundle associated to V and $\widehat{X}$. The fiber of $\mathcal{V}$ at $x \in X$ is the $\mathcal{L}_x$–twist of V, $\mathcal{V}_x = \mathcal{L}_x \underset{\mathrm{Aut}\, \mathcal{O}}{\times} V$.

A conformal vertex algebra is an $\mathrm{Aut}\, \mathcal{O}$–module, which has a filtration $V_{\leq i} := \oplus_{k=0}^{i} V_k$ by finite-dimensional $\mathrm{Aut}\, \mathcal{O}$–submodules. We obtain the directed system $(\mathcal{V}_{\leq i})$ of the corresponding vector bundles of finite rank on X and embeddings $\mathcal{V}_{\leq i} \hookrightarrow \mathcal{V}_{\leq j}, i \leq j$. We will denote this system simply by $\mathcal{V}$, thinking of it as an inductive limit of bundles of finite rank. Likewise, by $\mathcal{V}^*$ we will understand the inverse system of bundles $(\mathcal{V}_{\leq i})^*$ and surjections $(\mathcal{V}_{\leq j})^* \twoheadrightarrow (\mathcal{V}^i)^*$, thinking of it as a projective limit of bundles of finite rank.

The $\mathrm{Aut}\, \mathcal{O}$–bundle $\widehat{X}$ on X above carries an action of the Lie algebra $\mathrm{Der}\, \mathcal{O}$, which is compatible with the $\mathrm{Aut}\, \mathcal{O}$–action and simply transitive, i.e., $\mathrm{Der}\, \mathcal{O} \widehat{\otimes} \mathcal{O}_{\widehat{X}} \simeq \Theta_{\widehat{X}}$. Since V is a $\mathrm{Der}\, \mathcal{O}$–module, by the general construction of §6.1 we obtain a flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega$ on $\mathcal{V}$. Locally, ∇ can be written as $d + L_{-1} \otimes dz$, where d the deRham differential. The connection ∇ on $\mathcal{V}$ gives us a connection ∇^* on the dual bundle $\mathcal{V}^*$. To be precise, $\mathcal{V}$ is not an inductive limit of flat bundles, since $\nabla(\mathcal{V}_{\leq i}) \subset \mathcal{V}_{i+1} \otimes \Omega$, and likewise, $\mathcal{V}^*$ is not a projective limit of flat bundles (it is a $\mathcal{D}$–module on X, which is not quasicoherent as an $\mathcal{O}$–module).

Now let us restrict $\mathcal{V}^*$ to the punctured disc $D_x^\times = \operatorname{Spec} \mathcal{K}_x$ around $x \in X$. We want to define an $\operatorname{End} \mathcal{V}_x$–valued meromorphic section $\mathcal{Y}_x$ of $\mathcal{V}^*$ on $D_x^\times$. Pick a formal coordinate z at x. With this choice, identify $\mathcal{V}_x$ with V and trivialize $\mathcal{V}|_{D_x}$. We define our section $\mathcal{Y}_x$ in this trivialization through its matrix coefficients: to each triple $c \in V$, $c' \in V^*$ and a section $F : D_x \to V$ we need to assign a meromorphic function on the disc, which is $\mathbb{C}$–linear in c and c' and $\mathcal{O}$–linear in F. It suffices to assign such a function to triples c, c', F, where F is the constant section equal to $A \in V$ with respect to our trivialization. We set this function to be equal to $\langle c'|Y(A,z)|c\rangle$.

THEOREM 4.1. — *The section $\mathcal{Y}_x$ defined this way is canonical, i.e., independent of the choice of formal coordinate z at x. Moreover, $\mathcal{Y}_x$ is horizontal: $\nabla^* \mathcal{Y}_x = 0$.*

Thus, we see that a conformal vertex algebra gives rise to a vector bundle on any smooth curve and a canonical horizontal section of the restriction of this bundle to the neighborhood of each point with values in the endomorphisms of the corresponding twist of V. The fact that this section is canonical is equivalent to formula (16). Flatness follows from the formula $\partial_z Y(B,z) = Y(L_{-1}B, z)$, which holds in any vertex algebra.

REMARK 4.2. — Let $\mathcal{V}^{\boxtimes n}$ be the n–fold external tensor power of $\mathcal{V}$. Consider the n–point functions (10). According to Proposition 2.7, these are elements of $\mathbb{C}[[z_1, \dots, z_n]][(z_i - z_j)^{-1}]_{i \neq j}$. Trivializing the restriction of $\mathcal{V}^{\boxtimes n}$ to $D_x^n = \operatorname{Spec} \mathbb{C}[[z_1, \dots, z_n]]$ using the coordinate z, we obtain a $\mathcal{V}_x^*$–valued section of $(\mathcal{V}^{\boxtimes n})^*|_{D_x^n}$ with poles on the diagonals. Then this section is canonical, i.e., independent of the choice of z, and horizontal.

4.3. **Examples.**

If W is an $\operatorname{Aut} \mathcal{O}$–stable subspace of V, it gives rise to a subbundle $\mathcal{W}$ of $\mathcal{V}$. The universal section $\mathcal{Y}_x$ then gives rise to an $\operatorname{End} \mathcal{V}_x$–valued section of $\mathcal{W}^*|_{D_x^\times}$. This way one obtains many familiar geometric objects.

Suppose $A \in V_\Delta$ satisfies $L_n \cdot A = 0, L_0 A = \Delta A$. Then subspace $\mathbb{C}A \subset V$ is a one-dimensional $\operatorname{Aut} \mathcal{O}$ submodule of V. The line bundle associated to this module is nothing but the line bundle $\Omega^{-\Delta}$ of Δ–differentials on X. Thus we obtain a line subbundle $\mathcal{L}_A$ of $\mathcal{V}$. Our section $\mathcal{Y}_x$ therefore gives us an $\operatorname{End} \mathcal{V}_x$–valued section of $\mathcal{L}_A^{-1} = \Omega^\Delta$. In other words, the $\operatorname{End} \mathcal{V}_x$–valued Δ–differential $Y(A,z)(dz)^\Delta$ does not depend on the choice of formal coordinate z at x.

Next, consider the Heisenberg vertex algebra $V = \pi$, with conformal structure given by the vector $\frac{1}{2} b_{-1}^2 + b_{-2}$. For each smooth curve X, π gives rise to a vector bundle that we denote by Π. The first piece π_0 of our filtration on $\pi = \mathbb{C}[b_n]_{n<0}$ is the one-dimensional subspace spanned by the vacuum vector 1. This is a trivial representation of the group $\operatorname{Aut} \mathcal{O}$. Hence it gives rise to a trivial subbundle $\mathcal{O}_X \subset \Pi$. The next piece in the filtration,

$\pi_{\leq 1} = \mathbb{C}1 \oplus \mathbb{C}b_{-1}$, gives rise to a rank two subbundle of Π, which we denote by $\mathcal{B}$. It dual bundle is an extension

$$0 \to \Omega \to \mathcal{B}^* \to \mathcal{O}_X \to 0. \tag{17}$$

Our universal section $\mathcal{Y}_x$ gives rise to an $\operatorname{End}\Pi_x$–valued section of $\mathcal{B}^*|_{D_x^\times}$, which projects onto the section $\operatorname{End}\Pi_x$–valued section of $\mathcal{O}_X$ equal to Id (this follows from the vacuum axiom). Explicit computation shows that the space of sections of $\mathcal{B}^*$ which project onto the section 1 of $\mathcal{O}_X$ (equivalently, splittings of (17)) is canonically isomorphic to the space of *affine connections* (they may also be described as affine structures, see [Gu]). Thus, we conclude that the field $b(z)$ (or, more precisely, the expression $\partial_z + b(z)$) transforms as an $\operatorname{End}\Pi_x$–valued affine connection on $D_x^\times$. Similarly, we obtain connections on G–bundles from the subspace $V_k(\mathfrak{g})_{\leq 1}$ of the Kac-Moody vertex algebra $V_k(\mathfrak{g})$.

Let V be a conformal vertex algebra with central charge c. It follows from the definition that the vector space $\mathbb{C}|0\rangle \oplus \mathbb{C}\omega$ is $\operatorname{Aut}\mathcal{O}$–stable. Hence it gives rise to a rank two subbundle $\mathcal{T}_c$ of $\mathcal{V}$. Its dual bundle is an extension

$$0 \to \Omega^2 \to \mathcal{T}_c^* \to \mathcal{O}_X \to 0. \tag{18}$$

The universal section $\mathcal{Y}_x$ gives rise to an $\operatorname{End}\mathcal{V}_x$–valued section of $\mathcal{T}_c^*$ over the punctured disc, which projects onto $\mathrm{Id} \in \operatorname{End}\mathcal{V}_x \otimes \mathcal{O}_x$. Explicit calculation shows that the space of sections of $\mathcal{T}_c^*$ projecting onto $1 \in \Gamma(\mathcal{O}_X)$ (equivalently, splittings of (18)) is isomorphic to the space of self–adjoint second order differential operators $\rho : \Omega^{-\frac{1}{2}} \to \Omega^{\frac{3}{2}}$ with constant symbol $\dfrac{c}{6}$. Locally, such an operator can be written as $\frac{c}{6}\partial_z^2 + q(z)$. Operators of this form with symbol 1 are known as *projective connections* (they may also be described as projective structures, see [Gu]). Thus, we see that for $c \neq 0$ the field $T(z)$ (called the stress tensor in conformal field theory), or more precisely, the expression $\partial_z^2 + \frac{6}{c}T(z)$ transforms as an $\operatorname{End}\mathcal{V}_x$–valued projective connection on the punctured disc.

4.4. General twisting property.

We have seen in §4.2 that the vertex operation Y is in some sense invariant under the $\operatorname{Aut}\mathcal{O}$–action (see formula (16)). Therefore Y gives rise to a well-defined operation on the twist of V by any $\operatorname{Aut}\mathcal{O}$–torsor. This "twisting property" follows from the fact that $\operatorname{Aut}\mathcal{O}$ acts on V by "internal symmetries", that is by exponentiation of Fourier coefficients of the vertex operator $Y(\omega, z)$. It turns out that vertex algebras exhibit a similar twisting property with respect to any group $\mathcal{G}$ of internal symmetry, i.e., a group (or more generally, an ind-group) obtained by exponentiation of Fourier coefficients of vertex operators. Using associativity, one can obtain an analogue of formula (16) for the transformations of Y under the action of $\mathcal{G}$. This formula means that we get a well-defined operation on the twist of V by any $\mathcal{G}$–torsor.

For example, let V be a vertex algebra with a $\widehat{\mathfrak{g}}$*–structure* of level $k \neq -h^\vee$, i.e., one equipped with a homomorphism $V_k(\mathfrak{g}) \to V$, whose image is not contained in $\mathbb{C}|0\rangle$.

Recall that $V_k(\mathfrak{g})$ is a conformal vertex algebra if $k \neq -h^\vee$ (see §3.2). Given such a homomorphism $V_k(\mathfrak{g}) \to V$, the Fourier coefficients of the corresponding fields $Y(\omega, z)$ and $Y(J^a, z)$ generate an action of the Lie algebra $Vir \ltimes \widehat{\mathfrak{g}}$ on V. Suppose that the action of its Lie subalgebra $\mathrm{Der}_0\, \mathcal{O} \ltimes \mathfrak{g}(\mathcal{O})$ can be exponentiated to an action of the group $\mathrm{Aut}\, \mathcal{O} \ltimes G(\mathcal{O})$ on V (then we say that this group acts on V by internal symmetries). Let $\mathcal{P}$ be a G–bundle on a smooth curve X. Denote by $\widehat{\mathcal{P}}$ the principal $\mathrm{Aut}\, \mathcal{O} \ltimes G(\mathcal{O})$–bundle over X, whose fiber at x consists of pairs (z, s), where z is a formal coordinate at x and s is a trivialization of $\mathcal{P}|_{D_x}$. Let $\mathcal{V}^{\mathcal{P}}$ be the $\widehat{\mathcal{P}}$–twist of V. Then the vertex operation Y on V gives rise to a canonical section $\mathcal{Y}_x^{\mathcal{P}}$ of $(\mathcal{V}^{\mathcal{P}})^*|_{D_x^\times}$ with values in $\mathrm{End}\, \mathcal{V}_x^{\mathcal{P}}$.

5. CONFORMAL BLOCKS

5.1. The results of the previous section allow us to assign to a conformal vertex algebra a vector bundle with connection on any smooth curve X, and a family of local structures on it. We can now associate to any compact curve X an invariant of the vertex algebra structure.

DEFINITION 5.1. — *Let V be a conformal vertex algebra, X a smooth projective curve, and x a point of X. A linear functional φ on $\mathcal{V}_x$ is called a* conformal block *if $\varphi(\mathcal{Y}_x \cdot A) \in \Gamma(D_x^\times, \mathcal{V}^*)$ can be extended to a regular section of $\mathcal{V}^*$ on $X \backslash x$ for all $A \in \mathcal{V}_x$.*

The set of conformal blocks is a vector subspace of $\mathcal{V}_x^*$, denoted by $C(X, x, V)$.

Let $\varphi \in C(X, x, V)$, and $A \in \mathcal{V}_x$. Denote by φ_A the corresponding section of $\mathcal{V}^*$ over $X \backslash x$. If W is an $\mathrm{Aut}\, \mathcal{O}$–stable subspace of V, then φ_A can be projected onto a section of $\mathcal{W}^*$ over $X \backslash x$. Note that according to the vacuum axiom, $\varphi_{|0\rangle}$ is actually a regular section of $\mathcal{V}^*$ over the whole X, and so is its projection. In particular, taking as W the two-dimensional subspace of V spanned by $|0\rangle$ and ω, we assign to each conformal block a projective connection on X (if $c \neq 0$). Denote $\varphi(\cdot |0\rangle)$ as $\langle \cdot \rangle_\varphi$. If we choose a formal coordinate z at x, then we can write this projective connection as $\partial_z^2 + \frac{6}{c} \langle T(z) \rangle_\varphi$. If $c = 0$, then we obtain a quadratic differential $\langle T(z) \rangle_\varphi dz^2$.

REMARK 5.2. — Let V be a vertex algebra with $\widehat{\mathfrak{g}}$–structure of level $k \neq -h^\vee$, and $\mathcal{P}$ a G–bundle on X. A conformal block *twisted by* $\mathcal{P}$ is by definition a linear functional φ on $\mathcal{V}_x^{\mathcal{P}}$, such that $\varphi(\mathcal{Y}_x^{\mathcal{P}} \cdot A)$ (see §4.4) can be extended to a regular section of $(\mathcal{V}^{\mathcal{P}})^*$ on $X \backslash x$ for all $A \in \mathcal{V}_x^{\mathcal{P}}$. We denote the corresponding space by $C^{\mathcal{P}}(X, x, V)$.

5.2. Examples.

In the case of the Kac-Moody vertex algebra $V_k(\mathfrak{g})$ the Definition 5.1 can be simplified. In this case the subspace $(\mathfrak{g} \otimes t^{-1}) v_k$ is $\mathrm{Aut}\, \mathcal{O}$–invariant, and therefore we have a surjection $(\mathcal{V}_k(\mathfrak{g}))^* \twoheadrightarrow \mathfrak{g}^* \otimes \Omega$. Hence for each $\varphi \in C(X, x, \pi)$, $A \in \mathcal{V}_x$, we obtain a regular $\mathfrak{g}^*$–valued one-form on $X \backslash x$, whose restriction to $D_x^\times$ is $\varphi(J(z) \cdot A) dz$, where $J(z) = \sum_a J_a \otimes J^a(z)$.

Now observe that any $f = \sum f_n z^n \in \mathfrak{g} \otimes \mathcal{K}_x$ gives rise to an operator on $\mathcal{V}_k(\mathfrak{g})_x$, $\widetilde{f} = \mathrm{Res}_x(f, J(z))dz = \sum f_n b_n$, which does not depend on the choice of the coordinate z. Since by definition $\varphi(J(z) \cdot A)dz$ extends to a regular one-form on $X \backslash x$, we obtain from the residue theorem that $\varphi(\widetilde{f} \cdot A) = 0$ for all $f \in \mathfrak{g}_{x,\mathrm{out}} = \mathfrak{g} \otimes \mathbb{C}[X \backslash x]$. Hence we obtain a map from $C(X, x, V_k(\mathfrak{g}))$ to the space of $\mathfrak{g}_{x,\mathrm{out}}$–invariant functionals on $\mathcal{V}_k(\mathfrak{g})_x$.

LEMMA 5.3. — The space of conformal blocks $C(X, x, V_k(\mathfrak{g}))$ is isomorphic to the space of $\mathfrak{g}_{x,\mathrm{out}}$–invariant functionals on $\mathcal{V}_k(\mathfrak{g})_x$.

Thus we recover the common definition of conformal blocks as $\mathfrak{g}_{x,\mathrm{out}}$–invariants. They have been extensively studied recently because of the relation to the moduli spaces of G–bundles on curves (see §6.4 below). More generally, we have: $C^{\mathcal{P}}(X, x, V_k(\mathfrak{g})) = \mathrm{Hom}_{\mathfrak{g}^{\mathcal{P}}_{x,\mathrm{out}}}(\mathcal{V}_k(\mathfrak{g})^{\mathcal{P}}_x, \mathbb{C})$, where $\mathfrak{g}^{\mathcal{P}}_{x,\mathrm{out}}$ is the Lie algebra of sections of $\mathcal{P} \underset{G}{\times} \mathfrak{g}$ over $X \backslash x$.

The situation becomes more subtle in the case of the Virasoro vertex algebra Vir_c. The analogue of $\mathfrak{g}_{x,\mathrm{out}}$ is then the Lie algebra $\mathrm{Vect}(X \backslash x)$ of vector fields on $X \backslash x$. By analogy with the Kac-Moody case, we would like to define a homomorphism $\mathrm{Vect}(X \backslash x) \to \mathrm{End}\, \mathcal{V}ir_{c,x}$ sending $\xi(z)\partial_z \in \mathrm{Vect}(X \backslash x)$ to $\mathrm{Res}_x\, \xi(z) T(z) dz$. But it is not clear whether this map is independent of the choice of formal coordinate z, unless $c = 0$ (when the field $T(z)$ transforms as a quadratic differential). According to §4.3, if $c \neq 0$, then $\partial_z^2 + \frac{6}{c} T(z)$ is naturally a section of a rank two bundle $\mathcal{T}_c^*$. Hence it can be paired with sections of the rank two bundle $\mathcal{T}_c \otimes \Omega$ (we could split this bundle by choosing a projective structure on X, but we prefer not to do that). One can show that the sheaf $\widehat{\mathcal{T}}_c = (\mathcal{T}_c \otimes \Omega)/d\mathcal{O}$, which is an extension of Θ by $\Omega/d\mathcal{O}$, has a canonical Lie algebra structure (see [Wi, BS]), such that $Vir_x = \Gamma(D_x^\times, \widehat{\mathcal{T}}_c)$ is isomorphic to the Virasoro algebra and acts on $\mathcal{V}ir_{c,x}$. The map $\Gamma(X \backslash x, \widehat{\mathcal{T}}_c) \to Vir_x$ factors through $\mathrm{Vect}(X \backslash x)$, and hence the above map $\mathrm{Vect}(X \backslash x) \to \mathrm{End}\, \mathcal{V}ir_{c,x}$ is well-defined. One can then show that the space of conformal blocks $C(X, x, \mathrm{Vir}_c)$ is isomorphic to the space of $\mathrm{Vect}(X \backslash x)$–invariant functionals on $\mathcal{V}ir_{c,x}$.

5.3. General invariance condition.

For general vertex algebras the analogue of the above invariance condition is defined as follows. Let $\mathcal{V} \stackrel{\nabla}{\to} \mathcal{V} \otimes \Omega$, where $\mathcal{V} \otimes \Omega$ is placed in degree 0, be the de Rham complex of the flat vector bundle $\mathcal{V}$, considered as a complex of sheaves on the curve X. Here ∇ is the connection operator defined in §4.2 (recall that $\nabla = d + L_{-1} \otimes dz$). For $\Sigma \subset X$, denote by $H^0_{\mathrm{dR}}(\Sigma, \mathcal{V} \otimes \Omega)$ the degree 0 cohomology of the restriction of this complex to Σ. If Σ is affine or $D_x^\times$, then $H^0_{\mathrm{dR}}(\Sigma, \mathcal{V} \otimes \Omega)$ is simply the quotient of $\Gamma(\Sigma, \mathcal{V} \otimes \Omega)$ by the image of ∇.

There is a linear map $\gamma : \Gamma(D_x^\times, \mathcal{V} \otimes \Omega) \to \mathrm{End}\, \mathcal{V}_x$ sending $\mu \in \Gamma(D_x^\times, \mathcal{V} \otimes \Omega)$ to a linear operator $O_\mu = \mathrm{Res}_x \langle \mathcal{Y}_x, \mu \rangle$ on $\mathcal{V}_x$. Since the residue of a total derivative vanishes, γ factors through $U(\mathcal{V}_x) = H^0_{\mathrm{dR}}(D_x^\times, \mathcal{V} \otimes \Omega)$. If we choose a formal coordinate z at x, then we can identify $U(\mathcal{V}_x)$ with $U(V)$, the completion of the span of all Fourier coefficients of all vertex operators $Y(A,z), A \in V$. According to formula (9), $U(\mathcal{V}_x)$ is a Lie algebra. Note that in the case of Kac-Moody and Virasoro vertex algebras, $U(\mathcal{V}_x)$ contains $\widehat{\mathfrak{g}}_x$ and Vir_x, respectively, as Lie subalgebras.

Given an open $\Sigma \subset X$, we have a canonical map $H^0_{\mathrm{dR}}(\Sigma, \mathcal{V} \otimes \Omega) \to U(\mathcal{V}_x)$. Denote its image by $U_\Sigma(\mathcal{V}_x)$. It follows from the Beilinson-Drinfeld chiral algebra formalism (see §7) that $U_\Sigma(\mathcal{V}_x)$ is a Lie subalgebra of $U(\mathcal{V}_x)$. The image of $U_\Sigma(\mathcal{V}_x)$ in $\mathrm{End}\,\mathcal{V}_x$ is the span of all operators O_μ for $\mu \in \Gamma(\Sigma, \mathcal{V} \otimes \Omega)$. The residue theorem implies:

Proposition 5.4. — *The space of conformal blocks $C(X,x,V)$ coincides with the space of $U_{X\backslash x}(\mathcal{V}_x)$–invariant functionals on $\mathcal{V}_x$.*

Therefore the dual space to the space of conformal blocks is the *space of coinvariants* $H(X,x,V) = \mathcal{V}_x / U_{X\backslash x}(\mathcal{V}_x) \cdot \mathcal{V}_x$.

The definition of the spaces $C(X,x,V)$ and $H(X,x,V)$ can be generalized to the case of multiple (distinct) points $x_1, \ldots x_n$ on the curve X, at which we "insert" arbitrary conformal V–modules $M_1, \ldots, M_n$. A V–module M is called conformal if the gradation operator on M coincides, up to a shift, with the Fourier coefficient L_0^M of the field $Y_M(\omega, z)$. Assume for simplicity that the eigenvalues of L_0^M on all modules M_i are integers. Then each M_i is an $\mathrm{Aut}\,\mathcal{O}$–module, and so we can attach to it a vector bundle $\mathcal{M}_i$ on X with a flat connection (a general conformal V–module gives rise to a vector bundle with connection on the space of pairs (x, τ_x), where $x \in X$ and τ_x is a non-zero tangent vector to X at x).

The space of conformal blocks $C_V(X,(x_i),(M_i))_{i=1}^n$ is then by definition the space of linear functionals φ on $\mathcal{M}_{1,x_1} \otimes \cdots \otimes \mathcal{M}_{n,x_n}$, such that for any $A_i \in \mathcal{M}_{i,x_i}, i = 1, \ldots, n$, there exists a section of $\mathcal{V}^*$ over $X \setminus \{x_1, \ldots, x_n\}$, whose restriction to each $D_{x_i}^\times$ equals

$$\varphi(A_1 \otimes \cdots \otimes \mathcal{Y}_{M_i} \cdot A_i \otimes \cdots \otimes A_n).$$

Informally, one can say that the local sections of $\mathcal{V}^*$ over the discs around the points, obtained by acting with vertex operators at those points, can be "glued together" into a single meromorphic section of $\mathcal{V}^*$. There is a canonical isomorphism

$$C_V(X,(x_1,\ldots,x_n,y),(M_1,\ldots,M_n,V)) \simeq C_V(X,(x_1,\ldots,x_n),(M_1,\ldots,M_n)), \tag{19}$$

given by $\varphi \mapsto \varphi|_{M_1 \otimes \cdots \otimes M_N \otimes |0\rangle}$.

Remark 5.5. — Let V be a rational vertex algebra (see §3.4). Then we obtain a functor from the category of pointed algebraic curves with insertions of simple V–modules at the points to the category of vector spaces, which sends $(X,(x_i),(M_i))$ to $C_V(X,(x_i),(M_i))_{i=1}^n$. This is a version of *modular functor* corresponding to V [Se, Ga, Z2]. It is known that in the case of rational vertex algebras $L_k(\mathfrak{g})$ or $L_{c(p,q)}$ the spaces of conformal blocks are finite-dimensional. Moreover, the functor can be extended to the category of pointed stable curves with insertions. It then satisfies a factorization property, which expresses the space of conformal blocks associated to a singular curve with a double point in terms of conformal blocks associated to its normalization. The same is expected to be true for general rational vertex algebras.

5.4. **Functional realization.**

The spaces $C_V(X,(x_i),(M_i))_{i=1}^n$ with varying points $x_1,\dots,x_n$ can be organized into a vector bundle on $\overset{\circ}{X}{}^n := X^n\backslash\Delta$ (where Δ is the union of all diagonals) with a flat connection, which is a subbundle of $(\mathcal{M}_1\boxtimes\ldots\boxtimes\mathcal{M}_n)^*$. Consider for example the spaces $C_V(X,(x_i),(V))_{i=1}^n$. By (19), all of them are canonically isomorphic to each other and to $C_V(X,x,V)$. Hence the corresponding bundles of conformal blocks are canonically trivialized. For $\varphi\in C_V(X,x,V)$, let φ_n denote the corresponding element of $C_V(X,(x_i),(V))_{i=1}^n$. Let $A_i(x_i)$ be local sections of $\mathcal{V}$ near x_i. Evaluating our conformal blocks on them, we obtain what physicists call the chiral correlation functions

$$\varphi_n(A_1(x_1)\otimes\ldots\otimes A_n(x_n))\sim\langle A_1(x_1)\ldots A_n(x_n)\rangle_\varphi. \tag{20}$$

Moreover, we can explicitly describe these functions when x_i's are very close to each other, in the neighborhood of a point $x\in X$: if we choose a formal coordinate z at x and denote the coordinate of the point x_i by z_i, then

$$\varphi_n(A_1(z_1)\otimes\ldots\otimes A_n(z_n))=\varphi(Y(A_1,z_1)\ldots Y(A_n,z_n)|0\rangle). \tag{21}$$

We obtain that the restriction of the n–point chiral correlation function to $\overset{\circ}{D}{}^n_x = D^n_x\backslash\Delta := \operatorname{Spec}\mathbb{C}[[z_1,\dots,z_n]][(z_i-z_j)^{-1}]_{i\neq j}$ coincides with the n–point function corresponding to the functional φ introduced in §2.7.

According to Remark 4.2, the matrix elements given by the right hand side of formula (21) give rise to a horizontal section of $(\mathcal{V}^{\boxtimes n})^*$ on $\overset{\circ}{D}{}^n_x$. Therefore by Proposition 2.7 we obtain an embedding $\mathcal{V}^*_x\hookrightarrow\Gamma_\nabla(\overset{\circ}{D}{}^n_x,(\mathcal{V}^{\boxtimes n})^*)$, where Γ_∇ stands for the space of horizontal sections. If $\varphi\in\mathcal{V}^*_x$ is a conformal block, then Definition 5.1 implies that the image of $\varphi\in C_V(X,x,V)$ in $\Gamma_\nabla(\overset{\circ}{D}{}^n_x,(\mathcal{V}^{\boxtimes n})^*)$ extends to a horizontal section φ_n of $(\mathcal{V}^{\boxtimes n})^*$ over $\overset{\circ}{X}{}^n$. Thus, we obtain an embedding $C_V(X,x,V)\hookrightarrow\Gamma(\overset{\circ}{X}{}^n,(\mathcal{V}^{\boxtimes n})^*) = \Gamma_\nabla(X^n,j_*j^*(\mathcal{V}^{\boxtimes n})^*)$, where $j:\overset{\circ}{X}{}^n\hookrightarrow X^n$.

Imposing the bootstrap conditions on the diagonals from Proposition 2.7, we obtain a quotient $\mathcal{V}^*_n$ of $j_*j^*(\mathcal{V}^{\boxtimes n})^*$ (the precise definition of $\mathcal{V}^*_n$ uses the operation $\mathcal{Y}^{(2)}$ introduced in Theorem 7.1). Since by construction our sections satisfy the bootstrap conditions, we have embeddings $\mathcal{V}^*_x\hookrightarrow\Gamma_\nabla(D^n_x,\mathcal{V}^*_n)$ and $C_V(X,x,V)\hookrightarrow\Gamma_\nabla(X^n,\mathcal{V}^*_n)$. A remarkable fact is that these maps are isomorphisms for all $n>1$. Thus, using correlation functions we obtain *functional realizations* of the (dual space of) a vertex algebra and its spaces of conformal blocks.

REMARK 5.6. — Note that neither $(\mathcal{V}^{\boxtimes n})^*$ nor $\mathcal{V}^*_n$ is quasicoherent as an $\mathcal{O}_{X^n}$–module. It is more convenient to work with the dual sheaves $\mathcal{V}^{\boxtimes n}$ and $\mathcal{V}_n$, which are quasicoherent.

The sheaf $\mathcal{V}_n$ on X^n is defined by Beilinson and Drinfeld in their construction of factorization algebra corresponding to $\mathcal{V}$ (see §7). Note that the space $\Gamma_\nabla(X, \mathcal{V}_n^*)$ is dual to the top deRham cohomology $H^n_{\text{dR}}(X, \mathcal{V}_n \otimes \Omega_{X^n})$ (which is therefore isomorphic to $H(X, x, V)$).

In the case of Heisenberg algebra, the functional realization can be simplified, because we can use the sections of the exterior products of the canonical bundle Ω instead of the bundle Π^* (here we choose $\frac{1}{2}b_{-1}^2$ as the conformal vector, so that $b(z)dz$ transforms as a one-form). Namely, as in §2.7 we assign to $\varphi \in \Pi_x^*$ the collection of polydifferentials

$$\varphi(b(z_1)\dots b(z_n)|0\rangle)dz_1\dots dz_n.$$

These are sections of $\Omega^{\boxtimes n}(2\Delta)$ over D_x^n, which are symmetric and satisfy the bootstrap condition (12). Let Ω_∞ be the sheaf on X, whose sections over $U \subset X$ are collections of sections of $\Omega^{\boxtimes n}(2\Delta)$ over U^n, for $n \geq 0$, satisfying these conditions (the bootstrap condition makes sense globally because of the identification $\Omega^{\boxtimes 2}(2\Delta)/\Omega^{\boxtimes 2}(\Delta) \simeq \mathcal{O}$). Then we have canonical isomorphisms $\Pi_x^* \simeq \Gamma(D_x, \Omega_\infty), C(X, x, \pi) \simeq \Gamma(X, \Omega_\infty)$.

The same construction can be applied to the Kac-Moody vertex algebras, see [BD1].

Remark 5.7. — If we consider conformal blocks with general V–module insertions, then the horizontal sections of the corresponding flat bundle of conformal blocks over $\overset{\circ}{X}{}^n$ will have non-trivial monodromy around the diagonals. For example, in the case of the Kac-Moody vertex algebra $V_k(\mathfrak{g})$ and $X = \mathbb{P}^1$, these sections are solutions of the Knizhnik-Zamolodchikov equations, and the monodromy matrices are given by the R–matrices of the quantum group $U_q(\mathfrak{g})$, see [TK, SV2].

6. SHEAVES OF CONFORMAL BLOCKS ON MODULI SPACES

In the previous section we associated the space of conformal blocks to a vertex algebra V, a smooth curve X and a point x of X. Now we want to understand the behavior of these spaces as we move x along X and vary the complex structure on X. More precisely, we wish to organize the spaces of conformal blocks into a sheaf on the moduli space $\mathfrak{M}_{g,1}$ of smooth pointed curves of genus g (here and below by moduli space we mean the corresponding moduli stack in the smooth topology; $\mathcal{D}$–modules on algebraic stacks are defined in [BB, BD2]). Actually, for technical reasons we prefer to work with the spaces of coinvariants (also called covacua), which are dual to the spaces of conformal blocks. One can construct in a straightforward way a quasicoherent sheaf on $\mathfrak{M}_{g,1}$, whose fiber at (X, x) is the space of coinvariants attached to (X, x). But in fact this sheaf also carries a structure of (twisted) $\mathcal{D}$–module on $\mathfrak{M}_{g,1}$, i.e., we can canonically identify the (projectivizations of) the spaces of coinvariants attached to infinitesimally nearby points of $\mathfrak{M}_{g,1}$ (actually, this $\mathcal{D}$–module always descends to $\mathfrak{M}_g$). The key fact used in the proof of this statement is the "Virasoro uniformization" of $\mathfrak{M}_{g,1}$. Namely, the Lie algebra $\operatorname{Der}\mathcal{K}$

acts transitively on the moduli space of triples (X, x, z), where (X, x) are as above and z is a formal coordinate at x; this action is obtained by "gluing" X from D_x and $X \backslash x$ (see Theorem 6.1).

In the case when the vertex algebra V is rational, the corresponding twisted $\mathcal{D}$–module on $\mathfrak{M}_g$ is believed to be coherent, i.e., isomorphic to the sheaf of sections of a vector bundle of finite rank equipped with a projectively flat connection (this picture was first suggested by Friedan and Shenker [FrS]). This is known to be true in the case of the Kac-Moody vertex algebra $L_k(\mathfrak{g})$ [TUY] (see also [So]) and the minimal models of the Virasoro algebra [BFM]. Moreover, in those cases the $\mathcal{D}$–module on $\mathfrak{M}_g$ can be extended to a $\mathcal{D}$–module with regular singularities on the Deligne-Mumford compactification $\overline{\mathfrak{M}}_g$, and the dimensions of the fibers at the boundary are equal to those in $\mathfrak{M}_g$. This allows one to compute these dimensions by "Verlinde formula" from the dimensions of the spaces of coinvariants attached to $\mathbb{P}^1$ with three points. The latter numbers, called the fusion rules, can in turn be found from the matrix of the modular transformation $\tau \mapsto -1/\tau$ acting on the space of characters (see Theorem 3.2). The same pattern is believed to hold for other rational vertex algebras, but as far as I know this has not been proved in general.

The construction of the $\mathcal{D}$–module structure on the sheaf of coinvariants is a special case of the general formalism of localization of modules over Harish-Chandra pairs, due to Beilinson-Bernstein [BB], which we now briefly recall.

6.1. **Generalities on localization.**

A *Harish–Chandra pair* is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is a Lie algebra, K is a Lie group, such that $\mathfrak{k} = \operatorname{Lie} K$ is embedded into $\mathfrak{g}$, and an action Ad of K on $\mathfrak{g}$ compatible with the adjoint action of K on $\mathfrak{k} \subset \mathfrak{g}$ and the action of $\mathfrak{k}$ on $\mathfrak{g}$.

A $(\mathfrak{g}, K)$–*action* on a scheme Z is the data of an action of $\mathfrak{g}$ on Z (that is, a homomorphism $\rho : \mathfrak{g} \to \Theta_Z$), together with an action of K on Z, such that (1) the differential of the K–action is the restriction of the $\mathfrak{g}$–action to $\mathfrak{k}$, and (2) $\rho(Ad_k(a)) = k\rho(a)k^{-1}$. A $(\mathfrak{g}, K)$–action is called *transitive* if the map $\mathfrak{g} \otimes \mathcal{O}_Z \to \Theta_Z$ is surjective, and *simply transitive* if this map is an isomorphism.

A $(\mathfrak{g}, K)$–*structure* on a scheme S is a principal K–bundle $\pi : Z \to S$ together with a simply transitive $(\mathfrak{g}, K)$–action on Z which extends the fiberwise action of K.

An example of $(\mathfrak{g}, K)$–structure is the homogeneous space $S = G/K$, where G is a finite–dimensional Lie group, $\mathfrak{g} = \operatorname{Lie} G$, K is a Lie subgroup of G, and we take the obvious right action of $(\mathfrak{g}, K)$ on $\widehat{X} = G$. In the finite–dimensional setting, any space with a $(\mathfrak{g}, K)$–structure is locally of this form. However, in the infinite-dimensional setting this is no longer so. An example is the $(\operatorname{Der} \mathcal{O}, \operatorname{Aut} \mathcal{O})$–structure $Z = \widehat{X} \to X$ on a smooth curve X, which we used above. This example can be generalized to the case when S is an arbitrary smooth scheme. Define $\widehat{S}$ to be the scheme of pairs $(x, \vec{t}_x)$, where $x \in S$ and $\vec{t}_x$

is a system of formal coordinates at x. Set $\mathfrak{g} = \mathrm{Der}\,\mathbb{C}[[z_1, \dots, z_n]]$, $K = \mathrm{Aut}\,\mathbb{C}[[z_1, \dots, z_n]]$. Then $\widehat{S}$ is a $(\mathfrak{g}, K)$–structure on S, first considered by Gelfand, Kazhdan and Fuchs [GKF].

Let V be a $(\mathfrak{g}, K)$–module, i.e., a vector space together with actions of $\mathfrak{g}$ and K satisfying the obvious compatibility condition. Then we define a flat vector bundle $\mathcal{V}$ on any variety S with a $(\mathfrak{g}, K)$–structure Z. Namely, as a vector bundle, $\mathcal{V} = Z \underset{K}{\times} V$. Since by assumption the $\mathfrak{g}$–action on Z is simply transitive, it gives rise to a flat connection on the trivial vector bundle $Z \times V$ over Z, which descends to $\mathcal{V}$, because the actions of K and $\mathfrak{g}$ are compatible. In the special case of the $(\mathrm{Der}\,\mathcal{O}, \mathrm{Aut}\,\mathcal{O})$–structure $\widehat{X}$ over a smooth curve X we obtain the flat bundle $\mathcal{V}$ on X from §4.2.

Now consider the case when the $\mathfrak{g}$–action is transitive but not simply transitive (there are stabilizers). Then we can still construct a vector bundle $\mathcal{V}$ on S equipped with a K–equivariant action of the Lie algebroid $\mathfrak{g}_Z = \mathfrak{g} \otimes \mathcal{O}_Z$, but we cannot obtain an action of Θ_Z on $\mathcal{V}$ because the map $a : \mathfrak{g}_Z \to \Theta_Z$ is no longer an isomorphism. Nevertheless, Θ_Z will act on any sheaf, on which $\mathrm{Ker}\, a$ acts by 0. In particular, the sheaf $V \otimes \mathcal{O}_Z / \mathrm{Ker}\, a \cdot (V \otimes \mathcal{O}_Z) \simeq \mathcal{D}_Z \otimes_{U\mathfrak{g}} V$ gives rise to a K–equivariant $\mathcal{D}$–module on Z, and hence to a $\mathcal{D}$–module on S, denoted $\Delta(V)$ and called the *localization* of V on S.

More generally, suppose that V is a module over a Lie algebra $\mathfrak{l}$, which contains $\mathfrak{g}$ as a Lie subalgebra and carries a compatible K–action. Suppose also that we are given a K–equivariant Lie algebra subsheaf $\widetilde{\mathfrak{l}}$ of the constant sheaf of Lie algebras $\mathfrak{l} \otimes \mathcal{O}_Z$, which is preserved by the natural action of the Lie algebroid $\mathfrak{g}_Z$. Then if $\widetilde{\mathfrak{l}}$ contains $\mathrm{Ker}\, a$, the sheaf $V \otimes \mathcal{O}_Z / \widetilde{\mathfrak{l}} \cdot (V \otimes \mathcal{O}_Z)$ is a K–equivariant $\mathcal{D}$–module on Z, which descends to a $\mathcal{D}$–module $\widetilde{\Delta}(V)$ on S.

Let us describe the fibers of $\widetilde{\Delta}(V)$ (considered as an $\mathcal{O}_S$–module). For $s \in S$, let Z_s be the fiber of Z at s, and $\mathfrak{l}_s = Z_s \underset{K}{\times} \mathfrak{l}$, $\mathcal{V}_s = Z_s \underset{K}{\times} V$. The fibers of $\widetilde{\mathfrak{l}}$ at the points of $Z_s \subset M$ give rise to a well-defined Lie subalgebra $\widetilde{\mathfrak{l}}_s$ of $\mathfrak{l}_s$. Then the fiber of $\widetilde{\Delta}(V)$ at $s \in S$ is canonically isomorphic to the space of coinvariants $\mathcal{V}_s / \widetilde{\mathfrak{l}}_s \cdot \mathcal{V}_s$.

6.2. **Localization on the moduli space of curves.**

We apply the above formalism in the case when S is the moduli space $\mathfrak{M}_{g,1}$ of smooth pointed curves of genus $g > 1$, and Z is the moduli space $\widehat{\mathfrak{M}}_{g,1}$ of triples (X, x, z), where $(X, x) \in \mathfrak{M}_{g,1}$ and z is a formal coordinate at x. Clearly, $\widehat{\mathfrak{M}}_{g,1}$ is an $\mathrm{Aut}\,\mathcal{O}$–bundle over $\mathfrak{M}_{g,1}$.

THEOREM 6.1 (cf. [ADKP, BS, Ko, TUY]). — *$\widehat{\mathfrak{M}}_{g,1}$ carries a transitive action of* $\mathrm{Der}\,\mathcal{K}$ *compatible with the* $\mathrm{Aut}\,\mathcal{O}$*–action along the fibers.*

The action of the corresponding ind–group $\mathrm{Aut}\,\mathcal{K}$ is defined by the "gluing" construction. If (X, x, z) is an R–point of $\widehat{\mathfrak{M}}_{g,1}$, and $\rho \in \mathrm{Aut}\,\mathcal{K}(R)$, we construct a new R–point (X_ρ, x_ρ, z_ρ) of $\widehat{\mathfrak{M}}_{g,1}$ by "gluing" the formal neighborhood of x in X with $X \backslash x$ with a "twist" by ρ.

Now we are in the situation of §6.1, with $\mathfrak{g} = \operatorname{Der} \mathcal{K}$ and $K = \operatorname{Aut} \mathcal{O}$. Denote by $\mathcal{A}$ the Lie algebroid $\operatorname{Der} \mathcal{K} \widehat{\otimes} \mathcal{O}_{\widehat{\mathfrak{M}}_{g,1}}$ on $\widehat{\mathfrak{M}}_{g,1}$. By construction, the kernel of the corresponding homomorphism $a : \mathcal{A} \to \Theta$ is the subsheaf $\mathcal{A}_{\text{out}}$ of $\mathcal{A}$, whose fiber at $(X, x) \in \mathfrak{M}_{g,1}$ is $\operatorname{Vect}(X \backslash x)$. Applying the construction of §6.1 to $V = \operatorname{Vir}_0$ (the Virasoro vertex algebra with $c = 0$), we obtain a $\mathcal{D}$–module $\Delta(\operatorname{Vir}_0)$ on $\mathfrak{M}_{g,1}$, whose fibers are the spaces of coinvariants $\operatorname{Vir}_0 / \operatorname{Vect}(X \backslash x) \cdot \operatorname{Vir}_0$.

More generally, let V be an arbitrary conformal vertex algebra with central charge 0. Then it is a module over the Lie algebra $\mathfrak{l} = U(V)$ from §5.3. Let $\widetilde{\mathfrak{l}} = \mathcal{U}(V)_{\text{out}}$ be the subsheaf of $U(V) \widehat{\otimes} \mathcal{O}_{\widehat{\mathfrak{M}}_{g,1}}$, whose fiber at $(X, x, z) \in \mathfrak{M}_{g,1}$ equals $U_{X \backslash x}(V)$ (see §5.3). Note that $U(V)$ contains $\operatorname{Der} \mathcal{K}$, and $\mathcal{U}(V)_{\text{out}}$ contains $\mathcal{A}_{\text{out}}$. Moreover, we have:

LEMMA 6.2. — $\mathcal{U}(V)_{\text{out}}$ is preserved by the action of the Lie algebroid $\mathcal{A}$.

This is equivalent to the statement that $U_{X \backslash x}(V)$ and $U_{X_\rho \backslash x_\rho}(V)$ are conjugate by ρ, which follows from the fact that formula (16) is valid for any $\rho \in \operatorname{Aut} \mathcal{K}$.

So we are again in the situation of §6.1, and hence we obtain a $\mathcal{D}$–module $\widetilde{\Delta}(V)$ on $\mathfrak{M}_{g,1}$, whose fiber at (X, x) is precisely the space of coinvariants $H(X, x, V) = \mathcal{V}_x / U_{X \backslash x}(\mathcal{V}_x) \cdot \mathcal{V}_x$ (see §5.3), which is what we wanted.

In the case of vertex algebras with non-zero central charge, we need to modify the general construction of §6.1 as follows. Suppose that $\mathfrak{g}$ has a central extension $\widehat{\mathfrak{g}}$ which splits over $\mathfrak{k}$, and such that the extension

$$0 \to \mathcal{O}_Z \to \widehat{\mathfrak{g}} \otimes \mathcal{O}_Z \to \mathfrak{g}_Z \to 0 \tag{22}$$

splits over the kernel of $a : \mathfrak{g}_Z \to \Theta_Z$. Then the quotient $\mathfrak{g}'_Z$ of $\widehat{\mathfrak{g}} \otimes \mathcal{O}_Z$ by $\operatorname{Ker} a$ is an extension of Θ_Z by $\mathcal{O}_Z$, with a natural Lie algebroid structure. The enveloping algebra of this Lie algebroid is a sheaf $\mathcal{D}'_Z$ of twisted differential operators (TDO) on Z, see [BB]. Further, $\mathfrak{g}'_Z$ descends to a Lie algebroid on S, and gives rise to a sheaf of TDO $\mathcal{D}'_S$.

Now if V is a $(\widehat{\mathfrak{g}}, K)$–module, we obtain a $\mathcal{D}'_Z$–module $\mathcal{D}'_Z \otimes_{U\widehat{\mathfrak{g}}} V$. By construction, it is K–equivariant, and hence descends to a $\mathcal{D}'_S$–module on S. More generally, suppose that $\widehat{\mathfrak{g}}$ is a Lie subalgebra of $\widehat{\mathfrak{l}}$, and we are given a subsheaf $\widetilde{\mathfrak{l}}$ of $\widehat{\mathfrak{l}} \otimes \mathcal{O}_Z$, containing $\operatorname{Ker} a$ and preserved by the action of $\widehat{\mathfrak{g}} \otimes \mathcal{O}_Z$. Then the sheaf $V \otimes \mathcal{O}_Z / \widetilde{\mathfrak{l}} \cdot V \otimes \mathcal{O}_Z$ is a K–equivariant $\mathcal{D}'_Z$–module on Z, which descends to a $\mathcal{D}'_S$–module (still denoted by $\widetilde{\Delta}(V)$). Its fibers are the coinvariants $\mathcal{V}_s / \widetilde{\mathfrak{l}}_s \cdot \mathcal{V}_s$.

Let V be a conformal vertex algebra with an arbitrary central charge c. Then by the residue theorem the corresponding sequence (22) with $\widehat{\mathfrak{g}} = Vir$ splits over $\operatorname{Ker} a = \mathcal{A}_{\text{out}}$. Hence there exists a TDO sheaf $\mathcal{D}_c$ on $\mathfrak{M}_{g,1}$ and a $\mathcal{D}_c$–module $\widetilde{\Delta}(V)$ on $\mathfrak{M}_{g,1}$ whose fiber at (X, x) is the space of coinvariants $H(X, x, V)$. Actually, $\widetilde{\Delta}(V)$ descends to $\mathfrak{M}_g$. Explicit computation shows that the sheaf of differential operators on the determinant line bundle over $\mathfrak{M}_g$ corresponding to the sheaf of relative λ–differentials on the universal curve is $\mathcal{D}_{c(\lambda)}$, where $c(\lambda) = -12\lambda^2 + 12\lambda - 2$ (see [BS, BFM]).

It is straightforward to generalize the above construction to the case of multiple points with arbitrary V–module insertions $M_1, \ldots, M_n$. In that case we obtain a twisted $\mathcal{D}$–module on the moduli space $\mathfrak{M}'_{g,n}$ of n–pointed curves with non-zero tangent vectors at the points.

6.3. **Localization on other moduli spaces.**

In the previous section we constructed twisted $\mathcal{D}$–modules on the moduli spaces of pointed curves using the interior action of the Harish-Chandra pair $(Vir, \operatorname{Aut}\mathcal{O})$ on conformal vertex algebras and its modules. This construction can be generalized to the case of other moduli spaces if we consider interior actions of other Harish–Chandra pairs.

For example, consider the Kac-Moody vertex algebra $V_k(\mathfrak{g})$. It carries an action of the Harish-Chandra pair $(\widehat{\mathfrak{g}}, G(\mathcal{O}))$, where G is the simply-connected group with Lie algebra $\mathfrak{g}$. The corresponding moduli space is the moduli space $\mathfrak{M}_G(X)$ of G–bundles on a curve X. For $x \in X$, let $\widehat{\mathfrak{M}}_G(X)$ be the moduli spaces of pairs $(\mathcal{P}, s)$, where $\mathcal{P}$ is a G–bundle on X, and s is a trivialization of $\mathcal{P}|_{D_x}$. Then $\widehat{\mathfrak{M}}_G(X)$ is a principal $G(\mathcal{O}_x)$–bundle over $\mathfrak{M}_G(X)$, equipped with a compatible $\mathfrak{g}(\mathcal{K}_x)$–action. The latter is given by the gluing construction similar to that explained in the proof of Theorem 6.1.

Now we can apply the construction of §6.1 to the $(\widehat{\mathfrak{g}}_x, G(\mathcal{O}_x))$–module $\mathcal{V}_k(\mathfrak{g})_x$, and obtain a twisted $\mathcal{D}$–module on $\mathfrak{M}_G(X)$. Its fiber at $\mathcal{P} \in \mathfrak{M}_G(X)$ is the space of coinvariants $\mathcal{V}_k(\mathfrak{g})_x/\mathfrak{g}^{\mathcal{P}}_{x,\mathrm{out}} \cdot \mathcal{V}_k(\mathfrak{g})_x$, where $\mathfrak{g}^{\mathcal{P}}_{x,\mathrm{out}}$ is the Lie algebra of sections of $\mathcal{P} \underset{G}{\times} \mathfrak{g}$ over $X \backslash x$.

More generally, let V be a vertex algebra equipped with a $\widehat{\mathfrak{g}}$–structure of level $k \neq -h^\vee$ (see §4.4). Then V carries an action of the Harish-Chandra pair $(\widehat{\mathfrak{g}}_x, G(\mathcal{O}_x))$. In the same way as in the case of the Virasoro algebra, we attach to V a twisted $\mathcal{D}$–module on $\mathfrak{M}_G(X)$, whose fiber at $\mathcal{P}$ is the twisted space of coinvariants $H^{\mathcal{P}}(X, x, V)$, dual to the space $C^{\mathcal{P}}(X, x, V)$ defined in Remark 5.2.

For the Heisenberg vertex algebra π, the corresponding moduli space is the Jacobian $J(X)$ of X. The bundle $\widehat{J}(X)$ over $J(X)$, which parameterizes line bundles together with trivializations on D_x, carries a transitive action of the abelian Lie algebra $\mathcal{K}$. Applying the above construction, we assign to any conformal vertex algebra equipped with an embedding $\pi \to V$ a twisted $\mathcal{D}$–module on $J(X)$, whose fiber at $\mathcal{L} \in J(X)$ is the twisted space of coinvariants $H^{\mathcal{L}}(X, x, V)$. The corresponding TDO sheaf is the sheaf of differential operators acting on the theta line bundle on $J(X)$.

By considering the action of the semi-direct product $Vir \ltimes \widehat{\mathfrak{g}}$ on V we obtain a twisted $\mathcal{D}$-module on the moduli space of curves and G–bundles on them. We can further generalize the construction by inserting modules at points of the curve, etc.

6.4. **Local and global structure of moduli spaces.**

In the simplest cases the twisted $\mathcal{D}$–modules obtained by the above localization construction are the twisted sheaves $\mathcal{D}$ themselves (considered as left modules over themselves). For example, the sheaf $\Delta(V_k(\mathfrak{g}))$ on $\mathfrak{M}_G(X)$ is the sheaf $\mathcal{D}_k$ of differential operators acting on the kth power of the determinant line bundle $\mathcal{L}_G$ on $\mathfrak{M}_G(X)$ (the ample generator of the Picard group of $\mathfrak{M}_G(X)$). The dual space to the stalk of $\mathcal{D}_k$ at $\mathcal{P} \in \mathfrak{M}_G(X)$ is canonically identified with the space of sections of $\mathcal{L}_G^{\otimes k}$ on the formal neighborhood of $\mathcal{P}$ in $\mathfrak{M}_G(X)$. But according to our construction, this space is also isomorphic to the space of conformal blocks $C^{\mathcal{P}}(X, x, V_k(\mathfrak{g}))$. In particular, the coordinate ring of the formal deformation space of a G–bundle $\mathcal{P}$ on a curve X is isomorphic to $C^{\mathcal{P}}(X, x, V_0(\mathfrak{g}))$. Thus, using the description of conformal blocks in terms of correlation functions (see §5.4), we obtain a realization of the formal deformation space and a line bundle on it in terms of polydifferentials on powers of the curve X satisfying bootstrap conditions on the diagonals (see [BG, BD2, Gi]). On the other hand, if we replace X by D_x, the corresponding space of polydifferentials becomes $V_k(\mathfrak{g})^*$ (more precisely, its twist by the torsor of formal coordinates at x). Therefore the vertex algebra $V_k(\mathfrak{g})$ may be viewed as the local object responsible for deformations of G–bundles on curves.

Similarly, the Virasoro vertex algebra Vir_c and the Heisenberg vertex algebra π may be viewed as the local objects responsible for deformations of curves and line bundles on curves, respectively.

Optimistically, one may hope that any one–parameter family of "Verma module type" vertex algebras (such as Vir_c or $V_k(\mathfrak{g})$) is related to a moduli space of curves with some additional structures. It would be interesting to identify the deformation problems related to the most interesting examples of such vertex algebras (for instance, the $\mathcal{W}$–algebras of §3.7), and to construct the corresponding global moduli spaces. One can attach a "formal moduli space" to any vertex algebra V by taking the double quotient of the ind–group, whose Lie algebra is $U(V)$, by its "in" and "out" subgroups. But this moduli space is very big, even in the familiar examples of Virasoro and Kac-Moody algebras. The question is to construct a smaller formal moduli space with a line bundle, whose space of sections is isomorphic to the space of conformal blocks $C(X, x, V)$.

While "Verma module type" vertex algebras are related to the local structure of moduli spaces, rational vertex algebras (see §3.4) can be used to describe the global structure.

For instance, the space of conformal blocks $C(X, x, L_k(\mathfrak{g})) \simeq \mathrm{Hom}_{\mathfrak{g}_{\mathrm{out}}}(L_k(\mathfrak{g}), \mathbb{C})$, is isomorphic to $\Gamma_k = \Gamma(\mathfrak{M}_G(X), \mathcal{L}_G^{\otimes k})$, see [BL, Fa, KNR]. The space $C(X, x, L_k(\mathfrak{g}))$ satisfies a factorization property: its dimension does not change under degenerations of the curve into curves with nodal singularities. This property allows one to find this dimension by computing the spaces of conformal blocks in the case of $\mathbb{P}^1$ and three points with $L_k(\mathfrak{g})$–module insertions (fusion rules). The resulting formula for $\dim C(X, x, L_k) = \dim \Gamma_k$ is called the Verlinde formula, see [TUY, So].

Since $\mathcal{L}_G$ is ample, we can recover the moduli space of semi-stable G–bundles on X as the Proj of the graded ring $\oplus_{k\geq 0}\Gamma_{Nk} = \oplus_{k\geq 0}C(X,x,L_{Nk}(\mathfrak{g}))$ for large N (the ring of "non-abelian theta functions"). Thus, if we could define the product on conformal blocks in a natural way, we would obtain a description of the moduli space. Using the correlation function description of conformal blocks Feigin and Stoyanovsky [FS] have identified the space $C(X,x,L_k(\mathfrak{g}))$ with the space of sections of a line bundle on the power of X satisfying certain conditions. For example, when $\mathfrak{g} = \mathfrak{sl}_2$ it is the space of sections of the line bundle $\Omega(2nx)^{\boxtimes nk}$ over X^{nk}, which are symmetric and vanish on the diagonals of codimension k (here $x \in X$, and $n \geq g$). In these terms the multiplication $\Gamma_k \times \Gamma_l \rightarrow \Gamma_{k+l}$ is just the composition of exterior tensor product and symmetrization of these sections. Thus we obtain an explicit description of the coordinate ring of the moduli space of semi-stable SL_2–bundles. On the other hand, replacing in the above description X by D_x, we obtain a functional realization of $L_k(\mathfrak{sl}_2)^*$.

Similarly, the space of conformal blocks corresponding to the lattice vertex superalgebra $V_{\sqrt{N}\mathbb{Z}}$ (see §3.3) may be identified with the space of theta functions of order N on $J(X)$, so we obtain the standard "functional realization" of the Jacobian $J(X)$.

6.5. **Critical level.**

When $k = -h^\vee$ (which is called the critical level), the vertex algebra $V_k(\mathfrak{g})$ is not conformal. Nevertheless, $V_{-h^\vee}(\mathfrak{g})$ still carries an action of $\operatorname{Der}\mathcal{O}$ which satisfies the same properties as for non-critical levels. Therefore we can still attach to $V_{-h^\vee}$ a twisted $\mathcal{D}$–module on $\mathfrak{M}_G(X)$ (but we cannot vary the curve). This $\mathcal{D}$–module is just the sheaf $\mathcal{D}'$ of differential operators acting on the square root of the canonical bundle on $\mathfrak{M}_G(X)$ (see [BD2]).

Recall from §3.6 that the center of a vertex algebra V is the coset vertex algebra of the pair (V,V). This is a commutative vertex subalgebra of V. The center of $V_k(\mathfrak{g})$ is simply its subspace of $\mathfrak{g}[[z]]$–invariants. It is easy to show that this subspace equals $\mathbb{C}v_k$ if $k \neq -h^\vee$. In order to describe the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $V_{-h^\vee}(\mathfrak{g})$ we need the concept of *opers*, due to Beilinson and Drinfeld [BD2].

Let G_{ad} be the adjoint group of $\mathfrak{g}$, and B_{ad} be its Borel subgroup. By definition, a $\mathfrak{g}$–oper on a smooth curve X is a G_{ad}–bundle on X with a (flat) connection ∇ and a reduction to B_{ad}, satisfying a certain transversality condition [BD2]. For example, an $\mathfrak{sl}_2$–oper is the same as a projective connection. The set of $\mathfrak{g}$–opers on X is the set of points of an affine space, denoted $\operatorname{Op}_{\mathfrak{g}}(X)$, which is a torsor over $\oplus_{i=1}^{\ell}\Gamma(X,\Omega^{d_i+1})$, where d_i's are the exponents of $\mathfrak{g}$. Denote by $A_{\mathfrak{g}}(X)$ the ring of functions on $\operatorname{Op}_{\mathfrak{g}}(X)$.

The above definition can also be applied when X is replaced by the disc $D = \operatorname{Spec}\mathbb{C}[[z]]$ (see [DS]). The corresponding ring $A_{\mathfrak{g}}(D)$ has a natural $(\operatorname{Der}\mathcal{O},\operatorname{Aut}\mathcal{O})$–action, and is therefore a commutative vertex algebra (see §2.2). It is isomorphic to a limit of the $\mathcal{W}$–vertex algebra $\mathcal{W}_k(\mathfrak{g})$ as $k \rightarrow \infty$ (see §3.7), and because of that it is called the classical

$\mathcal{W}$–algebra corresponding to $\mathfrak{g}$. For example, $A_{\mathfrak{sl}_2}(D)$ is a limit of Vir_c as $c \to \infty$. The following result was conjectured by Drinfeld and proved in [FF3].

THEOREM 6.3. — *The center* $\mathfrak{z}(\widehat{\mathfrak{g}})$ *of* $V_{-h^\vee}(\mathfrak{g})$ *is isomorphic, as a commutative vertex algebra with* $\mathrm{Der}\,\mathcal{O}$*–action, to* $A_{^L\mathfrak{g}}(D)$*, where* $^L\mathfrak{g}$ *is the Langlands dual Lie algebra to* $\mathfrak{g}$.

In fact, more is true: both of these commutative vertex algebras carry what can be called Poisson vertex algebra (or coisson algebra, in the terminology of [BD3]) structures, which are preserved by this isomorphism.

The localization of $\mathfrak{z}(\widehat{\mathfrak{g}})$ on $\mathfrak{M}_G(X)$ is the $\mathcal{O}$–module $H(X,x,\mathfrak{z}(\widehat{\mathfrak{g}}))\otimes\mathcal{O}_{\mathfrak{M}_G(X)}$ (since $\widehat{\mathfrak{g}}$ does not preserve $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset V_{-h^\vee}(\mathfrak{g})$, this sheaf does not carry a $\mathcal{D}$–module structure). By functoriality, the embedding $\mathfrak{z}(\widehat{\mathfrak{g}}) \to V_{-h^\vee}(\mathfrak{g})$ of vertex algebras gives rise to a homomorphism of $\mathcal{O}$–modules $H(X,x,\mathfrak{z}(\widehat{\mathfrak{g}})) \otimes \mathcal{O}_{\mathfrak{M}_G(X)} \to \mathcal{D}'$. Theorem 6.3 implies that $H(X,x,\mathfrak{z}(\widehat{\mathfrak{g}})) \simeq H(X,x,A_{^L\mathfrak{g}}(D))$. As shown in [BD3], the space of coinvariants of a commutative vertex algebra A is canonically isomorphic to the ring of functions on $\Gamma_\nabla(X,\mathrm{Spec}\,\mathcal{A})$. In our case we obtain: $\Gamma_\nabla(X,\mathrm{Spec}\,\mathcal{A}_{^L\mathfrak{g}}) \simeq \mathrm{Op}_{^L\mathfrak{g}}(X)$, and so $H(X,x,A_{^L\mathfrak{g}}(D)) \simeq A_{^L\mathfrak{g}}(X)$. Passing to global sections we obtain:

COROLLARY 6.4. — *There is a homomorphism of algebras* $A_{^L\mathfrak{g}}(X) \to \Gamma(\mathfrak{M}_G(X),\mathcal{D}')$.

Beilinson and Drinfeld have shown in [BD2] that this map is an embedding, and if G is simply-connected, it is an isomorphism. In that case, each ^LG–oper ρ gives us a point of the spectrum of the commutative algebra $\Gamma(\mathfrak{M}_G(X),\mathcal{D}')$, and hence a homomorphism $\widetilde{\rho} : \Gamma(\mathfrak{M}_G(X),\mathcal{D}') \to \mathbb{C}$. The $\mathcal{D}'$–module $\mathcal{D}'/(\mathcal{D}'\cdot m_{\widetilde{\rho}})$ on $\mathfrak{M}_G(X)$, where $m_{\widetilde{\rho}}$ is the kernel of $\widetilde{\rho}$, is holonomic. It is shown in [BD2] that the corresponding untwisted $\mathcal{D}$–module is a Hecke eigensheaf attached to ρ by the geometric Langlands correspondence.

6.6. **Chiral deRham complex.**

Another application of the general localization pattern of §6.1 is the recent construction, due to Malikov, Schechtman and Vaintrob [MSV], of a sheaf of vertex superalgebras on any smooth scheme S called the chiral deRham complex. Under certain conditions, this sheaf has a purely even counterpart, the sheaf of chiral differential operators. Here we give a brief review of the construction of these sheaves.

Let Γ_N be the Weyl algebra associated to the symplectic vector space $\mathbb{C}((t))^N \oplus (\mathbb{C}((t))dt)^N$. It has topological generators $a_{i,n}, a^*_{i,n}, i = 1,\ldots,N; n \in \mathbb{Z}$, with relations

$$[a_{i,n}, a^*_{j,m}] = \delta_{i,j}\delta_{n,-m}, \qquad [a_{i,n}, a_{j,m}] = [a^*_{i,n}, a^*_{j,m}] = 0.$$

Denote by H_N the Fock representation of Γ_N generated by the vector $|0\rangle$, satisfying $a_{i,n}|0\rangle = 0, n \geq 0; a^*_{i,m}|0\rangle = 0, m > 0$. This is a vertex algebra, such that

$$Y(a_{i,-1}|0\rangle, z) = a(z) = \sum_{n\in\mathbb{Z}} a_{i,n} z^{-n-1}, \qquad Y(a^*_{i,0}|0\rangle, z) = a^*(z) = \sum_{n\in\mathbb{Z}} a^*_{i,n} z^{-n}.$$

Let $\bigwedge_N^\bullet$ be the fermionic analogue of H_N defined in the same way as in §3.7. This is the Fock module over the Clifford algebra with generators $\psi_{i,n}, \psi^*_{i,n}, i = 1, \ldots, N, n \in \mathbb{Z}$, equipped with an additional $\mathbb{Z}$–gradation. It is shown in [MSV] that the vertex algebra structure on H_N may be extended to $\widehat{H}_N = H_N \underset{\mathbb{C}[a^*_{i,0}]}{\otimes} \mathbb{C}[[a^*_{i,0}]]$. Let $\widehat{\Omega}^\bullet_N = \widehat{H}_N \otimes \bigwedge^\bullet_N$. It carries a differential $d = \operatorname{Res} \sum_i a_i(z)\psi^*_i(z)$, which makes it into a complex.

Denote by W_N (resp., $\mathcal{O}_N$, Ω^1_N) the topological Lie algebra of vector fields (resp., ring of functions, module of differentials) on $\operatorname{Spec} \mathbb{C}[[t_1, \ldots, t_N]]$. Define a map $W_N \to U(\widehat{\Omega}^\bullet_N)$ (the completion of the Lie algebra of Fourier coefficients of fields from $\widehat{\Omega}^\bullet_N$) sending

$$f(t_i)\partial_{t_j} \mapsto \operatorname{Res}\left(: f(a^*_i(z))a_j(z) : + \sum_{k=1}^N : (\partial_{t_k} f)(a^*_i(z))\psi^*_k(z)\psi_j(z) : \right).$$

Explicit computation shows that it is a homomorphism of Lie algebras (this is an example of "supersymmetric cancellation of anomalies") [FF2, MSV]. We obtain a W_N–action on $\widehat{\Omega}^\bullet_N$, which can be exponentiated to an action of the Harish-Chandra pair $(W_N, \operatorname{Aut} \mathcal{O}_N)$. Furthermore, because the $(W_N, \operatorname{Aut} \mathcal{O}_N)$–action comes from the residues of vertex operators, it preserves the vertex algebra structure on V (so the group $\operatorname{Aut} \mathcal{O}_N$ is an example of the group of internal symmetries from §4.4). Recall from §6.1 that any smooth scheme S of dimension N carries a canonical $(W_N, \operatorname{Aut} \mathcal{O}_N)$–structure $\widehat{S}$. Applying the construction of §6.1, we attach to $\widehat{\Omega}^\bullet_N$ a $\mathcal{D}$–module on S. The sheaf of horizontal sections of this $\mathcal{D}$–module is a sheaf of vertex superalgebras (with a structure of complex) on S. This is the *chiral deRham complex* of S, introduced in [MSV]. For its applications to mirror symmetry and elliptic cohomology, see [BoL, Bo1].

Now consider a purely bosonic analogue of this construction (i.e., replace $\widehat{\Omega}_N$ with $\widehat{H}_N$). The Lie algebra $U(\widehat{H}_N)$ of Fourier coefficients of the fields from $\widehat{H}_N$ has a filtration $A^{\leq i}_N, i \geq 0$, by the order of $a_{i,n}$'s. Note that $A^{\leq 1}_N$ is a Lie subalgebra of $U(\widehat{H}_N)$ and A^0_N is its (abelian) ideal. Denote by $\mathcal{T}_N$ the Lie algebra $A^{\leq 1}_N / A^0_N$. Intuitively, $U(\widehat{H}_N)$ is the algebra of differential operators on the topological vector space $\mathbb{C}((t))^N$ with the standard filtration, $A^{(0)}$ (resp., $\mathcal{T}_N$) is the space of functions (resp., the Lie algebra of vector fields) on it, and $\widehat{H}_N$ is the module of "delta–functions supported on $\mathbb{C}[[t]]^N$".

In contrast to the finite-dimensional case, the extension

$$0 \to A^0_N \to A^{\leq 1}_N \to \mathcal{T}_N \to 0$$

does not split [FF2]. Because of that, the naive map $W_N \to U(\widehat{H}_N)$ sending $f(t_i)\partial_{t_j}$ to $\operatorname{Res} : f(a^*_i(z))a_j(z) :$ is not a Lie algebra homomorphism. Instead, we obtain an extension

$$0 \to \Omega^1_N / d\mathcal{O}_N \to \widetilde{W}_N \to W_N \to 0 \tag{23}$$

of W_N by its module $\Omega^1_N / d\mathcal{O}_N$, which is embedded into $A^{\leq 1}_N$ by the formula $g(t_i)dt_j \mapsto \operatorname{Res} g(a^*_i(z))\partial_z a^*_j(z)$. If the sequence (23) were split, we would obtain a $(W_N, \operatorname{Aut} \mathcal{O}_N)$–action on $\widehat{H}_N$ and associate to $\widehat{H}_N$ a $\mathcal{D}$–module on S in the same way as above.

But in reality the extension (23) does not split for $N > 1$ (note that $\Omega^1_1/d\mathcal{O}_1 = 0$). Hence in general we only have an action on $\widehat{H}_N$ of the Harish-Chandra pair $(\widetilde{W}_N, \widetilde{\mathrm{Aut}\mathcal{O}_N})$, where $\widetilde{\mathrm{Aut}\mathcal{O}_N}$ is an extension of $\mathrm{Aut}\,\mathcal{O}_N$ by the additive group $\Omega^1_N/d\mathcal{O}_N$. This action again preserves the vertex algebra structure on V. To apply the construction of §6.1, we need a lifting of the $(W_N, \mathrm{Aut}\,\mathcal{O}_N)$–structure $\widehat{S}$ on S to a $(\widetilde{W}_N, \widetilde{\mathrm{Aut}\mathcal{O}_N})$–structure. Such liftings form a *gerbe* on S (in the analytic topology) with the lien $\Omega^1_S/d\mathcal{O}_S \simeq \Omega^2_{S,\mathrm{cl}}$, the sheaf of closed holomorphic two-forms on S. The equivalence class of this gerbe in $H^2(S, \Omega^2_{S,\mathrm{cl}})$ is computed in [GMS]. If this class equals 0, then we can lift $\widehat{S}$ to an $(\widetilde{W}_N, \widetilde{\mathrm{Aut}\mathcal{O}_N})$–structure $\widetilde{S}$ on S, and the set of isomorphism classes of such liftings is an $H^1(S, \Omega^2_{S,\mathrm{cl}})$–torsor. Applying the construction of §6.1, we can then attach to $\widehat{H}_N$ a $\mathcal{D}$–module on S. The sheaf of horizontal sections of this $\mathcal{D}$–module is a sheaf of vertex algebras on S. This is the sheaf of *chiral differential operators* on S (corresponding to $\widetilde{S}$) introduced in [MSV, GMS]. As shown in [GMS], the construction becomes more subtle in Zariski topology; in this case one naturally obtains a gerbe with the lien $\Omega^2_S \to \Omega^3_{S,\mathrm{cl}}$.

In the case when S is the flag manifold G/B of a simple Lie algebra $\mathfrak{g}$, one can construct the sheaf of chiral differential operators in a more direct way. The action of $\mathfrak{g}$ on the flag manifold G/B gives rise to a homomorphism $\mathfrak{g}((t)) \to \mathcal{T}_N$, where $N = \dim G/B$. It is shown in [Wa, FF1, FF2] that it can be lifted to a homomorphism $\widehat{\mathfrak{g}} \to A^{\leq 1}_N$ of level $-h^\vee$ (Wakimoto realization). Moreover, it gives rise to a homomorphism of vertex algebras $V_{-h^\vee}(\mathfrak{g}) \to H_N$. Hence we obtain an embedding of the constant subalgebra $\mathfrak{g}$ of $\widehat{\mathfrak{g}}$ into $\widetilde{W}_N$, and a $(\mathfrak{g}, B)$–action on $\widehat{H}_N$. On the other hand, G/B has a natural $(\mathfrak{g}, B)$–structure, namely G. Applying the localization construction to the Harish-Chandra pair $(\mathfrak{g}, B)$ instead of $(\widetilde{W}_N, \widetilde{\mathrm{Aut}\mathcal{O}_N})$, we obtain the sheaf of chiral differential operators on G/B – note that it is uniquely defined in this case (see [MSV]).

7. CHIRAL ALGEBRAS

In §4.2, we gave a coordinate independent description of the vertex operation $Y(\cdot, z)$. The formalism of chiral algebras invented by Beilinson and Drinfeld [BD3] (see [G] for a review) is based on a coordinate–free realization of the operator product expansion (see §2.6). In the definition of $\mathcal{Y}_x$ we acted by $Y(A, z)$ on B, placed at the fixed point $x \in X$. In order to define the OPE invariantly, we need to let x move along X as well. This is one as follows. Recall that to each conformal vertex algebra V we have assigned a vector bundle $\mathcal{V}$ on any smooth curve X. Choose a formal coordinate z at $x \in X$, and use it to trivialize $\mathcal{V}|_{D_x}$. Then $\mathcal{V} \boxtimes \mathcal{V}(\infty\Delta)$ is a sheaf on $D^2_x = \mathrm{Spec}\,\mathbb{C}[[z, w]]$ associated to the $\mathbb{C}[[z, w]]$–module $V \otimes V[[z, w]][(z-w)^{-1}]$. Let $\Delta_!\mathcal{V}$ be a sheaf on D^2_x associated to the $\mathbb{C}[[z, w]]$–module $V[[z, w]][(z-w)^{-1}]/V[[z, w]]$. Recall from §4.2 that we have a flat connection on $\mathcal{V}$. Hence $\mathcal{V}$ is a vector bundle with a flat connection, i.e., a $\mathcal{D}$–module, on D_x. The sheaves $\mathcal{V} \boxtimes \mathcal{V}(\infty\Delta)$ and $\Delta_!\mathcal{V}$ are not vector bundles, but they have natural structures of $\mathcal{D}$–modules on D_x^2 (independent of the choice of z). Because of that, we switch to the language of $\mathcal{D}$–modules. The following result is proved in the same way as in Theorem 4.1 (see also [HL]).

THEOREM 7.1. — *Define a map* $\mathcal{Y}_x^{(2)} : \mathcal{V} \boxtimes \mathcal{V}(\infty\Delta) \to \Delta_! \mathcal{V}$ *by the formula*

$$\mathcal{Y}_x^{(2)}(f(z,w)A \boxtimes B) = f(z,w)Y(A, z-w)\cdot B \quad \operatorname{mod} V[[z,w]].$$

Then $\mathcal{Y}_x^{(2)}$ *is a homomorphism of* $\mathcal{D}$*–modules, which is independent of the choice of the coordinate* z*.*

At this point it is convenient to pass to the right $\mathcal{D}$–module on X corresponding to the left $\mathcal{D}$–module $\mathcal{V}$. Recall that if $\mathcal{F}$ is a left $\mathcal{D}$ module on a smooth variety Z, then $\mathcal{F}^r := \mathcal{F} \otimes \Omega_Z$, where Ω_Z denotes the canonical sheaf on Z, is a right $\mathcal{D}$–module on Z. Denote $\Delta : X \hookrightarrow X^2, j : X^2 \backslash \Delta(X) \hookrightarrow X^2$.

COROLLARY 7.2. — *The vertex algebra structure on* V *gives rise to a homomorphism of right* $\mathcal{D}$*–modules on* X^2*,* $\mu : j^* j_* \mathcal{V}^r \boxtimes \mathcal{V}^r \to \Delta_! \mathcal{V}^r$ *satisfying the following conditions:*

- (skew-symmetry) $\mu(f(x,y)A \boxtimes B) = -\sigma_{12} \circ \mu(f(y,x)B \boxtimes A)$;
- (Jacobi identity) $\mu(\mu(f(x,y,z)A \boxtimes B) \boxtimes C) + \sigma_{123} \circ \mu(\mu(f(y,z,x)B \boxtimes C) \boxtimes A)$
$$+\sigma_{123}^{-1} \circ \mu(\mu(f(z,x,y)C \boxtimes A) \boxtimes B) = 0.$$
- (vacuum) *we are given an embedding* $\Omega \hookrightarrow \mathcal{V}^r$ *compatible with the natural homomorphism* $j_* j^* \Omega \boxtimes \Omega \to \Delta_! \Omega$*.*

Beilinson and Drinfeld define a *chiral algebra* on a curve X as a right $\mathcal{D}$–module $\mathcal{A}$ on X equipped with a homomorphism $j_* j^* \mathcal{A} \boxtimes \mathcal{A} \to \Delta_! \mathcal{A}$ satisfying the conditions of Corollary 7.2 (see [BD3, G]).

The axioms of chiral algebra readily imply that for any chiral algebra $\mathcal{A}$, its deRham cohomology $H^\bullet_{\mathrm{dR}}(\Sigma, \mathcal{A})$ is a (graded) Lie algebra for any open $\Sigma \subset X$. In particular, for an affine open $\Sigma \subset X$ we obtain the result mentioned in §5.3 that $H^0_{\mathrm{dR}}(\Sigma, \mathcal{V}^r)$ is a Lie algebra.

Define the right $\mathcal{D}$–module $\mathcal{V}_2^r$ on X^2 as the kernel of the homomorphism μ, and let $\mathcal{V}_2 = \mathcal{V}_2^r \otimes \Omega_{X^2}^{-1}$ be the corresponding left $\mathcal{D}$–module. There is a canonical isomorphism $H(X,x,V) \simeq H^2_{\mathrm{dR}}(X^2, \mathcal{V}_2^r)$ (see [G], Prop. 5.1). It is dual to the isomorphism $C(X,x,V) \simeq \Gamma_\nabla(X^2, \mathcal{V}_2^*)$ mentioned in §5.4.

Beilinson and Drinfeld give a beautiful description of chiral algebras as *factorization algebras*. By definition, a factorization algebra is a collection of quasicoherent $\mathcal{O}$–modules $\mathcal{F}_n$ on $X^n, n \geq 1$, satisfying a factorization condition, which essentially means that the fiber of $\mathcal{F}_n$ at $(x_1, \ldots, x_n)$ is isomorphic to $\otimes_{s \in S} (\mathcal{F}_1)_s$, where $S = \{x_1, \ldots, x_n\}$. For example, $j^* \mathcal{F}_2 = j^*(\mathcal{F}_1 \boxtimes \mathcal{F}_1)$ and $\Delta^* \mathcal{F}_2 = \mathcal{F}_1$. Intuitively, such a collection may be viewed as an $\mathcal{O}$–module on the *Ran space* $\mathcal{R}(X)$ of all finite non-empty subsets of X. In addition, it is required that $\mathcal{F}_1$ has a global section (unit) satisfying natural properties. It is proved in [BD3] that under these conditions each $\mathcal{F}_n$ is automatically a left $\mathcal{D}$–module. Moreover, the right $\mathcal{D}_X$–module $\mathcal{F}_1^r = \mathcal{F}_1 \otimes \Omega_X$ acquires a canonical structure of chiral algebra, with μ given by the composition $j^* j_* \mathcal{F}_1^r \boxtimes \mathcal{F}_1^r = j^* j_* \mathcal{F}_2^r \to \Delta_! \Delta^! \mathcal{F}_2^r = \Delta_! \mathcal{F}^r$. In fact, there is an equivalence between the categories of factorization algebras and chiral algebras on X [BD2].

For a chiral algebra $\mathcal{V}^r$, the corresponding sheaf $\mathcal{F}_n$ on X^n is the sheaf $\mathcal{V}_n$ mentioned in §5.4 (its dual is the sheaf of chiral correlation functions). Using these sheaves, Beilinson and Drinfeld define "chiral homology" $H_i^{\mathrm{ch}}(X, \mathcal{V}^r), i \geq 0$, of a chiral algebra $\mathcal{V}^r$. Intuitively, this is (up to a change of cohomological dimension) the deRham cohomology of the factorization algebra corresponding to $\mathcal{V}$, considered as a $\mathcal{D}$–module on the Ran space $\mathcal{R}(X)$. The 0th chiral homology of $\mathcal{V}^r$ is isomorphic to $H^n_{\mathrm{dR}}(X^n, \mathcal{V}_n \otimes \Omega_{X^n})$ for all $n > 1$ and hence to the space of coinvariants $H(X, x, V)$. The full chiral homology functor may therefore be viewed as a derived coinvariants functor.

As an example, we sketch the Beilinson-Drinfeld construction of the factorization algebra corresponding to the Kac-Moody vertex algebra $V_0(\mathfrak{g})$ [BD2] (see also [G]). Let Gr_n be the ind-scheme over X^n whose fiber $Gr_{x_1,\dots,x_n}$ over $(x_1, \dots, x_n) \in X$ is the moduli space of pairs $(\mathcal{P}, t)$, where $\mathcal{P}$ is a G–bundle on X, and t is its trivialization on $X \backslash \{x_1, \dots, x_n\}$ (e.g., the fiber of Gr_1 over any $x \in X$ is isomorphic to the affine Grassmannian $G(\mathcal{K})/G(\mathcal{O})$). Let e be the section of Gr_n corresponding to the trivial G–bundle, and $\mathcal{A}_{x_1 \dots, x_n}$ be the space of delta–functions on $Gr_{x_1,\dots,x_n}$ supported at e. These are fibers of a left $\mathcal{D}$–module $\mathcal{A}_n$ on X^n. Claim: $\{\mathcal{A}_n\}$ form a factorization algebra. The factorization property for $\{\mathcal{A}_n\}$ follows from the factorization property of the ind-schemes Gr_n: namely, $Gr_{x_1,\dots,x_n} = \prod_{s \in S} Gr_s$, where $S = \{x_1, \dots, x_n\}$. The chiral algebra corresponding to the factorization algebra $\{\mathcal{A}_n\}$ is $\mathcal{V}_0(\mathfrak{g})^r$. To obtain the factorization algebra corresponding to $\mathcal{V}_k(\mathfrak{g})^r$ with $k \neq 0$, one needs to make a twist by a line bundle on Gr_1.

A spectacular application of the above formalism is the Beilinson-Drinfeld construction of the *chiral Hecke algebra* associated to a simple Lie algebra $\mathfrak{g}$ and a negative integral level $k < -h^\vee$. The corresponding vertex algebra $S_k(\mathfrak{g})$ is a module over $\widehat{\mathfrak{g}} \times {}^L G$: $S_k(\mathfrak{g}) = \oplus_{\lambda \in {}^L P^+} M_{\lambda,k} \otimes V_\lambda^*$, where V_λ is a finite-dimensional ${}^L G$–module with highest weight λ, and $M_{\lambda,k}$ is the irreducible highest weight $\widehat{\mathfrak{g}}$–module of level k with highest weight $(k + h^\vee)\lambda$. Here ${}^L G$ is the group of adjoint type corresponding to ${}^L\mathfrak{g}$, and ${}^L P^+$ is the set of its dominant weights; we identify the dual $\mathfrak{h}^*$ of the Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} = {}^L\mathfrak{h}^*$ using the normalized invariant inner product on $\mathfrak{g}$. In particular, $S_k(\mathfrak{g})$ is a module over $V_k(\mathfrak{g}) = M_{0,k}$. Note that using results of [KL] one can identify the tensor category of ${}^L G$–modules with a subcategory of the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$–modules of level k, so that V_λ corresponds to $M_{\lambda,k}$. The analogues of $S_k(\mathfrak{g})$ when $\mathfrak{g}$ is replaced by an abelian Lie algebra with a non-degenerate inner product are lattice vertex algebras V_L from §3.4.

As a chiral algebra, the chiral Hecke algebra is the right $\mathcal{D}$–module $\mathcal{S}_k(\mathfrak{g})^r$ on X corresponding to the twist of $S_k(\mathfrak{g})$ by $\widehat{X}$. The corresponding factorization algebra is constructed analogously to the above construction of $V_k(\mathfrak{g})$ using certain irreducible $\mathcal{D}$–modules on Gr_1 corresponding to V_λ; see [G] for the construction in the abelian case (it is also possible to construct a vertex algebra structure on $S_k(\mathfrak{g})$ directly, using results of [KL]). Moreover, for any flat ${}^LG$–bundle $\mathcal{E}$ on X, the twist of $\mathcal{S}_k(\mathfrak{g})^r$ by $\mathcal{E}$ with respect to the ${}^LG$–action on $S_k(\mathfrak{g})$ is also a chiral algebra on X. Conjecturally, the complex of sheaves on $\mathfrak{M}_G(X)$ obtained by localization of this chiral algebra (whose fibers are its chiral homologies) is closely related to the automorphic $\mathcal{D}$–module that should be attached to $\mathcal{E}$ by the geometric Langlands correspondence.

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